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Journal of Approximation Theory

Journal of Approximation Theory 161 (2009) 142-173

www.elsevier.com/locate/jat

Interpolation approximations based on Gauss–Lobatto–Legendre–Birkhoff quadrature

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> Received 30 May 2007; received in revised form 31 July 2008; accepted 7 August 2008 Available online 13 November 2008

> > Communicated by Jóseph Szabados

Abstract

We derive in this paper the asymptotic estimates of the nodes and weights of the *Gauss–Lobatto–Legendre–Birkhoff* (GLLB) quadrature formula, and obtain optimal error estimates for the associated GLLB interpolation in Jacobi weighted Sobolev spaces. We also present a user-oriented implementation of the pseudospectral methods based on the GLLB quadrature nodes for Neumann problems. This approach allows an *exact* imposition of Neumann boundary conditions, and is as efficient as the pseudospectral methods based on Gauss–Lobatto quadrature for PDEs with Dirichlet boundary conditions. (© 2008 Elsevier Inc. All rights reserved.

Keywords: GLLB quadrature rule; Collocation method; Neumann problems; Asymptotic estimates; Interpolation errors

1. Introduction

In a recent work, Ezzirani and Guessab [8] proposed a fast algorithm for computing the nodes and weights of some Gauss–Birkhoff type quadrature formulae (cf. [19,20]) with mixed boundary conditions. One special rule of particular importance, termed as the

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Gauss-Lobatto-Legendre-Birkhoff (GLLB) quadrature, takes the form

$$\int_{-1}^{1} \phi(x) dx \sim \phi'(-1)\omega_{-} + \sum_{j=1}^{N-1} \phi(x_j)\omega_j + \phi'(1)\omega_{+}, \qquad (1.1)$$

which is exact for all polynomials of degree 2N - 1, and whose interior nodes are zeros of the quasi-orthogonal polynomial (cf. [25]) formed by a linear combination of the Jacobi polynomials $J_{N-1}^{(2,2)}(x)$ and $J_{N-3}^{(2,2)}(x)$. Motivated by [8], our first intention is to study the asymptotic behaviors of the nodes and weights of (1.1). Two important results are

$$x_j \cong \cos \frac{(2j+1/2)\pi}{2N+1}, \quad 1 \le j \le N-1,$$
 (1.2)

and

$$\omega_j \cong \frac{\pi}{N} \sin \theta_j, \quad \theta_j = \arccos x_j, \quad 1 \le j \le N - 1.$$
 (1.3)

With the aid of these estimates, we are able to analyze the GLLB interpolation errors:

- . . -

$$\begin{aligned} \|\mathcal{I}_{N}u - u\|_{L^{2}(I)} + N^{-1} \|(1 - x^{2})^{1/2} (\mathcal{I}_{N}u - u)'\|_{L^{2}(I)} \\ &\leq c N^{-m} \|(1 - x^{2})^{(m-2)/2} u^{(m)}\|_{L^{2}(I)}, \end{aligned}$$
(1.4)

where I = (-1, 1) and \mathcal{I}_N is the Gauss–Birkhoff interpolation operator associated with the GLLB points. Similar to the approximation results on the Legendre–Gauss–Lobatto interpolation obtained in [15,16], the estimate (1.4) is optimal.

The GLLB quadrature formula involves derivative values $\phi'(\pm 1)$ at the endpoints. Hence its nodes can be a natural choice of the preassigned interpolation points for many important problems, such as numerical integration and integral equations with the data of derivatives at the endpoints, etc. In particular, it plays an important role in spectral approximations of second-order elliptic problems with Neumann boundary conditions [8]. As we know, the numerical solutions of such problems given by the commonly used Galerkin method with any fixed mode N, do not fulfill the Neumann boundary conditions exactly. Thereby, we prefer to collocation methods oftentimes, whose solutions satisfy the physical boundary conditions. Whereas, in a collocation method, a proper choice of the collocation points is crucial in terms of accuracy, stability and ease of treating the boundary conditions (BCs), especially, the treatment of BCs is even more critical due to the global feature of spectral method [18,21]. The GLLB collocation approximation takes the boundary conditions into account in such a way that the resulting collocation systems have diagonal mass matrices, and therefore leads to explicit time discretization for time-dependent problems [8]. However, like the collocation method based on Gauss–Lobatto points [17,5,2,13, 3,24,9], the GLLB differentiation matrices are ill-conditioned. Thus a suitable preconditioned iteration solver is preferable in actual computations. We construct in this paper an efficient finitedifference preconditioning for one-dimensional second-order elliptic equations. We also present a user-oriented implementation and an error analysis of the GLLB pseudospectral methods based on variational formulations.

The rest of the paper is organized as follows. In Section 2, we introduce the GLLB quadrature formula, and then investigate the asymptotic behaviors of its nodes and weights in Section 3. With the aid of asymptotic estimates and some other approximation results, we are able to analyze the GLLB interpolation errors in Section 4. We give a user-oriented description of the

implementation of the GLLB collocation method, and present some illustrative numerical results in Section 5. The final section is for some concluding remarks.

2. The GLLB quadrature formula

In this section, we introduce the Gauss-Lobatto-Legendre-Birkhoff quadrature formula.

2.1. Preliminaries

We start with some notations to be used in the subsequent sections.

2.1.1. Notations

- Let I := (-1, 1), and $\chi(x) \in L^1(I)$ be a generic positive weight function defined in I. For any integer $r \ge 0$, the weighted Sobolev space $H_{\chi}^r(I)$ is defined as usual with the inner product, semi-norm and norm denoted by $(u, v)_{r,\chi}, |v|_{r,\chi}$ and $||v||_{r,\chi}$, respectively. In particular, $L_{\chi}^2(I) = H_{\chi}^0(I), (u, v)_{\chi} = (u, v)_{0,\chi}$ and $||v||_{\chi} = ||v||_{0,\chi}$. For any real $r > 0, H_{\chi}^r(I)$ and its norm are defined by space interpolation as in Admas [1]. In cases where no confusion would arise, χ may be dropped from the notations, whenever $\chi \equiv 1$.
- Let $\omega^{\alpha}(x) = (1-x^2)^{\alpha}$, $x \in I$ with $\alpha > -1$ be the Gegenbauer weight function. In particular, denote $\omega(x) = 1 x^2$.
- Let \mathbb{N} be the set of all non-negative integers. For any $N \in \mathbb{N}$, let \mathbb{P}_N be the set of all algebraic polynomials of degree at most N. Without loss of generality, we assume that $N \ge 4$ throughout this paper.
- Denote by c a generic positive constant, which is independent of any function and N (mode in series expansion).
- For two sequences $\{z_n\}$ and $\{w_n\}$ with $w_n \neq 0$, the expression $z_n \sim w_n$ means $|z_n|/|w_n| \rightarrow c \neq 0$ as $n \rightarrow \infty$. In particular, $z_n \cong w_n$ means $z_n/w_n \approx 1$ for $n \gg 1$, or $z_n/w_n \rightarrow 1$ for $n \rightarrow \infty$.
- For simplicity, we sometimes denote $\partial_x^l v = \frac{d^l v}{dx^l} = v^{(l)}$, for any integer $l \ge 1$.

2.1.2. Jacobi polynomials

We recall some relevant properties of the Jacobi polynomials, denoted by $J_n^{(\alpha,\beta)}(x)$, $x \in I$ with $\alpha, \beta > -1$ and $n \ge 0$, which are mutually orthogonal with respect to the Jacobi weight $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$:

$$\int_{-1}^{1} J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) \mathrm{d}x = \gamma_n^{(\alpha,\beta)} \delta_{mn}, \qquad (2.1)$$

where δ_{mn} is the Knonecker symbol, and

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$
(2.2)

In the subsequent discussions, we will mainly use two special types of Jacobi polynomials. The first one is $J_n^{(2,2)}(x)$, defined by the three-term recursive relation (see, e.g., Szegö [23]):

$$n(n+4)J_n^{(2,2)}(x) = (n+2)(2n+3)xJ_{n-1}^{(2,2)}(x) - (n+1)(n+2)J_{n-2}^{(2,2)}(x),$$

$$J_0^{(2,2)}(x) = 1, \qquad J_1^{(2,2)}(x) = 3x, \quad x \in [-1, 1].$$
(2.3)

The coefficient of leading term of $J_n^{(2,2)}(x)$ is

$$K_n^{(2,2)} = \frac{(2n+4)!}{2^n n! (n+4)!}.$$
(2.4)

Moreover, we have

$$J_n^{(2,2)}(-x) = (-1)^n J_n^{(2,2)}(x), \qquad J_n^{(2,2)}(1) = \frac{(n+1)(n+2)}{2}.$$
(2.5)

The corresponding monic polynomial is defined by dividing the leading coefficient in (2.4):

$$P_n(x) := \lambda_n J_n^{(2,2)}(x) \quad \text{with } \lambda_n = \left(K_n^{(2,2)}\right)^{-1}.$$
(2.6)

As a direct consequence of (2.3) and (2.6),

$$P_{n+1}(x) = x P_n(x) - a_n P_{n-1}(x) \quad \text{with } a_n = \frac{n(n+4)}{(2n+3)(2n+5)},$$
(2.7)

where $P_{-1} = 0$ and $P_0 = 1$. By (2.1)–(2.2), the normalized Jacobi polynomials

$$\widetilde{P}_n(x) := \widetilde{\lambda}_n J_n^{(2,2)}(x) \quad \text{with } \widetilde{\lambda}_n = \left(\gamma_n^{(2,2)}\right)^{-1/2},$$
(2.8)

satisfy $\|\widetilde{P}_n\|_{\omega^2}^2 = 1.$

We will also utilize the Legendre polynomials, denoted by $L_n(x)$, which are mutually orthogonal with respect to the unit weight $\omega = 1$. Note that the constant of orthogonality and leading coefficient respectively are

$$\gamma_n^{(0,0)} = \|L_n\|^2 = \frac{2}{2n+1}, \qquad K_n^{(0,0)} = \frac{(2n)!}{2^n (n!)^2}.$$
 (2.9)

Recall that $L_n(\pm 1) = (\pm 1)^n$ and $L'_n(\pm 1) = \frac{1}{2}(-1)^{n-1}n(n+1)$.

2.2. GLLB quadrature formula

The GLLB quadrature formula can be derived from Theorem 4.2 of Ezzirani and Guessab [8], which belongs to a Birkhoff type rule (see, e.g., [19,20]).

Theorem 2.1. Let $\{P_n\}$ be the monic Jacobi polynomials defined in (2.6). Consider the quasiorthogonal polynomial:

$$Q_{N-1}(x) = P_{N-1}(x) + b_N P_{N-3}(x), \quad N \ge 2,$$
(2.10)

where

$$b_N = -\frac{(N-1)(N+2)(2N^2+2N+3)}{(2N-1)(2N+1)(2N^2-2N-3)} + \frac{\sqrt{12(N-1)(N+2)(N^2+N-3)}}{(2N-1)(2N^2-2N-3)}$$

$$\coloneqq -b_N^I + b_N^{\mathbb{I}}.$$
 (2.11)

Then we have that

① For $N \ge 4$,

$$\frac{1}{4} - \frac{1}{N} < -b_N < \frac{(N+2)(N+3)}{(2N-1)(2N+1)}.$$
(2.12)

The N-1 zeros of Q_{N-1}(x), denoted by {x_j}^{N-1}_{j=1}, are distinct, real and all located in (-1, 1).
 There exists a unique set of weights {ω_j}^N_{j=0} such that

$$\int_{-1}^{1} \phi(x) dx = \phi'(-1)\omega_0 + \sum_{j=1}^{N-1} \phi(x_j)\omega_j + \phi'(1)\omega_N \coloneqq \mathcal{S}_N[\phi], \quad \forall \phi \in \mathbb{P}_{2N-1}.(2.13)$$

④ The interior weights are all positive and explicitly expressed by

$$\omega_j = \frac{A_N}{P_{N-2}(x_j)Q'_{N-1}(x_j)} \frac{1}{(1-x_j^2)^2}, \quad 1 \le j \le N-1,$$
(2.14)

with

$$A_{N} = \left(1 - \frac{b_{N}}{a_{N-2}}\right) \frac{\lambda_{N-2}^{2}}{\tilde{\lambda}_{N-2}^{2}},$$
(2.15)

where the constants a_{N-2} , b_N , λ_{N-2} and $\tilde{\lambda}_{N-2}$ are defined in (2.7), (2.11), (2.6) and (2.8), respectively.

Proof. The proof is based on a reassembly and extension of some relevant results in [8]. For clarity, we first present a one-to-one correspondence between the notations of [8] and those of this paper:

$$ln = N - 1, \quad d\hat{\sigma} = (1 - x^2)^2 dx, \quad \hat{\pi}_k = P_k, \quad \widetilde{\pi}_k = \widetilde{P}_k, \quad q_{n,2} = Q_{N-1}, \quad (2.16)$$

$$\beta_k = a_k, \quad \hat{s}_{N-1} = b_N, \quad \beta_{n-1} = a_{N-2} - b_N, \quad J_n^*(\hat{\sigma}) = \mathbb{J}_{N-1}.$$

We next derive the expression of b_N . By Theorem 4.2 and 4.3 of [8], we find that

$$a_{N-2} - b_N = \beta_{N-2}$$

= $\frac{-\sqrt{12(N-1)(N+2)(N^2+N-3)} + (N+2)(2N^2-4N+3)}{(2N-1)(2N^2-2N-3)}$

Hence, a direct computation leads to

 $b_N = a_{N-2} - \beta_{N-2} \stackrel{(2.7)}{=} \text{RHS of } (2.11).$

① The special values b_j (j = 1, 2, 3) can be computed directly via (2.11). To obtain the bounds for b_N with $N \ge 4$, we first prove that

$$-b_N < \frac{(N+2)(N+3)}{(2N-1)(2N+1)}.$$
(2.17)

Since $-b_N = b_N^I - b_N^{\mathbb{I}} < b_N^I$, it suffices to verify

$$W_1 := (N+2)(N+3)(2N^2 - 2N - 3) - (N-1)(N+2)(2N^2 + 2N + 3) > 0.$$

A direct calculation yields

 $W_1 = 2(N-3)(N+2)(2N+1).$

Hence (2.17) is valid for $N \ge 4$, and it remains to prove

$$-b_N > \frac{1}{N} - \frac{1}{4}.$$
 (2.18)

Clearly, $-\sqrt{N^2 + N - 3} > -\sqrt{N^2 + N - 2} = -\sqrt{(N - 1)(N + 2)}$ and $2N(N - 1) > 2N^2 - 2N - 3$. Therefore,

$$\begin{split} -b_N &> \frac{(N-1)(N+2)(2N^2+2N+3)-(2N+1)\sqrt{12}(N-1)(N+2)}{(2N-1)(2N+1)(2N^2-2N-3)} \\ &> \frac{(N-1)(N+2)\left[(2N^2+2N+3)-4(2N+1)\right]}{(2N-1)(2N+1)(2N^2-2N-3)} \\ &> \frac{(N-1)(N+2)(2N^2-6N-8)}{2N(N-1)(2N-1)(2N+1)} \\ &> \frac{(N+2)(N-4)(N+1)}{2N(2N-1)(2N+1)} \\ &> \frac{(N+2)(N-4)}{2N(2N-1)} > \frac{N-4}{4N} = \frac{1}{4} - \frac{1}{N}. \end{split}$$

Thus, a combination of (2.17) and (2.18) leads to (2.12).

⁽²⁾ Observe from (2.12) that for $N \ge 3$, $-b_N > 0$. Therefore, by Theorem 4.1 of [8], the quasi-orthogonal polynomial can be expressed as a determinant

$$Q_{N-1}(x) = P_{N-1}(x) + b_N P_{N-3}(x) = \det \left(x \mathbb{I}_{N-1} - \mathbb{J}_{N-1} \right),$$
(2.19)

where \mathbb{I}_{N-1} is an identity matrix, and \mathbb{J}_{N-1} is a symmetric tri-diagonal matrix of order N-1,

$$\mathbb{J}_{N-1} = \begin{bmatrix}
0 & \sqrt{a_1} & & & \\
\sqrt{a_1} & 0 & \sqrt{a_2} & & & \\
& \ddots & \ddots & \ddots & & \\
& & \sqrt{a_{N-3}} & 0 & \sqrt{a_{N-2} - b_N} \\
& & & & \sqrt{a_{N-2} - b_N} & 0
\end{bmatrix}$$
(2.20)

with a_n and b_N given in (2.7) and (2.11), respectively. Therefore, all the zeros of Q_{N-1} are real and distinct.

③ Theorem 4.2 of [8] reveals that with such a choice of b_N , all the zeros of Q_{N-1} are distributed in (-1, 1), and the quadrature formula (2.13) is unique. Moreover, the interior weights $\{\omega_j\}_{j=1}^{N-1}$ are all positive.

⁽⁴⁾ We now consider the expressions of the weights. By Formula (38) of [8],

$$(1 - x_j^2)^2 \omega_j = \frac{A_N^*}{K_N^*(x_j)}, \quad 1 \le j \le N - 1,$$
(2.21)

where $A_N^* = 1 - b_N / a_{N-2}$, and

$$K_N^*(x_j) = A_N^* \sum_{k=0}^{N-3} \widetilde{P}_k^2(x_j) + \widetilde{P}_{N-2}^2(x_j).$$
(2.22)

To simplify $K_N^*(x_j)$, we recall the Christoffel–Darboux formula (see, e.g., Szegö [23]):

$$\widetilde{P}_{n+1}'(x)\widetilde{P}_n(x) - \widetilde{P}_n'(x)\widetilde{P}_{n+1}(x) = \frac{d_{n+1}}{d_n}\sum_{k=0}^n \widetilde{P}_k^2(x) > 0, \quad \forall x \in [-1,1],$$
(2.23)

where d_k is the leading coefficient $\widetilde{P}_k(x)$, and by (2.6) and (2.8),

$$d_k = \frac{\lambda_k}{\lambda_k}, \qquad \widetilde{P}_k(x) = d_k P_k(x). \tag{2.24}$$

.

Hence,

$$\begin{aligned} Q'_{N-1}(x)\widetilde{P}_{N-2}(x) & -\widetilde{P}'_{N-2}(x)Q_{N-1}(x) \stackrel{(2.10)}{=} \left[P'_{N-1}(x)\widetilde{P}_{N-2}(x) - \widetilde{P}'_{N-2}(x)P_{N-1}(x) \right] \\ & - b_N \left[\widetilde{P}'_{N-2}(x)P_{N-3}(x) - P'_{N-3}(x)\widetilde{P}_{N-2}(x) \right] \\ \stackrel{(2.24)}{=} \frac{1}{d_{N-1}} \left[\widetilde{P}'_{N-1}(x)\widetilde{P}_{N-2}(x) - \widetilde{P}'_{N-2}(x)\widetilde{P}_{N-1}(x) \right] \\ & - \frac{b_N}{d_{N-3}} \left[\widetilde{P}'_{N-2}(x)\widetilde{P}_{N-3}(x) - \widetilde{P}'_{N-3}(x)\widetilde{P}_{N-2}(x) \right] \\ \stackrel{(2.23)}{=} \frac{1}{d_{N-2}} \sum_{k=0}^{N-2} \widetilde{P}_k^2(x) - \frac{b_N d_{N-2}}{d_{N-3}^2} \sum_{k=0}^{N-3} \widetilde{P}_k^2(x) \\ & = \frac{1}{d_{N-2}} \left\{ \left[1 - \frac{b_N d_{N-2}^2}{d_{N-3}^2} \right] \sum_{k=0}^{N-3} \widetilde{P}_k^2(x) + \widetilde{P}_{N-2}^2(x) \right\} \\ \stackrel{(2.26)}{=} \frac{1}{d_{N-2}} \left\{ \left[1 - \frac{b_N}{a_{N-2}} \right] \sum_{k=0}^{N-3} \widetilde{P}_k^2(x) + \widetilde{P}_{N-2}^2(x) \right\}. \end{aligned}$$

In the last step, we used the identity

$$a_{N-2} = \frac{\lambda_{N-3}^2}{\lambda_{N-2}^2} \frac{\lambda_{N-2}^2}{\lambda_{N-3}^2} = \frac{d_{N-3}^2}{d_{N-2}^2},$$
(2.26)

which can be verified directly by the definition of the constants. Thus, taking $x = x_j$ in (2.25) and using the fact: $Q_{N-1}(x_j) = 0$, $1 \le j \le N - 1$, gives

$$Q_{N-1}'(x_j)\widetilde{P}_{N-2}(x_j) \stackrel{(2.25)}{=} \frac{1}{d_{N-2}} \left\{ A_N^* \sum_{k=0}^{N-3} \widetilde{P}_k^2(x_j) + \widetilde{P}_{N-2}^2(x_j) \right\} = \frac{1}{d_{N-2}} K_N^*(x_j), \ (2.27)$$

which implies that

$$K_N^*(x_j) \stackrel{(2,24)}{=} d_{N-2}^2 Q'_{N-1}(x_j) P_{N-2}(x_j) \stackrel{(2,24)}{=} \frac{\widetilde{\lambda}_{N-2}^2}{\lambda_{N-2}^2} Q'_{N-1}(x_j) P_{N-2}(x_j).$$

Plugging it into (2.21) leads to (2.14)–(2.15). \Box

- **Remark 2.1.** ① The choice of b_N so that all zeros of $Q_{N-1}(x)$ are in (-1, 1), is unique, and thereby uniquely determines the quadrature rule. It is seen from (2.12) that b_N is uniformly bounded with $-b_N \cong 1/4$, whose behavior is depicted in Fig. 3.1 (left).
- (2) The formulae (2.19)–(2.20) indicate that the quadrature nodes $\{x_j\}_{j=1}^{N-1}$ are eigenvalues of the symmetric tridiagonal matrix \mathbb{J}_{N-1} , and therefore, can be evaluated by using some standard methods (e.g., the QR-algorithm) as with the classical Gauss quadrature (see, e.g., [11]).



Fig. 3.1. Left: the constant $-b_N$ and its bounds (cf. (2.10)–(2.11)) against N. Right: asymptotic behavior of Δ_M^{\min} and Δ_N^{\max} against N (right).

Making use of Formula (41) of [8], the weights $\{\omega_j\}_{j=1}^{N-1}$ can be computed from the first

- component of the orthonormal eigenvectors of \mathbb{J}_{N-1} . (a) Alternative to the eigen-method, the nodes $\{x_j\}_{j=1}^{N-1}$ can be located by a root-finding method, say Newton-Raphson iteration, which turns out to be more efficient for a quadrature rule of higher order. A good initial approximation might be $\{x_j^{(0)}\}_{j=1}^{N-1}$ given in (3.24). Accordingly, the weights $\{\omega_j\}_{j=1}^{N-1}$ can be computed via the compact formulae (2.14)–(2.15).
- ④ It is worthwhile to point out that the quadrature nodes and weights are symmetric

$$x_j + x_{N-j} = 0, \quad \omega_j = \omega_{N-j}, \quad j = 1, 2, \dots, N-1.$$
 (2.28)

Therefore, the computational cost can be halved.

(5) Let h_0 and h_N be the Lagrangian polynomials given in (5.6) of this paper. The boundary weights

$$\omega_0 = \int_{-1}^1 h_0(x) \mathrm{d}x, \qquad \omega_N = \int_{-1}^1 h_N(x) \mathrm{d}x.$$

One verifies that $\omega_0 = -\omega_N$, and an explicit evaluation of the above integrals by using the relevant properties of Jacobi polynomials leads to the formula for ω_0 in Theorem 4.2 of [8].

The GLLB quadrature formula (2.13) enjoys the same degree of exactness as the Gauss-Legendre-Lobatto (GLL) quadrature (with N + 1 nodes), but in contrast to GLL, GLLB includes the boundary derivative values $\phi'(\pm 1)$, which is thereby suitable for *exactly* imposing Neuman boundary conditions (see Section 5).

3. Asymptotic properties of the nodes and weights

In this section, we make a quantitative asymptotic estimate of the nodes and weights in the GLLB quadrature formula (2.13). These properties will play an essential role in our subsequent analysis of the GLLB interpolation errors.

3.1. Asymptotic property of the nodes

We start with the interlacing property. Theorem 4.1 of [20] reveals that the zeros of Q_{N-1} interlace with those of Q_{N-2} . The following theorem indicates that the zeros of Q_{N-1} also interlace with those of \widetilde{P}_{N-2} , which can be proved in a fashion similar to that in [20]. For integrity, we provide the proof below.

Theorem 3.1. Let $\{x_j\}_{j=1}^{N-1}$ and $\{y_k\}_{k=1}^{N-2}$ be the zeros of $Q_{N-1}(x)$ and $\widetilde{P}_{N-2}(x)$, respectively. Then we have

$$-1 = y_0 < x_1 < y_1 < x_2 < y_2 < \dots < y_{N-2} < x_{N-1} < 1 = y_{N-1}.$$
(3.1)

Proof. We take three steps to complete the proof.

Step I: Show that

$$Q_{N-1}(y_j)Q_{N-1}(y_{j+1}) < 0, \qquad j = 1, 2, \dots N - 3,$$
(3.2)

equivalently to say, between two consecutive zeros of $\widetilde{P}_{N-2}(x)$, there exists at least one zero of $Q_{N-1}(x)$. To justify this, we first deduce that

$$Q'_{N-1}(x)\widetilde{P}_{N-2}(x) - \widetilde{P}'_{N-2}(x)Q_{N-1}(x) > 0, \quad \forall x \in [-1, 1],$$
(3.3)

which is a consequence of (2.7), (2.25) and (2.12), since

$$d_{N-2} > 0, \qquad 1 - \frac{b_N}{a_{N-2}} > 1 + \left(\frac{1}{4} - \frac{1}{N}\right)\frac{1}{a_{N-2}} \ge 1, \quad \forall N \ge 4.$$

Thanks to $\widetilde{P}_{N-2}(y_k) = 0$, taking $x = y_j$, y_{j+1} in (3.3) yields

$$\widetilde{P}'_{N-2}(y_j)Q_{N-1}(y_j) < 0, \quad \widetilde{P}'_{N-2}(y_{j+1})Q_{N-1}(y_{j+1}) < 0, \tag{3.4}$$

which certainly implies

$$\widetilde{P}_{N-2}'(y_j)\widetilde{P}_{N-2}'(y_{j+1})\cdot Q_{N-1}(y_j)Q_{N-1}(y_{j+1}) > 0.$$

Therefore, to prove (3.2), it suffices to check that

$$\widetilde{P}_{N-2}'(y_j)\widetilde{P}_{N-2}'(y_{j+1}) < 0.$$
(3.5)

Notice that

$$\widetilde{P}_{N-2}(x) = d_{N-2} \prod_{k=1}^{N-2} (x - y_k),$$

and consequently,

$$\widetilde{P}_{N-2}'(y_j) = d_{N-2} \prod_{k=1}^{j-1} (y_j - y_k) \cdot \prod_{k=j+1}^{N-2} (y_j - y_k) = (-1)^{N-j-2} d_{N-2} \cdot C_1,$$

$$\widetilde{P}_{N-2}'(y_{j+1}) = (-1)^{N-j-3} d_{N-2} \cdot C_2,$$
(3.6)

where C_1 and C_2 are two positive constants. Hence

$$\widetilde{P}_{N-2}'(y_j)\widetilde{P}_{N-2}'(y_{j+1}) = (-1)^{2N-2j-5} \cdot d_{N-2}^2 C_1 C_2 < 0, \quad 1 \le j \le N-3.$$
(3.7)

This validates (3.5) and thereby (3.2) follows.

Step II: At this point, it remains to consider the possibility of possessing zeros of $Q_{N-1}(x)$ in subintervals $(y_{N-2}, 1)$ and $(-1, y_1)$. Clearly, by the first identity of (3.6), the largest zero y_{N-2} satisfies $\widetilde{P}'_{N-2}(y_{N-2}) > 0$. Thus, by (3.4),

$$Q_{N-1}(y_{N-2}) < 0. (3.8)$$

On the other hand, we have

$$Q_{N-1}(1) = P_{N-1}(1) + b_N P_{N-3}(1) > 0, (3.9)$$

which is due to a direct calculation using (2.5)–(2.6) and (2.12):

$$\frac{P_{N-1}(1)}{P_{N-3}(1)} + b_N = \frac{(N+2)(N+3)}{(2N-1)(2N+1)} + b_N > 0,$$

and $P_{N-3}(1) > 0$. The sign of (3.8)–(3.9) indicates that there is at least one zero of $Q_{N-1}(x)$ in the interval $(y_{N-2}, 1)$. Using the symmetry of the zeros (see [8,23]):

$$x_j = -x_{N-j}, \quad j = 1, \dots, N-1; \quad y_k = -y_{N-1-k}, \quad k = 1, \dots, N-2,$$
 (3.10)

we deduce that the interval $(-1, y_1)$ also contains at least one zero of $Q_{N-1}(x)$.

Final step: A combination of the previous statements reaches the conclusion: each of the N - 1 subintervals $\{(y_j, y_{j+1})\}_{j=0}^{N-2} (y_0 = -1, y_{N-1} = 1)$ contains at least one of the N - 1 zeros of $Q_{N-1}(x)$, and therefore there can only be a unique one in each subinterval. \Box

To visualize the above interlacing property, we denote

$$\Delta_N^{\min} = \min_{1 \le j \le N-1} (y_j - x_j), \qquad \Delta_N^{\max} = \max_{1 \le j \le N-1} (y_j - x_j), \quad \forall N \ge 4.$$
(3.11)

According to Theorem 3.1, we expect to see that

$$\Delta_N^{\max} > \Delta_N^{\min} > 0, \quad \forall N \ge 4, \quad \lim_{N \to \infty} \Delta_N^{\min} = \lim_{N \to \infty} \Delta_N^{\max} = 0, \tag{3.12}$$

which is illustrated by Fig. 3.1(right).

In the analysis of interpolation error, we need more precise asymptotic estimates of the GLLB quadrature nodes. For this purpose, hereafter, we assume that the zeros $\{x_j\}_{j=1}^{N-1}$ of $Q_{N-1}(x)$ are arranged in descending order. We make the change of variables

$$x = \cos \theta, \quad \theta \in [0, \pi], \qquad \theta_j = \cos^{-1} x_j, \quad j = 1, 2, \dots, N - 1.$$
 (3.13)

The main result is stated in the following theorem.

Theorem 3.2. Let $\{\theta_j\}_{j=1}^{N-1}$ be the same as in (3.13). Then

$$\theta_j \in I_j := (\widetilde{\theta}_{j-1}, \widetilde{\theta}_j) \subset (0, \pi), \quad 1 \le j \le N - 1,$$
(3.14)

where

$$\widetilde{\theta}_j = \frac{2j + \frac{3}{2}}{2N+1}\pi, \quad 0 \le j \le N-1.$$
(3.15)

In other words,

$$0 < \widetilde{\theta}_0 < \theta_1 < \widetilde{\theta}_1 < \theta_2 < \widetilde{\theta}_2 < \dots < \widetilde{\theta}_{N-2} < \theta_{N-1} < \widetilde{\theta}_{N-1} < \pi.$$
(3.16)

Proof. By the intermediate mean-value theorem, (3.14) is equivalent to

$$Q_{N-1}(\cos\tilde{\theta}_{j-1})Q_{N-1}(\cos\tilde{\theta}_{j}) < 0, \quad j = 1, 2, \dots, N-1.$$
 (3.17)

To prove this result, we first recall Formula (8.21.10) of Szegö [23]:

$$J_{n}^{(2,2)}(\cos\theta) = n^{-\frac{1}{2}}G(\theta)\cos\left(\left(n+\frac{5}{2}\right)\theta + \gamma\right) + O(n^{-\frac{3}{2}}), \quad \theta \in (0,\pi),$$
(3.18)

where

$$G(\theta) = \pi^{-\frac{1}{2}} \left(\sin \frac{\theta}{2} \right)^{-\frac{5}{2}} \left(\cos \frac{\theta}{2} \right)^{-\frac{5}{2}} = 4\sqrt{\frac{2}{\pi}} (\sin \theta)^{-\frac{5}{2}}, \quad \gamma = -\frac{5}{4}\pi.$$
(3.19)

Hence, using (2.6), (2.10) and (3.18) gives

$$Q_{N-1}(\cos\theta) \stackrel{(2.10)}{=} P_{N-1}(\cos\theta) + b_N P_{N-3}(\cos\theta) \stackrel{(2.6)}{=} \lambda_{N-1} J_{N-1}^{(2,2)}(\cos\theta) + b_N \lambda_{N-3} J_{N-3}^{(2,2)}(\cos\theta) \stackrel{(3.18)}{=} G(\theta) \left\{ (N-1)^{-\frac{1}{2}} \lambda_{N-1} \cos\left(\left(N+\frac{3}{2}\right)\theta + \gamma\right) + (N-3)^{-\frac{1}{2}} b_N \lambda_{N-3} \cos\left(\left(N-\frac{1}{2}\right)\theta + \gamma\right) \right\} + \left\{ \lambda_{N-1} O((N-1)^{-\frac{3}{2}}) + b_N \lambda_{N-3} O((N-3)^{-\frac{3}{2}}) \right\} := H_N(\theta) + R_N.$$
(3.20)

Notice that $\tilde{\theta}_j$ in (3.15) solves the equation

$$\left(N+\frac{1}{2}\right)\widetilde{\theta}_j+\gamma=\left(j-\frac{1}{2}\right)\pi,\quad 0\leq j\leq N-1,$$

which implies

$$\left(N+\frac{3}{2}\right)\widetilde{\theta}_{j}+\gamma=\left(j-\frac{1}{2}\right)\pi+\widetilde{\theta}_{j},\quad \left(N-\frac{1}{2}\right)\widetilde{\theta}_{j}+\gamma=\left(j-\frac{1}{2}\right)\pi-\widetilde{\theta}_{j}.$$

Therefore, taking $\theta = \tilde{\theta}_j$ in (3.20) and using trigonometric identities, gives

$$H_{N}(\widetilde{\theta}_{j}) = G(\widetilde{\theta}_{j}) \left\{ (N-1)^{-\frac{1}{2}} \lambda_{N-1} \cos\left(\left(j-\frac{1}{2}\right)\pi + \widetilde{\theta}_{j}\right) + (N-3)^{-\frac{1}{2}} b_{N} \lambda_{N-3} \cos\left(\left(j-\frac{1}{2}\right)\pi - \widetilde{\theta}_{j}\right) \right\}$$
$$= (-1)^{j} \left(\sin\widetilde{\theta}_{j}\right) G(\widetilde{\theta}_{j}) \left\{ (N-1)^{-\frac{1}{2}} \lambda_{N-1} - (N-3)^{-\frac{1}{2}} b_{N} \lambda_{N-3} \right\}$$
$$:= (-1)^{j} S_{N}(\widetilde{\theta}_{j}).$$
(3.21)

Since $-b_N > 0$, $\lambda_k > 0$ and $\theta_j \in (0, \pi)$, we have that $S_N(\tilde{\theta}_j) > 0$, and thereby

$$\operatorname{sgn}\left(H_N(\widetilde{\theta}_j)\right) = (-1)^j, \quad 0 \le j \le N - 1.$$
(3.22)

Moreover, by (2.12), (3.19) and (3.21),

$$\begin{aligned} |H_N(\widetilde{\theta}_j)| &\geq 2^{\frac{5}{2}} \pi^{-\frac{1}{2}} \left(\sin \widetilde{\theta}_j \right)^{-\frac{3}{2}} \left\{ (N-1)^{-\frac{1}{2}} \lambda_{N-1} + \left(\frac{1}{4} - \frac{1}{N} \right) (N-3)^{-\frac{1}{2}} \lambda_{N-3} \right\} \\ &\geq c N^{-\frac{1}{2}} (\lambda_{N-1} + \lambda_{N-3}), \end{aligned}$$



Fig. 3.2. Left: the error $\widetilde{\Delta}_N$ (solid line) and $N^{-1.9}$ ("o") against N. Right: asymptotic behavior of $\delta_N^{(1)}$ (solid line) and $\delta_N^{(2)}$ ("o") (cf. (3.43)) against N.

which, together with the fact $-b_N \cong \frac{1}{4}$, implies

$$|R_N| \le cN^{-\frac{3}{2}}(\lambda_{N-1} + \lambda_{N-3}) \le cN^{-1}|H_N(\widetilde{\theta}_j)|.$$
(3.23)

Hence, a combination of (3.20)- (3.23) leads to that

$$\operatorname{sgn}\left(Q_{N-1}(\cos\widetilde{\theta}_j)\right) = \operatorname{sgn}\left(H_N(\widetilde{\theta}_j)\right) = (-1)^j, \quad 0 \le j \le N-1, \quad \forall N \gg 1,$$

which implies (3.17) and therefore, there exists at least one zero of $Q_{N-1}(\cos \theta)$ in each subinterval $I_j = (\tilde{\theta}_{j-1}, \tilde{\theta}_j), 1 \le j \le N-1$. Because the number of zeros equals to the number of subintervals, there exists exactly one zero in I_j . \Box

As a consequence of Theorem 3.2, a good asymptotic approximation to the zeros $\{x_j\}_{j=0}^{N-1}$ might be

$$x_j = \cos \theta_j \cong x_j^{(0)} := \cos \left(\frac{\tilde{\theta}_{j-1} + \tilde{\theta}_j}{2}\right) = \cos \frac{(2j+1/2)\pi}{2N+1}, \quad 1 \le j \le N-1.$$
 (3.24)

To illustrate this property numerically, we denote

$$\widetilde{\Delta}_N = \max_{1 \le j \le N-1} \left| x_j - x_j^{(0)} \right|.$$

We plot in Fig. 3.2 (left) the error $\widetilde{\Delta}_N$ (solid line) and the reference value $N^{-1.9}$ (" \circ ") against N, which indicates that $\widetilde{\Delta}_N$ decays algebraically at a rate of $N^{-1.9}$, and predicts that

$$x_j = \cos\frac{(2j+1/2)\pi}{2N+1} + O(N^{-1.9}), \quad 1 \le j \le N-1.$$
(3.25)

3.2. Asymptotic properties of the weights

We establish below an important result concerning the asymptotic behaviors of the interior quadrature weights $\{\omega_j\}_{j=1}^{N-1}$ in (2.13).

Theorem 3.3. Let $\{\theta_j\}_{j=1}^{N-1}$ be the zeros of the quadrature (trigonometric) polynomial $Q_{N-1}(\cos \theta)$. Then for any $N \gg 1$,

$$\omega_j \cong \frac{\pi}{N} \sin \theta_j, \quad j = 1, 2, \dots, N - 1.$$
(3.26)

Proof. We rewrite the weights as

$$\omega_{j} \stackrel{(2.14)}{=} \frac{A_{N}}{(1-x_{j}^{2})^{2}} \frac{1}{P_{N-2}(x_{j})Q'_{N-1}(x_{j})} \\ \stackrel{(2.6)}{=} \frac{A_{N}}{\lambda_{N-2}\lambda_{N-3}} \frac{1}{(1-x_{j}^{2})^{2}} \frac{1}{J_{N-2}^{(2,2)}(x_{j})W'_{N}(x_{j})},$$
(3.27)

where

$$W_N(x) := \frac{\lambda_{N-1}}{\lambda_{N-3}} J_{N-1}^{(2,2)}(x) + b_N J_{N-3}^{(2,2)}(x) = \lambda_{N-3}^{-1} Q_{N-1}(x).$$
(3.28)

To prove (3.26), it suffices to study the asymptotic behaviors of the constant, $J_{N-2}^{(2,2)}(x_j)$ and $W'_N(x_j)$ in (3.27). For clarity, we split the rest of the proof into three steps.

Step I: Let us first estimate the constants in (3.27)–(3.28). A direct calculation using (2.6)–(2.7) and (2.12) leads to

$$a_{N-2} \cong \frac{1}{4}, \qquad -b_N \cong \frac{1}{4}, \qquad \frac{1}{\tilde{\lambda}_{N-2}^2} \cong \frac{16}{N}, \qquad \frac{\lambda_{N-2}}{\lambda_{N-3}} = \frac{(N-2)(N+2)}{N(2N-1)} \cong \frac{1}{2},$$
(3.29)

which, together with (2.15), implies

$$\frac{A_N}{\lambda_{N-2}\lambda_{N-3}} = \left(1 - \frac{b_N}{a_{N-2}}\right) \frac{\lambda_{N-2}}{\lambda_{N-3}\tilde{\lambda}_{N-2}^2} \cong \frac{16}{N}.$$
(3.30)

Step II: Let

$$\Theta_{N,j} = \left(N + \frac{1}{2}\right)\theta_j - \frac{5}{4}\pi.$$
(3.31)

We next show that

$$\sin \Theta_{N,j} \cong 0, \quad \forall N \gg 1. \tag{3.32}$$

Since $Q_{N-1}(x_i) = 0$, we have

$$W_N(\cos \theta_j) \stackrel{(3.28)}{=} \lambda_{N-3}^{-1} Q_{N-1}(\cos \theta_j) = \lambda_{N-3}^{-1} Q_{N-1}(x_j) = 0.$$

Hence, by (3.18),

$$0 = W_N(\cos \theta_j) = \frac{\lambda_{N-1}}{\lambda_{N-3}} J_{N-1}^{(2,2)}(\cos \theta_j) + b_N J_{N-3}^{(2,2)}(\cos \theta_j)$$

$$\stackrel{(3.18)}{=} (N-3)^{-\frac{1}{2}} G(\theta_j) \left\{ c_N \cos \left(\Theta_{N,j} + \theta_j \right) + b_N \cos \left(\Theta_{N,j} - \theta_j \right) \right\} + O(N^{-\frac{3}{2}})$$

$$= (N-3)^{-\frac{1}{2}} G(\theta_j) \left\{ (b_N - c_N) \sin \theta_j \sin \Theta_{N,j} + (b_N + c_N) \cos \theta_j \cos \Theta_{N,j} \right\} + O(N^{-\frac{3}{2}}),$$

where $G(\theta_i)$ is given in (3.19), and by (3.29),

$$c_N = \frac{\lambda_{N-1}}{\lambda_{N-3}} \left(\frac{N-3}{N-1}\right)^{\frac{1}{2}} \cong \frac{\lambda_{N-1}}{\lambda_{N-2}} \frac{\lambda_{N-2}}{\lambda_{N-3}} \cong \frac{1}{4}.$$
(3.33)

Consequently,

$$0 = (b_N - c_N)\sin\theta_j\sin\theta_{N,j} + (b_N + c_N)\cos\theta_j\cos\theta_{N,j} + O(N^{-1}\sin^{5/2}\theta_j).$$
(3.34)

On the other hand, using (3.29) and (3.33) leads to

$$b_N + c_N \cong 0, \quad b_N - c_N \cong -\frac{1}{2}, \quad \text{for } N \gg 1.$$
 (3.35)

Since $\sin \theta_j \neq 0 (> 0)$, the desired result (3.32) follows from (3.34).

Step III: Applying Formula (8.8.1) of [23] (note: this formula may be derived by differentiating (3.18) and using (3.28)) to $W'_N(x)$, and using trigonometric identities, yields

$$\frac{\mathrm{d}W_N}{\mathrm{d}\theta}(\cos\theta_j) = (N-3)^{\frac{1}{2}}G(\theta_j) \left\{ -c_N \sin(\Theta_{N,j} + \theta_j) - b_N \sin(\Theta_{N,j} - \theta_j) + (N\sin\theta_j)^{-1}O(1) \right\} \\
= (N-3)^{\frac{1}{2}}G(\theta_j) \left\{ (b_N - c_N) \sin\theta_j \cos\Theta_{N,j} - (b_N + c_N) \cos\theta_j \sin\Theta_{N,j} + (N\sin\theta_j)^{-1}O(1) \right\}.$$
(3.36)

Hence, by (3.32) and (3.35),

$$\frac{\mathrm{d}W_N}{\mathrm{d}\theta}(\cos\theta_j) \cong -\frac{1}{2}\sqrt{N}G(\theta_j)\sin\theta_j\cos\Theta_{N,j}, \quad 1 \le j \le N-1,$$
(3.37)

which implies

$$W_N'(x_j) = \left. \frac{\mathrm{d}W_N}{\mathrm{d}\theta} (\cos\theta_j) \frac{\mathrm{d}\theta}{\mathrm{d}x} \right|_{\theta=\theta_j} = -\frac{\mathrm{d}W_N}{\mathrm{d}\theta} (\cos\theta_j) \frac{1}{\sin\theta_j} = \frac{\sqrt{N}}{2} G(\theta_j) \cos\theta_{N,j}.$$
(3.38)

On the other hand, by (3.18),

$$J_{N-2}^{(2,2)}(\cos \theta_j) \cong N^{-\frac{1}{2}} G(\theta_j) \cos \Theta_{N,j}, \quad 1 \le j \le N-1.$$
(3.39)

Multiplying (3.38) by (3.39) yields

$$J_{N-2}^{(2,2)}(x_j)W_N'(x_j) \cong \frac{1}{2}G^2(\theta_j)\cos^2\Theta_{N,j} \cong \frac{1}{2}G^2(\theta_j),$$
(3.40)

where in the last step, we used the fact:

$$\cos^2 \Theta_{N,j} \stackrel{(3.32)}{\cong} 1, \quad \forall N \gg 1.$$
(3.41)

Finally, thanks to

$$\frac{1}{(1-x_j^2)^2} \stackrel{(3.13)}{=} \frac{1}{\sin^4 \theta_j}, \qquad G^2(\theta_j) \stackrel{(3.19)}{=} \frac{2^5}{\pi} \frac{1}{\sin^5 \theta_j},$$

the desired result (3.26) follows from (3.27), (3.30) and (3.40). \Box

As a direct consequence of (3.24) and (3.26), we derive the following explicit asymptotic expression:

$$\omega_j \cong \omega_j^{(0)} \coloneqq \frac{\pi}{N} \sin \frac{(2j+1/2)\pi}{2N+1}, \quad 1 \le j \le N-1.$$
(3.42)

(A))

To examine how good it is, we set

$$\delta_N^{(1)} = \frac{\pi}{N} \max_{1 \le j \le N-1} \left\{ \frac{\sin \theta_j}{\omega_j} \right\}, \qquad \delta_N^{(2)} = \frac{\pi}{N} \max_{1 \le j \le N-1} \left\{ \frac{\omega_j^{(0)}}{\omega_j} \right\}.$$

By Theorem 3.3, we expect to see that

$$\delta_N^{(i)} \cong 1, \quad i = 1, 2, \ \forall N \gg 1, \tag{3.43}$$

which indeed can be visualized from Fig. 3.2(right).

So far, we have derived two asymptotic estimates for the GLLB nodes and weights (cf. (3.24) and (3.42)), which will be useful for the analysis of the GLLB interpolation error in the forthcoming section.

4. GLLB interpolation error estimates

This section is devoted to the analysis of the GLLB interpolation errors in Sobolev norms, which will be used for the error analysis of GLLB pseudospectral methods. We first state the main result, and then present the ingredients for the proof including some inequalities and orthogonal projections. Finally, we give the proof of the interpolation errors.

4.1. The main result

We begin with the definition of the GLLB interpolation operator associated with the GLLB quadrature formula. The GLLB interpolant $\mathcal{I}_N u \in \mathbb{P}_N$, satisfies

$$(\mathcal{I}_N u)'(\pm 1) = u'(\pm 1), \quad (\mathcal{I}_N u)(x_j) = u(x_j), \quad 1 \le j \le N - 1.$$
 (4.1)

Denote the weight functions by $\omega(x) = 1 - x^2$ and $\omega^{\alpha}(x) = (1 - x^2)^{\alpha}$. Let $\bar{\alpha} = \max\{0, -[\alpha] - 1\}$ with $[\alpha]$ being the largest integer $\leq \alpha$. To describe the error, we introduce the weighted Sobolev space

$$B^m_{\alpha}(I) := \left\{ u \in L^2(I) : \partial_x^k u \in L^2_{\omega^{\alpha+k}}(I), \ \bar{\alpha}+1 \le k \le m \right\} \cap H^{\bar{\alpha}}(I), \quad m \ge 0,$$

with the norm

$$\|u\|_{B^{m}_{\alpha}(I)} = \left(\|u\|^{2}_{H^{\bar{\alpha}}(I)} + \sum_{k=\bar{\alpha}+1}^{m} \|\partial^{k}_{x}u\|^{2}_{\omega^{\alpha+k}}\right)^{\frac{1}{2}}.$$

The main result on the GLLB interpolation error is stated as follows.

Theorem 4.1. For any $u \in B^m_{-2}(I)$ and $m \ge 2$,

$$N\|\mathcal{I}_{N}u - u\| + \|\partial_{x}(\mathcal{I}_{N}u - u)\|_{\omega} \le cN^{1-m}\|\partial_{x}^{m}u\|_{\omega^{m-2}}.$$
(4.2)

4.2. Preparations for the proof

The main ingredients for the proof Theorem 4.1 consist of the asymptotic estimates (cf. Theorem 3.2 and 3.3), several inequalities and the approximation property of one specific orthogonal projection to be stated below.

4.2.1. Some inequalities

For notational convenience, we define the discrete inner product and discrete norm associated with the GLLB quadrature formula as

$$\langle u, v \rangle_N = \mathcal{S}_N[u \cdot v], \qquad \|v\|_N = \langle v, v \rangle_N^{\frac{1}{2}}$$

where $S_N[\cdot]$ represents the finite sum in (2.13). Clearly, the exactness (2.13) implies that

$$\langle \phi, \psi \rangle_N = (\phi, \psi), \quad \forall \phi \cdot \psi \in \mathbb{P}_{2N-1}.$$
 (4.3)

We have the following equivalence between the continuous and discrete norms over the polynomial space \mathbb{P}_N , which will be useful in the analysis of interpolation error, and the GLLB pseudospectral methods for nonlinear problems.

Lemma 4.1. For any integer $N \ge 4$,

$$\|\phi\| \le \|\phi\|_N \le \sqrt{1 + C_N} \|\phi\| \le 3 \|\phi\|, \quad \forall \phi \in \mathbb{P}_N,$$

$$(4.4)$$

where

$$C_N = \left(1 + \frac{3}{N-2}\right) \left(1 + \frac{3}{N-1}\right) \left(1 + \frac{3}{N}\right).$$

Proof. We first prove that (4.4) holds for the Legendre polynomial L_N . That is

$$\gamma_N^{(0,0)} = \|L_N\|^2 \le \langle L_N, L_N \rangle_N \le (1+C_N) \|L_N\|^2 = (1+C_N)\gamma_N^{(0,0)}.$$
(4.5)

For this purpose, set

$$\psi(x) = L_N^2(x) - \left(K_N^{(0,0)}\right)^2 (1-x^2)^2 Q_{N-1}(x) P_{N-3}(x).$$

Since the leading coefficient of $Q_{N-1} \cdot P_{N-3}$ is one, we deduce from (2.9) that $\psi \in \mathbb{P}_{2N-1}$. Hence, using the fact $Q_{N-1}(x_j) = 0$ and the orthogonality, gives

$$\langle L_N, L_N \rangle_N = \langle 1, \psi \rangle_N \stackrel{(4.3)}{=} (1, \psi) = \int_{-1}^1 \psi(x) dx \begin{pmatrix} \frac{(2.10)}{=} & \int_{-1}^1 L_N^2(x) dx - b_N \left(K_N^{(0,0)} \right)^2 \int_{-1}^1 P_{N-3}^2(x) (1-x^2)^2 dx \\ = & \gamma_N^{(0,0)} - b_N \left(\lambda_{N-3} K_N^{(0,0)} \right)^2 \gamma_{N-3}^{(2,2)} \\ = & \left(1 - b_N \left(\lambda_{N-3} K_N^{(0,0)} \right)^2 \gamma_{N-3}^{(2,2)} \left(\gamma_N^{(0,0)} \right)^{-1} \right) \gamma_N^{(0,0)}.$$
(4.6)

Using (2.2)–(2.6) and (2.9) to work out the constants, yields

$$\left(\lambda_{N-3}K_N^{(0,0)}\right)^2\gamma_{N-3}^{(2,2)}\left(\gamma_N^{(0,0)}\right)^{-1} = \frac{(N+1)(2N-1)(2N+1)}{N(N-1)(N-2)}.$$

Furthermore, by (2.12), we have that for $N \ge 4$,

$$0 < -b_N \left(\lambda_{N-3} K_N^{(0,0)}\right)^2 \gamma_{N-3}^{(2,2)} \left(\gamma_N^{(0,0)}\right)^{-1} < \frac{(N+1)(N+2)(N+3)}{(N-2)(N-1)N} = C_N.$$
(4.7)

Accordingly, a combination of (4.6)–(4.7) leads to (4.5).

Next, for any $\phi \in \mathbb{P}_N$, we write

$$\phi(x) = \sum_{n=0}^{N} \hat{\phi}_n L_n(x) = \sum_{n=0}^{N-1} \hat{\phi}_n L_n(x) + \hat{\phi}_N L_N(x) \coloneqq \Phi(x) + \hat{\phi}_N L_N(x),$$

and therefore

$$\|\phi\|^{2} = \sum_{n=0}^{N} \hat{\phi}_{n}^{2} \gamma_{n}^{(0,0)} = \|\Phi\|^{2} + \hat{\phi}_{N}^{2} \gamma_{N}^{(0,0)}.$$

It is clear that $\Phi^2(x) \in \mathbb{P}_{2N-2}$, and so by (4.3) and the orthogonality of Legendre polynomials,

$$\langle \Phi, L_N \rangle_N = (\Phi, L_N) = 0,$$

which implies

$$\begin{split} \|\phi\|_{N}^{2} &= \langle \Phi, \Phi \rangle_{N} + 2\hat{\phi}_{N} \langle \Phi, L_{N} \rangle_{N} + \hat{\phi}_{N}^{2} \langle L_{N}, L_{N} \rangle_{N} = \|\Phi\|^{2} + \hat{\phi}_{N}^{2} \langle L_{N}, L_{N} \rangle_{N} \\ &= \sum_{n=0}^{N-1} \hat{\phi}_{n}^{2} \gamma_{n}^{(0,0)} + \hat{\phi}_{N}^{2} \langle L_{N}, L_{N} \rangle_{N} \,. \end{split}$$

Finally, applying (4.5) leads to the desired result. \Box

The following Bernstein-Markov type inequality holds for polynomials in the finitedimensional space:

$$X_N = \{ \phi \in \mathbb{P}_N : \phi'(\pm 1) = 0 \}.$$
(4.8)

Lemma 4.2. For any $\phi \in X_N$,

$$\|\phi'\|_{\omega} \le \sqrt{N(N+1)} \|\phi\| \le (N+1) \|\phi\|.$$
(4.9)

Proof. Let

$$\psi_n(x) = L_n(x) + e_n L_{n+2}(x), \qquad e_n = -\frac{n(n+1)}{(n+2)(n+3)}.$$
(4.10)

Since $L'_n(\pm 1) = \frac{1}{2}(-1)^{n-1}n(n+1)$, we have that $\{\psi_n\}_{n=0}^{N-2}$ forms a basis for X_N . Consequently, for any $\phi \in X_N$,

$$\phi(x) = \sum_{n=0}^{N-2} \tilde{\phi}_n \psi_n(x) = \sum_{n=0}^{N-2} \tilde{\phi}_n L_n(x) + \sum_{n=2}^{N} e_{n-2} \tilde{\phi}_{n-2} L_n(x)$$

= $\tilde{\phi}_0 L_0(x) + \tilde{\phi}_1 L_1(x) + \sum_{n=2}^{N-2} (\tilde{\phi}_n + e_{n-2} \tilde{\phi}_{n-2}) L_n(x)$
+ $e_{N-3} \tilde{\phi}_{N-3} L_{N-1}(x) + e_{N-2} \tilde{\phi}_{N-2} L_N(x).$ (4.11)

By the recursive relation

$$L'_k(x) = \frac{1}{2}(k+1)J^{(1,1)}_{k-1}(x), \quad k \ge 1,$$

and (4.10),

$$\begin{split} \phi'(x) &= \sum_{n=0}^{N-2} \tilde{\phi}_n \psi'_n(x) = \frac{1}{2} \sum_{n=0}^{N-3} (n+2) \tilde{\phi}_{n+1} J_n^{(1,1)}(x) + \frac{1}{2} \sum_{n=1}^{N-1} (n+2) e_{n-1} \tilde{\phi}_{n-1} J_n^{(1,1)}(x) \\ &= \tilde{\phi}_1 J_0^{(1,1)}(x) + \frac{1}{2} \sum_{n=1}^{N-3} (n+2) (\tilde{\phi}_{n+1} + e_{n-1} \tilde{\phi}_{n-1}) J_n^{(1,1)}(x) \\ &+ \frac{1}{2} N e_{N-3} \tilde{\phi}_{N-3} J_{N-2}^{(1,1)}(x) + \frac{1}{2} (N+1) e_{N-2} \tilde{\phi}_{N-2} J_{N-1}^{(1,1)}(x). \end{split}$$

Using the orthogonality (2.1)–(2.2) gives

$$\begin{aligned} \|\phi\|^2 &= \tilde{\phi}_0^2 \gamma_0^{(0,0)} + \tilde{\phi}_1^2 \gamma_1^{(0,0)} + \sum_{n=2}^{N-2} (\tilde{\phi}_n + e_{n-2} \tilde{\phi}_{n-2})^2 \gamma_n^{(0,0)} \\ &+ e_{N-3}^2 \tilde{\phi}_{N-3}^2 \gamma_{N-1}^{(0,0)} + e_{N-2}^2 \tilde{\phi}_{N-2}^2 \gamma_N^{(0,0)}, \end{aligned}$$

and

$$\begin{split} \|\phi'\|_{\omega}^{2} &= \tilde{\phi}_{1}^{2} \gamma_{0}^{(1,1)} + \frac{1}{4} \sum_{n=1}^{N-3} (n+2)^{2} (\tilde{\phi}_{n+1} + e_{n-1} \tilde{\phi}_{n-1})^{2} \gamma_{n}^{(1,1)} \\ &+ \frac{1}{4} N^{2} e_{N-3}^{2} \tilde{\phi}_{N-3}^{2} \gamma_{N-2}^{(1,1)} + \frac{1}{4} (N+1)^{2} e_{N-2}^{2} \tilde{\phi}_{N-2}^{2} \gamma_{N-1}^{(1,1)} \\ &= \tilde{\phi}_{1}^{2} \gamma_{0}^{(1,1)} + \frac{1}{4} \sum_{n=2}^{N-2} (n+1)^{2} (\tilde{\phi}_{n} + e_{n-2} \tilde{\phi}_{n-2})^{2} \gamma_{n-1}^{(1,1)} \\ &+ \frac{1}{4} N^{2} e_{N-3}^{2} \tilde{\phi}_{N-3}^{2} \gamma_{N-2}^{(1,1)} + \frac{1}{4} (N+1)^{2} e_{N-2}^{2} \tilde{\phi}_{N-2}^{2} \gamma_{N-1}^{(1,1)}. \end{split}$$

In view of the above facts, we deduce from (2.2) that

$$\|\phi'\|_{\omega}^{2} \leq \frac{1}{4} \max_{1 \leq n \leq N} \left\{ (n+1)^{2} \gamma_{n-1}^{(1,1)} (\gamma_{n}^{(0,0)})^{-1} \right\} \|\phi\|^{2} = N(N+1) \|\phi\|^{2}.$$

This implies the desired result. \Box

In the preceding analysis, we will also use the following Poincaré inequality (see, e.g., [5]).

Lemma 4.3. For any $u \in H^1(I)$,

$$||u|| \le c(||u'|| + |\bar{u}|), \quad where \ \bar{u} = \int_{-1}^{1} u(x) \mathrm{d}x.$$
 (4.12)

4.2.2. Orthogonal projections

We first consider the orthogonal projection $\pi_N : L^2(I) \to \mathbb{P}_N$ such that for any $u \in L^2(I)$,

$$(\pi_N u - u, \phi) = 0, \quad \forall \phi \in \mathbb{P}_N.$$
(4.13)

The following result can be found in [10,14].

Lemma 4.4. For any $u \in B_0^m(I)$ and $m \ge 0$,

$$\|\pi_N u - u\| \le c N^{-m} \|\partial_x^m u\|_{\omega^m}.$$
(4.14)

We now turn to the second orthogonal projection. For simplicity, we denote

$$\partial_x^{-1} v(x) = \int_{-1}^x v(y) dy, \qquad \bar{\partial}_x^{-1} v(x) = -\int_x^1 v(y) dy, \qquad \bar{v} = \int_{-1}^1 v(x) dx.$$

Define the space

$$X := \left\{ v : v \in H^2(I), v'(\pm 1) = 0 \right\}, \quad X^0 := \left\{ v \in X : \bar{v} = 0 \right\}, \quad X^0_N := \mathbb{P}_N \cap X^0.$$

Consider the orthogonal projection $\pi_N^{1,0}: X^0 \to X_N^0$, defined by

$$a(\pi_N^{1,0}v - v, \phi) = 0, \quad \forall \phi \in X_N^0,$$
(4.15)

where the bilinear form $a(u, v) := (\partial_x u, \partial_x v)$. Recall that for real $\mu \ge 0$, the Sobolev space $H^{\mu}(I)$ and its norm $\|\cdot\|_{\mu}$ are defined by space interpolation as in [1].

Lemma 4.5. For any $v \in X^0 \cap B^m_{-2}(I)$ with $m \ge 2$,

$$\|\pi_N^{1,0}v - v\|_{\mu} \le cN^{\mu-m} \|\partial_x^m v\|_{\omega^{m-2}}, \quad 0 \le \mu \le 1.$$
(4.16)

Proof. We first consider the case $\mu = 1$. Let

$$v^{*}(x) = \xi + \partial_{x}v(1)x + \partial_{x}^{-1}\bar{\partial}_{x}^{-1}(\pi_{N-2}\partial_{x}^{2}v)(x)$$
(4.17)

where the constant ξ is chosen such that $\overline{v^*} = \overline{v}$. Clearly, $\partial_x v^*(1) = \partial_x v(1)$ and by (4.13) with $\phi = 1$,

$$\partial_x v^*(-1) = \partial_x v(1) - \int_I \pi_{N-2} \partial_x^2 v(x) \mathrm{d}x = \partial_x v(1) - \int_I \partial_x^2 v(x) \mathrm{d}x = \partial_x v(-1).$$
(4.18)

For clarity, let us denote $g = v^* - v$. Thanks to the above facts, we have from (4.13) and (4.14) that

$$\begin{aligned} |v^{*} - v|_{1}^{2} &= (\partial_{x}(v^{*} - v), \partial_{x}g) = -(\partial_{x}^{2}(v^{*} - v), g) \\ &= -(\pi_{N-2}\partial_{x}^{2}v - \partial_{x}^{2}v, g) = (\pi_{N-2}\partial_{x}^{2}v - \partial_{x}^{2}v, \pi_{N-2}g - g) \\ &\leq \|\pi_{N-2}\partial_{x}^{2}v - \partial_{x}^{2}v\|\|\pi_{N-2}g - g\| \leq cN^{1-m}\|\partial_{x}^{m}v\|_{\omega^{m-2}}\|\partial_{x}g\|_{\omega} \\ &\leq cN^{1-m}\|\partial_{x}^{m}v\|_{\omega^{m-2}}|v^{*} - v|_{1}. \end{aligned}$$
(4.19)

Moreover, by the Poincaré inequality (4.12) and (4.15),

$$\begin{aligned} \|\pi_N^{1,0}v - v\|_1^2 &\leq c |\pi_N^{1,0}v - v|_1^2 = ca(\pi_N^{1,0}v - v, \pi_N^{1,0}v - v) \\ &= ca(\pi_N^{1,0}v - v, v^* - v) \leq c |\pi_N^{1,0}v - v|_1 |v^* - v|_1. \end{aligned}$$

In view of this fact, we have from (4.19) that

$$\|\pi_N^{1,0}v - v\|_1 \le c|v^* - v|_1 \le cN^{1-m} \|\partial_x^m v\|_{\omega^{m-2}}.$$
(4.20)

We next prove the case $\mu = 0$ by using a duality argument (see, e.g., [6]). For any $f \in L^2(I)$, we consider an auxiliary problem. It is to find $w \in X^0$ such that

$$a(w, z) = (f, z), \quad \forall z \in X^0.$$
 (4.21)

By the Poincaré inequality (cf. (4.12)) and Lax–Milgram Lemma, the problem (4.21) has a unique solution with the regularity

$$\|w\|_2 \le c\|f\|. \tag{4.22}$$

Taking $z = \pi_N^{1,0} v - v$ in (4.21), we deduce from (4.15), (4.20) and (4.22) that

$$\begin{split} |(\pi_N^{1,0}v - v, f)| &= |a(\pi_N^{1,0}v - v, w)| = |a(\pi_N^{1,0}v - v, \pi_N^{1,0}w - w)| \\ &\leq |\pi_N^{1,0}v - v|_1 |\pi_N^{1,0}w - w|_1 \leq cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|\partial_x^2 w\| \\ &\leq cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|f\|. \end{split}$$

Consequently

$$\|\pi_N^{1,0}v - v\| = \sup_{\substack{f \in L^2(I)\\ f \neq 0}} \frac{|(\pi_N^{1,0}v - v, f)|}{\|f\|} \le cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}}.$$
(4.23)

Finally, we get the desired result by (4.20), (4.23) and space interpolation.

With the aid of the previous preparations, we are able to derive the following important result.

Theorem 4.2. There exists an operator $\pi_N^1 : X \to X_N$, such that $\overline{\pi_N^1 v} = \overline{v}$, and

$$a(\pi_N^1 v - v, \phi) = 0, \quad \forall \phi \in X_N.$$

$$(4.24)$$

Moreover, for any $v \in X \cap B^m_{-2}(I)$ *with* $m \ge 2$ *,*

$$\|\pi_N^1 v - v\|_{\mu} \le c N^{\mu-m} \|\partial_x^m v\|_{\omega^{m-2}}, \quad 0 \le \mu \le 1.$$
(4.25)

Proof. For any $v \in X$, since $v - \overline{v}/2 \in X^0$, we define

$$\pi_N^1 v(x) = \pi_N^{1,0} (v(x) - \bar{v}/2) + \bar{v}/2.$$

One verifies readily that $\pi_N^1 v \in X_N$ and $\overline{\pi_N^1 v} = \overline{v}$. Moreover, by (4.15),

$$a(\pi_N^1 v - v, \phi) = a(\pi_N^{1,0}(v - \bar{v}/2) - (v - \bar{v}/2), \phi - \bar{\phi}/2) = 0, \quad \forall \phi \in X_N.$$

Moreover, by (4.16) and the fact $r \ge 2$,

$$\begin{aligned} \|\pi_N^1 v - v\|_{\mu} &= \|\pi_N^{1,0}(v - \bar{v}/2) - (v - \bar{v}/2)\|_{\mu} \le c N^{\mu-m} \|\partial_x^m (v - \bar{v}/2)\|_{\omega^{m-2}} \\ &\le c N^{\mu-m} \|\partial_x^m v\|_{\omega^{m-2}}. \end{aligned}$$

This leads to (4.25).

4.3. Continuity of the GLLB interpolation operator

In order to prove Theorem 4.1, it is essential to show that \mathcal{I}_N is a continuous operator from X to $L^2(I)$, as stated below.

Lemma 4.6. For any $v \in X$,

$$\|\mathcal{I}_N v\| \le c(\|v\| + N^{-1} \|\partial_x v\|_{\omega}), \tag{4.26}$$

where the weight function $\omega(x) = 1 - x^2$.

Proof. In this proof, we mainly use Theorems 3.2 and 3.3. Let $x = \cos \theta$ and $\hat{v}(\theta) = v(\cos \theta)$. Then by (2.13) and (4.4) and Theorem 3.3,

$$\|\mathcal{I}_N v\|^2 \le \|\mathcal{I}_N v\|_N^2 = \sum_{j=1}^{N-1} v^2(x_j)\omega_j \le cN^{-1} \sum_{j=1}^{N-1} \hat{v}^2(\theta_j) \sin \theta_j.$$
(4.27)

By using an inequality of space interpolation (see formula (13.7) of [2]), we know that for any $f \in H^1(a, b)$,

$$\max_{a \le x \le b} |f(x)| \le c \left(\frac{1}{\sqrt{b-a}} \|f\|_{L^2(a,b)} + \sqrt{b-a} \|\partial_x f\|_{L^2(a,b)} \right).$$
(4.28)

Now, let $\tilde{\theta}_0$, $\tilde{\theta}_{N-1}$ and I_j be the same as in Theorem 3.2. Denote $a_0 = \tilde{\theta}_0$ and $a_1 = \tilde{\theta}_{N-1}$. Then by (4.27) and (4.28),

$$\begin{split} \left\|\mathcal{I}_{N}v\right\|^{2} &\stackrel{(4.27)}{\leq} c N^{-1} \sum_{j=1}^{N-1} \sup_{\theta \in \tilde{I}_{j}} \left|\hat{v}(\theta)\sqrt{\sin\theta}\right|^{2} \\ &\stackrel{(4.28)}{\leq} c \sum_{j=1}^{N-1} \left(\left\|\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}(\tilde{I}_{j})}^{2} + N^{-2} \left\|\partial_{\theta}\left(\hat{v}(\theta)\sqrt{\sin\theta}\right)\right\|_{L^{2}(\tilde{I}_{j})}^{2}\right) \\ &\leq c \left(\left\|\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}(0,\pi)}^{2} + N^{-2} \left\|\partial_{\theta}\left(\hat{v}(\theta)\sqrt{\sin\theta}\right)\right\|_{L^{2}([a_{0},a_{1}])}^{2}\right) \\ &\leq c \left(\left\|\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}(0,\pi)}^{2} + N^{-2} \left\|\hat{v}(\theta)(\sin\theta)^{-1/2}\right\|_{L^{2}([a_{0},a_{1}])}^{2} \\ &+ N^{-2} \left\|\partial_{\theta}\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}(0,\pi)}^{2} + \sup_{a_{0}\leq\theta\leq a_{1}} \frac{1}{N^{2}\sin^{2}\theta} \cdot \left\|\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}([a_{0},a_{1}])}^{2} \\ &+ N^{-2} \left\|\partial_{\theta}\hat{v}(\theta)\sqrt{\sin\theta}\right\|_{L^{2}(0,\pi)}^{2} \right). \end{split}$$

Since Theorem 3.2 implies $\sup_{a_0 \le \theta \le a_1} \frac{1}{N^2 \sin^2 \theta} \le c$, we transform θ back to *x*, and obtain

$$\left\|\mathcal{I}_N v\right\|^2 \le c \left(\left\|v\right\|^2 + N^{-2} \left\|\partial_x v\right\|_{\omega}^2\right),$$

which ends the proof. \Box

4.4. Proof of Theorem 4.1

Let

$$v_N^*(x) = v(-1) + (1+x)\partial_x v(1) + \partial_x^{-1} \bar{\partial}_x^{-1} (\pi_{N-2} \partial_x^2 v)(x) \in \mathbb{P}_N.$$

Like (4.18), we have $v_N^*(-1) = v(-1)$ and $\partial_x v_N^*(\pm 1) = \partial_x v(\pm 1)$. Thus, by (4.13),

$$(v_N^* - v, \phi) = (\partial_x^2 (v_N^* - v), \partial_x^{-1} \bar{\partial}_x^{-1} \phi)$$

= $(\pi_{N-2} \partial_x^2 v - \partial_x^2 v, \partial_x^{-1} \bar{\partial}_x^{-1} \phi) = 0, \quad \forall \phi \in \mathbb{P}_{N-4}.$ (4.29)

A similar argument as in the derivation of (4.19) leads to

$$|v_N^* - v|_1 \le c N^{1-m} \|\partial_x^m v\|_{\omega^{m-2}}.$$
(4.30)

Denote $g = v_N^* - v$. Then by (4.14), (4.29) and (4.30),

$$\begin{split} \|v_N^* - v\|^2 &= (v_N^* - v, g) = (v_N^* - v, \partial_x \bar{\partial}_x^{-1} g) = (v_N^* - v, \partial_x \bar{\partial}_x^{-1} g - \partial_x \pi_{N-3} \bar{\partial}_x^{-1} g) \\ &= (\partial_x (v_N^* - v), \pi_{N-3} \bar{\partial}_x^{-1} g - \bar{\partial}_x^{-1} g) \le |v_N^* - v|_1 \|\pi_{N-3} \bar{\partial}_x^{-1} g - \bar{\partial}_x^{-1} g\| \\ &\le c N^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|g\|_{\omega} \le c N^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|v_N^* - v\|, \end{split}$$

which implies

$$\|v_N^* - v\| \le cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}}.$$
(4.31)

Since $\mathcal{I}_N v_N^* = v_N^*$ and $v - v_N^* \in X$, we have from Lemma 4.6, (4.30) and (4.31) that

$$\begin{aligned} \|\mathcal{I}_N v - v_N^*\| &= \|\mathcal{I}_N (v_N^* - v)\| \le c(\|v_N^* - v\| + N^{-1} \|\partial_x (v_N^* - v)\|_{\omega}) \\ &\le c(\|v_N^* - v\| + N^{-1} |v_N^* - v|_1) \le cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}}. \end{aligned}$$
(4.32)

By the above and Lemma 4.2,

$$\|\partial_{x}(\mathcal{I}_{N}v - v_{N}^{*})\|_{\omega} \le cN \|\mathcal{I}_{N}v - v_{N}^{*}\| \le cN^{1-m} \|\partial_{x}^{m}v\|_{\omega^{m-2}}.$$
(4.33)

Finally, a combination of (4.30)–(4.33) leads to that

$$N\|\mathcal{I}_{N}v - v\| + \|\partial_{x}(\mathcal{I}_{N}v - v)\|_{\omega} \le N\|\mathcal{I}_{N}v - v_{N}^{*}\| + \|\partial_{x}(\mathcal{I}_{N}v - v_{N}^{*})\|_{\omega} + N\|v_{N}^{*} - v\| + \|\partial_{x}(v_{N}^{*} - v)\|_{\omega} \le cN^{1-m}\|\partial_{x}^{m}v\|_{\omega^{m-2}}.$$

This ends the proof.

4.5. Corollaries

We present below two mathematical consequences.

Corollary 4.1. For any $\phi \in X_M$ and $\psi \in X_L$, we have

$$\|\mathcal{I}_N \phi\|_N \le c(1 + MN^{-1}) \|\phi\|, \tag{4.34}$$

$$|\langle \phi, \psi \rangle_N| \le c(1 + MN^{-1})(1 + LN^{-1}) \|\phi\| \|\psi\|.$$
(4.35)

Proof. Using (4.4) and Lemmas 4.2 and 4.6 gives

$$\|\mathcal{I}_N \phi\|_N \le c \|\mathcal{I}_N \phi\| \le c (\|\phi\| + N^{-1} \|\partial_x \phi\|_{\omega}) \le c (1 + MN^{-1}) \|\phi\|,$$

and

$$\begin{split} |\langle \phi, \psi \rangle_N | &= |\langle \mathcal{I}_N \phi, \mathcal{I}_N \psi \rangle_N | \le \|\mathcal{I}_N \phi\|_N \|\mathcal{I}_N \psi\|_N \\ &\le c(1 + MN^{-1})(1 + LN^{-1}) \|\phi\| \|\psi\|. \quad \Box \end{split}$$

Corollary 4.2. For any $v \in X \cap B^m_{-2}(I)$, $m \ge 2$ and any $\phi \in X_N$,

$$|(v,\phi) - \langle v,\phi \rangle_N| \le cN^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|\phi\|_N.$$

$$(4.36)$$

Proof. By (4.3), (4.4), and Theorem 4.1,

$$\begin{aligned} |(v,\phi) - \langle v,\phi\rangle_N | &\leq |(v,\phi) - (\pi_{N-1}^1 v,\phi)| + |\left\langle \pi_{N-1}^1 v,\phi\right\rangle_N - \langle \mathcal{I}_N v,\phi\rangle_N | \\ &\leq c \left(\|\pi_{N-1}^1 v - v\| + \|\mathcal{I}_N v - v\| \right) \|\phi\| \\ &\leq c N^{-m} \|\partial_x^m v\|_{\omega^{m-2}} \|\phi\|_N. \end{aligned}$$

This result is useful in numerical analysis of the related pseudospectral scheme. \Box

5. GLLB pseudospectral methods and error estimates

Among several different versions of spectral approximations, pseudospectral methods are commonly used and more preferable in industrial codes owing to its ease of implementations and treatment of nonlinear problems. Most existing literature concerning this method is based on the collocation points that are identified as (generalized) Gauss–Lobatto quadrature formulae [24,9, 21,18]. In a pseudospectral method, the choice of collocation points is crucial for the stability and treatment of boundary conditions [21]. The GLLB quadrature formula (2.13) involves first-order derivative values at the endpoints, which allows a natural and exact imposition of Neumann boundary conditions. Based on this, Ezzirani and Guessab [8] proposed a GLLB collocation method for some model elliptic equations, and showed that this method leads to a resulting discrete system with a diagonal mass matrix, and thereby can be used to introduce explicit resolutions in the lumped mass method for the time-dependent problems.

The main purposes of this section are two-folds: (i) to present a user-oriented implementation of the GLLB psuedospectral methods based on pure collocation and variational formulations, and (ii) to make an error analysis of the GLLB pseudospectral method based on a (discrete) variational formulation. We will first restrict our attentions to one-dimensional problems, and then follow with multi-dimensional cases. It should be pointed out that the methods go beyond these model equations.

5.1. GLLB collocation method

Consider the elliptic equation:

$$\begin{cases} \mathbb{L}[u](x) := -u''(x) + b(x)u(x) = f(x), & b(x) \ge 0, \ x \in I = (-1, 1), \\ u'(\pm 1) = g_{\pm}, \end{cases}$$
(5.1)

where in the case of $b(x) \equiv 0$, the problem admits a solution only provided that the given data satisfy the compatibility

$$\int_{-1}^{1} f(x) dx = g_{-} - g_{+}.$$
(5.2)

Let $\{x_j\}_{j=0}^N$ be the GLLB quadrature nodes. The GLLB collocation approximation to (5.1) is to find $u_N \in \mathbb{P}_N$ such that

$$\begin{cases} \mathbb{L}[u_N](x_j) = -u_N''(x_j) + b(x_j)u_N(x_j) = F(x_j), & 1 \le j \le N-1, \\ u_N'(\pm 1) = g_{\pm}, \end{cases}$$
(5.3)

where F(x) is a consistent approximation to f(x) with F(x) = f(x) for $b(x) \neq 0$, and the case for $b(x) \equiv 0$ to be specified below.

We see that like the LGL collocation method for Dirichlet problems, the numerical solution satisfies the boundary conditions exactly.

Remark 5.1. In the case of $b(x) \equiv 0$, the scheme (5.3) is reduced to

$$-u_N''(x_j) = F(x_j), \quad 1 \le j \le N - 1; \quad u_N'(\pm 1) = g_{\pm}.$$
(5.4)

Let $(\mathbb{I}_{N-2}^{L}F)(x) \in \mathbb{P}_{N-2}$ be the Lagrange interpolation polynomial of F associated with the interior GLLB nodes $\{x_j\}_{j=1}^{N-1}$. Since $u''_N \in \mathbb{P}_{N-2}$, the scheme (5.4) implies

$$-u_N''(x) = \left(\mathbb{I}_{N-2}^L F\right)(x), \quad x \in I; \quad u_N'(\pm 1) = g_{\pm}$$

Thus, a direct integration leads to

$$\int_{-1}^{1} \left(\mathbb{I}_{N-2}^{L} F \right)(x) \mathrm{d}x = g_{-} - g_{+}.$$
(5.5)

This means that (5.4) has a solution as long as the above compatibility holds. However, since f, g_+ and g_- are given, the compatibility is not valid if we take F(x) = f(x). To meet the condition (5.5), we may follow the idea of Guo (cf. page 558 of [12]) to take

$$F(x) = f(x) - \frac{1}{2}S_N\left[\mathbb{I}_{N-2}^L f\right] + \frac{1}{2}(g_- - g_+),$$

where the functional $S_N[\cdot]$ is defined in (2.13). It is clear that for $N \ge 3$, we have

$$\left(\mathbb{I}_{N-2}^{L}F\right)(x) = \left(\mathbb{I}_{N-2}^{L}f\right)(x) - \frac{1}{2}\mathcal{S}_{N}\left[\mathbb{I}_{N-2}^{L}f\right] + \frac{1}{2}(g_{-}-g_{+}).$$

By virtue of (2.13) and (5.2), we have that

$$\int_{-1}^{1} \left(\mathbb{I}_{N-2}^{L} F \right)(x) dx = \int_{-1}^{1} \left(\mathbb{I}_{N-2}^{L} f \right)(x) dx - \mathcal{S}_{N} \left[\mathbb{I}_{N-2}^{L} f \right] + g_{-} - g_{+} = g_{-} - g_{+}.$$

We now examine the matrix form of (5.3). Let $\{h_j\}_{j=0}^N \subseteq \mathbb{P}_N$ be the set of Lagrangian polynomials associated with the GLLB points:

$$\begin{aligned} &h'_0(-1) = 1, \quad h'_0(1) = h_0(x_k) = 0, & 1 \le k \le N - 1, \\ &h'_j(\pm 1) = 0, \quad h_j(x_k) = \delta_{jk}, & 1 \le j, k \le N - 1, \\ &h'_N(1) = 1, \quad h'_N(-1) = h_N(x_k) = 0, & 1 \le k \le N - 1, \end{aligned}$$

$$(5.6)$$

whose explicit expressions are given in Appendix A. It is clear that $\{h_j\}_{j=0}^N \subseteq \mathbb{P}_N$ and spans the polynomial space \mathbb{P}_N . Under this nodal basis, we write

$$u_N(x) = g_- h_0(x) + g_+ h_N(x) + \sum_{k=1}^{N-1} a_k h_k(x) \in \mathbb{P}_N,$$
(5.7)

and determine the unknowns $\{a_k\}_{k=1}^{N-1}$ by (5.3), i.e., the system

$$\mathbf{A}_{c}\vec{\mathbf{a}} \coloneqq \left(-\mathbf{D}_{in}^{(2)} + \mathbf{B}\right)\vec{\mathbf{a}} = \vec{\mathbf{b}},\tag{5.8}$$

where

$$d_{jk}^{(2)} = h_k''(x_j), \quad \mathbf{D}_{in}^{(2)} = \left(d_{jk}^{(2)}\right)_{1 \le j,k \le N-1}, \mathbf{B} = \text{diag}\left(b(x_1), b(x_2), \dots, b(x_{N-1})\right), \quad \vec{\mathbf{a}} = (a_1, a_2, \dots, a_{N-1})^{\mathrm{T}}, \vec{\mathbf{b}} = (f(x_1), \dots, f(x_{N-1}))^{\mathrm{T}} + \left(d_{10}^{(2)}, d_{20}^{(2)}, \dots, d_{(N-1)0}^{(2)}\right)^{\mathrm{T}} g_{-} + \left(d_{1N}^{(2)}, d_{2N}^{(2)}, \dots, d_{(N-1)N}^{(2)}\right)^{\mathrm{T}} g_{+}.$$
(5.9)

As with the usual collocation schemes, the GLLB method is easy to implement once the associated differentiation matrices are pre-computed. The following lemma provides a recursive way to evaluate the differentiation matrices.

Lemma 5.1. Let

$$d_{jk}^{(l)} = h_k^{(l)}(x_j) = \frac{\mathrm{d}^l h_k}{\mathrm{d}x^l}(x_j), \quad \mathbf{D}^{(l)} = \left(d_{jk}^{(l)}\right)_{0 \le j,k \le N}, \quad l \ge 0.$$
(5.10)

Then we have

$$\begin{cases} \mathbf{D}^{(l+1)} = \mathbf{D}^{(1)} \times \widetilde{\mathbf{D}}^{(l)}, & l \ge 0, \\ \mathbf{D}^{(0)} = \widetilde{\mathbf{D}}^{(0)} = \mathbf{I}_{N+1}, \end{cases}$$
(5.11)

where \mathbf{I}_{N+1} is the identity matrix of order N + 1, and $\widetilde{\mathbf{D}}^{(l)}$ is identical to $\mathbf{D}^{(l)}$ except that the first and last rows of $\mathbf{D}^{(l)}$ are replaced by $\begin{pmatrix} d_{00}^{(l+1)}, d_{01}^{(l+1)}, \ldots, d_{0N}^{(l+1)} \end{pmatrix}$ and $\begin{pmatrix} d_{N0}^{(l+1)}, d_{N1}^{(l+1)}, \ldots, d_{NN}^{(l+1)} \end{pmatrix}$, respectively.

Proof. For any $\phi \in \mathbb{P}_N$, we have that

$$\phi(x) = \phi'(-1)h_0(x) + \sum_{j=1}^{N-1} \phi(x_j)h_j(x) + \phi'(1)h_N(x) \in \mathbb{P}_N$$

which implies that

$$\phi'(x) = \phi'(-1)h'_0(x) + \sum_{j=1}^{N-1} \phi(x_j)h'_j(x) + \phi'(1)h'_N(x) \in \mathbb{P}_N.$$

Let $x_0 = -1$ and $x_N = 1$. Taking $\phi(x) = h_k^{(l)}(x)$ and $x = x_i$ leads to that for all $0 \le i, k \le N$ and $l \ge 0$,

$$d_{ik}^{(l+1)} = h_k^{(l+1)}(x_i) = h_k^{(l+1)}(x_0)h_0'(x_i) + \sum_{j=1}^{N-1} h_k^{(l)}(x_j)h_j'(x_i) + h_k^{(l+1)}(x_N)h_N'(x_i)$$

= $d_{i0}^{(1)}d_{0k}^{(l+1)} + \sum_{j=1}^{N-1} d_{ij}^{(1)}d_{jk}^{(l)} + d_{iN}^{(1)}d_{Nk}^{(l+1)},$ (5.12)

which implies the desired result. \Box

As a consequence of this lemma, it suffices to evaluate the first-order differentiation matrix $D^{(1)}$ and values $h_j^{(l+1)}(\pm 1)$ for $l \ge 1$ and $0 \le j \le N$ to compute higher-order differentiation matrices.

Although the collocation scheme (5.3) is easy to implement, the GLLB differentiation matrix $D^{(2)}$ is full with $\text{Cond}(D^{(2)}) \sim N^4$, and thereby when N is large, the accuracy of nodes and of entries of $D^{(2)}$ are subject to significant roundoff errors. To overcome this trouble, it is advisable to construct a preconditioning for the system (5.8), and use a suitable iteration solver [4,7]. As a matter of fact, Canuto [4] proposed a finite-difference preconditioning for the LGL collocation method for Neumann problems, but it cannot be applied to this context directly. However, with a different treatment of the boundary conditions, we are able to derive an optimal preconditioner as that in [4] (for the LGL collocation method).

To this end, we assume that $\{x_j\}_{j=0}^N$ are arranged in an ascending order, and let

$$\delta_j = x_{j+1} - x_j, \quad u_j = u(x_j), \quad u''_j = u''(x_j), \quad f_j = f(x_j).$$

Taking the Neumann boundary conditions into account, we discretize u_1'' and u_{N-1}'' as

$$u_1'' \approx \frac{u_2 - u_1 + \delta_1 u_0'}{\delta_1(\delta_1/2 + \delta_0)}, \qquad u_{N-1}'' \approx \frac{u_{N-2} - u_{N-1} - \delta_{N-2} u_N'}{\delta_{N-2}(\delta_{N-2}/2 + \delta_{N-1})},$$
(5.13)

and use centered differences for the interiors $\{u_j''\}_{j=2}^{N-2}$. More precisely, the finite-difference approximation to (5.1) reads

$$\begin{cases} \frac{u_1 - u_2}{\delta_1(\delta_1/2 + \delta_0)} + b_1 u_1 = f_1 + \frac{g_-}{\delta_1/2 + \delta_0}, \\ -\frac{2u_{j-1}}{\delta_{j-1}[\delta_j + \delta_{j+1}]} + \left[\frac{2}{\delta_j\delta_{j-1}} + b_j\right] u_j - \frac{2u_{j+1}}{\delta_j[\delta_j + \delta_{j+1}]} = f_j, \\ \frac{2 \le j \le N - 2,}{u_{N-1} - u_{N-2}} \\ \frac{u_{N-1} - u_{N-2}}{\delta_{N-2}(\delta_{N-2}/2 + \delta_{N-1})} + b_{N-1}u_{N-1} = f_{N-1} - \frac{g_+}{\delta_{N-2}/2 + \delta_{N-1}}. \end{cases}$$
(5.14)

Denote by \mathbf{A}_d the coefficient matrix of the above system. Table 5.1 contains the spectral radii (i.e., the largest modulus of the eigenvalues) of \mathbf{A}_c (cf. (5.8)) and $\mathbf{A}_d^{-1}\mathbf{A}_c$ with b = 1 (in columns 2–3) and $b(x) = 100 + \sin(100\pi x)$ (in columns 4–5). The eigenvalues of two matrices are real, positive and distinct in all cases. We also point out that the smallest modulus of the eigenvalues of $\mathbf{A}_d^{-1}\mathbf{A}_c$ is one for both cases, while that of \mathbf{A}_c is one for b = 1 and 100 for $b(x) = 100 + \sin(100\pi x)$. The eigenvalues of the preconditioned matrix $\mathbf{A}_d^{-1}\mathbf{A}_c$ lie in the interval $[1, \pi^2/4]$, as shown in Table 5.1, and therefore the system (5.8) can be solved efficiently by an iteration solver.

N	$\rho\left(\mathbf{A_{c}}\right)$	$\rho\left(\mathbf{A}_{\mathrm{d}}^{-1}\mathbf{A}_{\mathrm{c}}\right)$	$\rho\left(\mathbf{A_{c}}\right)$	$\rho\left(\mathbf{A}_{d}^{-1}\mathbf{A}_{c}\right)$
8	47.2	1.68	146.3	1.1
16	633.8	2.11	732.8	1.58
32	10202.0	2.30	10301.0	2.10
64	166178.9	2.39	166277.9	2.33
128	2691955.4	2.43	2692054.2	2.41
256	43374170.2	2.45	43374269.2	2.44
512	696563431.7	2.46	696563530.7	2.46

Table 5.1 The spectral radii of A_c and $A_d^{-1}A_c$.

5.2. GLLB pseudospectral method in a variational form

The collocation scheme (5.3) is based on a *strong form* of the original equation, while it is more preferable to implement the GLLB pseudospectral method in a *discrete weak form*. For simplicity, we consider (5.1) with a constant coefficient and homogeneous Neumann boundary conditions, i.e., $b(x) \equiv b$ and $g_{\pm} = 0$. The GLLB pseudospectral scheme reads

$$\begin{cases} \text{Find } u_N \in X_N \text{ such that} \\ \left\langle u'_N, v'_N \right\rangle_N + \left\langle bu_N, v_N \right\rangle_N = \left\langle f, v_N \right\rangle_N, \quad \forall v_N \in X_N. \end{cases}$$
(5.15)

Remark 5.2. Unlike the collocation method based on Legendre–Gauss–Lobotto points for Dirichlet boundary conditions, the GLLB scheme (5.3) (with $b(x) \equiv b$ and $g_{\pm} = 0$) is not equivalent to the pseudospectral scheme (5.15). Indeed, multiplying (5.3) with F(x) = f(x) by $v_N(x_j)\omega_j$ and summing the result for j = 1, 2, ..., N-1, we may rewrite the GLLB collocation scheme as

$$\begin{cases} \text{Find } u_N \in X_N \text{ such that} \\ \left\langle u'_N, v'_N \right\rangle_N + \left\langle bu_N, v_N \right\rangle_N = \left\langle f, v_N \right\rangle_N + \mathcal{R}_N, \quad \forall v_N \in X_N, \end{cases}$$
(5.16)

where

$$\mathcal{R}_{N} = (\mathbb{L}[u])'(-1)v_{N}(-1)\omega_{0} + (\mathbb{L}[u])'(1)v_{N}(1)\omega_{N} - f'(-1)v_{N}(-1)\omega_{0} - f'(1)v_{N}(1)\omega_{N} = -(u_{N}'''(-1) - (bu_{N})'(-1) + f'(-1))v_{N}(-1)\omega_{0} - (u_{N}'''(1) - (bu_{N})'(1) + f'(1))v_{N}(1)\omega_{N}.$$
(5.17)

Hence, up to a *boundary residual*, two schemes are equivalent. \Box

We now examine the matrix form of the system (5.15). As usual, we may choose the nodal basis $\{h_k\}_{k=1}^{N-1}$ for X_N . By (5.6), one verifies that the coefficient matrix under this basis is

$$\mathbf{A}_{\mathbf{p}} \coloneqq \left(\mathbf{D}_{\text{in}}^{(1)}\right)^T \mathbf{W} \mathbf{D}_{\text{in}}^{(1)} + b \mathbf{W}.$$
(5.18)

where $\mathbf{D}_{in}^{(1)} = \left(d_{jk}^{(1)}\right)_{1 \le j,k \le N-1}$ (cf. (5.10)), and $\mathbf{W} = \text{diag}(\omega_1, \omega_2, \dots, \omega_{N-1})$. We see that $\mathbf{A}_{\mathbf{p}}$ is full and ill-conditioned as the GLLB collocation system (5.8) (cf. Table 5.2), so it subjects to similar roundoff errors (cf. Fig. 5.1).

Table 5.2 Condition numbers

N		b = 0.01	b = 1	b = 100		
32	Ac	1.25e+06	1.24e+04	1.19e+02		
32	Ap	4.33e+05	4.40e + 03	8.01e+01		
32	$\mathbf{A}_{\mathbf{s}}^{\mathbf{r}}$	50.20	1.999	198.7		
64	Ac	1.95e + 07	1.93e + 05	1.83e+03		
64	Ap	3.43e + 06	3.50e + 04	6.72e+02		
64	$\mathbf{A}_{\mathbf{s}}^{\mathbf{r}}$	50.20	1.999	199.9		
128	Ac	3.08e + 08	3.05e + 06	2.88e+04		
128	Ap	2.74e + 07	2.79e+05	5.55e+03		
128	$\mathbf{A}_{\mathbf{s}}^{\mathbf{r}}$	50.20	1.999	199.9		
256	Ac	4.90e + 09	4.85e + 07	4.59e+05		
256	$\mathbf{A}_{\mathbf{p}}$	2.19e + 08	2.24e + 06	4.51e+04		
256	$\mathbf{A}_{\mathbf{S}}^{\mathbf{r}}$	50.20	1.999	199.9		



Fig. 5.1. Convergence rate (L^2 -errors vs. N) for b = 1 (left) and b = 100 (right).

In fact, it is more advisable to use a modal basis and perform the GLLB method in the frequency space. Using this approach, the spectral linear system will be sparse and well-conditioned. As with [22], we define the basis function as a "compact" combination of the Legendre polynomials

$$\phi_0(x) = 1, \qquad \phi_n(x) := d_n(L_n(x) + e_n L_{n+2}(x)), \quad n \ge 1,$$
(5.19)

where

$$d_n = \sqrt{\frac{(n+2)(n+3)}{2n(n+1)(2n+3)}}, \qquad e_n = -\frac{n(n+1)}{(n+2)(n+3)}.$$

Since $L'_n(\pm 1) = \frac{1}{2}(-1)^{n+1}n(n+1)$, one can verify readily that $\phi'_n(\pm 1) = 0$, and

$$X_N = \operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_{N-2}\}.$$

Moreover, as shown in Appendix B, we have

$$\phi'_n(x) = (1 - x^2) P_{n-1}(x) = \widetilde{\lambda}_{n-1}(1 - x^2) J_{n-1}^{(2,2)}(x), \quad n \ge 1,$$
(5.20)

which, together with (2.1), (2.8) and (4.3), leads to

$$s_{jk} \coloneqq \left\langle \phi'_k, \phi'_j \right\rangle_N = \left(\phi'_k, \phi'_j \right) = \begin{cases} 1, & 1 \le j = k \le N - 2, \\ 0, & \text{otherwise.} \end{cases}$$
(5.21)

Meanwhile, using (4.3), (5.19) and the orthogonality of the Legendre polynomials yields

$$m_{jk} := \langle \phi_k, \phi_j \rangle_N = \begin{cases} d_{N-2}^2 \left(\|L_{N-2}\|^2 + e_{N-2}^2 \langle L_N, L_N \rangle_N \right), & k = j = N-2 \\ (\phi_k, \phi_j), & \text{otherwise}, \end{cases}$$

\$\neq 0, & only if \$k = j\$ or \$k = j \pm 2\$.

Hence, by setting $u_N(x) = \sum_{n=0}^{N-2} \hat{u}_n \phi_n(x)$, and

$$\hat{\mathbf{u}} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-2})^{\mathrm{T}}, \quad f_j = \langle f, \phi_j \rangle_N, \quad \hat{\mathbf{f}} = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-2})^{\mathrm{T}},$$

the system (5.15) under the basis (5.19) becomes

$$\mathbf{A}_{\mathbf{s}}\hat{\mathbf{u}} := \left[\operatorname{diag}(0, 1, \dots, 1) + b(m_{jk})_{0 \le j,k \le N-2}\right]\hat{\mathbf{u}} = \hat{\mathbf{f}}.$$
(5.22)

We see that A_s is pentadiagonal with three nonzero diagonals, which can be decoupled into two tridiagonal sub-matrices, and inverted efficiently as in [22]. Moreover, the entries of the coefficient matrix can be evaluated *exactly*.

We can also prove that the condition number of \mathbf{A}_s does not depend on N. To show this, we define the discrete l^2 -inner product and norm, i.e., for any two vectors of length N-1, $\langle \mathbf{u}, \mathbf{v} \rangle_{l^2} := \sum_{j=0}^{N-2} \mathbf{u}_j \mathbf{v}_j$ and $\|\mathbf{u}\|_{l^2} = \langle \mathbf{u}, \mathbf{u} \rangle_{l^2}^{1/2}$. By using (5.21), (4.3), the definition of \mathbf{A}_s , (4.4) and the Poincaré inequality (4.12) successively, we derive that

$$\begin{aligned} \|\hat{\mathbf{u}}\|_{l^{2}}^{2} &= \|u_{N}'\|^{2} + \hat{u}_{0}^{2} \leq \langle u_{N}', u_{N}' \rangle_{N} + b \langle u_{N}, u_{N} \rangle_{N} + \hat{u}_{0}^{2} = \langle \mathbf{A}_{s} \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle_{l^{2}} + \hat{u}_{0}^{2} \\ &= \|u_{N}'\|_{N}^{2} + b \|u_{N}\|_{N}^{2} + \hat{u}_{0}^{2} \leq \|u_{N}'\|^{2} + 9b \|u_{N}\|^{2} + \hat{u}_{0}^{2} \\ &\leq (1 + cb) \|u_{N}'\|^{2} + cb |\bar{u}_{N}|^{2} + \hat{u}_{0}^{2} \\ &\leq (1 + cb) (\|u_{N}'\|^{2} + \hat{u}_{0}^{2}) = \|\hat{\mathbf{u}}\|_{l^{2}}^{2}, \end{aligned}$$
(5.23)

where in the last step, we used the fact $\bar{u}_N = \int_{-1}^1 u_N(x) dx = 2\hat{u}_0$. Therefore, we claim that $\text{Cond}(\mathbf{A}_s) \le 1 + cb$, which is also verified numerically by Table 5.2.

We compare in Table 5.2 the condition numbers of the matrices A_c , A_p and A_s resulting from the pure collocation method (PCOL, cf. (5.8)), pseudospectral method with nodal basis (PSND, cf. (5.18)) and pseudospectral method with modal basis (PSMD, cf. (5.22)), respectively. We see that for various *b* and *N*, the condition numbers of the system (5.22) are small and independent of *N* (also cf. (5.23)), while those of the collocation scheme and pseudospectral method using a nodal basis grow like $O(N^4)$.

To check the accuracy, we take b = 1 and $u(x) = \cos^2(12\pi x)$ as the exact solution. The discrete L^2 -errors against various N are plotted at the left of Fig. 5.1, which indicates an exponential convergence rate. We also see that PSND and PSMD methods can provide better numerical results than PCOL method. To exam the effect of roundoff errors, we take b = 100 and $u(x) = \cos^2(8\pi x)$ as an exact solution. We observe from Fig. 5.1(right) that the effect of roundoff errors is much more severe in PCOL method. Indeed, like the Gauss-Lobatto pseudospectral method, both PSND and PSMD have higher accuracy, when compared to PCOL. However, in multi-dimensional cases, PSMD (based on tensor product of the basis functions) is

more preferable, since it involves sparse and well-conditioned systems, which can be inverted very efficiently by using the matrix decomposition techniques (cf. [22]).

5.3. Error estimates

In this section, we apply the approximation results established in the previous section to analyze the errors of the GLLB pseudospectral scheme (5.15).

Theorem 5.1. Let u_N and u be respectively the solutions of (5.15) and (5.1) with b(x) = b > 0and $g_{\pm} = 0$. If $u \in X \cap B^m_{-2}(I)$ and $f \in B^s_{-2}(I)$ with integers $m, s \ge 2$, then

$$\|u - u_N\|_{\mu} \le c(N^{\mu - m} \|\partial_x^m u\|_{\omega^{m-2}} + N^{-s} \|\partial_x^s f\|_{\omega^{s-2}}), \quad \mu = 0, 1.$$
(5.24)

Proof. For simplicity, let $e_N = \pi_N^1 u - u_N$ and $E_v(\phi) = (v, \phi) - \langle v, \phi \rangle_N$. By (4.3), (5.1) and (5.15),

$$(\partial_x(u-u_N), \partial_x\phi) + b((u, \phi) - \langle u_N, \phi \rangle_N) = E_f(\phi), \quad \forall \phi \in X_N.$$

Thanks to (4.24), we can rewrite the above equation as

$$(\partial_x e_N, \partial_x \phi) + b \langle e_N, \phi \rangle_N = b \left(\left\langle \pi_N^1 u, \phi \right\rangle_N - (u, \phi) \right) + E_f(\phi), \quad \forall \phi \in X_N.$$
(5.25)

Using (4.4), Theorems 4.1 and 4.2, and Corollary 4.2 leads to

$$\begin{split} \left| \left\langle \pi_N^1 u, \phi \right\rangle_N - (u, \phi) \right| &\leq \left| \left\langle \pi_N^1 u - \mathcal{I}_N u, \phi \right\rangle_N \right| + |E_u(\phi)| \\ &\leq \|\pi_N^1 u - \mathcal{I}_N u\| \|\phi\| + |E_u(\phi)| \leq c N^{-m} \|\partial_x^m u\|_{\omega^{m-2}} \|\phi\|, \end{split}$$

and

$$|E_f(\phi)| \le N^{-s} \|\partial_x^s f\|_{\omega^{s-2}} \|\phi\|.$$

Hence, taking $\phi = e_N$ in (5.25) and using (4.4), we reach that

$$|e_N|_1 + ||e_N|| \le c(N^{-m} ||\partial_x^m u||_{\omega^{m-2}} + N^{-s} ||\partial_x^s f||_{\omega^{s-2}}).$$

Finally, using Theorem 4.2 again leads to the desired result. \Box

Remark 5.3. Theorem 4.2 of [4] presents an analysis estimate in L^2 -norm of a modified LGL collocation method for (5.1) (with b = 1) with a convergence order $O(N^{2-m})$. Obviously, the estimate (5.24) improve the existing result essentially, and seems optimal (with the order $O(N^{-m})$).

6. Concluding remarks

In the foregoing discussions, we restrict our attentions to one-dimensional linear problems, but the ideas and techniques go beyond these equations. An interesting extension is on the construction and analysis of a quadrature formula and the associated pseudospectral approximations for mixed boundary conditions $\alpha_{\pm}u'(\pm 1) + \beta_{\pm}u(\pm 1)$ where α_{\pm} and β_{\pm} are constants such that

$$-u'' + bu = f, \quad \text{in} \ (-1, 1); \quad \alpha_{\pm}u'(\pm 1) + \beta_{\pm}u(\pm 1) = g_{\pm}, \tag{6.1}$$

is well-posed. We will report this topic in our future work.

In summary, we have derived in this paper the asymptotic properties of the nodes and weights of the GLLB quadrature formula, and obtained optimal estimates for the GLLB interpolation errors. We also presented a detailed implementation of the associated GLLB pseudospectral method for second-order PDEs.

Acknowledgments

The work of the first author is partially supported by a Startup Grant of NTU, Singapore MOE Grant T207B2202, and Singapore NRF2007IDM-IDM002-010.

The work of the second author is supported in part by NSF of China No. 10871131, Science and Technology Commission of Shanghai Municipality, Grant No. 075105118, Shanghai Leading Academic Discipline Project No. S30405 and Fund for E-institute of Shanghai Universities No. E03004.

Appendix A. Expressions of the Lagrangian basis

For notational simplicity, let $q(x) := Q_{N-1}(x)$ be the quadrature polynomial given in (2.10), and define

$$q_j(x) := \frac{1}{Q'_{N-1}(x_j)} \frac{Q_{N-1}(x)}{x - x_j} = \frac{1}{q'(x_j)} \frac{q(x)}{x - x_j} \in \mathbb{P}_{N-2}, \quad 1 \le j \le N - 1.$$
(A.1)

We see that $\{q_j\}_{j=1}^{N-1}$ are the Lagrangian basis associated with the interior points $\{x_j\}_{j=1}^{N-1}$ of the GLLB quadrature. This matter with (5.6) implies that the corresponding basis functions can be expressed as follows.

• Boundary basis functions :

$$h_0(x) = \frac{\left[q'(1)(x-1) - q(1)\right]q(x)}{q(-1)q'(1) - 2q'(1)q'(-1) - q'(-1)q(1)},$$

$$h_N(x) = \frac{\left[q'(-1)(x+1) - q(-1)\right]q(x)}{q(1)q'(-1) + 2q'(1)q'(-1) - q'(1)q(-1)}.$$
(A.2)

• Interior basis functions:

$$h_j(x) = \frac{x^2 + ax + b}{x_j^2 + ax_j + b} q_j(x), \quad 1 \le j \le N - 1,$$
(A.3)

where

$$a = \frac{2q_{j}(1)q'_{j}(-1) + 2q'_{j}(1)q_{j}(-1)}{q'_{j}(1)q_{j}(-1) - 2q'_{j}(1)q'_{j}(-1) - q_{j}(1)q'_{j}(-1)},$$

$$b = \frac{3q_{j}(1)q'_{j}(-1) - 3q'_{j}(1)q_{j}(-1) + 2q'_{j}(1)q'_{j}(-1) - 4q_{j}(1)q_{j}(-1)}{q'_{j}(1)q_{j}(-1) - 2q'_{j}(1)q'_{j}(-1) - q_{j}(1)q'_{j}(-1)}.$$
(A.4)

Appendix B. The derivation of (5.20)

We first recall the formula (see, e.g., [23]):

$$\partial_x J_n^{(\alpha,\beta)}(x) = \frac{1}{2} (n+\alpha+\beta+1) J_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \ge 1.$$
(B.1)

Using (B.1) and the expressions of d_n and e_n gives

$$\begin{split} \phi_n'(x) &= d_n \left(\frac{1}{2} (n+1) J_{n-1}^{(1,1)}(x) + \frac{1}{2} (n+3) e_n J_{n+1}^{(1,1)}(x) \right) \\ &= \frac{1}{2} d_n (n+1) \left(J_{n-1}^{(1,1)}(x) - \frac{n}{n+2} J_{n+1}^{(1,1)}(x) \right). \end{split}$$

By (B.1) and formula (4.5.5) of [23],

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}J_n^{(1,1)}(x) = \frac{(n+1)(n+3)}{2n+3}\left(J_{n-1}^{(1,1)}(x) - \frac{n}{n+2}J_{n+1}^{(1,1)}(x)\right).$$

A combination of the above two facts leads to

$$\phi'_n(x) = \frac{2n+3}{4} d_n (1-x^2) J_{n-1}^{(2,2)}(x) = \widetilde{\lambda}_{n-1} (1-x^2) J_{n-1}^{(2,2)}(x).$$

Indeed, the normalized factor d_n is chosen such that $\{\phi_n\}$ is orthonormal in $L^2_{\omega^2}(I)$.

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