# Interpolation approximations based on Gauss-Lobatto-Legendre-Birkhoff quadrature 

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#### Abstract

We derive in this paper the asymptotic estimates of the nodes and weights of the Gauss-Lobatto-Legendre-Birkhoff (GLLB) quadrature formula, and obtain optimal error estimates for the associated GLLB interpolation in Jacobi weighted Sobolev spaces. We also present a user-oriented implementation of the pseudospectral methods based on the GLLB quadrature nodes for Neumann problems. This approach allows an exact imposition of Neumann boundary conditions, and is as efficient as the pseudospectral methods based on Gauss-Lobatto quadrature for PDEs with Dirichlet boundary conditions.


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## 1. Introduction

In a recent work, Ezzirani and Guessab [8] proposed a fast algorithm for computing the nodes and weights of some Gauss-Birkhoff type quadrature formulae (cf. [19,20]) with mixed boundary conditions. One special rule of particular importance, termed as the

[^0]Gauss-Lobatto-Legendre-Birkhoff (GLLB) quadrature, takes the form

$$
\begin{equation*}
\int_{-1}^{1} \phi(x) \mathrm{d} x \sim \phi^{\prime}(-1) \omega_{-}+\sum_{j=1}^{N-1} \phi\left(x_{j}\right) \omega_{j}+\phi^{\prime}(1) \omega_{+}, \tag{1.1}
\end{equation*}
$$

which is exact for all polynomials of degree $2 N-1$, and whose interior nodes are zeros of the quasi-orthogonal polynomial (cf. [25]) formed by a linear combination of the Jacobi polynomials $J_{N-1}^{(2,2)}(x)$ and $J_{N-3}^{(2,2)}(x)$. Motivated by [8], our first intention is to study the asymptotic behaviors of the nodes and weights of (1.1). Two important results are

$$
\begin{equation*}
x_{j} \cong \cos \frac{(2 j+1 / 2) \pi}{2 N+1}, \quad 1 \leq j \leq N-1, \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega_{j} \cong \frac{\pi}{N} \sin \theta_{j}, \quad \theta_{j}=\arccos x_{j}, \quad 1 \leq j \leq N-1 . \tag{1.3}
\end{equation*}
$$

With the aid of these estimates, we are able to analyze the GLLB interpolation errors:

$$
\begin{align*}
& \left\|\mathcal{I}_{N} u-u\right\|_{L^{2}(I)}+N^{-1}\left\|\left(1-x^{2}\right)^{1 / 2}\left(\mathcal{I}_{N} u-u\right)^{\prime}\right\|_{L^{2}(I)} \\
& \quad \leq c N^{-m}\left\|\left(1-x^{2}\right)^{(m-2) / 2} u{ }^{(m)}\right\|_{L^{2}(I)}, \tag{1.4}
\end{align*}
$$

where $I=(-1,1)$ and $\mathcal{I}_{N}$ is the Gauss-Birkhoff interpolation operator associated with the GLLB points. Similar to the approximation results on the Legendre-Gauss-Lobatto interpolation obtained in [15,16], the estimate (1.4) is optimal.

The GLLB quadrature formula involves derivative values $\phi^{\prime}( \pm 1)$ at the endpoints. Hence its nodes can be a natural choice of the preassigned interpolation points for many important problems, such as numerical integration and integral equations with the data of derivatives at the endpoints, etc. In particular, it plays an important role in spectral approximations of second-order elliptic problems with Neumann boundary conditions [8]. As we know, the numerical solutions of such problems given by the commonly used Galerkin method with any fixed mode $N$, do not fulfill the Neumann boundary conditions exactly. Thereby, we prefer to collocation methods oftentimes, whose solutions satisfy the physical boundary conditions. Whereas, in a collocation method, a proper choice of the collocation points is crucial in terms of accuracy, stability and ease of treating the boundary conditions ( BCs ), especially, the treatment of BCs is even more critical due to the global feature of spectral method [18,21]. The GLLB collocation approximation takes the boundary conditions into account in such a way that the resulting collocation systems have diagonal mass matrices, and therefore leads to explicit time discretization for time-dependent problems [8]. However, like the collocation method based on Gauss-Lobatto points [17,5,2,13, $3,24,9]$, the GLLB differentiation matrices are ill-conditioned. Thus a suitable preconditioned iteration solver is preferable in actual computations. We construct in this paper an efficient finitedifference preconditioning for one-dimensional second-order elliptic equations. We also present a user-oriented implementation and an error analysis of the GLLB pseudospectral methods based on variational formulations.

The rest of the paper is organized as follows. In Section 2, we introduce the GLLB quadrature formula, and then investigate the asymptotic behaviors of its nodes and weights in Section 3. With the aid of asymptotic estimates and some other approximation results, we are able to analyze the GLLB interpolation errors in Section 4. We give a user-oriented description of the
implementation of the GLLB collocation method, and present some illustrative numerical results in Section 5. The final section is for some concluding remarks.

## 2. The GLLB quadrature formula

In this section, we introduce the Gauss-Lobatto-Legendre-Birkhoff quadrature formula.

### 2.1. Preliminaries

We start with some notations to be used in the subsequent sections.

### 2.1.1. Notations

- Let $I:=(-1,1)$, and $\chi(x) \in L^{1}(I)$ be a generic positive weight function defined in $I$. For any integer $r \geq 0$, the weighted Sobolev space $H_{\chi}^{r}(I)$ is defined as usual with the inner product, semi-norm and norm denoted by $(u, v)_{r, \chi},|v|_{r, \chi}$ and $\|v\|_{r, \chi}$, respectively. In particular, $L_{\chi}^{2}(I)=H_{\chi}^{0}(I),(u, v)_{\chi}=(u, v)_{0, \chi}$ and $\|v\|_{\chi}=\|v\|_{0, \chi}$. For any real $r>0, H_{\chi}^{r}(I)$ and its norm are defined by space interpolation as in Admas [1]. In cases where no confusion would arise, $\chi$ may be dropped from the notations, whenever $\chi \equiv 1$.
- Let $\omega^{\alpha}(x)=\left(1-x^{2}\right)^{\alpha}, x \in I$ with $\alpha>-1$ be the Gegenbauer weight function. In particular, denote $\omega(x)=1-x^{2}$.
- Let $\mathbb{N}$ be the set of all non-negative integers. For any $N \in \mathbb{N}$, let $\mathbb{P}_{N}$ be the set of all algebraic polynomials of degree at most $N$. Without loss of generality, we assume that $N \geq 4$ throughout this paper.
- Denote by $c$ a generic positive constant, which is independent of any function and $N$ (mode in series expansion).
- For two sequences $\left\{z_{n}\right\}$ and $\left\{w_{n}\right\}$ with $w_{n} \neq 0$, the expression $z_{n} \sim w_{n}$ means $\left|z_{n}\right| /\left|w_{n}\right| \rightarrow$ $c(\neq 0)$ as $n \rightarrow \infty$. In particular, $z_{n} \cong w_{n}$ means $z_{n} / w_{n} \approx 1$ for $n \gg 1$, or $z_{n} / w_{n} \rightarrow 1$ for $n \rightarrow \infty$.
- For simplicity, we sometimes denote $\partial_{x}^{l} v=\frac{\mathrm{d}^{l} v}{\mathrm{~d} x^{l}}=v^{(l)}$, for any integer $l \geq 1$.


### 2.1.2. Jacobi polynomials

We recall some relevant properties of the Jacobi polynomials, denoted by $J_{n}^{(\alpha, \beta)}(x), x \in I$ with $\alpha, \beta>-1$ and $n \geq 0$, which are mutually orthogonal with respect to the Jacobi weight $\omega^{\alpha, \beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}:$

$$
\begin{equation*}
\int_{-1}^{1} J_{n}^{(\alpha, \beta)}(x) J_{m}^{(\alpha, \beta)}(x) \omega^{\alpha, \beta}(x) \mathrm{d} x=\gamma_{n}^{(\alpha, \beta)} \delta_{m n} \tag{2.1}
\end{equation*}
$$

where $\delta_{m n}$ is the Knonecker symbol, and

$$
\begin{equation*}
\gamma_{n}^{(\alpha, \beta)}=\frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2 n+\alpha+\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1)} . \tag{2.2}
\end{equation*}
$$

In the subsequent discussions, we will mainly use two special types of Jacobi polynomials. The first one is $J_{n}^{(2,2)}(x)$, defined by the three-term recursive relation (see, e.g., Szegö [23]):

$$
\begin{align*}
& n(n+4) J_{n}^{(2,2)}(x)=(n+2)(2 n+3) x J_{n-1}^{(2,2)}(x)-(n+1)(n+2) J_{n-2}^{(2,2)}(x), \\
& J_{0}^{(2,2)}(x)=1, \quad J_{1}^{(2,2)}(x)=3 x, \quad x \in[-1,1] . \tag{2.3}
\end{align*}
$$

The coefficient of leading term of $J_{n}^{(2,2)}(x)$ is

$$
\begin{equation*}
K_{n}^{(2,2)}=\frac{(2 n+4)!}{2^{n} n!(n+4)!} . \tag{2.4}
\end{equation*}
$$

Moreover, we have

$$
\begin{equation*}
J_{n}^{(2,2)}(-x)=(-1)^{n} J_{n}^{(2,2)}(x), \quad J_{n}^{(2,2)}(1)=\frac{(n+1)(n+2)}{2} . \tag{2.5}
\end{equation*}
$$

The corresponding monic polynomial is defined by dividing the leading coefficient in (2.4):

$$
\begin{equation*}
P_{n}(x):=\lambda_{n} J_{n}^{(2,2)}(x) \quad \text { with } \lambda_{n}=\left(K_{n}^{(2,2)}\right)^{-1} \tag{2.6}
\end{equation*}
$$

As a direct consequence of (2.3) and (2.6),

$$
\begin{equation*}
P_{n+1}(x)=x P_{n}(x)-a_{n} P_{n-1}(x) \quad \text { with } a_{n}=\frac{n(n+4)}{(2 n+3)(2 n+5)}, \tag{2.7}
\end{equation*}
$$

where $P_{-1}=0$ and $P_{0}=1$. By (2.1)-(2.2), the normalized Jacobi polynomials

$$
\begin{equation*}
\widetilde{P}_{n}(x):=\tilde{\lambda}_{n} J_{n}^{(2,2)}(x) \quad \text { with } \tilde{\lambda}_{n}=\left(\gamma_{n}^{(2,2)}\right)^{-1 / 2} \tag{2.8}
\end{equation*}
$$

satisfy $\left\|\widetilde{P}_{n}\right\|_{\omega^{2}}^{2}=1$.
We will also utilize the Legendre polynomials, denoted by $L_{n}(x)$, which are mutually orthogonal with respect to the unit weight $\omega=1$. Note that the constant of orthogonality and leading coefficient respectively are

$$
\begin{equation*}
\gamma_{n}^{(0,0)}=\left\|L_{n}\right\|^{2}=\frac{2}{2 n+1}, \quad K_{n}^{(0,0)}=\frac{(2 n)!}{2^{n}(n!)^{2}} \tag{2.9}
\end{equation*}
$$

Recall that $L_{n}( \pm 1)=( \pm 1)^{n}$ and $L_{n}^{\prime}( \pm 1)=\frac{1}{2}(-1)^{n-1} n(n+1)$.

### 2.2. GLLB quadrature formula

The GLLB quadrature formula can be derived from Theorem 4.2 of Ezzirani and Guessab [8], which belongs to a Birkhoff type rule (see, e.g., $[19,20]$ ).

Theorem 2.1. Let $\left\{P_{n}\right\}$ be the monic Jacobi polynomials defined in (2.6). Consider the quasiorthogonal polynomial:

$$
\begin{equation*}
Q_{N-1}(x)=P_{N-1}(x)+b_{N} P_{N-3}(x), \quad N \geq 2, \tag{2.10}
\end{equation*}
$$

where

$$
\begin{align*}
b_{N} & =-\frac{(N-1)(N+2)\left(2 N^{2}+2 N+3\right)}{(2 N-1)(2 N+1)\left(2 N^{2}-2 N-3\right)}+\frac{\sqrt{12(N-1)(N+2)\left(N^{2}+N-3\right)}}{(2 N-1)\left(2 N^{2}-2 N-3\right)} \\
& :=-b_{N}^{I}+b_{N}^{\mathbb{I}} . \tag{2.11}
\end{align*}
$$

Then we have that
(1) For $N \geq 4$,

$$
\begin{equation*}
\frac{1}{4}-\frac{1}{N}<-b_{N}<\frac{(N+2)(N+3)}{(2 N-1)(2 N+1)} \tag{2.12}
\end{equation*}
$$

(2) The $N-1$ zeros of $Q_{N-1}(x)$, denoted by $\left\{x_{j}\right\}_{j=1}^{N-1}$, are distinct, real and all located in $(-1,1)$.
(3) There exists a unique set of weights $\left\{\omega_{j}\right\}_{j=0}^{N}$ such that

$$
\begin{equation*}
\int_{-1}^{1} \phi(x) \mathrm{d} x=\phi^{\prime}(-1) \omega_{0}+\sum_{j=1}^{N-1} \phi\left(x_{j}\right) \omega_{j}+\phi^{\prime}(1) \omega_{N}:=\mathcal{S}_{N}[\phi], \quad \forall \phi \in \mathbb{P}_{2 N-1} \tag{2.13}
\end{equation*}
$$

(4) The interior weights are all positive and explicitly expressed by

$$
\begin{equation*}
\omega_{j}=\frac{A_{N}}{P_{N-2}\left(x_{j}\right) Q_{N-1}^{\prime}\left(x_{j}\right)} \frac{1}{\left(1-x_{j}^{2}\right)^{2}}, \quad 1 \leq j \leq N-1, \tag{2.14}
\end{equation*}
$$

with

$$
\begin{equation*}
A_{N}=\left(1-\frac{b_{N}}{a_{N-2}}\right) \frac{\lambda_{N-2}^{2}}{\tilde{\lambda}_{N-2}^{2}} \tag{2.15}
\end{equation*}
$$

where the constants $a_{N-2}, b_{N}, \lambda_{N-2}$ and $\tilde{\lambda}_{N-2}$ are defined in (2.7), (2.11), (2.6) and (2.8), respectively.

Proof. The proof is based on a reassembly and extension of some relevant results in [8]. For clarity, we first present a one-to-one correspondence between the notations of [8] and those of this paper:

$$
\begin{align*}
& \ln \leftrightharpoons N-1, \quad \mathrm{~d} \hat{\sigma} \leftrightharpoons\left(1-x^{2}\right)^{2} \mathrm{~d} x, \quad \hat{\pi}_{k} \leftrightharpoons P_{k}, \quad \tilde{\pi}_{k} \leftrightharpoons \widetilde{P}_{k}, \quad q_{n, 2} \leftrightharpoons Q_{N-1}  \tag{2.16}\\
& \beta_{k} \leftrightharpoons a_{k}, \quad \hat{s}_{N-1} \leftrightharpoons b_{N}, \quad \beta_{n-1} \leftrightharpoons a_{N-2}-b_{N}, \quad J_{n}^{*}(\hat{\sigma}) \leftrightharpoons \mathbb{J}_{N-1}
\end{align*}
$$

We next derive the expression of $b_{N}$. By Theorem 4.2 and 4.3 of [8], we find that

$$
\begin{aligned}
& a_{N-2}-b_{N}=\beta_{N-2} \\
& \quad=\frac{-\sqrt{12(N-1)(N+2)\left(N^{2}+N-3\right)}+(N+2)\left(2 N^{2}-4 N+3\right)}{(2 N-1)\left(2 N^{2}-2 N-3\right)}
\end{aligned}
$$

Hence, a direct computation leads to

$$
b_{N}=a_{N-2}-\beta_{N-2} \stackrel{(2.7)}{=} \text { RHS of (2.11). }
$$

(1) The special values $b_{j}(j=1,2,3)$ can be computed directly via (2.11). To obtain the bounds for $b_{N}$ with $N \geq 4$, we first prove that

$$
\begin{equation*}
-b_{N}<\frac{(N+2)(N+3)}{(2 N-1)(2 N+1)} \tag{2.17}
\end{equation*}
$$

Since $-b_{N}=b_{N}^{I}-b_{N}^{\mathbb{I}}<b_{N}^{I}$, it suffices to verify

$$
W_{1}:=(N+2)(N+3)\left(2 N^{2}-2 N-3\right)-(N-1)(N+2)\left(2 N^{2}+2 N+3\right)>0
$$

A direct calculation yields

$$
W_{1}=2(N-3)(N+2)(2 N+1) .
$$

Hence (2.17) is valid for $N \geq 4$, and it remains to prove

$$
\begin{equation*}
-b_{N}>\frac{1}{N}-\frac{1}{4} \tag{2.18}
\end{equation*}
$$

Clearly, $-\sqrt{N^{2}+N-3}>-\sqrt{N^{2}+N-2}=-\sqrt{(N-1)(N+2)}$ and $2 N(N-1)>$ $2 N^{2}-2 N-3$. Therefore,

$$
\begin{aligned}
-b_{N} & >\frac{(N-1)(N+2)\left(2 N^{2}+2 N+3\right)-(2 N+1) \sqrt{12}(N-1)(N+2)}{(2 N-1)(2 N+1)\left(2 N^{2}-2 N-3\right)} \\
& >\frac{(N-1)(N+2)\left[\left(2 N^{2}+2 N+3\right)-4(2 N+1)\right]}{(2 N-1)(2 N+1)\left(2 N^{2}-2 N-3\right)} \\
& >\frac{(N-1)(N+2)\left(2 N^{2}-6 N-8\right)}{2 N(N-1)(2 N-1)(2 N+1)} \\
& >\frac{(N+2)(N-4)(N+1)}{2 N(2 N-1)(2 N+1)} \\
& >\frac{(N+2)(N-4)}{2 N(2 N-1)}>\frac{N-4}{4 N}=\frac{1}{4}-\frac{1}{N} .
\end{aligned}
$$

Thus, a combination of (2.17) and (2.18) leads to (2.12).
(2) Observe from (2.12) that for $N \geq 3,-b_{N}>0$. Therefore, by Theorem 4.1 of [8], the quasi-orthogonal polynomial can be expressed as a determinant

$$
\begin{equation*}
Q_{N-1}(x)=P_{N-1}(x)+b_{N} P_{N-3}(x)=\operatorname{det}\left(x \mathbb{I}_{N-1}-\mathbb{J}_{N-1}\right), \tag{2.19}
\end{equation*}
$$

where $\mathbb{I}_{N-1}$ is an identity matrix, and $\mathbb{J}_{N-1}$ is a symmetric tri-diagonal matrix of order $N-1$,

$$
\mathbb{J}_{N-1}=\left[\begin{array}{ccccc}
0 & \sqrt{a_{1}} & & &  \tag{2.20}\\
\sqrt{a_{1}} & 0 & \sqrt{a_{2}} & & \\
& \ddots & \ddots & \ddots & \\
& & \sqrt{a_{N-3}} & 0 & \sqrt{a_{N-2}-b_{N}} \\
& & & \sqrt{a_{N-2}-b_{N}} & 0
\end{array}\right]
$$

with $a_{n}$ and $b_{N}$ given in (2.7) and (2.11), respectively. Therefore, all the zeros of $Q_{N-1}$ are real and distinct.
(3) Theorem 4.2 of [8] reveals that with such a choice of $b_{N}$, all the zeros of $Q_{N-1}$ are distributed in $(-1,1)$, and the quadrature formula (2.13) is unique. Moreover, the interior weights $\left\{\omega_{j}\right\}_{j=1}^{N-1}$ are all positive.
(4) We now consider the expressions of the weights. By Formula (38) of [8],

$$
\begin{equation*}
\left(1-x_{j}^{2}\right)^{2} \omega_{j}=\frac{A_{N}^{*}}{K_{N}^{*}\left(x_{j}\right)}, \quad 1 \leq j \leq N-1, \tag{2.21}
\end{equation*}
$$

where $A_{N}^{*}=1-b_{N} / a_{N-2}$, and

$$
\begin{equation*}
K_{N}^{*}\left(x_{j}\right)=A_{N}^{*} \sum_{k=0}^{N-3} \widetilde{P}_{k}^{2}\left(x_{j}\right)+\widetilde{P}_{N-2}^{2}\left(x_{j}\right) \tag{2.22}
\end{equation*}
$$

To simplify $K_{N}^{*}\left(x_{j}\right)$, we recall the Christoffel-Darboux formula (see, e.g., Szegö [23]):

$$
\begin{equation*}
\widetilde{P}_{n+1}^{\prime}(x) \widetilde{P}_{n}(x)-\widetilde{P}_{n}^{\prime}(x) \widetilde{P}_{n+1}(x)=\frac{d_{n+1}}{d_{n}} \sum_{k=0}^{n} \widetilde{P}_{k}^{2}(x)>0, \quad \forall x \in[-1,1], \tag{2.23}
\end{equation*}
$$

where $d_{k}$ is the leading coefficient $\widetilde{P}_{k}(x)$, and by (2.6) and (2.8),

$$
\begin{equation*}
d_{k}=\frac{\widetilde{\lambda}_{k}}{\lambda_{k}}, \quad \widetilde{P}_{k}(x)=d_{k} P_{k}(x) \tag{2.24}
\end{equation*}
$$

Hence,

$$
\begin{align*}
Q_{N-1}^{\prime}(x) \widetilde{P}_{N-2}(x) & -\widetilde{P}_{N-2}^{\prime}(x) Q_{N-1}(x) \stackrel{(2.10)}{=}\left[P_{N-1}^{\prime}(x) \widetilde{P}_{N-2}(x)-\widetilde{P}_{N-2}^{\prime}(x) P_{N-1}(x)\right] \\
& -b_{N}\left[\widetilde{P}_{N-2}^{\prime}(x) P_{N-3}(x)-P_{N-3}^{\prime}(x) \widetilde{P}_{N-2}(x)\right] \\
& \stackrel{(2.24)}{=} \frac{1}{d_{N-1}}\left[\widetilde{P}_{N-1}^{\prime}(x) \widetilde{P}_{N-2}(x)-\widetilde{P}_{N-2}^{\prime}(x) \widetilde{P}_{N-1}(x)\right] \\
& -\frac{b_{N}}{d_{N-3}}\left[\widetilde{P}_{N-2}^{\prime}(x) \widetilde{P}_{N-3}(x)-\widetilde{P}_{N-3}^{\prime}(x) \widetilde{P}_{N-2}(x)\right] \\
& \stackrel{(2.23)}{=} \frac{1}{d_{N-2}} \sum_{k=0}^{N-2} \widetilde{P}_{k}^{2}(x)-\frac{b_{N} d_{N-2}}{d_{N-3}^{2}} \sum_{k=0}^{N-3} \widetilde{P}_{k}^{2}(x)  \tag{2.25}\\
& =\frac{1}{d_{N-2}}\left\{\left[1-\frac{b_{N} d_{N-2}^{2}}{d_{N-3}^{2}}\right] \sum_{k=0}^{N-3} \widetilde{P}_{k}^{2}(x)+\widetilde{P}_{N-2}^{2}(x)\right\} \\
& \stackrel{(2.26)}{=} \frac{1}{d_{N-2}}\left\{\left[1-\frac{b_{N}}{a_{N-2}}\right] \sum_{k=0}^{N-3} \widetilde{P}_{k}^{2}(x)+\widetilde{P}_{N-2}^{2}(x)\right\} .
\end{align*}
$$

In the last step, we used the identity

$$
\begin{equation*}
a_{N-2}=\frac{\tilde{\lambda}_{N-3}^{2}}{\tilde{\lambda}_{N-2}^{2}} \frac{\lambda_{N-2}^{2}}{\lambda_{N-3}^{2}}=\frac{d_{N-3}^{2}}{d_{N-2}^{2}} \tag{2.26}
\end{equation*}
$$

which can be verified directly by the definition of the constants. Thus, taking $x=x_{j}$ in (2.25) and using the fact: $Q_{N-1}\left(x_{j}\right)=0,1 \leq j \leq N-1$, gives

$$
\begin{equation*}
Q_{N-1}^{\prime}\left(x_{j}\right) \widetilde{P}_{N-2}\left(x_{j}\right) \stackrel{(2.25)}{=} \frac{1}{d_{N-2}}\left\{A_{N}^{*} \sum_{k=0}^{N-3} \widetilde{P}_{k}^{2}\left(x_{j}\right)+\widetilde{P}_{N-2}^{2}\left(x_{j}\right)\right\}=\frac{1}{d_{N-2}} K_{N}^{*}\left(x_{j}\right) \tag{2.27}
\end{equation*}
$$

which implies that

$$
K_{N}^{*}\left(x_{j}\right) \stackrel{(2.24)}{=} d_{N-2}^{2} Q_{N-1}^{\prime}\left(x_{j}\right) P_{N-2}\left(x_{j}\right) \stackrel{(2.24)}{=} \frac{\tilde{\lambda}_{N-2}^{2}}{\lambda_{N-2}^{2}} Q_{N-1}^{\prime}\left(x_{j}\right) P_{N-2}\left(x_{j}\right)
$$

Plugging it into (2.21) leads to (2.14)-(2.15).
Remark 2.1. (1) The choice of $b_{N}$ so that all zeros of $Q_{N-1}(x)$ are in $(-1,1)$, is unique, and thereby uniquely determines the quadrature rule. It is seen from (2.12) that $b_{N}$ is uniformly bounded with $-b_{N} \cong 1 / 4$, whose behavior is depicted in Fig. 3.1 (left).
(2) The formulae (2.19)-(2.20) indicate that the quadrature nodes $\left\{x_{j}\right\}_{j=1}^{N-1}$ are eigenvalues of the symmetric tridiagonal matrix $\mathbb{J}_{N-1}$, and therefore, can be evaluated by using some standard methods (e.g., the QR-algorithm) as with the classical Gauss quadrature (see, e.g., [11]).


Fig. 3.1. Left: the constant $-b_{N}$ and its bounds (cf. (2.10)-(2.11)) against $N$. Right: asymptotic behavior of $\Delta_{N}^{\min }$ and $\Delta_{N}^{\max }$ against $N$ (right).

Making use of Formula (41) of [8], the weights $\left\{\omega_{j}\right\}_{j=1}^{N-1}$ can be computed from the first component of the orthonormal eigenvectors of $\mathbb{J}_{N-1}$.
(3) Alternative to the eigen-method, the nodes $\left\{x_{j}\right\}_{j=1}^{N-1}$ can be located by a root-finding method, say Newton-Raphson iteration, which turns out to be more efficient for a quadrature rule of higher order. A good initial approximation might be $\left\{x_{j}^{(0)}\right\}_{j=1}^{N-1}$ given in (3.24). Accordingly, the weights $\left\{\omega_{j}\right\}_{j=1}^{N-1}$ can be computed via the compact formulae (2.14)-(2.15).
(4) It is worthwhile to point out that the quadrature nodes and weights are symmetric

$$
\begin{equation*}
x_{j}+x_{N-j}=0, \quad \omega_{j}=\omega_{N-j}, \quad j=1,2, \ldots, N-1 \tag{2.28}
\end{equation*}
$$

Therefore, the computational cost can be halved.
(5) Let $h_{0}$ and $h_{N}$ be the Lagrangian polynomials given in (5.6) of this paper. The boundary weights

$$
\omega_{0}=\int_{-1}^{1} h_{0}(x) \mathrm{d} x, \quad \omega_{N}=\int_{-1}^{1} h_{N}(x) \mathrm{d} x .
$$

One verifies that $\omega_{0}=-\omega_{N}$, and an explicit evaluation of the above integrals by using the relevant properties of Jacobi polynomials leads to the formula for $\omega_{0}$ in Theorem 4.2 of [8].
The GLLB quadrature formula (2.13) enjoys the same degree of exactness as the Gauss-Legendre-Lobatto (GLL) quadrature (with $N+1$ nodes), but in contrast to GLL, GLLB includes the boundary derivative values $\phi^{\prime}( \pm 1)$, which is thereby suitable for exactly imposing Neumman boundary conditions (see Section 5).

## 3. Asymptotic properties of the nodes and weights

In this section, we make a quantitative asymptotic estimate of the nodes and weights in the GLLB quadrature formula (2.13). These properties will play an essential role in our subsequent analysis of the GLLB interpolation errors.

### 3.1. Asymptotic property of the nodes

We start with the interlacing property. Theorem 4.1 of [20] reveals that the zeros of $Q_{N-1}$ interlace with those of $Q_{N-2}$. The following theorem indicates that the zeros of $Q_{N-1}$ also
interlace with those of $\widetilde{P}_{N-2}$, which can be proved in a fashion similar to that in [20]. For integrity, we provide the proof below.

Theorem 3.1. Let $\left\{x_{j}\right\}_{j=1}^{N-1}$ and $\left\{y_{k}\right\}_{k=1}^{N-2}$ be the zeros of $Q_{N-1}(x)$ and $\widetilde{P}_{N-2}(x)$, respectively. Then we have

$$
\begin{equation*}
-1=y_{0}<x_{1}<y_{1}<x_{2}<y_{2}<\cdots<y_{N-2}<x_{N-1}<1=y_{N-1} \tag{3.1}
\end{equation*}
$$

Proof. We take three steps to complete the proof.
Step I: Show that

$$
\begin{equation*}
Q_{N-1}\left(y_{j}\right) Q_{N-1}\left(y_{j+1}\right)<0, \quad j=1,2, \ldots N-3, \tag{3.2}
\end{equation*}
$$

equivalently to say, between two consecutive zeros of $\widetilde{P}_{N-2}(x)$, there exists at least one zero of $Q_{N-1}(x)$. To justify this, we first deduce that

$$
\begin{equation*}
Q_{N-1}^{\prime}(x) \widetilde{P}_{N-2}(x)-\widetilde{P}_{N-2}^{\prime}(x) Q_{N-1}(x)>0, \quad \forall x \in[-1,1] \tag{3.3}
\end{equation*}
$$

which is a consequence of (2.7), (2.25) and (2.12), since

$$
d_{N-2}>0, \quad 1-\frac{b_{N}}{a_{N-2}}>1+\left(\frac{1}{4}-\frac{1}{N}\right) \frac{1}{a_{N-2}} \geq 1, \quad \forall N \geq 4
$$

Thanks to $\widetilde{P}_{N-2}\left(y_{k}\right)=0$, taking $x=y_{j}, y_{j+1}$ in (3.3) yields

$$
\begin{equation*}
\widetilde{P}_{N-2}^{\prime}\left(y_{j}\right) Q_{N-1}\left(y_{j}\right)<0, \quad \widetilde{P}_{N-2}^{\prime}\left(y_{j+1}\right) Q_{N-1}\left(y_{j+1}\right)<0 \tag{3.4}
\end{equation*}
$$

which certainly implies

$$
\widetilde{P}_{N-2}^{\prime}\left(y_{j}\right) \widetilde{P}_{N-2}^{\prime}\left(y_{j+1}\right) \cdot Q_{N-1}\left(y_{j}\right) Q_{N-1}\left(y_{j+1}\right)>0
$$

Therefore, to prove (3.2), it suffices to check that

$$
\begin{equation*}
\widetilde{P}_{N-2}^{\prime}\left(y_{j}\right) \widetilde{P}_{N-2}^{\prime}\left(y_{j+1}\right)<0 \tag{3.5}
\end{equation*}
$$

Notice that

$$
\widetilde{P}_{N-2}(x)=d_{N-2} \prod_{k=1}^{N-2}\left(x-y_{k}\right)
$$

and consequently,

$$
\begin{align*}
& \widetilde{P}_{N-2}^{\prime}\left(y_{j}\right)=d_{N-2} \prod_{k=1}^{j-1}\left(y_{j}-y_{k}\right) \cdot \prod_{k=j+1}^{N-2}\left(y_{j}-y_{k}\right)=(-1)^{N-j-2} d_{N-2} \cdot C_{1}  \tag{3.6}\\
& \widetilde{P}_{N-2}^{\prime}\left(y_{j+1}\right)=(-1)^{N-j-3} d_{N-2} \cdot C_{2}
\end{align*}
$$

where $C_{1}$ and $C_{2}$ are two positive constants. Hence

$$
\begin{equation*}
\widetilde{P}_{N-2}^{\prime}\left(y_{j}\right) \widetilde{P}_{N-2}^{\prime}\left(y_{j+1}\right)=(-1)^{2 N-2 j-5} \cdot d_{N-2}^{2} C_{1} C_{2}<0, \quad 1 \leq j \leq N-3 \tag{3.7}
\end{equation*}
$$

This validates (3.5) and thereby (3.2) follows.
Step II: At this point, it remains to consider the possibility of possessing zeros of $Q_{N-1}(x)$ in subintervals $\left(y_{N-2}, 1\right)$ and $\left(-1, y_{1}\right)$. Clearly, by the first identity of (3.6), the largest zero $y_{N-2}$ satisfies $\widetilde{P}_{N-2}^{\prime}\left(y_{N-2}\right)>0$. Thus, by (3.4),

$$
\begin{equation*}
Q_{N-1}\left(y_{N-2}\right)<0 . \tag{3.8}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
Q_{N-1}(1)=P_{N-1}(1)+b_{N} P_{N-3}(1)>0, \tag{3.9}
\end{equation*}
$$

which is due to a direct calculation using (2.5)-(2.6) and (2.12):

$$
\frac{P_{N-1}(1)}{P_{N-3}(1)}+b_{N}=\frac{(N+2)(N+3)}{(2 N-1)(2 N+1)}+b_{N}>0
$$

and $P_{N-3}(1)>0$. The sign of (3.8)-(3.9) indicates that there is at least one zero of $Q_{N-1}(x)$ in the interval $\left(y_{N-2}, 1\right)$. Using the symmetry of the zeros (see $[8,23]$ ):

$$
\begin{equation*}
x_{j}=-x_{N-j}, \quad j=1, \ldots, N-1 ; \quad y_{k}=-y_{N-1-k}, \quad k=1, \ldots, N-2, \tag{3.10}
\end{equation*}
$$

we deduce that the interval $\left(-1, y_{1}\right)$ also contains at least one zero of $Q_{N-1}(x)$.
Final step: A combination of the previous statements reaches the conclusion: each of the $N-1$ subintervals $\left\{\left(y_{j}, y_{j+1}\right)\right\}_{j=0}^{N-2}\left(y_{0}=-1, y_{N-1}=1\right)$ contains at least one of the $N-1$ zeros of $Q_{N-1}(x)$, and therefore there can only be a unique one in each subinterval.

To visualize the above interlacing property, we denote

$$
\begin{equation*}
\Delta_{N}^{\min }=\min _{1 \leq j \leq N-1}\left(y_{j}-x_{j}\right), \quad \Delta_{N}^{\max }=\max _{1 \leq j \leq N-1}\left(y_{j}-x_{j}\right), \quad \forall N \geq 4 \tag{3.11}
\end{equation*}
$$

According to Theorem 3.1, we expect to see that

$$
\begin{equation*}
\Delta_{N}^{\max }>\Delta_{N}^{\min }>0, \quad \forall N \geq 4, \quad \lim _{N \rightarrow \infty} \Delta_{N}^{\min }=\lim _{N \rightarrow \infty} \Delta_{N}^{\max }=0 \tag{3.12}
\end{equation*}
$$

which is illustrated by Fig. 3.1(right).
In the analysis of interpolation error, we need more precise asymptotic estimates of the GLLB quadrature nodes. For this purpose, hereafter, we assume that the zeros $\left\{x_{j}\right\}_{j=1}^{N-1}$ of $Q_{N-1}(x)$ are arranged in descending order. We make the change of variables

$$
\begin{equation*}
x=\cos \theta, \quad \theta \in[0, \pi], \quad \theta_{j}=\cos ^{-1} x_{j}, \quad j=1,2, \ldots, N-1 . \tag{3.13}
\end{equation*}
$$

The main result is stated in the following theorem.
Theorem 3.2. Let $\left\{\theta_{j}\right\}_{j=1}^{N-1}$ be the same as in (3.13). Then

$$
\begin{equation*}
\theta_{j} \in I_{j}:=\left(\widetilde{\theta}_{j-1}, \tilde{\theta}_{j}\right) \subset(0, \pi), \quad 1 \leq j \leq N-1 \tag{3.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{\theta}_{j}=\frac{2 j+\frac{3}{2}}{2 N+1} \pi, \quad 0 \leq j \leq N-1 \tag{3.15}
\end{equation*}
$$

In other words,

$$
\begin{equation*}
0<\widetilde{\theta}_{0}<\theta_{1}<\widetilde{\theta}_{1}<\theta_{2}<\widetilde{\theta}_{2}<\cdots<\widetilde{\theta}_{N-2}<\theta_{N-1}<\widetilde{\theta}_{N-1}<\pi . \tag{3.16}
\end{equation*}
$$

Proof. By the intermediate mean-value theorem, (3.14) is equivalent to

$$
\begin{equation*}
Q_{N-1}\left(\cos \widetilde{\theta}_{j-1}\right) Q_{N-1}\left(\cos \widetilde{\theta}_{j}\right)<0, \quad j=1,2, \ldots, N-1 . \tag{3.17}
\end{equation*}
$$

To prove this result, we first recall Formula (8.21.10) of Szegö [23]:

$$
\begin{equation*}
J_{n}^{(2,2)}(\cos \theta)=n^{-\frac{1}{2}} G(\theta) \cos \left(\left(n+\frac{5}{2}\right) \theta+\gamma\right)+O\left(n^{-\frac{3}{2}}\right), \quad \theta \in(0, \pi) \tag{3.18}
\end{equation*}
$$

where

$$
\begin{equation*}
G(\theta)=\pi^{-\frac{1}{2}}\left(\sin \frac{\theta}{2}\right)^{-\frac{5}{2}}\left(\cos \frac{\theta}{2}\right)^{-\frac{5}{2}}=4 \sqrt{\frac{2}{\pi}}(\sin \theta)^{-\frac{5}{2}}, \quad \gamma=-\frac{5}{4} \pi \tag{3.19}
\end{equation*}
$$

Hence, using (2.6), (2.10) and (3.18) gives

$$
\begin{align*}
& Q_{N-1}(\cos \theta) \stackrel{(2.10)}{=} P_{N-1}(\cos \theta)+b_{N} P_{N-3}(\cos \theta) \\
& \stackrel{(2.6)}{=} \lambda_{N-1} J_{N-1}^{(2,2)}(\cos \theta)+b_{N} \lambda_{N-3} J_{N-3}^{(2,2)}(\cos \theta) \\
& \stackrel{(3.18)}{=} G(\theta)\left\{(N-1)^{-\frac{1}{2}} \lambda_{N-1} \cos \left(\left(N+\frac{3}{2}\right) \theta+\gamma\right)\right. \\
&\left.+(N-3)^{-\frac{1}{2}} b_{N} \lambda_{N-3} \cos \left(\left(N-\frac{1}{2}\right) \theta+\gamma\right)\right\} \\
&+\left\{\lambda_{N-1} O\left((N-1)^{-\frac{3}{2}}\right)+b_{N} \lambda_{N-3} O\left((N-3)^{-\frac{3}{2}}\right)\right\} \\
&:= H_{N}(\theta)+R_{N} . \tag{3.20}
\end{align*}
$$

Notice that $\tilde{\theta}_{j}$ in (3.15) solves the equation

$$
\left(N+\frac{1}{2}\right) \tilde{\theta}_{j}+\gamma=\left(j-\frac{1}{2}\right) \pi, \quad 0 \leq j \leq N-1,
$$

which implies

$$
\left(N+\frac{3}{2}\right) \widetilde{\theta}_{j}+\gamma=\left(j-\frac{1}{2}\right) \pi+\widetilde{\theta}_{j}, \quad\left(N-\frac{1}{2}\right) \tilde{\theta}_{j}+\gamma=\left(j-\frac{1}{2}\right) \pi-\widetilde{\theta}_{j} .
$$

Therefore, taking $\theta=\widetilde{\theta}_{j}$ in (3.20) and using trigonometric identities, gives

$$
\begin{align*}
H_{N}\left(\widetilde{\theta}_{j}\right)= & G\left(\widetilde{\theta}_{j}\right)\left\{(N-1)^{-\frac{1}{2}} \lambda_{N-1} \cos \left(\left(j-\frac{1}{2}\right) \pi+\widetilde{\theta}_{j}\right)\right. \\
& \left.+(N-3)^{-\frac{1}{2}} b_{N} \lambda_{N-3} \cos \left(\left(j-\frac{1}{2}\right) \pi-\widetilde{\theta}_{j}\right)\right\} \\
= & (-1)^{j}\left(\sin \widetilde{\theta}_{j}\right) G\left(\widetilde{\theta}_{j}\right)\left\{(N-1)^{-\frac{1}{2}} \lambda_{N-1}-(N-3)^{-\frac{1}{2}} b_{N} \lambda_{N-3}\right\} \\
:= & (-1)^{j} S_{N}\left(\widetilde{\theta}_{j}\right) . \tag{3.21}
\end{align*}
$$

Since $-b_{N}>0, \lambda_{k}>0$ and $\theta_{j} \in(0, \pi)$, we have that $S_{N}\left(\tilde{\theta}_{j}\right)>0$, and thereby

$$
\begin{equation*}
\operatorname{sgn}\left(H_{N}\left(\widetilde{\theta}_{j}\right)\right)=(-1)^{j}, \quad 0 \leq j \leq N-1 . \tag{3.22}
\end{equation*}
$$

Moreover, by (2.12), (3.19) and (3.21),

$$
\begin{aligned}
\left|H_{N}\left(\widetilde{\theta}_{j}\right)\right| & \geq 2^{\frac{5}{2}} \pi^{-\frac{1}{2}}\left(\sin \tilde{\theta}_{j}\right)^{-\frac{3}{2}}\left\{(N-1)^{-\frac{1}{2}} \lambda_{N-1}+\left(\frac{1}{4}-\frac{1}{N}\right)(N-3)^{-\frac{1}{2}} \lambda_{N-3}\right\} \\
& \geq c N^{-\frac{1}{2}}\left(\lambda_{N-1}+\lambda_{N-3}\right)
\end{aligned}
$$



Fig. 3.2. Left: the error $\widetilde{\Delta}_{N}$ (solid line) and $N^{-1.9}$ ("०") against $N$. Right: asymptotic behavior of $\delta_{N}^{(1)}$ (solid line) and $\delta_{N}^{(2)}(" \circ ")($ cf. (3.43)) against $N$.
which, together with the fact $-b_{N} \cong \frac{1}{4}$, implies

$$
\begin{equation*}
\left|R_{N}\right| \leq c N^{-\frac{3}{2}}\left(\lambda_{N-1}+\lambda_{N-3}\right) \leq c N^{-1}\left|H_{N}\left(\tilde{\theta}_{j}\right)\right| . \tag{3.23}
\end{equation*}
$$

Hence, a combination of (3.20)- (3.23) leads to that

$$
\operatorname{sgn}\left(Q_{N-1}\left(\cos \tilde{\theta}_{j}\right)\right)=\operatorname{sgn}\left(H_{N}\left(\tilde{\theta}_{j}\right)\right)=(-1)^{j}, \quad 0 \leq j \leq N-1, \quad \forall N \gg 1,
$$

which implies (3.17) and therefore, there exists at least one zero of $Q_{N-1}(\cos \theta)$ in each subinterval $I_{j}=\left(\widetilde{\theta}_{j-1}, \widetilde{\theta}_{j}\right), 1 \leq j \leq N-1$. Because the number of zeros equals to the number of subintervals, there exists exactly one zero in $I_{j}$.

As a consequence of Theorem 3.2, a good asymptotic approximation to the zeros $\left\{x_{j}\right\}_{j=0}^{N-1}$ might be

$$
\begin{equation*}
x_{j}=\cos \theta_{j} \cong x_{j}^{(0)}:=\cos \left(\frac{\widetilde{\theta}_{j-1}+\widetilde{\theta}_{j}}{2}\right)=\cos \frac{(2 j+1 / 2) \pi}{2 N+1}, \quad 1 \leq j \leq N-1 \tag{3.24}
\end{equation*}
$$

To illustrate this property numerically, we denote

$$
\widetilde{\Delta}_{N}=\max _{1 \leq j \leq N-1}\left|x_{j}-x_{j}^{(0)}\right| .
$$

We plot in Fig. 3.2 (left) the error $\widetilde{\Delta}_{N}$ (solid line) and the reference value $N^{-1.9}$ ("o") against $N$, which indicates that $\widetilde{\Delta}_{N}$ decays algebraically at a rate of $N^{-1.9}$, and predicts that

$$
\begin{equation*}
x_{j}=\cos \frac{(2 j+1 / 2) \pi}{2 N+1}+O\left(N^{-1.9}\right), \quad 1 \leq j \leq N-1 . \tag{3.25}
\end{equation*}
$$

### 3.2. Asymptotic properties of the weights

We establish below an important result concerning the asymptotic behaviors of the interior quadrature weights $\left\{\omega_{j}\right\}_{j=1}^{N-1}$ in (2.13).

Theorem 3.3. Let $\left\{\theta_{j}\right\}_{j=1}^{N-1}$ be the zeros of the quadrature (trigonometric) polynomial $Q_{N-1}(\cos \theta)$. Then for any $N \gg 1$,

$$
\begin{equation*}
\omega_{j} \cong \frac{\pi}{N} \sin \theta_{j}, \quad j=1,2, \ldots, N-1 \tag{3.26}
\end{equation*}
$$

Proof. We rewrite the weights as

$$
\begin{align*}
\omega_{j} & \stackrel{(2.14)}{=} \frac{A_{N}}{\left(1-x_{j}^{2}\right)^{2}} \frac{1}{P_{N-2}\left(x_{j}\right) Q_{N-1}^{\prime}\left(x_{j}\right)} \\
& \stackrel{(2.6)}{=} \frac{A_{N}}{(2.10)} \frac{1}{\lambda_{N-2} \lambda_{N-3}} \frac{1}{\left(1-x_{j}^{2}\right)^{2}} \frac{1}{J_{N-2}^{(2,2)}\left(x_{j}\right) W_{N}^{\prime}\left(x_{j}\right)}, \tag{3.27}
\end{align*}
$$

where

$$
\begin{equation*}
W_{N}(x):=\frac{\lambda_{N-1}}{\lambda_{N-3}} J_{N-1}^{(2,2)}(x)+b_{N} J_{N-3}^{(2,2)}(x)=\lambda_{N-3}^{-1} Q_{N-1}(x) . \tag{3.28}
\end{equation*}
$$

To prove (3.26), it suffices to study the asymptotic behaviors of the constant, $J_{N-2}^{(2,2)}\left(x_{j}\right)$ and $W_{N}^{\prime}\left(x_{j}\right)$ in (3.27). For clarity, we split the rest of the proof into three steps.
Step I: Let us first estimate the constants in (3.27)-(3.28). A direct calculation using (2.6)-(2.7) and (2.12) leads to

$$
\begin{equation*}
a_{N-2} \cong \frac{1}{4}, \quad-b_{N} \cong \frac{1}{4}, \quad \frac{1}{\widetilde{\lambda}_{N-2}^{2}} \cong \frac{16}{N}, \quad \frac{\lambda_{N-2}}{\lambda_{N-3}}=\frac{(N-2)(N+2)}{N(2 N-1)} \cong \frac{1}{2} \tag{3.29}
\end{equation*}
$$

which, together with (2.15), implies

$$
\begin{equation*}
\frac{A_{N}}{\lambda_{N-2} \lambda_{N-3}}=\left(1-\frac{b_{N}}{a_{N-2}}\right) \frac{\lambda_{N-2}}{\lambda_{N-3} \widetilde{\lambda}_{N-2}^{2}} \cong \frac{16}{N} . \tag{3.30}
\end{equation*}
$$

Step II: Let

$$
\begin{equation*}
\Theta_{N, j}=\left(N+\frac{1}{2}\right) \theta_{j}-\frac{5}{4} \pi . \tag{3.31}
\end{equation*}
$$

We next show that

$$
\begin{equation*}
\sin \Theta_{N, j} \cong 0, \quad \forall N \gg 1 \tag{3.32}
\end{equation*}
$$

Since $Q_{N-1}\left(x_{j}\right)=0$, we have

$$
W_{N}\left(\cos \theta_{j}\right) \stackrel{(3.28)}{=} \lambda_{N-3}^{-1} Q_{N-1}\left(\cos \theta_{j}\right)=\lambda_{N-3}^{-1} Q_{N-1}\left(x_{j}\right)=0 .
$$

Hence, by (3.18),

$$
\begin{aligned}
0= & W_{N}\left(\cos \theta_{j}\right)=\frac{\lambda_{N-1}}{\lambda_{N-3}} J_{N-1}^{(2,2)}\left(\cos \theta_{j}\right)+b_{N} J_{N-3}^{(2,2)}\left(\cos \theta_{j}\right) \\
\stackrel{(3.18)}{=} & (N-3)^{-\frac{1}{2}} G\left(\theta_{j}\right)\left\{c_{N} \cos \left(\Theta_{N, j}+\theta_{j}\right)+b_{N} \cos \left(\Theta_{N, j}-\theta_{j}\right)\right\}+O\left(N^{-\frac{3}{2}}\right) \\
= & (N-3)^{-\frac{1}{2}} G\left(\theta_{j}\right)\left\{\left(b_{N}-c_{N}\right) \sin \theta_{j} \sin \Theta_{N, j}\right. \\
& \left.+\left(b_{N}+c_{N}\right) \cos \theta_{j} \cos \Theta_{N, j}\right\}+O\left(N^{-\frac{3}{2}}\right),
\end{aligned}
$$

where $G\left(\theta_{j}\right)$ is given in (3.19), and by (3.29),

$$
\begin{equation*}
c_{N}=\frac{\lambda_{N-1}}{\lambda_{N-3}}\left(\frac{N-3}{N-1}\right)^{\frac{1}{2}} \cong \frac{\lambda_{N-1}}{\lambda_{N-2}} \frac{\lambda_{N-2}}{\lambda_{N-3}} \cong \frac{1}{4} . \tag{3.33}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
0=\left(b_{N}-c_{N}\right) \sin \theta_{j} \sin \Theta_{N, j}+\left(b_{N}+c_{N}\right) \cos \theta_{j} \cos \Theta_{N, j}+O\left(N^{-1} \sin ^{5 / 2} \theta_{j}\right) \tag{3.34}
\end{equation*}
$$

On the other hand, using (3.29) and (3.33) leads to

$$
\begin{equation*}
b_{N}+c_{N} \cong 0, \quad b_{N}-c_{N} \cong-\frac{1}{2}, \quad \text { for } N \gg 1 \tag{3.35}
\end{equation*}
$$

Since $\sin \theta_{j} \neq 0(>0)$, the desired result (3.32) follows from (3.34).
Step III: Applying Formula (8.8.1) of [23] (note: this formula may be derived by differentiating (3.18) and using (3.28)) to $W_{N}^{\prime}(x)$, and using trigonometric identities, yields

$$
\begin{align*}
& \frac{\mathrm{d} W_{N}}{\mathrm{~d} \theta}\left(\cos \theta_{j}\right) \\
& \quad=(N-3)^{\frac{1}{2}} G\left(\theta_{j}\right)\left\{-c_{N} \sin \left(\Theta_{N, j}+\theta_{j}\right)-b_{N} \sin \left(\Theta_{N, j}-\theta_{j}\right)+\left(N \sin \theta_{j}\right)^{-1} O(1)\right\} \\
& \quad=(N-3)^{\frac{1}{2}} G\left(\theta_{j}\right)\left\{\left(b_{N}-c_{N}\right) \sin \theta_{j} \cos \Theta_{N, j}-\left(b_{N}+c_{N}\right) \cos \theta_{j} \sin \Theta_{N, j}\right. \\
& \left.\quad+\left(N \sin \theta_{j}\right)^{-1} O(1)\right\} . \tag{3.36}
\end{align*}
$$

Hence, by (3.32) and (3.35),

$$
\begin{equation*}
\frac{\mathrm{d} W_{N}}{\mathrm{~d} \theta}\left(\cos \theta_{j}\right) \cong-\frac{1}{2} \sqrt{N} G\left(\theta_{j}\right) \sin \theta_{j} \cos \Theta_{N, j}, \quad 1 \leq j \leq N-1, \tag{3.37}
\end{equation*}
$$

which implies

$$
\begin{equation*}
W_{N}^{\prime}\left(x_{j}\right)=\left.\frac{\mathrm{d} W_{N}}{\mathrm{~d} \theta}\left(\cos \theta_{j}\right) \frac{\mathrm{d} \theta}{\mathrm{~d} x}\right|_{\theta=\theta_{j}}=-\frac{\mathrm{d} W_{N}}{\mathrm{~d} \theta}\left(\cos \theta_{j}\right) \frac{1}{\sin \theta_{j}}=\frac{\sqrt{N}}{2} G\left(\theta_{j}\right) \cos \Theta_{N, j} . \tag{3.38}
\end{equation*}
$$

On the other hand, by (3.18),

$$
\begin{equation*}
J_{N-2}^{(2,2)}\left(\cos \theta_{j}\right) \cong N^{-\frac{1}{2}} G\left(\theta_{j}\right) \cos \Theta_{N, j}, \quad 1 \leq j \leq N-1 . \tag{3.39}
\end{equation*}
$$

Multiplying (3.38) by (3.39) yields

$$
\begin{equation*}
J_{N-2}^{(2,2)}\left(x_{j}\right) W_{N}^{\prime}\left(x_{j}\right) \cong \frac{1}{2} G^{2}\left(\theta_{j}\right) \cos ^{2} \Theta_{N, j} \cong \frac{1}{2} G^{2}\left(\theta_{j}\right), \tag{3.40}
\end{equation*}
$$

where in the last step, we used the fact:

$$
\begin{equation*}
\cos ^{2} \Theta_{N, j} \stackrel{(3.32)}{\cong} 1, \quad \forall N \gg 1 . \tag{3.41}
\end{equation*}
$$

Finally, thanks to

$$
\frac{1}{\left(1-x_{j}^{2}\right)^{2}} \stackrel{(3.13)}{=} \frac{1}{\sin ^{4} \theta_{j}}, \quad G^{2}\left(\theta_{j}\right) \stackrel{(3.19)}{=} \frac{2^{5}}{\pi} \frac{1}{\sin ^{5} \theta_{j}}
$$

the desired result (3.26) follows from (3.27), (3.30) and (3.40).

As a direct consequence of (3.24) and (3.26), we derive the following explicit asymptotic expression:

$$
\begin{equation*}
\omega_{j} \cong \omega_{j}^{(0)}:=\frac{\pi}{N} \sin \frac{(2 j+1 / 2) \pi}{2 N+1}, \quad 1 \leq j \leq N-1 \tag{3.42}
\end{equation*}
$$

To examine how good it is, we set

$$
\delta_{N}^{(1)}=\frac{\pi}{N} \max _{1 \leq j \leq N-1}\left\{\frac{\sin \theta_{j}}{\omega_{j}}\right\}, \quad \delta_{N}^{(2)}=\frac{\pi}{N} \max _{1 \leq j \leq N-1}\left\{\frac{\omega_{j}^{(0)}}{\omega_{j}}\right\}
$$

By Theorem 3.3, we expect to see that

$$
\begin{equation*}
\delta_{N}^{(i)} \cong 1, \quad i=1,2, \forall N \gg 1 \tag{3.43}
\end{equation*}
$$

which indeed can be visualized from Fig. 3.2(right).
So far, we have derived two asymptotic estimates for the GLLB nodes and weights (cf. (3.24) and (3.42)), which will be useful for the analysis of the GLLB interpolation error in the forthcoming section.

## 4. GLLB interpolation error estimates

This section is devoted to the analysis of the GLLB interpolation errors in Sobolev norms, which will be used for the error analysis of GLLB pseudospectral methods. We first state the main result, and then present the ingredients for the proof including some inequalities and orthogonal projections. Finally, we give the proof of the interpolation errors.

### 4.1. The main result

We begin with the definition of the GLLB interpolation operator associated with the GLLB quadrature formula. The GLLB interpolant $\mathcal{I}_{N} u \in \mathbb{P}_{N}$, satisfies

$$
\begin{equation*}
\left(\mathcal{I}_{N} u\right)^{\prime}( \pm 1)=u^{\prime}( \pm 1), \quad\left(\mathcal{I}_{N} u\right)\left(x_{j}\right)=u\left(x_{j}\right), \quad 1 \leq j \leq N-1 \tag{4.1}
\end{equation*}
$$

Denote the weight functions by $\omega(x)=1-x^{2}$ and $\omega^{\alpha}(x)=\left(1-x^{2}\right)^{\alpha}$. Let $\bar{\alpha}=$ $\max \{0,-[\alpha]-1\}$ with $[\alpha]$ being the largest integer $\leq \alpha$. To describe the error, we introduce the weighted Sobolev space

$$
B_{\alpha}^{m}(I):=\left\{u \in L^{2}(I): \partial_{x}^{k} u \in L_{\omega^{\alpha+k}}^{2}(I), \bar{\alpha}+1 \leq k \leq m\right\} \cap H^{\bar{\alpha}}(I), \quad m \geq 0
$$

with the norm

$$
\|u\|_{B_{\alpha}^{m}(I)}=\left(\|u\|_{H^{\bar{\alpha}}(I)}^{2}+\sum_{k=\bar{\alpha}+1}^{m}\left\|\partial_{x}^{k} u\right\|_{\omega^{\alpha+k}}^{2}\right)^{\frac{1}{2}} .
$$

The main result on the GLLB interpolation error is stated as follows.
Theorem 4.1. For any $u \in B_{-2}^{m}(I)$ and $m \geq 2$,

$$
\begin{equation*}
N\left\|\mathcal{I}_{N} u-u\right\|+\left\|\partial_{x}\left(\mathcal{I}_{N} u-u\right)\right\|_{\omega} \leq c N^{1-m}\left\|\partial_{x}^{m} u\right\|_{\omega^{m-2}} \tag{4.2}
\end{equation*}
$$

### 4.2. Preparations for the proof

The main ingredients for the proof Theorem 4.1 consist of the asymptotic estimates (cf. Theorem 3.2 and 3.3), several inequalities and the approximation property of one specific orthogonal projection to be stated below.

### 4.2.1. Some inequalities

For notational convenience, we define the discrete inner product and discrete norm associated with the GLLB quadrature formula as

$$
\langle u, v\rangle_{N}=\mathcal{S}_{N}[u \cdot v], \quad\|v\|_{N}=\langle v, v\rangle_{N}^{\frac{1}{2}},
$$

where $\mathcal{S}_{N}[\cdot]$ represents the finite sum in (2.13). Clearly, the exactness (2.13) implies that

$$
\begin{equation*}
\langle\phi, \psi\rangle_{N}=(\phi, \psi), \quad \forall \phi \cdot \psi \in \mathbb{P}_{2 N-1} \tag{4.3}
\end{equation*}
$$

We have the following equivalence between the continuous and discrete norms over the polynomial space $\mathbb{P}_{N}$, which will be useful in the analysis of interpolation error, and the GLLB pseudospectral methods for nonlinear problems.

Lemma 4.1. For any integer $N \geq 4$,

$$
\begin{equation*}
\|\phi\| \leq\|\phi\|_{N} \leq \sqrt{1+C_{N}}\|\phi\| \leq 3\|\phi\|, \quad \forall \phi \in \mathbb{P}_{N} \tag{4.4}
\end{equation*}
$$

where

$$
C_{N}=\left(1+\frac{3}{N-2}\right)\left(1+\frac{3}{N-1}\right)\left(1+\frac{3}{N}\right) .
$$

Proof. We first prove that (4.4) holds for the Legendre polynomial $L_{N}$. That is

$$
\begin{equation*}
\gamma_{N}^{(0,0)}=\left\|L_{N}\right\|^{2} \leq\left\langle L_{N}, L_{N}\right\rangle_{N} \leq\left(1+C_{N}\right)\left\|L_{N}\right\|^{2}=\left(1+C_{N}\right) \gamma_{N}^{(0,0)} . \tag{4.5}
\end{equation*}
$$

For this purpose, set

$$
\psi(x)=L_{N}^{2}(x)-\left(K_{N}^{(0,0)}\right)^{2}\left(1-x^{2}\right)^{2} Q_{N-1}(x) P_{N-3}(x)
$$

Since the leading coefficient of $Q_{N-1} \cdot P_{N-3}$ is one, we deduce from (2.9) that $\psi \in \mathbb{P}_{2 N-1}$. Hence, using the fact $Q_{N-1}\left(x_{j}\right)=0$ and the orthogonality, gives

$$
\begin{align*}
\left\langle L_{N}, L_{N}\right\rangle_{N} & =\langle 1, \psi\rangle_{N} \stackrel{(4.3)}{=}(1, \psi)=\int_{-1}^{1} \psi(x) \mathrm{d} x \\
& \stackrel{(2.10)}{=} \int_{-1}^{1} L_{N}^{2}(x) \mathrm{d} x-b_{N}\left(K_{N}^{(0,0)}\right)^{2} \int_{-1}^{1} P_{N-3}^{2}(x)\left(1-x^{2}\right)^{2} \mathrm{~d} x \\
& =\gamma_{N}^{(0,0)}-b_{N}\left(\lambda_{N-3} K_{N}^{(0,0)}\right)^{2} \gamma_{N-3}^{(2,2)} \\
& =\left(1-b_{N}\left(\lambda_{N-3} K_{N}^{(0,0)}\right)^{2} \gamma_{N-3}^{(2,2)}\left(\gamma_{N}^{(0,0)}\right)^{-1}\right) \gamma_{N}^{(0,0)} . \tag{4.6}
\end{align*}
$$

Using (2.2)-(2.6) and (2.9) to work out the constants, yields

$$
\left(\lambda_{N-3} K_{N}^{(0,0)}\right)^{2} \gamma_{N-3}^{(2,2)}\left(\gamma_{N}^{(0,0)}\right)^{-1}=\frac{(N+1)(2 N-1)(2 N+1)}{N(N-1)(N-2)}
$$

Furthermore, by (2.12), we have that for $N \geq 4$,

$$
\begin{equation*}
0<-b_{N}\left(\lambda_{N-3} K_{N}^{(0,0)}\right)^{2} \gamma_{N-3}^{(2,2)}\left(\gamma_{N}^{(0,0)}\right)^{-1}<\frac{(N+1)(N+2)(N+3)}{(N-2)(N-1) N}=C_{N} . \tag{4.7}
\end{equation*}
$$

Accordingly, a combination of (4.6)-(4.7) leads to (4.5).
Next, for any $\phi \in \mathbb{P}_{N}$, we write

$$
\phi(x)=\sum_{n=0}^{N} \hat{\phi}_{n} L_{n}(x)=\sum_{n=0}^{N-1} \hat{\phi}_{n} L_{n}(x)+\hat{\phi}_{N} L_{N}(x):=\Phi(x)+\hat{\phi}_{N} L_{N}(x),
$$

and therefore

$$
\|\phi\|^{2}=\sum_{n=0}^{N} \hat{\phi}_{n}^{2} \gamma_{n}^{(0,0)}=\|\Phi\|^{2}+\hat{\phi}_{N}^{2} \gamma_{N}^{(0,0)}
$$

It is clear that $\Phi^{2}(x) \in \mathbb{P}_{2 N-2}$, and so by (4.3) and the orthogonality of Legendre polynomials,

$$
\left\langle\Phi, L_{N}\right\rangle_{N}=\left(\Phi, L_{N}\right)=0,
$$

which implies

$$
\begin{aligned}
\|\phi\|_{N}^{2} & =\langle\Phi, \Phi\rangle_{N}+2 \hat{\phi}_{N}\left\langle\Phi, L_{N}\right\rangle_{N}+\hat{\phi}_{N}^{2}\left\langle L_{N}, L_{N}\right\rangle_{N}=\|\Phi\|^{2}+\hat{\phi}_{N}^{2}\left\langle L_{N}, L_{N}\right\rangle_{N} \\
& =\sum_{n=0}^{N-1} \hat{\phi}_{n}^{2} \gamma_{n}^{(0,0)}+\hat{\phi}_{N}^{2}\left\langle L_{N}, L_{N}\right\rangle_{N}
\end{aligned}
$$

Finally, applying (4.5) leads to the desired result.
The following Bernstein-Markov type inequality holds for polynomials in the finitedimensional space:

$$
\begin{equation*}
X_{N}=\left\{\phi \in \mathbb{P}_{N}: \phi^{\prime}( \pm 1)=0\right\} . \tag{4.8}
\end{equation*}
$$

Lemma 4.2. For any $\phi \in X_{N}$,

$$
\begin{equation*}
\left\|\phi^{\prime}\right\|_{\omega} \leq \sqrt{N(N+1)}\|\phi\| \leq(N+1)\|\phi\| . \tag{4.9}
\end{equation*}
$$

Proof. Let

$$
\begin{equation*}
\psi_{n}(x)=L_{n}(x)+e_{n} L_{n+2}(x), \quad e_{n}=-\frac{n(n+1)}{(n+2)(n+3)} . \tag{4.10}
\end{equation*}
$$

Since $L_{n}^{\prime}( \pm 1)=\frac{1}{2}(-1)^{n-1} n(n+1)$, we have that $\left\{\psi_{n}\right\}_{n=0}^{N-2}$ forms a basis for $X_{N}$. Consequently, for any $\phi \in X_{N}$,

$$
\begin{align*}
\phi(x)= & \sum_{n=0}^{N-2} \tilde{\phi}_{n} \psi_{n}(x)=\sum_{n=0}^{N-2} \tilde{\phi}_{n} L_{n}(x)+\sum_{n=2}^{N} e_{n-2} \tilde{\phi}_{n-2} L_{n}(x) \\
= & \tilde{\phi}_{0} L_{0}(x)+\tilde{\phi}_{1} L_{1}(x)+\sum_{n=2}^{N-2}\left(\tilde{\phi}_{n}+e_{n-2} \tilde{\phi}_{n-2}\right) L_{n}(x) \\
& +e_{N-3} \tilde{\phi}_{N-3} L_{N-1}(x)+e_{N-2} \tilde{\phi}_{N-2} L_{N}(x) . \tag{4.11}
\end{align*}
$$

By the recursive relation

$$
L_{k}^{\prime}(x)=\frac{1}{2}(k+1) J_{k-1}^{(1,1)}(x), \quad k \geq 1
$$

and (4.10),

$$
\begin{aligned}
\phi^{\prime}(x)= & \sum_{n=0}^{N-2} \tilde{\phi}_{n} \psi_{n}^{\prime}(x)=\frac{1}{2} \sum_{n=0}^{N-3}(n+2) \tilde{\phi}_{n+1} J_{n}^{(1,1)}(x)+\frac{1}{2} \sum_{n=1}^{N-1}(n+2) e_{n-1} \tilde{\phi}_{n-1} J_{n}^{(1,1)}(x) \\
= & \tilde{\phi}_{1} J_{0}^{(1,1)}(x)+\frac{1}{2} \sum_{n=1}^{N-3}(n+2)\left(\tilde{\phi}_{n+1}+e_{n-1} \tilde{\phi}_{n-1}\right) J_{n}^{(1,1)}(x) \\
& +\frac{1}{2} N e_{N-3} \tilde{\phi}_{N-3} J_{N-2}^{(1,1)}(x)+\frac{1}{2}(N+1) e_{N-2} \tilde{\phi}_{N-2} J_{N-1}^{(1,1)}(x) .
\end{aligned}
$$

Using the orthogonality (2.1)-(2.2) gives

$$
\begin{aligned}
\|\phi\|^{2}= & \tilde{\phi}_{0}^{2} \gamma_{0}^{(0,0)}+\tilde{\phi}_{1}^{2} \gamma_{1}^{(0,0)}+\sum_{n=2}^{N-2}\left(\tilde{\phi}_{n}+e_{n-2} \tilde{\phi}_{n-2}\right)^{2} \gamma_{n}^{(0,0)} \\
& +e_{N-3}^{2} \tilde{\phi}_{N-3}^{2} \gamma_{N-1}^{(0,0)}+e_{N-2}^{2} \tilde{\phi}_{N-2}^{2} \gamma_{N}^{(0,0)},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\phi^{\prime}\right\|_{\omega}^{2}= & \tilde{\phi}_{1}^{2} \gamma_{0}^{(1,1)}+\frac{1}{4} \sum_{n=1}^{N-3}(n+2)^{2}\left(\tilde{\phi}_{n+1}+e_{n-1} \tilde{\phi}_{n-1}\right)^{2} \gamma_{n}^{(1,1)} \\
& +\frac{1}{4} N^{2} e_{N-3}^{2} \tilde{\phi}_{N-3}^{2} \gamma_{N-2}^{(1,1)}+\frac{1}{4}(N+1)^{2} e_{N-2}^{2} \tilde{\phi}_{N-2}^{2} \gamma_{N-1}^{(1,1)} \\
= & \tilde{\phi}_{1}^{2} \gamma_{0}^{(1,1)}+\frac{1}{4} \sum_{n=2}^{N-2}(n+1)^{2}\left(\tilde{\phi}_{n}+e_{n-2} \tilde{\phi}_{n-2}\right)^{2} \gamma_{n-1}^{(1,1)} \\
& +\frac{1}{4} N^{2} e_{N-3}^{2} \tilde{\phi}_{N-3}^{2} \gamma_{N-2}^{(1,1)}+\frac{1}{4}(N+1)^{2} e_{N-2}^{2} \tilde{\phi}_{N-2}^{2} \gamma_{N-1}^{(1,1)} .
\end{aligned}
$$

In view of the above facts, we deduce from (2.2) that

$$
\left\|\phi^{\prime}\right\|_{\omega}^{2} \leq \frac{1}{4} \max _{1 \leq n \leq N}\left\{(n+1)^{2} \gamma_{n-1}^{(1,1)}\left(\gamma_{n}^{(0,0)}\right)^{-1}\right\}\|\phi\|^{2}=N(N+1)\|\phi\|^{2}
$$

This implies the desired result.
In the preceding analysis, we will also use the following Poincaré inequality (see, e.g., [5]).
Lemma 4.3. For any $u \in H^{1}(I)$,

$$
\begin{equation*}
\|u\| \leq c\left(\left\|u^{\prime}\right\|+|\bar{u}|\right), \quad \text { where } \bar{u}=\int_{-1}^{1} u(x) \mathrm{d} x . \tag{4.12}
\end{equation*}
$$

### 4.2.2. Orthogonal projections

We first consider the orthogonal projection $\pi_{N}: L^{2}(I) \rightarrow \mathbb{P}_{N}$ such that for any $u \in L^{2}(I)$,

$$
\begin{equation*}
\left(\pi_{N} u-u, \phi\right)=0, \quad \forall \phi \in \mathbb{P}_{N} \tag{4.13}
\end{equation*}
$$

The following result can be found in $[10,14]$.

Lemma 4.4. For any $u \in B_{0}^{m}(I)$ and $m \geq 0$,

$$
\begin{equation*}
\left\|\pi_{N} u-u\right\| \leq c N^{-m}\left\|\partial_{x}^{m} u\right\|_{\omega^{m}} \tag{4.14}
\end{equation*}
$$

We now turn to the second orthogonal projection. For simplicity, we denote

$$
\partial_{x}^{-1} v(x)=\int_{-1}^{x} v(y) \mathrm{d} y, \quad \bar{\partial}_{x}^{-1} v(x)=-\int_{x}^{1} v(y) \mathrm{d} y, \quad \bar{v}=\int_{-1}^{1} v(x) \mathrm{d} x .
$$

Define the space

$$
X:=\left\{v: v \in H^{2}(I), v^{\prime}( \pm 1)=0\right\}, \quad X^{0}:=\{v \in X: \bar{v}=0\}, \quad X_{N}^{0}:=\mathbb{P}_{N} \cap X^{0} .
$$

Consider the orthogonal projection $\pi_{N}^{1,0}: X^{0} \rightarrow X_{N}^{0}$, defined by

$$
\begin{equation*}
a\left(\pi_{N}^{1,0} v-v, \phi\right)=0, \quad \forall \phi \in X_{N}^{0} \tag{4.15}
\end{equation*}
$$

where the bilinear form $a(u, v):=\left(\partial_{x} u, \partial_{x} v\right)$. Recall that for real $\mu \geq 0$, the Sobolev space $H^{\mu}(I)$ and its norm $\|\cdot\|_{\mu}$ are defined by space interpolation as in [1].

Lemma 4.5. For any $v \in X^{0} \cap B_{-2}^{m}(I)$ with $m \geq 2$,

$$
\begin{equation*}
\left\|\pi_{N}^{1,0} v-v\right\|_{\mu} \leq c N^{\mu-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}, \quad 0 \leq \mu \leq 1 . \tag{4.16}
\end{equation*}
$$

Proof. We first consider the case $\mu=1$. Let

$$
\begin{equation*}
v^{*}(x)=\xi+\partial_{x} v(1) x+\partial_{x}^{-1} \bar{\partial}_{x}^{-1}\left(\pi_{N-2} \partial_{x}^{2} v\right)(x) \tag{4.17}
\end{equation*}
$$

where the constant $\xi$ is chosen such that $\overline{v^{*}}=\bar{v}$. Clearly, $\partial_{x} v^{*}(1)=\partial_{x} v(1)$ and by (4.13) with $\phi=1$,

$$
\begin{equation*}
\partial_{x} v^{*}(-1)=\partial_{x} v(1)-\int_{I} \pi_{N-2} \partial_{x}^{2} v(x) \mathrm{d} x=\partial_{x} v(1)-\int_{I} \partial_{x}^{2} v(x) \mathrm{d} x=\partial_{x} v(-1) . \tag{4.18}
\end{equation*}
$$

For clarity, let us denote $g=v^{*}-v$. Thanks to the above facts, we have from (4.13) and (4.14) that

$$
\begin{align*}
\left|v^{*}-v\right|_{1}^{2} & =\left(\partial_{x}\left(v^{*}-v\right), \partial_{x} g\right)=-\left(\partial_{x}^{2}\left(v^{*}-v\right), g\right) \\
& =-\left(\pi_{N-2} \partial_{x}^{2} v-\partial_{x}^{2} v, g\right)=\left(\pi_{N-2} \partial_{x}^{2} v-\partial_{x}^{2} v, \pi_{N-2} g-g\right) \\
& \leq\left\|\pi_{N-2} \partial_{x}^{2} v-\partial_{x}^{2} v\right\|\left\|\pi_{N-2} g-g\right\| \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\left\|\partial_{x} g\right\|_{\omega} \\
& \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\left|v^{*}-v\right|_{1} . \tag{4.19}
\end{align*}
$$

Moreover, by the Poincaré inequality (4.12) and (4.15),

$$
\begin{aligned}
\left\|\pi_{N}^{1,0} v-v\right\|_{1}^{2} & \leq c\left|\pi_{N}^{1,0} v-v\right|_{1}^{2}=c a\left(\pi_{N}^{1,0} v-v, \pi_{N}^{1,0} v-v\right) \\
& =c a\left(\pi_{N}^{1,0} v-v, v^{*}-v\right) \leq c\left|\pi_{N}^{1,0} v-v\right|_{1}\left|v^{*}-v\right|_{1} .
\end{aligned}
$$

In view of this fact, we have from (4.19) that

$$
\begin{equation*}
\left\|\pi_{N}^{1,0} v-v\right\|_{1} \leq c\left|v^{*}-v\right|_{1} \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} \tag{4.20}
\end{equation*}
$$

We next prove the case $\mu=0$ by using a duality argument (see, e.g., [6]). For any $f \in L^{2}(I)$, we consider an auxiliary problem. It is to find $w \in X^{0}$ such that

$$
\begin{equation*}
a(w, z)=(f, z), \quad \forall z \in X^{0} \tag{4.21}
\end{equation*}
$$

By the Poincaré inequality (cf. (4.12)) and Lax-Milgram Lemma, the problem (4.21) has a unique solution with the regularity

$$
\begin{equation*}
\|w\|_{2} \leq c\|f\| \tag{4.22}
\end{equation*}
$$

Taking $z=\pi_{N}^{1,0} v-v$ in (4.21), we deduce from (4.15), (4.20) and (4.22) that

$$
\begin{aligned}
\left|\left(\pi_{N}^{1,0} v-v, f\right)\right| & =\left|a\left(\pi_{N}^{1,0} v-v, w\right)\right|=\left|a\left(\pi_{N}^{1,0} v-v, \pi_{N}^{1,0} w-w\right)\right| \\
& \leq\left|\pi_{N}^{1,0} v-v\right|_{1}\left|\pi_{N}^{1,0} w-w\right|_{1} \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\left\|\partial_{x}^{2} w\right\| \\
& \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\|f\| .
\end{aligned}
$$

Consequently

$$
\begin{equation*}
\left\|\pi_{N}^{1,0} v-v\right\|=\sup _{\substack{f \in L^{2}(I) \\ f \neq 0}} \frac{\left|\left(\pi_{N}^{1,0} v-v, f\right)\right|}{\|f\|} \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} \tag{4.23}
\end{equation*}
$$

Finally, we get the desired result by (4.20), (4.23) and space interpolation.
With the aid of the previous preparations, we are able to derive the following important result.
Theorem 4.2. There exists an operator $\pi_{N}^{1}: X \rightarrow X_{N}$, such that $\overline{\pi_{N}^{1} v}=\bar{v}$, and

$$
\begin{equation*}
a\left(\pi_{N}^{1} v-v, \phi\right)=0, \quad \forall \phi \in X_{N} \tag{4.24}
\end{equation*}
$$

Moreover, for any $v \in X \cap B_{-2}^{m}(I)$ with $m \geq 2$,

$$
\begin{equation*}
\left\|\pi_{N}^{1} v-v\right\|_{\mu} \leq c N^{\mu-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}, \quad 0 \leq \mu \leq 1 \tag{4.25}
\end{equation*}
$$

Proof. For any $v \in X$, since $v-\bar{v} / 2 \in X^{0}$, we define

$$
\pi_{N}^{1} v(x)=\pi_{N}^{1,0}(v(x)-\bar{v} / 2)+\bar{v} / 2
$$

One verifies readily that $\pi_{N}^{1} v \in X_{N}$ and $\overline{\pi_{N}^{1} v}=\bar{v}$. Moreover, by (4.15),

$$
a\left(\pi_{N}^{1} v-v, \phi\right)=a\left(\pi_{N}^{1,0}(v-\bar{v} / 2)-(v-\bar{v} / 2), \phi-\bar{\phi} / 2\right)=0, \quad \forall \phi \in X_{N}
$$

Moreover, by (4.16) and the fact $r \geq 2$,

$$
\begin{aligned}
\left\|\pi_{N}^{1} v-v\right\|_{\mu} & =\left\|\pi_{N}^{1,0}(v-\bar{v} / 2)-(v-\bar{v} / 2)\right\|_{\mu} \leq c N^{\mu-m}\left\|\partial_{x}^{m}(v-\bar{v} / 2)\right\|_{\omega^{m-2}} \\
& \leq c N^{\mu-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}
\end{aligned}
$$

This leads to (4.25).

### 4.3. Continuity of the GLLB interpolation operator

In order to prove Theorem 4.1, it is essential to show that $\mathcal{I}_{N}$ is a continuous operator from $X$ to $L^{2}(I)$, as stated below.

Lemma 4.6. For any $v \in X$,

$$
\begin{equation*}
\left\|\mathcal{I}_{N} v\right\| \leq c\left(\|v\|+N^{-1}\left\|\partial_{x} v\right\|_{\omega}\right) \tag{4.26}
\end{equation*}
$$

where the weight function $\omega(x)=1-x^{2}$.
Proof. In this proof, we mainly use Theorems 3.2 and 3.3. Let $x=\cos \theta$ and $\hat{v}(\theta)=v(\cos \theta)$. Then by (2.13) and (4.4) and Theorem 3.3,

$$
\begin{equation*}
\left\|\mathcal{I}_{N} v\right\|^{2} \leq\left\|\mathcal{I}_{N} v\right\|_{N}^{2}=\sum_{j=1}^{N-1} v^{2}\left(x_{j}\right) \omega_{j} \leq c N^{-1} \sum_{j=1}^{N-1} \hat{v}^{2}\left(\theta_{j}\right) \sin \theta_{j} . \tag{4.27}
\end{equation*}
$$

By using an inequality of space interpolation (see formula (13.7) of [2]), we know that for any $f \in H^{1}(a, b)$,

$$
\begin{equation*}
\max _{a \leq x \leq b}|f(x)| \leq c\left(\frac{1}{\sqrt{b-a}}\|f\|_{L^{2}(a, b)}+\sqrt{b-a}\left\|\partial_{x} f\right\|_{L^{2}(a, b)}\right) . \tag{4.28}
\end{equation*}
$$

Now, let $\widetilde{\theta}_{0}, \widetilde{\theta}_{N-1}$ and $I_{j}$ be the same as in Theorem 3.2. Denote $a_{0}=\widetilde{\theta}_{0}$ and $a_{1}=\widetilde{\theta}_{N-1}$. Then by (4.27) and (4.28),

$$
\begin{aligned}
&\left\|\mathcal{I}_{N} v\right\|^{2} \stackrel{(4.27)}{\leq} c N^{-1} \sum_{j=1}^{N-1} \sup _{\theta \in \bar{I}_{j}}|\hat{v}(\theta) \sqrt{\sin \theta}|^{2} \\
& \stackrel{(4.28)}{\leq} c \sum_{j=1}^{N-1}\left(\|\hat{v}(\theta) \sqrt{\sin \theta}\|_{L^{2}\left(\bar{I}_{j}\right)}^{2}+N^{-2}\left\|\partial_{\theta}(\hat{v}(\theta) \sqrt{\sin \theta})\right\|_{L^{2}\left(\bar{I}_{j}\right)}^{2}\right) \\
& \leq c\left(\|\hat{v}(\theta) \sqrt{\sin \theta}\|_{L^{2}(0, \pi)}^{2}+N^{-2}\left\|\partial_{\theta}(\hat{v}(\theta) \sqrt{\sin \theta})\right\|_{L^{2}\left(\left[a_{0}, a_{1}\right]\right)}^{2}\right) \\
& \leq c\left(\|\hat{v}(\theta) \sqrt{\sin \theta}\|_{L^{2}(0, \pi)}^{2}+N^{-2}\left\|\hat{v}(\theta)(\sin \theta)^{-1 / 2}\right\|_{L^{2}\left(\left[a_{0}, a_{1}\right]\right)}^{2}\right. \\
&\left.+N^{-2}\left\|\partial_{\theta} \hat{v}(\theta) \sqrt{\sin \theta}\right\|_{L^{2}(0, \pi)}^{2}\right) \\
& \leq c\left(\|\hat{v}(\theta) \sqrt{\sin \theta}\|_{L^{2}(0, \pi)}^{2}+\sup _{a_{0} \leq \theta \leq a_{1}} \frac{1}{N^{2} \sin ^{2} \theta} \cdot\|\hat{v}(\theta) \sqrt{\sin \theta}\|_{L^{2}\left(\left[a_{0} \cdot a_{1}\right]\right)}^{2}\right. \\
&\left.+N^{-2}\left\|\partial_{\theta} \hat{v}(\theta) \sqrt{\sin \theta}\right\|_{L^{2}(0, \pi)}^{2}\right) .
\end{aligned}
$$

Since Theorem 3.2 implies $\sup _{a_{0} \leq \theta \leq a_{1}} \frac{1}{N^{2} \sin ^{2} \theta} \leq c$, we transform $\theta$ back to $x$, and obtain

$$
\left\|\mathcal{I}_{N} v\right\|^{2} \leq c\left(\|v\|^{2}+N^{-2}\left\|\partial_{x} v\right\|_{\omega}^{2}\right)
$$

which ends the proof.

### 4.4. Proof of Theorem 4.1

Let

$$
v_{N}^{*}(x)=v(-1)+(1+x) \partial_{x} v(1)+\partial_{x}^{-1} \bar{\partial}_{x}^{-1}\left(\pi_{N-2} \partial_{x}^{2} v\right)(x) \in \mathbb{P}_{N} .
$$

Like (4.18), we have $v_{N}^{*}(-1)=v(-1)$ and $\partial_{x} v_{N}^{*}( \pm 1)=\partial_{x} v( \pm 1)$. Thus, by (4.13),

$$
\begin{align*}
\left(v_{N}^{*}-v, \phi\right) & =\left(\partial_{x}^{2}\left(v_{N}^{*}-v\right), \partial_{x}^{-1} \bar{\partial}_{x}^{-1} \phi\right) \\
& =\left(\pi_{N-2} \partial_{x}^{2} v-\partial_{x}^{2} v, \partial_{x}^{-1} \bar{\partial}_{x}^{-1} \phi\right)=0, \quad \forall \phi \in \mathbb{P}_{N-4} . \tag{4.29}
\end{align*}
$$

A similar argument as in the derivation of (4.19) leads to

$$
\begin{equation*}
\left|v_{N}^{*}-v\right|_{1} \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} \tag{4.30}
\end{equation*}
$$

Denote $g=v_{N}^{*}-v$. Then by (4.14), (4.29) and (4.30),

$$
\begin{aligned}
\left\|v_{N}^{*}-v\right\|^{2} & =\left(v_{N}^{*}-v, g\right)=\left(v_{N}^{*}-v, \partial_{x} \bar{\partial}_{x}^{-1} g\right)=\left(v_{N}^{*}-v, \partial_{x} \bar{\partial}_{x}^{-1} g-\partial_{x} \pi_{N-3} \bar{\partial}_{x}^{-1} g\right) \\
& =\left(\partial_{x}\left(v_{N}^{*}-v\right), \pi_{N-3} \bar{\partial}_{x}^{-1} g-\bar{\partial}_{x}^{-1} g\right) \leq\left|v_{N}^{*}-v\right|_{1}\left\|\pi_{N-3} \bar{\partial}_{x}^{-1} g-\bar{\partial}_{x}^{-1} g\right\| \\
& \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\|g\|_{\omega} \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\left\|v_{N}^{*}-v\right\|,
\end{aligned}
$$

which implies

$$
\begin{equation*}
\left\|v_{N}^{*}-v\right\| \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} \tag{4.31}
\end{equation*}
$$

Since $\mathcal{I}_{N} v_{N}^{*}=v_{N}^{*}$ and $v-v_{N}^{*} \in X$, we have from Lemma 4.6, (4.30) and (4.31) that

$$
\begin{align*}
\left\|\mathcal{I}_{N} v-v_{N}^{*}\right\| & =\left\|\mathcal{I}_{N}\left(v_{N}^{*}-v\right)\right\| \leq c\left(\left\|v_{N}^{*}-v\right\|+N^{-1}\left\|\partial_{x}\left(v_{N}^{*}-v\right)\right\|_{\omega}\right) \\
& \leq c\left(\left\|v_{N}^{*}-v\right\|+N^{-1}\left|v_{N}^{*}-v\right|_{1}\right) \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} . \tag{4.32}
\end{align*}
$$

By the above and Lemma 4.2,

$$
\begin{equation*}
\left\|\partial_{x}\left(\mathcal{I}_{N} v-v_{N}^{*}\right)\right\|_{\omega} \leq c N\left\|\mathcal{I}_{N} v-v_{N}^{*}\right\| \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} \tag{4.33}
\end{equation*}
$$

Finally, a combination of (4.30)-(4.33) leads to that

$$
\begin{aligned}
& N\left\|\mathcal{I}_{N} v-v\right\|+\left\|\partial_{x}\left(\mathcal{I}_{N} v-v\right)\right\|_{\omega} \leq N\left\|\mathcal{I}_{N} v-v_{N}^{*}\right\|+\left\|\partial_{x}\left(\mathcal{I}_{N} v-v_{N}^{*}\right)\right\|_{\omega} \\
& \quad+N\left\|v_{N}^{*}-v\right\|+\left\|\partial_{x}\left(v_{N}^{*}-v\right)\right\|_{\omega} \leq c N^{1-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}} .
\end{aligned}
$$

This ends the proof.

### 4.5. Corollaries

We present below two mathematical consequences.
Corollary 4.1. For any $\phi \in X_{M}$ and $\psi \in X_{L}$, we have

$$
\begin{align*}
& \left\|\mathcal{I}_{N} \phi\right\|_{N} \leq c\left(1+M N^{-1}\right)\|\phi\|,  \tag{4.34}\\
& \left|\langle\phi, \psi\rangle_{N}\right| \leq c\left(1+M N^{-1}\right)\left(1+L N^{-1}\right)\|\phi\|\|\psi\| . \tag{4.35}
\end{align*}
$$

Proof. Using (4.4) and Lemmas 4.2 and 4.6 gives

$$
\left\|\mathcal{I}_{N} \phi\right\|_{N} \leq c\left\|\mathcal{I}_{N} \phi\right\| \leq c\left(\|\phi\|+N^{-1}\left\|\partial_{x} \phi\right\|_{\omega}\right) \leq c\left(1+M N^{-1}\right)\|\phi\|,
$$

and

$$
\begin{aligned}
\left|\langle\phi, \psi\rangle_{N}\right| & =\left|\left\langle\mathcal{I}_{N} \phi, \mathcal{I}_{N} \psi\right\rangle_{N}\right| \leq\left\|\mathcal{I}_{N} \phi\right\|_{N}\left\|\mathcal{I}_{N} \psi\right\|_{N} \\
& \leq c\left(1+M N^{-1}\right)\left(1+L N^{-1}\right)\|\phi\|\|\psi\| .
\end{aligned}
$$

Corollary 4.2. For any $v \in X \cap B_{-2}^{m}(I), m \geq 2$ and any $\phi \in X_{N}$,

$$
\begin{equation*}
\left|(v, \phi)-\langle v, \phi\rangle_{N}\right| \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\|\phi\|_{N} . \tag{4.36}
\end{equation*}
$$

Proof. By (4.3), (4.4), and Theorem 4.1,

$$
\begin{aligned}
\left|(v, \phi)-\langle v, \phi\rangle_{N}\right| & \leq\left|(v, \phi)-\left(\pi_{N-1}^{1} v, \phi\right)\right|+\left|\left\langle\pi_{N-1}^{1} v, \phi\right\rangle_{N}-\left\langle\mathcal{I}_{N} v, \phi\right\rangle_{N}\right| \\
& \leq c\left(\left\|\pi_{N-1}^{1} v-v\right\|+\left\|\mathcal{I}_{N} v-v\right\|\right)\|\phi\| \\
& \leq c N^{-m}\left\|\partial_{x}^{m} v\right\|_{\omega^{m-2}}\|\phi\|_{N} .
\end{aligned}
$$

This result is useful in numerical analysis of the related pseudospectral scheme.

## 5. GLLB pseudospectral methods and error estimates

Among several different versions of spectral approximations, pseudospectral methods are commonly used and more preferable in industrial codes owing to its ease of implementations and treatment of nonlinear problems. Most existing literature concerning this method is based on the collocation points that are identified as (generalized) Gauss-Lobatto quadrature formulae [24,9, $21,18]$. In a pseudospectral method, the choice of collocation points is crucial for the stability and treatment of boundary conditions [21]. The GLLB quadrature formula (2.13) involves first-order derivative values at the endpoints, which allows a natural and exact imposition of Neumann boundary conditions. Based on this, Ezzirani and Guessab [8] proposed a GLLB collocation method for some model elliptic equations, and showed that this method leads to a resulting discrete system with a diagonal mass matrix, and thereby can be used to introduce explicit resolutions in the lumped mass method for the time-dependent problems.

The main purposes of this section are two-folds: (i) to present a user-oriented implementation of the GLLB psuedospectral methods based on pure collocation and variational formulations, and (ii) to make an error analysis of the GLLB pseudospectral method based on a (discrete) variational formulation. We will first restrict our attentions to one-dimensional problems, and then follow with multi-dimensional cases. It should be pointed out that the methods go beyond these model equations.

### 5.1. GLLB collocation method

Consider the elliptic equation:

$$
\left\{\begin{array}{l}
\mathbb{L}[u](x):=-u^{\prime \prime}(x)+b(x) u(x)=f(x), \quad b(x) \geq 0, x \in I=(-1,1),  \tag{5.1}\\
u^{\prime}( \pm 1)=g_{ \pm},
\end{array}\right.
$$

where in the case of $b(x) \equiv 0$, the problem admits a solution only provided that the given data satisfy the compatibility

$$
\begin{equation*}
\int_{-1}^{1} f(x) \mathrm{d} x=g_{-}-g_{+} . \tag{5.2}
\end{equation*}
$$

Let $\left\{x_{j}\right\}_{j=0}^{N}$ be the GLLB quadrature nodes. The GLLB collocation approximation to (5.1) is to find $u_{N} \in \mathbb{P}_{N}$ such that

$$
\left\{\begin{array}{l}
\mathbb{L}\left[u_{N}\right]\left(x_{j}\right)=-u_{N}^{\prime \prime}\left(x_{j}\right)+b\left(x_{j}\right) u_{N}\left(x_{j}\right)=F\left(x_{j}\right), \quad 1 \leq j \leq N-1,  \tag{5.3}\\
u_{N}^{\prime}( \pm 1)=g_{ \pm},
\end{array}\right.
$$

where $F(x)$ is a consistent approximation to $f(x)$ with $F(x)=f(x)$ for $b(x) \neq 0$, and the case for $b(x) \equiv 0$ to be specified below.

We see that like the LGL collocation method for Dirichlet problems, the numerical solution satisfies the boundary conditions exactly.

Remark 5.1. In the case of $b(x) \equiv 0$, the scheme (5.3) is reduced to

$$
\begin{equation*}
-u_{N}^{\prime \prime}\left(x_{j}\right)=F\left(x_{j}\right), \quad 1 \leq j \leq N-1 ; \quad u_{N}^{\prime}( \pm 1)=g_{ \pm} . \tag{5.4}
\end{equation*}
$$

Let $\left(\mathbb{I}_{N-2}^{L} F\right)(x) \in \mathbb{P}_{N-2}$ be the Lagrange interpolation polynomial of $F$ associated with the interior GLLB nodes $\left\{x_{j}\right\}_{j=1}^{N-1}$. Since $u_{N}^{\prime \prime} \in \mathbb{P}_{N-2}$, the scheme (5.4) implies

$$
-u_{N}^{\prime \prime}(x)=\left(\mathbb{I}_{N-2}^{L} F\right)(x), \quad x \in I ; \quad u_{N}^{\prime}( \pm 1)=g_{ \pm}
$$

Thus, a direct integration leads to

$$
\begin{equation*}
\int_{-1}^{1}\left(\mathbb{I}_{N-2}^{L} F\right)(x) \mathrm{d} x=g_{-}-g_{+} \tag{5.5}
\end{equation*}
$$

This means that (5.4) has a solution as long as the above compatibility holds. However, since $f, g_{+}$and $g_{-}$are given, the compatibility is not valid if we take $F(x)=f(x)$. To meet the condition (5.5), we may follow the idea of Guo (cf. page 558 of [12]) to take

$$
F(x)=f(x)-\frac{1}{2} \mathcal{S}_{N}\left[\mathbb{I}_{N-2}^{L} f\right]+\frac{1}{2}\left(g_{-}-g_{+}\right)
$$

where the functional $\mathcal{S}_{N}[\cdot]$ is defined in (2.13). It is clear that for $N \geq 3$, we have

$$
\left(\mathbb{I}_{N-2}^{L} F\right)(x)=\left(\mathbb{I}_{N-2}^{L} f\right)(x)-\frac{1}{2} \mathcal{S}_{N}\left[\mathbb{I}_{N-2}^{L} f\right]+\frac{1}{2}\left(g_{-}-g_{+}\right)
$$

By virtue of (2.13) and (5.2), we have that

$$
\int_{-1}^{1}\left(\mathbb{I}_{N-2}^{L} F\right)(x) \mathrm{d} x=\int_{-1}^{1}\left(\mathbb{I}_{N-2}^{L} f\right)(x) \mathrm{d} x-\mathcal{S}_{N}\left[\mathbb{I}_{N-2}^{L} f\right]+g_{-}-g_{+}=g_{-}-g_{+} .
$$

We now examine the matrix form of (5.3). Let $\left\{h_{j}\right\}_{j=0}^{N} \subseteq \mathbb{P}_{N}$ be the set of Lagrangian polynomials associated with the GLLB points:

$$
\begin{array}{lll}
h_{0}^{\prime}(-1)=1, & h_{0}^{\prime}(1)=h_{0}\left(x_{k}\right)=0, & 1 \leq k \leq N-1, \\
h_{j}^{\prime}( \pm 1)=0, & h_{j}\left(x_{k}\right)=\delta_{j k}, & 1 \leq j, k \leq N-1,  \tag{5.6}\\
h_{N}^{\prime}(1)=1, & h_{N}^{\prime}(-1)=h_{N}\left(x_{k}\right)=0, & 1 \leq k \leq N-1,
\end{array}
$$

whose explicit expressions are given in Appendix A. It is clear that $\left\{h_{j}\right\}_{j=0}^{N} \subseteq \mathbb{P}_{N}$ and spans the polynomial space $\mathbb{P}_{N}$. Under this nodal basis, we write

$$
\begin{equation*}
u_{N}(x)=g_{-} h_{0}(x)+g_{+} h_{N}(x)+\sum_{k=1}^{N-1} a_{k} h_{k}(x) \in \mathbb{P}_{N}, \tag{5.7}
\end{equation*}
$$

and determine the unknowns $\left\{a_{k}\right\}_{k=1}^{N-1}$ by (5.3), i.e., the system

$$
\begin{equation*}
\mathbf{A}_{c} \overrightarrow{\mathbf{a}}:=\left(-\mathbf{D}_{\mathrm{in}}^{(2)}+\mathbf{B}\right) \overrightarrow{\mathbf{a}}=\overrightarrow{\mathbf{b}} \tag{5.8}
\end{equation*}
$$

where

$$
\begin{align*}
& d_{j k}^{(2)}=h_{k}^{\prime \prime}\left(x_{j}\right), \quad \mathbf{D}_{\text {in }}^{(2)}=\left(d_{j k}^{(2)}\right)_{1 \leq j, k \leq N-1}, \\
& \mathbf{B}=\operatorname{diag}\left(b\left(x_{1}\right), b\left(x_{2}\right), \ldots, b\left(x_{N-1}\right)\right), \quad \overrightarrow{\mathbf{a}}=\left(a_{1}, a_{2}, \ldots, a_{N-1}\right)^{\mathrm{T}}, \\
& \overrightarrow{\mathbf{b}}=\left(f\left(x_{1}\right), \ldots, f\left(x_{N-1}\right)\right)^{\mathrm{T}}+\left(d_{10}^{(2)}, d_{20}^{(2)}, \ldots, d_{(N-1) 0}^{(2)}\right)^{\mathrm{T}} g_{-}  \tag{5.9}\\
& +\left(d_{1 N}^{(2)}, d_{2 N}^{(2)}, \ldots, d_{(N-1) N}^{(2)}\right)^{\mathrm{T}} g_{+} .
\end{align*}
$$

As with the usual collocation schemes, the GLLB method is easy to implement once the associated differentiation matrices are pre-computed. The following lemma provides a recursive way to evaluate the differentiation matrices.

## Lemma 5.1. Let

$$
\begin{equation*}
d_{j k}^{(l)}=h_{k}^{(l)}\left(x_{j}\right)=\frac{\mathrm{d}^{l} h_{k}}{\mathrm{~d} x^{l}}\left(x_{j}\right), \quad \mathbf{D}^{(l)}=\left(d_{j k}^{(l)}\right)_{0 \leq j, k \leq N}, \quad l \geq 0 \tag{5.10}
\end{equation*}
$$

Then we have

$$
\left\{\begin{array}{l}
\mathbf{D}^{(l+1)}=\mathbf{D}^{(1)} \times \widetilde{\mathbf{D}}^{(l)}, \quad l \geq 0,  \tag{5.11}\\
\mathbf{D}^{(0)}=\widetilde{\mathbf{D}}^{(0)}=\mathbf{I}_{N+1},
\end{array}\right.
$$

where $\mathbf{I}_{N+1}$ is the identity matrix of order $N+1$, and $\widetilde{\mathbf{D}}^{(l)}$ is identical to $\mathbf{D}^{(l)}$ except that the first and last rows of $\mathbf{D}^{(l)}$ are replaced by $\left(d_{00}^{(l+1)}, d_{01}^{(l+1)}, \ldots, d_{0 N}^{(l+1)}\right)$ and $\left(d_{N 0}^{(l+1)}, d_{N 1}^{(l+1)}, \ldots, d_{N N}^{(l+1)}\right)$, respectively.
Proof. For any $\phi \in \mathbb{P}_{N}$, we have that

$$
\phi(x)=\phi^{\prime}(-1) h_{0}(x)+\sum_{j=1}^{N-1} \phi\left(x_{j}\right) h_{j}(x)+\phi^{\prime}(1) h_{N}(x) \in \mathbb{P}_{N},
$$

which implies that

$$
\phi^{\prime}(x)=\phi^{\prime}(-1) h_{0}^{\prime}(x)+\sum_{j=1}^{N-1} \phi\left(x_{j}\right) h_{j}^{\prime}(x)+\phi^{\prime}(1) h_{N}^{\prime}(x) \in \mathbb{P}_{N}
$$

Let $x_{0}=-1$ and $x_{N}=1$. Taking $\phi(x)=h_{k}^{(l)}(x)$ and $x=x_{i}$ leads to that for all $0 \leq i, k \leq N$ and $l \geq 0$,

$$
\begin{align*}
d_{i k}^{(l+1)} & =h_{k}^{(l+1)}\left(x_{i}\right)=h_{k}^{(l+1)}\left(x_{0}\right) h_{0}^{\prime}\left(x_{i}\right)+\sum_{j=1}^{N-1} h_{k}^{(l)}\left(x_{j}\right) h_{j}^{\prime}\left(x_{i}\right)+h_{k}^{(l+1)}\left(x_{N}\right) h_{N}^{\prime}\left(x_{i}\right) \\
& =d_{i 0}^{(1)} d_{0 k}^{(l+1)}+\sum_{j=1}^{N-1} d_{i j}^{(1)} d_{j k}^{(l)}+d_{i N}^{(1)} d_{N k}^{(l+1)} \tag{5.12}
\end{align*}
$$

which implies the desired result.
As a consequence of this lemma, it suffices to evaluate the first-order differentiation matrix $D^{(1)}$ and values $h_{j}^{(l+1)}( \pm 1)$ for $l \geq 1$ and $0 \leq j \leq N$ to compute higher-order differentiation matrices.

Although the collocation scheme (5.3) is easy to implement, the GLLB differentiation matrix $D^{(2)}$ is full with $\operatorname{Cond}\left(D^{(2)}\right) \sim N^{4}$, and thereby when $N$ is large, the accuracy of nodes and of entries of $D^{(2)}$ are subject to significant roundoff errors. To overcome this trouble, it is advisable to construct a preconditioning for the system (5.8), and use a suitable iteration solver [4,7]. As a matter of fact, Canuto [4] proposed a finite-difference preconditioning for the LGL collocation method for Neumann problems, but it cannot be applied to this context directly. However, with a different treatment of the boundary conditions, we are able to derive an optimal preconditioner as that in [4] (for the LGL collocation method).

To this end, we assume that $\left\{x_{j}\right\}_{j=0}^{N}$ are arranged in an ascending order, and let

$$
\delta_{j}=x_{j+1}-x_{j}, \quad u_{j}=u\left(x_{j}\right), \quad u_{j}^{\prime \prime}=u^{\prime \prime}\left(x_{j}\right), \quad f_{j}=f\left(x_{j}\right)
$$

Taking the Neumann boundary conditions into account, we discretize $u_{1}^{\prime \prime}$ and $u_{N-1}^{\prime \prime}$ as

$$
\begin{equation*}
u_{1}^{\prime \prime} \approx \frac{u_{2}-u_{1}+\delta_{1} u_{0}^{\prime}}{\delta_{1}\left(\delta_{1} / 2+\delta_{0}\right)}, \quad u_{N-1}^{\prime \prime} \approx \frac{u_{N-2}-u_{N-1}-\delta_{N-2} u_{N}^{\prime}}{\delta_{N-2}\left(\delta_{N-2} / 2+\delta_{N-1}\right)}, \tag{5.13}
\end{equation*}
$$

and use centered differences for the interiors $\left\{u_{j}^{\prime \prime}\right\}_{j=2}^{N-2}$. More precisely, the finite-difference approximation to (5.1) reads

$$
\left\{\begin{array}{l}
\frac{u_{1}-u_{2}}{\delta_{1}\left(\delta_{1} / 2+\delta_{0}\right)}+b_{1} u_{1}=f_{1}+\frac{g_{-}}{\delta_{1} / 2+\delta_{0}}  \tag{5.14}\\
-\frac{2 u_{j-1}}{\delta_{j-1}\left[\delta_{j}+\delta_{j+1}\right]}+\left[\frac{2}{\delta_{j} \delta_{j-1}}+b_{j}\right] u_{j}-\frac{2 u_{j+1}}{\delta_{j}\left[\delta_{j}+\delta_{j+1}\right]}=f_{j} \\
\frac{2 \leq j \leq N-2,}{u_{N-1}-u_{N-2}} \\
\frac{\delta_{N-2}\left(\delta_{N-2} / 2+\delta_{N-1}\right)}{\delta_{+}}+b_{N-1} u_{N-1}=f_{N-1}-\frac{g_{+}}{\delta_{N-2} / 2+\delta_{N-1}}
\end{array}\right.
$$

Denote by $\mathbf{A}_{d}$ the coefficient matrix of the above system. Table 5.1 contains the spectral radii (i.e., the largest modulus of the eigenvalues) of $\mathbf{A}_{c}$ (cf. (5.8)) and $\mathbf{A}_{\mathrm{d}}^{-1} \mathbf{A}_{\mathrm{c}}$ with $b=1$ (in columns $2-3$ ) and $b(x)=100+\sin (100 \pi x)$ (in columns $4-5$ ). The eigenvalues of two matrices are real, positive and distinct in all cases. We also point out that the smallest modulus of the eigenvalues of $\mathbf{A}_{d}^{-1} \mathbf{A}_{c}$ is one for both cases, while that of $\mathbf{A}_{c}$ is one for $b=1$ and 100 for $b(x)=100+\sin (100 \pi x)$. The eigenvalues of the preconditioned matrix $\mathbf{A}_{\mathrm{d}}^{-1} \mathbf{A}_{\mathrm{c}}$ lie in the interval [1, $\left.\pi^{2} / 4\right]$, as shown in Table 5.1, and therefore the system (5.8) can be solved efficiently by an iteration solver.

Table 5.1
The spectral radii of $\mathbf{A}_{\mathrm{c}}$ and $\mathbf{A}_{\mathrm{d}}^{-1} \mathbf{A}_{\mathrm{c}}$.

| $N$ | $\rho\left(\mathbf{A}_{\mathrm{c}}\right)$ | $\rho\left(\mathbf{A}_{\mathrm{d}}^{-1} \mathbf{A}_{\mathrm{c}}\right)$ | $\rho\left(\mathbf{A}_{\mathrm{c}}\right)$ | $\rho\left(\mathbf{A}_{\mathrm{d}}^{-1} \mathbf{A}_{\mathrm{c}}\right)$ |
| ---: | ---: | :--- | ---: | :--- |
| 8 | 47.2 | 1.68 | 146.3 | 1.1 |
| 16 | 633.8 | 2.11 | 732.8 | 1.58 |
| 32 | 10202.0 | 2.30 | 10301.0 | 2.10 |
| 64 | 166178.9 | 2.39 | 166277.9 | 2.33 |
| 128 | 2691955.4 | 2.43 | 2692054.2 | 2.41 |
| 256 | 43374170.2 | 2.45 | 43374269.2 | 2.44 |
| 512 | 696563431.7 | 2.46 | 696563530.7 | 2.46 |

### 5.2. GLLB pseudospectral method in a variational form

The collocation scheme (5.3) is based on a strong form of the original equation, while it is more preferable to implement the GLLB pseudospectral method in a discrete weak form. For simplicity, we consider (5.1) with a constant coefficient and homogeneous Neumann boundary conditions, i.e., $b(x) \equiv b$ and $g_{ \pm}=0$. The GLLB pseudospectral scheme reads

$$
\left\{\begin{array}{l}
\text { Find } u_{N} \in X_{N} \text { such that }  \tag{5.15}\\
\left\langle u_{N}^{\prime}, v_{N}^{\prime}\right\rangle_{N}+\left\langle b u_{N}, v_{N}\right\rangle_{N}=\left\langle f, v_{N}\right\rangle_{N}, \quad \forall v_{N} \in X_{N}
\end{array}\right.
$$

Remark 5.2. Unlike the collocation method based on Legendre-Gauss-Lobotto points for Dirichlet boundary conditions, the GLLB scheme (5.3) (with $b(x) \equiv b$ and $g_{ \pm}=0$ ) is not equivalent to the pseudospectral scheme (5.15). Indeed, multiplying (5.3) with $F(x)=f(x)$ by $v_{N}\left(x_{j}\right) \omega_{j}$ and summing the result for $j=1,2, \ldots, N-1$, we may rewrite the GLLB collocation scheme as

$$
\left\{\begin{array}{l}
\text { Find } u_{N} \in X_{N} \text { such that }  \tag{5.16}\\
\left\langle u_{N}^{\prime}, v_{N}^{\prime}\right\rangle_{N}+\left\langle b u_{N}, v_{N}\right\rangle_{N}=\left\langle f, v_{N}\right\rangle_{N}+\mathcal{R}_{N}, \quad \forall v_{N} \in X_{N},
\end{array}\right.
$$

where

$$
\begin{align*}
\mathcal{R}_{N}= & (\mathbb{L}[u])^{\prime}(-1) v_{N}(-1) \omega_{0}+(\mathbb{L}[u])^{\prime}(1) v_{N}(1) \omega_{N} \\
& -f^{\prime}(-1) v_{N}(-1) \omega_{0}-f^{\prime}(1) v_{N}(1) \omega_{N} \\
= & -\left(u_{N}^{\prime \prime \prime}(-1)-\left(b u_{N}\right)^{\prime}(-1)+f^{\prime}(-1)\right) v_{N}(-1) \omega_{0} \\
& -\left(u_{N}^{\prime \prime \prime}(1)-\left(b u_{N}\right)^{\prime}(1)+f^{\prime}(1)\right) v_{N}(1) \omega_{N} . \tag{5.17}
\end{align*}
$$

Hence, up to a boundary residual, two schemes are equivalent.
We now examine the matrix form of the system (5.15). As usual, we may choose the nodal basis $\left\{h_{k}\right\}_{k=1}^{N-1}$ for $X_{N}$. By (5.6), one verifies that the coefficient matrix under this basis is

$$
\begin{equation*}
\mathbf{A}_{\mathbf{p}}:=\left(\mathbf{D}_{\mathrm{in}}^{(1)}\right)^{T} \mathbf{W D}_{\mathrm{in}}^{(1)}+b \mathbf{W} \tag{5.18}
\end{equation*}
$$

where $\mathbf{D}_{\text {in }}^{(1)}=\left(d_{j k}^{(1)}\right)_{1 \leq j, k \leq N-1}$ (cf. (5.10)), and $\mathbf{W}=\operatorname{diag}\left(\omega_{1}, \omega_{2}, \ldots, \omega_{N-1}\right)$. We see that $\mathbf{A}_{\mathbf{p}}$ is full and ill-conditioned as the GLLB collocation system (5.8) (cf. Table 5.2), so it subjects to similar roundoff errors (cf. Fig. 5.1).

Table 5.2
Condition numbers.

| $N$ |  | $b=0.01$ | $b=1$ | $b=100$ |
| :--- | :--- | :--- | :--- | :--- |
| 32 | $\mathbf{A}_{\mathbf{c}}$ | $1.25 \mathrm{e}+06$ | $1.24 \mathrm{e}+04$ | $1.19 \mathrm{e}+02$ |
| 32 | $\mathbf{A}_{\mathrm{p}}$ | $4.33 \mathrm{e}+05$ | $4.40 \mathrm{e}+03$ | $8.01 \mathrm{e}+01$ |
| 32 | $\mathbf{A}_{s}$ | 50.20 | 1.999 | 198.7 |
| 64 | $\mathbf{A}_{\mathbf{c}}$ | $1.95 \mathrm{e}+07$ | $1.93 \mathrm{e}+05$ | $1.83 \mathrm{e}+03$ |
| 64 | $\mathbf{A}_{\mathrm{p}}$ | $3.43 \mathrm{e}+06$ | $3.50 \mathrm{e}+04$ | $6.72 \mathrm{e}+02$ |
| 64 | $\mathbf{A}_{\mathbf{s}}$ | 50.20 | 1.999 | 199.9 |
| 128 | $\mathbf{A}_{\mathbf{c}}$ | $3.08 \mathrm{e}+08$ | $3.05 \mathrm{e}+06$ | $2.88 \mathrm{e}+04$ |
| 128 | $\mathbf{A}_{\mathrm{p}}$ | $2.74 \mathrm{e}+07$ | $2.79 \mathrm{e}+05$ | $5.55 \mathrm{e}+03$ |
| 128 | $\mathbf{A}_{\mathbf{s}}$ | 50.20 | 1.999 | 199.9 |
| 256 | $\mathbf{A}_{\mathbf{c}}$ | $4.90 \mathrm{e}+09$ | $4.85 \mathrm{e}+07$ | $4.59 \mathrm{e}+05$ |
| 256 | $\mathbf{A}_{\mathrm{p}}$ | $2.19 \mathrm{e}+08$ | $2.24 \mathrm{e}+06$ | 1.999 |
| 256 | $\mathbf{A}_{s}$ | 50.20 |  | 199.9 |



Fig. 5.1. Convergence rate ( $L^{2}$-errors vs. $N$ ) for $b=1$ (left) and $b=100$ (right).
In fact, it is more advisable to use a modal basis and perform the GLLB method in the frequency space. Using this approach, the spectral linear system will be sparse and wellconditioned. As with [22], we define the basis function as a "compact" combination of the Legendre polynomials

$$
\begin{equation*}
\phi_{0}(x)=1, \quad \phi_{n}(x):=d_{n}\left(L_{n}(x)+e_{n} L_{n+2}(x)\right), \quad n \geq 1, \tag{5.19}
\end{equation*}
$$

where

$$
d_{n}=\sqrt{\frac{(n+2)(n+3)}{2 n(n+1)(2 n+3)}}, \quad e_{n}=-\frac{n(n+1)}{(n+2)(n+3)} .
$$

Since $L_{n}^{\prime}( \pm 1)=\frac{1}{2}(-1)^{n+1} n(n+1)$, one can verify readily that $\phi_{n}^{\prime}( \pm 1)=0$, and

$$
X_{N}=\operatorname{span}\left\{\phi_{0}, \phi_{1}, \ldots, \phi_{N-2}\right\} .
$$

Moreover, as shown in Appendix B, we have

$$
\begin{equation*}
\phi_{n}^{\prime}(x)=\left(1-x^{2}\right) P_{n-1}(x)=\tilde{\lambda}_{n-1}\left(1-x^{2}\right) J_{n-1}^{(2,2)}(x), \quad n \geq 1, \tag{5.20}
\end{equation*}
$$

which, together with (2.1), (2.8) and (4.3), leads to

$$
s_{j k}:=\left\langle\phi_{k}^{\prime}, \phi_{j}^{\prime}\right\rangle_{N}=\left(\phi_{k}^{\prime}, \phi_{j}^{\prime}\right)= \begin{cases}1, & 1 \leq j=k \leq N-2  \tag{5.21}\\ 0, & \text { otherwise }\end{cases}
$$

Meanwhile, using (4.3), (5.19) and the orthogonality of the Legendre polynomials yields

$$
\begin{aligned}
m_{j k} & :=\left\langle\phi_{k}, \phi_{j}\right\rangle_{N}=\left\{\begin{array}{l}
d_{N-2}^{2}\left(\left\|L_{N-2}\right\|^{2}+e_{N-2}^{2}\left\langle L_{N}, L_{N}\right\rangle_{N}\right), \quad k=j=N-2, \\
\left(\phi_{k}, \phi_{j}\right), \quad \text { otherwise },
\end{array}\right. \\
& \neq 0, \quad \text { only if } k=j \text { or } k=j \pm 2 .
\end{aligned}
$$

Hence, by setting $u_{N}(x)=\sum_{n=0}^{N-2} \hat{u}_{n} \phi_{n}(x)$, and

$$
\hat{\mathbf{u}}=\left(\hat{u}_{0}, \hat{u}_{1}, \ldots, \hat{u}_{N-2}\right)^{\mathrm{T}}, \quad f_{j}=\left\langle f, \phi_{j}\right\rangle_{N}, \quad \hat{\mathbf{f}}=\left(\hat{f}_{0}, \hat{f}_{1}, \ldots, \hat{f}_{N-2}\right)^{\mathrm{T}}
$$

the system (5.15) under the basis (5.19) becomes

$$
\begin{equation*}
\mathbf{A}_{\mathbf{s}} \hat{\mathbf{u}}:=\left[\operatorname{diag}(0,1, \ldots, 1)+b\left(m_{j k}\right)_{0 \leq j, k \leq N-2}\right] \hat{\mathbf{u}}=\hat{\mathbf{f}} \tag{5.22}
\end{equation*}
$$

We see that $\mathbf{A}_{\mathrm{s}}$ is pentadiagonal with three nonzero diagonals, which can be decoupled into two tridiagonal sub-matrices, and inverted efficiently as in [22]. Moreover, the entries of the coefficient matrix can be evaluated exactly.

We can also prove that the condition number of $\mathbf{A}_{\mathrm{s}}$ does not depend on $N$. To show this, we define the discrete $l^{2}$-inner product and norm, i.e., for any two vectors of length $N-1,\langle\mathbf{u}, \mathbf{v}\rangle_{l^{2}}:=$ $\sum_{j=0}^{N-2} \mathbf{u}_{j} \mathbf{v}_{j}$ and $\|\mathbf{u}\|_{l^{2}}=\langle\mathbf{u}, \mathbf{u}\rangle_{l^{2}}^{1 / 2}$. By using (5.21), (4.3), the definition of $\mathbf{A}_{s}$, (4.4) and the Poincaré inequality (4.12) successively, we derive that

$$
\begin{align*}
\|\hat{\mathbf{u}}\|_{l^{2}}^{2} & =\left\|u_{N}^{\prime}\right\|^{2}+\hat{u}_{0}^{2} \leq\left\langle u_{N}^{\prime}, u_{N}^{\prime}\right\rangle_{N}+b\left\langle u_{N}, u_{N}\right\rangle_{N}+\hat{u}_{0}^{2}=\left\langle\mathbf{A}_{s} \hat{\mathbf{u}}, \hat{\mathbf{u}}\right\rangle_{l^{2}}+\hat{u}_{0}^{2} \\
& =\left\|u_{N}^{\prime}\right\|_{N}^{2}+b\left\|u_{N}\right\|_{N}^{2}+\hat{u}_{0}^{2} \leq\left\|u_{N}^{\prime}\right\|^{2}+9 b\left\|u_{N}\right\|^{2}+\hat{u}_{0}^{2} \\
& \leq(1+c b)\left\|u_{N}^{\prime}\right\|^{2}+c b\left|\bar{u}_{N}\right|^{2}+\hat{u}_{0}^{2} \\
& \leq(1+c b)\left(\left\|u_{N}^{\prime}\right\|^{2}+\hat{u}_{0}^{2}\right)=\|\hat{\mathbf{u}}\|_{l^{2}}^{2}, \tag{5.23}
\end{align*}
$$

where in the last step, we used the fact $\bar{u}_{N}=\int_{-1}^{1} u_{N}(x) \mathrm{d} x=2 \hat{u}_{0}$. Therefore, we claim that $\operatorname{Cond}\left(\mathbf{A}_{s}\right) \leq 1+c b$, which is also verified numerically by Table 5.2.

We compare in Table 5.2 the condition numbers of the matrices $\mathbf{A}_{\mathrm{c}}, \mathbf{A}_{\mathrm{p}}$ and $\mathbf{A}_{\mathrm{s}}$ resulting from the pure collocation method (PCOL, cf. (5.8)), pseudospectral method with nodal basis (PSND, cf. (5.18)) and pseudospectral method with modal basis (PSMD, cf. (5.22)), respectively. We see that for various $b$ and $N$, the condition numbers of the system (5.22) are small and independent of $N$ (also cf. (5.23)), while those of the collocation scheme and pseudospectral method using a nodal basis grow like $O\left(N^{4}\right)$.

To check the accuracy, we take $b=1$ and $u(x)=\cos ^{2}(12 \pi x)$ as the exact solution. The discrete $L^{2}$-errors against various $N$ are plotted at the left of Fig. 5.1, which indicates an exponential convergence rate. We also see that PSND and PSMD methods can provide better numerical results than PCOL method. To exam the effect of roundoff errors, we take $b=100$ and $u(x)=\cos ^{2}(8 \pi x)$ as an exact solution. We observe from Fig. 5.1(right) that the effect of roundoff errors is much more severe in PCOL method. Indeed, like the Gauss-Lobatto pseudospectral method, both PSND and PSMD have higher accuracy, when compared to PCOL. However, in multi-dimensional cases, PSMD (based on tensor product of the basis functions) is
more preferable, since it involves sparse and well-conditioned systems, which can be inverted very efficiently by using the matrix decomposition techniques (cf. [22]).

### 5.3. Error estimates

In this section, we apply the approximation results established in the previous section to analyze the errors of the GLLB pseudospectral scheme (5.15).

Theorem 5.1. Let $u_{N}$ and $u$ be respectively the solutions of (5.15) and (5.1) with $b(x)=b>0$ and $g_{ \pm}=0$. If $u \in X \cap B_{-2}^{m}(I)$ and $f \in B_{-2}^{s}(I)$ with integers $m, s \geq 2$, then

$$
\begin{equation*}
\left\|u-u_{N}\right\|_{\mu} \leq c\left(N^{\mu-m}\left\|\partial_{x}^{m} u\right\|_{\omega^{m-2}}+N^{-s}\left\|\partial_{x}^{s} f\right\|_{\omega^{s-2}}\right), \quad \mu=0,1 \tag{5.24}
\end{equation*}
$$

Proof. For simplicity, let $e_{N}=\pi_{N}^{1} u-u_{N}$ and $E_{v}(\phi)=(v, \phi)-\langle v, \phi\rangle_{N}$. By (4.3), (5.1) and (5.15),

$$
\left(\partial_{x}\left(u-u_{N}\right), \partial_{x} \phi\right)+b\left((u, \phi)-\left\langle u_{N}, \phi\right\rangle_{N}\right)=E_{f}(\phi), \quad \forall \phi \in X_{N}
$$

Thanks to (4.24), we can rewrite the above equation as

$$
\begin{equation*}
\left(\partial_{x} e_{N}, \partial_{x} \phi\right)+b\left\langle e_{N}, \phi\right\rangle_{N}=b\left(\left\langle\pi_{N}^{1} u, \phi\right\rangle_{N}-(u, \phi)\right)+E_{f}(\phi), \quad \forall \phi \in X_{N} \tag{5.25}
\end{equation*}
$$

Using (4.4), Theorems 4.1 and 4.2, and Corollary 4.2 leads to

$$
\begin{aligned}
\left|\left\langle\pi_{N}^{1} u, \phi\right\rangle_{N}-(u, \phi)\right| & \leq\left|\left\langle\pi_{N}^{1} u-\mathcal{I}_{N} u, \phi\right\rangle_{N}\right|+\left|E_{u}(\phi)\right| \\
& \leq\left\|\pi_{N}^{1} u-\mathcal{I}_{N} u\right\|\|\phi\|+\left|E_{u}(\phi)\right| \leq c N^{-m}\left\|\partial_{x}^{m} u\right\|_{\omega^{m-2}}\|\phi\|
\end{aligned}
$$

and

$$
\left|E_{f}(\phi)\right| \leq N^{-s}\left\|\partial_{x}^{s} f\right\|_{\omega^{s-2}}\|\phi\|
$$

Hence, taking $\phi=e_{N}$ in (5.25) and using (4.4), we reach that

$$
\left|e_{N}\right|_{1}+\left\|e_{N}\right\| \leq c\left(N^{-m}\left\|\partial_{x}^{m} u\right\|_{\omega^{m-2}}+N^{-s}\left\|\partial_{x}^{s} f\right\|_{\omega^{s-2}}\right)
$$

Finally, using Theorem 4.2 again leads to the desired result.
Remark 5.3. Theorem 4.2 of [4] presents an analysis estimate in $L^{2}$-norm of a modified LGL collocation method for (5.1) (with $b=1$ ) with a convergence order $O\left(N^{2-m}\right)$. Obviously, the estimate (5.24) improve the existing result essentially, and seems optimal (with the order $O\left(N^{-m}\right)$ ).

## 6. Concluding remarks

In the foregoing discussions, we restrict our attentions to one-dimensional linear problems, but the ideas and techniques go beyond these equations. An interesting extension is on the construction and analysis of a quadrature formula and the associated pseudospectral approximations for mixed boundary conditions $\alpha_{ \pm} u^{\prime}( \pm 1)+\beta_{ \pm} u( \pm 1)$ where $\alpha_{ \pm}$and $\beta_{ \pm}$are constants such that

$$
\begin{equation*}
-u^{\prime \prime}+b u=f, \quad \text { in }(-1,1) ; \quad \alpha_{ \pm} u^{\prime}( \pm 1)+\beta_{ \pm} u( \pm 1)=g_{ \pm} \tag{6.1}
\end{equation*}
$$

is well-posed. We will report this topic in our future work.

In summary, we have derived in this paper the asymptotic properties of the nodes and weights of the GLLB quadrature formula, and obtained optimal estimates for the GLLB interpolation errors. We also presented a detailed implementation of the associated GLLB pseudospectral method for second-order PDEs.

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## Appendix A. Expressions of the Lagrangian basis

For notational simplicity, let $q(x):=Q_{N-1}(x)$ be the quadrature polynomial given in (2.10), and define

$$
\begin{equation*}
q_{j}(x):=\frac{1}{Q_{N-1}^{\prime}\left(x_{j}\right)} \frac{Q_{N-1}(x)}{x-x_{j}}=\frac{1}{q^{\prime}\left(x_{j}\right)} \frac{q(x)}{x-x_{j}} \in \mathbb{P}_{N-2}, \quad 1 \leq j \leq N-1 . \tag{A.1}
\end{equation*}
$$

We see that $\left\{q_{j}\right\}_{j=1}^{N-1}$ are the Lagrangian basis associated with the interior points $\left\{x_{j}\right\}_{j=1}^{N-1}$ of the GLLB quadrature. This matter with (5.6) implies that the corresponding basis functions can be expressed as follows.

- Boundary basis functions :

$$
\begin{align*}
& h_{0}(x)=\frac{\left[q^{\prime}(1)(x-1)-q(1)\right] q(x)}{q(-1) q^{\prime}(1)-2 q^{\prime}(1) q^{\prime}(-1)-q^{\prime}(-1) q(1)},  \tag{A.2}\\
& h_{N}(x)=\frac{\left[q^{\prime}(-1)(x+1)-q(-1)\right] q(x)}{q(1) q^{\prime}(-1)+2 q^{\prime}(1) q^{\prime}(-1)-q^{\prime}(1) q(-1)} .
\end{align*}
$$

- Interior basis functions:

$$
\begin{equation*}
h_{j}(x)=\frac{x^{2}+a x+b}{x_{j}^{2}+a x_{j}+b} q_{j}(x), \quad 1 \leq j \leq N-1, \tag{A.3}
\end{equation*}
$$

where

$$
\begin{align*}
a & =\frac{2 q_{j}(1) q_{j}^{\prime}(-1)+2 q_{j}^{\prime}(1) q_{j}(-1)}{q_{j}^{\prime}(1) q_{j}(-1)-2 q_{j}^{\prime}(1) q_{j}^{\prime}(-1)-q_{j}(1) q_{j}^{\prime}(-1)}, \\
b & =\frac{3 q_{j}(1) q_{j}^{\prime}(-1)-3 q_{j}^{\prime}(1) q_{j}(-1)+2 q_{j}^{\prime}(1) q_{j}^{\prime}(-1)-4 q_{j}(1) q_{j}(-1)}{q_{j}^{\prime}(1) q_{j}(-1)-2 q_{j}^{\prime}(1) q_{j}^{\prime}(-1)-q_{j}(1) q_{j}^{\prime}(-1)} . \tag{A.4}
\end{align*}
$$

## Appendix B. The derivation of (5.20)

We first recall the formula (see, e.g., [23]):

$$
\begin{equation*}
\partial_{x} J_{n}^{(\alpha, \beta)}(x)=\frac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{(\alpha+1, \beta+1)}(x), \quad n \geq 1 . \tag{B.1}
\end{equation*}
$$

Using (B.1) and the expressions of $d_{n}$ and $e_{n}$ gives

$$
\begin{aligned}
\phi_{n}^{\prime}(x) & =d_{n}\left(\frac{1}{2}(n+1) J_{n-1}^{(1,1)}(x)+\frac{1}{2}(n+3) e_{n} J_{n+1}^{(1,1)}(x)\right) \\
& =\frac{1}{2} d_{n}(n+1)\left(J_{n-1}^{(1,1)}(x)-\frac{n}{n+2} J_{n+1}^{(1,1)}(x)\right) .
\end{aligned}
$$

By (B.1) and formula (4.5.5) of [23],

$$
\left(1-x^{2}\right) \frac{\mathrm{d}}{\mathrm{~d} x} J_{n}^{(1,1)}(x)=\frac{(n+1)(n+3)}{2 n+3}\left(J_{n-1}^{(1,1)}(x)-\frac{n}{n+2} J_{n+1}^{(1,1)}(x)\right) .
$$

A combination of the above two facts leads to

$$
\phi_{n}^{\prime}(x)=\frac{2 n+3}{4} d_{n}\left(1-x^{2}\right) J_{n-1}^{(2,2)}(x)=\tilde{\lambda}_{n-1}\left(1-x^{2}\right) J_{n-1}^{(2,2)}(x)
$$

Indeed, the normalized factor $d_{n}$ is chosen such that $\left\{\phi_{n}\right\}$ is orthonormal in $L_{\omega^{2}}^{2}(I)$.

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