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On L^2-Stability Analysis of Time-Domain Acoustic Scattering Problems with Exact Nonreflecting Boundary Conditions

Bo Wang[1], Li-Lian Wang[2,*]

[1] *College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China*
[2] *Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore*

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Abstract. This paper is devoted to stability analysis of the acoustic wave equation exterior to a bounded scatterer, where the unbounded computational domain is truncated by the exact time-domain circular/spherical nonreflecting boundary condition (NRBC). Different from the usual energy method, we adopt an argument that leads to L^2-*a priori* estimates with minimum regularity requirement for the initial data and source term. This needs some delicate analysis of the involved NRBC. These results play an essential role in the error analysis of the interior solvers (e.g., finite-element/spectral-element/spectral methods) for the reduced scattering problems. We also apply the technique to analyze a time-domain waveguide problem.

AMS subject classifications: 65R10, 65N35, 65E05, 65M70

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1 Introduction

In this paper, we consider the time-domain acoustic scattering problem:

$$\partial_t^2 U = c^2 \Delta U + F, \quad \text{in } \Omega_\infty := \mathbb{R}^d \setminus \bar{D},\ t>0,\ d=2,3; \tag{1.1}$$

$$U = U_0, \quad \partial_t U = U_1, \quad \text{in } \Omega_\infty,\ t=0; \tag{1.2}$$

$$U = 0, \quad \text{on } \Gamma_D,\ t>0; \quad \partial_t U + c\partial_{\boldsymbol{n}} U = o(|\boldsymbol{x}|^{(1-d)/2}),\ |\boldsymbol{x}|\to\infty,\ t>0, \tag{1.3}$$

where D is a bounded obstacle (scatterer) with Lipschitz boundary Γ_D, $c>0$ is the wave speed, and $\boldsymbol{n}=\boldsymbol{x}/|\boldsymbol{x}|$. Assume that the data F, U_0 and U_1 are compactly supported in a 2D disk or a 3D ball B of radius b, which contains the obstacle D.

*Corresponding author. *Email addresses:* bowanghn@gmail.com (B. Wang), LiLian@ntu.edu.sg (L.L. Wang)

The acoustic wave propagates in the free space exterior to D, so the first important issue is to reduce the unbounded domain to a bounded domain. One efficient way is to set up an artificial boundary and impose a transparent/non-reflecting boundary condition (NRBC) thereon (see e.g., [8]). It is advantageous to use the exact NRBC, as it can be placed as close as possible to the scatterer, and the reduced problem, so as the discretized problem, can be best mimic to the continuous problem. Though such a NRBC is global in time and space in nature, fast and accurate numerical and/or semi-numerical means were developed for its evaluation and/or seamless integration with some solver in the reduced domain (see e.g., [3, 14, 15]).

This paper is largely concerned with the analysis of the reduced scattering problem by the exact circular/spherical NRBC. We remark that in [5,15], the usual energy method (i.e., testing the equation with $\partial_t U$) was used to obtain H^1-type estimates under strong regularity assumptions for the initial data and source term. Moreover, this approach did not lead to optimal L^2-estimates. In this paper, we resort to an argument in [4,7], which, together with a delicate analysis of the involved NRBC, leads to $L^\infty(L^2)$- and $L^2(L^2)$-*a priori* estimates for the reduced problem with a minimum regularity requirement for the initial data and source term. With this at our disposal, we can also analyze a waveguide problem considered in [18].

The paper is organized as follows. We present the reduced problem and carry out the *a priori* estimates in the forthcoming section. In the last section, we apply the argument to analyze a waveguide problem.

2 $L^\infty(L^2)$- and $L^2(L^2)$-a priori estimates

2.1 The reduced problem

We first reduce the scattering problem (1.1)-(1.3) to a bounded domain via the exact circular/spherical NRBC (see e.g., [3, 8, 15]), leading to

$$\partial_t^2 U = c^2 \Delta U + F, \qquad \text{in } \Omega := B \setminus \bar{D},\ t>0,\ d=2,3; \tag{2.1}$$

$$U = U_0, \quad \partial_t U = U_1, \qquad \text{in } \Omega,\ t=0; \qquad U=0, \quad \text{on } \Gamma_D,\ t>0; \tag{2.2}$$

$$\partial_r U - \mathcal{T}_d(U) = 0, \qquad \text{at } r=b,\ t>0, \tag{2.3}$$

where the time-domain DtN boundary condition at the artificial boundary $\Gamma_b := \partial B$, is given, in polar/spherical coordinates, by

$$\mathcal{T}_d(U) = \begin{cases} \Big(-\dfrac{1}{c}\dfrac{\partial U}{\partial t} - \dfrac{U}{2r}\Big)\Big|_{r=b} + \displaystyle\sum_{|n|=0}^{\infty} \sigma_n(t) * \widehat{U}_n(b,t) e^{\mathrm{i}n\phi}, & d=2, \\ \Big(-\dfrac{1}{c}\dfrac{\partial U}{\partial t} - \dfrac{U}{r}\Big)\Big|_{r=b} + \displaystyle\sum_{n=0}^{\infty}\sum_{|m|=0}^{n} \sigma_{n+1/2}(t) * \widehat{U}_{nm}(b,t)\, Y_n^m(\theta,\phi), & d=3. \end{cases} \tag{2.4}$$

Here, the kernel functions in the convolution are

$$\sigma_\nu(t) := \mathcal{L}^{-1}\left[\frac{s}{c} + \frac{1}{2b} + \frac{s}{c}\frac{K'_\nu(sb/c)}{K_\nu(sb/c)}\right], \quad \nu = n, n+1/2, \tag{2.5}$$

where K_ν is the modified Bessel function of the second kind of order ν (see e.g., [1,17]), and $\mathcal{L}^{-1}[h(s)]$ is the inverse Laplace transform of a Laplace transformable function $H(t)$:

$$h(s) = \mathcal{L}[H(t)](s) = \int_0^\infty H(t)e^{-st}dt, \quad s \in \mathbb{C}, \ \mathrm{Re}(s) > 0.$$

In (2.4), $\{Y_n^m\}$ are the spherical harmonics, which are orthonormal as defined in [10], and $\{\widehat{U}_n\}/\{\widehat{U}_{nm}\}$ are the Fourier/spherical harmonic expansion coefficients of $U|_{r=b}$. Recall that the convolution in (2.4) is defined as usual:

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Another useful alternative expression (cf. [15]) of $\mathcal{T}_d(U)$, where the temporal convolution is expressed in terms of expansion coefficients of $\partial_t U|_{r=b}$, is given as follows:

$$\mathcal{T}_d(U) = -\frac{1}{c}\frac{\partial U}{\partial t}\Big|_{r=b} + \frac{1}{c}\begin{cases} \displaystyle\sum_{|n|=0}^{\infty} \omega_n(t) * \partial_t \widehat{U}_n(b,t)e^{\mathrm{i}n\phi}, & d=2, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{|m|=0}^{n} \omega_{n+1/2}(t) * \partial_t \widehat{U}_{nm}(b,t)Y_n^m(\theta,\phi), & d=3, \end{cases} \tag{2.6}$$

where for $d = 2,3$,

$$\omega_\nu(t) = \mathcal{L}^{-1}\left[1 - \frac{(d-2)c}{2bs} + \frac{K'_\nu(sb/c)}{K_\nu(sb/c)}\right](t), \quad \nu = n, n+1/2. \tag{2.7}$$

It is clearly that

$$\omega_\nu(t) := \omega_\nu(t;d) := -\frac{(d-1)c}{2b} + c\int_0^t \sigma_\nu(\tau)d\tau, \tag{2.8}$$

and $\omega'_\nu(t) = c\sigma_\nu(t)$. Since $K_{-n}(z) = K_n(z)$ (see [1, Formula (9.6.6)]), it suffices to consider ω_n and σ_n with $n \geq 0$, for $d = 2$.

Remark 2.1. It is seen from (2.4) that the NRBC is global in time and space, due to the involvement of the convolution and Fourier/spherical harmonic expansions. We refer to [3,14,15] for fast and accurate methods for dealing with the inverse Laplace transform and temporal convolution, and also refer to [11,13] for techniques to overcome globleness of the exact boundary conditions in the context of time-harmonic scattering problems.

2.2 A priori estimates

We now intend to derive *a priori* estimates for the solution of the reduced problem (2.1)-(2.3). For this purpose, we first introduce some notation. Given a generic weight function w, let $H^r_w(\Omega)$ be the usual weighted Sobolev space of complex-valued functions as in Admas [2] with the norm $\|\cdot\|_{H^r_w(\Omega)}$. As usual, $L^2_w(\Omega)=H^0_w(\Omega)$ with inner product denoted by $(\cdot,\cdot)_{L^2_w(\Omega)}$. We drop the weight function, whenever $w\equiv 1$. To characterize the regularity in time, we also use e.g., the space $L^\infty(0,T;L^2(\Omega))$, as defined in [2].

We formulate the equation (2.1)-(2.3) in a weak form (in space). For any $t>0$, it is to find $U(\cdot,t)\in\mathcal{V}:=\{v\in H^1(\Omega): v|_{\Gamma_D}=0\}$ such that

$$\int_\Omega \partial_{tt}U\overline{V}d\boldsymbol{x}=-c^2\int_\Omega \nabla U\cdot\nabla\overline{V}d\boldsymbol{x}+c^2\int_{\Gamma_b}\mathcal{T}_d(U)\overline{V}d\gamma+\int_\Omega F\overline{V}d\boldsymbol{x},\quad \forall V\in\mathcal{V}. \tag{2.9}$$

Remark 2.2. It is important to point out that using the standard energy argument (i.e., taking $V=\partial_t U$ in (2.9), see [5,15]), we are able to derive the a priori estimates in the energy norm, that is, $\|\partial_t U\|_{L^\infty(0,T;L^2(\Omega))}+c\|\nabla U\|_{L^\infty(0,T;L^2(\Omega))}$. However, this requires strong regularity of the initial and boundary data, and does not lead to optimal L^2-estimates.

Hereafter, we take a different route that will lead to $L^\infty(L^2)$-*a priori* estimates for the reduced problem (2.1)-(2.3), with a minimum requirement for the regularity of the inputs. We essentially employ an argument due to Dupont [7] (also see Baker [4]), but significant care is needed to analyze the exact NRBCs. For this purpose, we first make necessary preparations.

Recall the Plancherel or Parseval identity for the Laplace transform (see e.g., [6, (2.46)]).

Lemma 2.1. *Let $s=s_1+\mathrm{i}s_2$ with $s_1,s_2\in\mathbb{R}$. If f,g are Laplace transformable, then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{L}[f](s)\mathcal{L}[\bar{g}](s)ds_2=\int_0^\infty e^{-2s_1t}f(t)\bar{g}(t)dt,\quad \forall s_1>\gamma, \tag{2.10}$$

where γ is the absissa of convergence for both f and g, and $\bar{g}$ is the complex conjugate of g.

For notational convenience, we introduce the modified spherical Bessel function (cf. [17]):

$$k_n(z)=\sqrt{\frac{2}{\pi z}}K_{n+1/2}(z),\qquad \text{so } \frac{k_n'(z)}{k_n(z)}=-\frac{1}{2z}+\frac{K'_{n+1/2}(z)}{K_{n+1/2}(z)}. \tag{2.11}$$

Then, by (2.7),

$$\omega_{n+1/2}(t)=\mathcal{L}^{-1}\left[1+\frac{k_n'(sb/c)}{k_n(sb/c)}\right](t),\quad n\ge 0. \tag{2.12}$$

We shall use the following property (see [5,15]).

Lemma 2.2. *Let $s=s_1+\mathrm{i}s_2$ with $s_1,s_2\in\mathbb{R}$. Then we have*

$$\mathrm{Re}\Big(\frac{Z_n'(sb/c)}{Z_n(sb/c)}\Big)\le 0,\quad \forall s_1>0, \tag{2.13}$$

where $Z_n(z)=K_n(z)$ or $k_n(z)$.

The following lemma is indispensable for the forthcoming analysis.

Lemma 2.3. *For any $v\in L^2(0,T)$ with $v(0)=0$, we have*

$$\mathrm{Re}\int_0^T\Big(\int_0^t[\sigma_\nu*v](\tau)d\tau\Big)\bar{v}(t)dt\le\frac{1}{c}\int_0^T|v(t)|^2dt+\frac{1}{4b}\Big|\int_0^T v(t)dt\Big|^2, \tag{2.14}$$

$$\mathrm{Re}\int_0^T\Big(\int_0^t[\omega_\nu*v'](\tau)d\tau\Big)\bar{v}(t)dt+\frac{(d-2)c}{4b}\Big|\int_0^T v(t)dt\Big|^2\le\int_0^T|v(t)|^2dt, \tag{2.15}$$

for $\nu=n,n+1/2$, where σ_ν and ω_ν are the convolution kernel functions.

Proof. Using the property of the Laplace transform:

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](s)=\frac{1}{s}\mathcal{L}[f](s),$$

we have

$$\mathcal{L}\left[\int_0^t[\sigma_\nu*v](\tau)d\tau\right](s)=\frac{1}{s}\mathcal{L}[\sigma_\nu*v](s)=\left[\frac{1}{c}+\frac{1}{2bs}+\frac{1}{c}\frac{K_\nu'(sb/c)}{K_\nu(sb/c)}\right]\mathcal{L}[v](s), \tag{2.16}$$

where in the last step, we used the property of the Laplace transform and the definition:

$$\sigma_\nu(t)=\mathcal{L}^{-1}\Big[\frac{s}{c}+\frac{1}{2b}+\frac{s}{c}\frac{K_\nu'(sb/c)}{K_\nu(sb/c)}\Big].$$

Let $\tilde{v}=v\mathbf{1}_{[0,T]}$ where $\mathbf{1}_{[0,T]}$ is the characteristic function of $[0,T]$. Using the Parseval identity (2.10), and Lemma 2.2, we have

$$\begin{aligned}
&\mathrm{Re}\int_0^T e^{-2s_1t}\Big(\int_0^t[\sigma_\nu*v](\tau)d\tau\Big)\bar{v}(t)dt=\mathrm{Re}\int_0^\infty e^{-2s_1t}\Big(\int_0^t[\sigma_\nu*\tilde{v}](\tau)d\tau\Big)\bar{\tilde{v}}(t)dt\\
=&\frac{1}{2\pi}\int_{-\infty}^\infty\mathrm{Re}\left[\frac{1}{c}+\frac{1}{2bs}+\frac{1}{c}\frac{K_\nu'(sb/c)}{K_\nu(sb/c)}\right]\mathcal{L}[\tilde{v}](s)\mathcal{L}[\bar{\tilde{v}}](s)ds_2\\
\le&\frac{1}{2\pi}\int_{-\infty}^\infty\mathrm{Re}\left[\frac{1}{c}+\frac{1}{2bs}\right]|\mathcal{L}[\tilde{v}](s)|^2ds_2\\
=&\frac{1}{c}\int_0^T e^{-2s_1t}|v(t)|^2dt+\frac{1}{2b}\mathrm{Re}\int_0^T e^{-2s_1t}\Big(\int_0^t v(\tau)d\tau\Big)\bar{v}(t)dt.
\end{aligned} \tag{2.17}$$

Letting $s_1\to0$, we obtain

$$\mathrm{Re}\int_0^T\Big(\int_0^t[\sigma_\nu*v](\tau)d\tau\Big)\bar{v}(t)dt\le\frac{1}{c}\int_0^T|v(t)|^2dt+\frac{1}{2b}\mathrm{Re}\int_0^T\Big(\int_0^t v(\tau)d\tau\Big)\bar{v}(t)dt. \tag{2.18}$$

Moreover, using integrate by parts yields

$$\int_0^T\Big(\int_0^t v(\tau)d\tau\Big)\bar{v}(t)dt=\left[\int_0^t v(\tau)d\tau\int_0^t\bar{v}(\tau)d\tau\right]_0^T-\int_0^T\Big(\int_0^t\bar{v}(\tau)d\tau\Big)v(t)dt,$$

which implies

$$\mathrm{Re}\int_0^T\Big(\int_0^t v(\tau)d\tau\Big)\bar{v}(t)dt=\frac{1}{2}\Big|\int_0^T v(t)dt\Big|^2. \tag{2.19}$$

Therefore, (2.14) follows from (2.18).

We now turn to the derivation of (2.15). By (2.8),

$$\omega_\nu(0)=-\frac{(d-1)c}{2b},\qquad \omega_\nu'(t)=c\sigma_\nu(t).$$

A direct calculation using integration by parts and the condition $v(0)=0$, leads to

$$\begin{aligned}\int_0^t[\omega_\nu*v'](\tau)d\tau&=\omega_\nu(0)\int_0^t v(\tau)d\tau+\int_0^t[\omega_\nu'*v](\tau)d\tau\\&=-\frac{(d-1)c}{2b}\int_0^t v(\tau)d\tau+\int_0^t[\omega_\nu'*v](\tau)d\tau\\&=-\frac{(d-1)c}{2b}\int_0^t v(\tau)d\tau+c\int_0^t[\sigma_\nu*v](\tau)d\tau.\end{aligned}$$

Therefore, by (2.14) and (2.19),

$$\begin{aligned}&\mathrm{Re}\int_0^T\Big(\int_0^t[\omega_\nu*v'](\tau)d\tau\Big)\bar{v}(t)dt\\&\le-\frac{(d-1)c}{4b}\Big|\int_0^T v(t)dt\Big|^2+\frac{c}{4b}\Big|\int_0^T v(t)dt\Big|^2+\int_0^T|v(t)|^2dt.\end{aligned}$$

This implies (2.15). □

With the above preparations, we are ready to derive the $L^\infty(L^2)$-*a priori* estimates by using an argument due to [4,7].

Theorem 2.1. *Let* U $(\in\mathcal{V}$ *for* $t>0)$ *be the solution of* (2.9). *If* $U_0,U_1\in L^2(\Omega)$ *and* $F\in L^1\big(0,T;L^2(\Omega)\big)$, *then the solution* $U\in L^\infty(0,T;L^2(\Omega))$. *Moreover, we have*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))}\le C\big(\|U_0\|_{L^2(\Omega)}+T\|U_1\|_{L^2(\Omega)}+T\|F\|_{L^1(0,T;L^2(\Omega))}\big), \tag{2.20}$$

and

$$\|U\|_{L^2((0,T)\times\Omega))}\le C\sqrt{T}\big(\|U_0\|_{L^2(\Omega)}+T\|U_1\|_{L^2(\Omega)}+T\|F\|_{L^1(0,T;L^2(\Omega))}\big), \tag{2.21}$$

where C *is a positive constant independent of* T,c *and any functions.*

Proof. Let $0<\xi\le T$, and define

$$\psi(\boldsymbol{x},t)=\int_t^\xi U(\boldsymbol{x},\tau)d\tau,\quad 0\le t\le\xi,\quad \boldsymbol{x}\in\Omega. \tag{2.22}$$

It is clear that

$$\psi(\boldsymbol{x},\xi)=0, \quad \frac{\partial \psi}{\partial t}(\boldsymbol{x},t)=-U(\boldsymbol{x},t). \tag{2.23}$$

Moreover, for any $\phi(\boldsymbol{x},t)\in L^2((0,\xi)\times\Omega)$, we have

$$\int_0^\xi \phi(\boldsymbol{x},t)\bar{\psi}(\boldsymbol{x},t)dt=\int_0^\xi \Big(\int_0^t \phi(\boldsymbol{x},\tau)d\tau\Big)\overline{U}(\boldsymbol{x},t)dt. \tag{2.24}$$

We show this identity below. Indeed, using integration by parts and (2.23), we have

$$\begin{aligned}\int_0^\xi \phi(\boldsymbol{x},t)\bar{\psi}(\boldsymbol{x},t)dt&=\int_0^\xi \Big(\phi(\boldsymbol{x},t)\int_t^\xi \overline{U}(\boldsymbol{x},\tau)d\tau\Big)dt\\ &=\int_0^\xi\int_t^\xi \overline{U}(\boldsymbol{x},\tau)d\tau d\Big(\int_0^t\phi(\boldsymbol{x},\tau)d\tau\Big)\\ &=\int_0^t \phi(\boldsymbol{x},\tau)d\tau\int_t^\xi \overline{U}(\boldsymbol{x},\tau)d\tau\Big|_0^\xi+\int_0^\xi\Big(\int_0^t\phi(\boldsymbol{x},\tau)d\tau\Big)\overline{U}(\boldsymbol{x},t)dt\\ &=\int_0^\xi\Big(\int_0^t\phi(\boldsymbol{x},\tau)d\tau\Big)\overline{U}(\boldsymbol{x},t)dt.\end{aligned}$$

Next, taking the test function $V=\psi$ in (2.9), leads to

$$\int_\Omega \partial_{tt}U\bar{\psi}d\boldsymbol{x}=-c^2\int_\Omega \nabla U\cdot\nabla\bar{\psi}d\boldsymbol{x}+\int_\Omega F\bar{\psi}d\boldsymbol{x}+c^2\int_{\Gamma_b}\mathcal{T}_d(U)\bar{\psi}d\gamma. \tag{2.25}$$

By (2.23),

$$\begin{aligned}\mathrm{Re}\int_0^\xi\int_\Omega \partial_{tt}U\bar{\psi}d\boldsymbol{x}dt&=\mathrm{Re}\int_\Omega\int_0^\xi\Big(\partial_t(\partial_t U\bar{\psi})+\partial_t U\overline{U}\Big)dtd\boldsymbol{x}\\ &=\mathrm{Re}\int_\Omega\Big((\partial_t U\bar{\psi})\Big|_0^\xi+\frac{1}{2}|U|^2\Big|_0^\xi\Big)d\boldsymbol{x}\\ &=\frac{1}{2}\|U(\cdot,\xi)\|^2_{L^2(\Omega)}-\frac{1}{2}\|U_0\|^2_{L^2(\Omega)}-\mathrm{Re}\int_\Omega U_1(\boldsymbol{x})\bar{\psi}(\boldsymbol{x},0)d\boldsymbol{x}.\end{aligned}$$

Thus, integrating (2.25) from $t=0$ to ξ and taking the real parts, yields

$$\begin{aligned}&\frac{1}{2}\|U(\cdot,\xi)\|^2_{L^2(\Omega)}-\frac{1}{2}\|U_0\|^2_{L^2(\Omega)}+\frac{c^2}{2}\int_\Omega\Big|\int_0^\xi \nabla U(\boldsymbol{x},t)dt\Big|^2d\boldsymbol{x}\\ =&\mathrm{Re}\int_\Omega U_1(\boldsymbol{x})\bar{\psi}(\boldsymbol{x},0)d\boldsymbol{x}+\mathrm{Re}\int_0^\xi\int_\Omega F\bar{\psi}d\boldsymbol{x}dt+c^2\mathrm{Re}\int_0^\xi\int_{\Gamma_b}\mathcal{T}_d(U)\bar{\psi}d\gamma dt.\end{aligned} \tag{2.26}$$

We derive from (2.22) and the Cauchy-Schwartz inequality that

$$\begin{aligned}&\mathrm{Re}\int_\Omega U_1(\boldsymbol{x})\bar{\psi}(\boldsymbol{x},0)d\boldsymbol{x}=\mathrm{Re}\int_\Omega U_1(\boldsymbol{x})\Big(\int_0^\xi\overline{U}(\boldsymbol{x},t)dt\Big)d\boldsymbol{x}\\ =&\mathrm{Re}\int_0^\xi\int_\Omega U_1(\boldsymbol{x})\overline{U}(\boldsymbol{x},t)d\boldsymbol{x}dt\le\|U_1\|_{L^2(\Omega)}\int_0^\xi\|U(\cdot,t)\|_{L^2(\Omega)}dt.\end{aligned} \tag{2.27}$$

Similarly, by (2.24), we have that for $0\le t\le\xi\le T$,

$$\begin{aligned}\mathrm{Re}\int_0^{\xi}\int_{\Omega}F\bar{\psi}d\boldsymbol{x}dt=&\mathrm{Re}\int_{\Omega}\int_0^{\xi}\Big(\int_0^t F(\boldsymbol{x},\tau)d\tau\Big)\overline{U}(\boldsymbol{x},t)d\boldsymbol{x}dt\\ =&\mathrm{Re}\int_0^{\xi}\int_0^t\int_{\Omega}F(\boldsymbol{x},\tau)\overline{U}(\boldsymbol{x},t)d\boldsymbol{x}d\tau dt\\ \le&\mathrm{Re}\int_0^{\xi}\Big(\int_0^t\|F(\cdot,\tau)\|_{L^2(\Omega)}d\tau\Big)\|U(\cdot,t)\|_{L^2(\Omega)}dt\\ \le&\mathrm{Re}\int_0^{\xi}\Big(\int_0^{\xi}\|F(\cdot,\tau)\|_{L^2(\Omega)}d\tau\Big)\|U(\cdot,t)\|_{L^2(\Omega)}dt\\ =&\Big(\int_0^{\xi}\|F(\cdot,t)\|_{L^2(\Omega)}dt\Big)\Big(\int_0^{\xi}\|U(\cdot,t)\|_{L^2(\Omega)}dt\Big).\end{aligned}\tag{2.28}$$

For the NRBC term, we consider the 3D case (2D case is similar). Using Lemma 2.3, we obtain

$$\begin{aligned}&\mathrm{Re}\int_0^{\xi}\int_{\Gamma_b}\mathcal{T}_d(U)\bar{\psi}d\gamma dt=-\frac{1}{c}\mathrm{Re}\int_{\Gamma_b}\int_0^{\xi}\frac{\partial U}{\partial t}\Big(\int_t^{\xi}\overline{U}(\cdot,\tau)d\tau\Big)dtd\gamma\\ &\qquad+\frac{1}{c}\sum_{n=0}^{\infty}\sum_{|m|=0}^{n}\mathrm{Re}\int_0^{\xi}\omega_{n+\frac12}(t)*\partial_t\widehat{U}_{nm}(b,t)\Big(\int_t^{\xi}\overline{\widehat{U}}_{nm}(b,\tau)d\tau\Big)dt\\ =&-\frac{1}{c}\int_{\Gamma_b}\int_0^{\xi}|U(\cdot,t)|^2dtd\gamma\\ &\qquad+\frac{1}{c}\sum_{n=0}^{\infty}\sum_{|m|=0}^{n}\mathrm{Re}\int_0^{\xi}\Big(\int_0^t\omega_{n+\frac12}(\tau)*\partial_\tau\widehat{U}_{nm}(b,\tau)d\tau\Big)\overline{\widehat{U}}_{nm}(b,t)dt\\ \le&-\frac{1}{c}\int_{\Gamma_b}\int_0^{\xi}|U(\cdot,t)|^2dtd\gamma+\frac{1}{c}\sum_{n=0}^{\infty}\sum_{|m|=0}^{n}\int_0^{\xi}|\widehat{U}_{nm}(b,t)|^2dt=0.\end{aligned}$$

Now, substituting (2.27)-(2.28) into (2.26), we have that for any $\xi\in[0,T]$,

$$\begin{aligned}&\frac12\|U(\cdot,\xi)\|^2_{L^2(\Omega)}+\frac{c^2}{2}\int_{\Omega}\Big|\int_0^{\xi}\nabla U(\boldsymbol{x},t)dt\Big|^2d\boldsymbol{x}\\ \le&\frac12\|U_0\|^2_{L^2(\Omega)}+\Big(\int_0^{\xi}\|F(\cdot,t)\|_{L^2(\Omega)}dt+\|U_1\|_{L^2(\Omega)}\Big)\int_0^{\xi}\|U(\cdot,t)\|_{L^2(\Omega)}dt.\end{aligned}\tag{2.29}$$

Taking L^∞-norm with respect to ξ on both sides of (2.29), yields

$$\|U\|^2_{L^\infty(0,T;L^2(\Omega))}\le\|U_0\|^2_{L^2(\Omega)}+2T\big(\|F\|_{L^1(0;T;L^2(\Omega))}+\|U_1\|_{L^2(\Omega)}\big)\|U\|_{L^\infty(0,T;L^2(\Omega))}.$$

Therefore, the estimate (2.20) follows directly from the Cauchy-Schwartz inequality.

Integrating (2.29) with respect to ξ over $(0,T)$ and using the Cauchy-Schwartz inequality, leads to

$$\|U\|^2_{L^2((0,T)\times\Omega)}\le T\|U_0\|^2_{L^2(\Omega)}+2T^{3/2}\big(\|F\|_{L^1(0;T;L^2(\Omega))}+\|U_1\|_{L^2(\Omega)}\big)\|U\|_{L^2((0,T)\times\Omega)}.$$

Using the Cauchy-Schwartz inequality again, we derive the L^2-bound (2.21). □

2.3 Regular scatterers

The previous analysis applies to a general bounded scatterer with Lipschitz boundary. Accordingly, the results pave the way for analyzing finite-element/spectral-element approximations to the reduced problem. However, if the scatterer is a disk/ball, it is ideal to formulate the problem in the polar/spherical coordinates. Moreover, the NRBC turns out to be local in the space of Fourier/spherical harmonic coefficients. This allows us to further reduce the problem of interest to a sequence of decoupled one-dimensional problems (see (2.32) below). We refer to [15] for the fast spectral-Galerkin solver under this notion and [11] for the time-harmonic case coupled with an efficient technique for dealing with irregular scatterers. The previous results do not imply the estimates below, but the argument can be applied.

Consider the reduced problem (2.1)-(2.3) with a regular scatterer:

$$\begin{aligned}&\partial_t^2 U = c^2\Delta U + F, \qquad \text{in } \Omega=\big\{\boldsymbol{x}\in\mathbb{R}^d : b_0<|\boldsymbol{x}|<b\big\},\ t>0,\ d=2,3;\\ &U=U_0,\ \partial_t U = U_1, \quad \text{in } \Omega,\ t=0;\\ &U|_{r=b_0}=0, \quad \big(\partial_r U - \mathcal{T}_d(U)\big)\big|_{r=b}=0,\ t>0,\end{aligned} \tag{2.30}$$

where $\mathcal{T}_d(U)$ is the time-domain DtN map as before and $b_0>0$. We expand the solution and given data in Fourier/spherical harmonic series, e.g.,

$$\big\{U,F,U_0,U_1\big\} = \sum_{|n|=0}^{\infty}\big\{\widehat{U}_n,\widehat{F}_n,\widehat{U}_{0,n},\widehat{U}_{1,n}\big\}e^{\mathrm{i}n\theta}. \tag{2.31}$$

Then the problem (2.30), after a polar (in 2-D) and spherical (in 3-D) transform, reduces to a sequence of one-dimensional problems (for brevity, we use u to denote the Fourier/spherical harmonic expansion coefficients of U, and likewise, we use u_0,u_1 and f to denote the expansion coefficients of U_0,U_1 and F, respectively):

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} - \frac{c^2}{r^{d-1}}\frac{\partial}{\partial r}\Big(r^{d-1}\frac{\partial u}{\partial r}\Big) + c^2\beta_n\frac{u}{r^2} = f, \quad b_0<r<b,\ t>0;\\ &u|_{t=0}=u_0, \quad \frac{\partial u}{\partial t}\Big|_{t=0}=u_1, \quad b_0<r<b; \quad u|_{r=b_0}=0,\ t>0;\\ &\Big(\frac{1}{c}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{d-1}{2r}u\Big)\Big|_{r=b} = \int_0^t \sigma_\nu(t-\tau)u(b,\tau)d\tau,\ t>0,\end{aligned} \tag{2.32}$$

where $\beta_n = n^2, n(n+1)$ and $\nu = n, n+1/2$ for $d=2,3$, respectively.

Hereafter, let $I=(b_0,b)$ and $\varpi = r^{d-1}$. We introduce the weighted space $L^2_\varpi(I)$ (of complex-valued functions), and denote the (weighted) norm by $\|\cdot\|_\varpi$ and the inner product by $(\cdot,\cdot)_\varpi$. The weak form for (2.32) is to find that $u(\cdot,t)\in V := \{\phi\in H^1_\varpi(I) : \phi(b_0)=0\}$, such that for all $t>0$ and $w\in V$,

$$\begin{aligned}&(\ddot{u},w)_\varpi+cb^{d-1}\dot{u}(b,t)w(b)+c^2(\partial_r u,\partial_r w)_\varpi+c^2\beta_n(ur^{-2},w)_\varpi\\&\qquad+c^2b^{d-1}\Big[\frac{(d-1)}{2b}u(b,t)-\sigma_\nu(t)*u(b,t)\Big]w(b)=(f,w)_\varpi,\quad t>0,\\&u(r,0)=u_0(r),\quad \dot{u}(r,0)=u_1(r),\quad r\in I,\end{aligned}\tag{2.33}$$

where $\ddot{u},\dot{u}$ denote the derivatives in time.

Like Theorem 2.1, we derive the *a prior* estimates for (2.33).

Theorem 2.2. *Let u be the solution of* (2.33). *If $u_0,u_1\in L^2_\varpi(I)$ and $f\in L^1(0,T;L^2_\varpi(I))$, then for all $T>0$, and each mode n,*

$$\|u\|_{L^\infty(0,T;L^2_\varpi(I))}\le C\big(\|u_0\|_{L^2_\varpi(I)}+T\|u_1\|_{L^2_\varpi(I)}+T\|f\|_{L^1(0,T;L^2_\varpi(I))}\big),\tag{2.34}$$

and

$$\|u\|_{L^2(0,T;L^2_\varpi(I))}\le C\sqrt{T}\big(\|u_0\|_{L^2_\varpi(I)}+T\|u_1\|_{L^2_\varpi(I)}+T\|f\|_{L^1(0,T;L^2_\varpi(I))}\big),\tag{2.35}$$

where C is a positive constant independent of T, c and any functions.

Proof. Like the proof of Theorem 2.1, taking $w=\int_t^\xi u(r,\tau)d\tau$ with $0<\xi\le T$ in (2.33), and integrating the resulted equation from 0 to ξ, we obtain that

$$\begin{aligned}&\frac12\|u(\cdot,\xi)\|^2_{L^2_\varpi(I)}-\frac12\|u(\cdot,0)\|^2_{L^2_\varpi(I)}+cb^{d-1}\int_0^\xi|u(b,t)|^2dt\\&\quad+\frac{c^2}{2}\int_{b_0}^b\Big|\int_0^\xi\partial_r u(r,t)dt\Big|^2\varpi(r)dr+\frac{c^2\beta_n}{2}\int_{b_0}^b r^{d-3}\Big|\int_0^\xi u(r,t)dt\Big|^2dr\\&\quad+c^2b^{d-1}\mathrm{Re}\int_0^\xi\Big[\frac{d-1}{2b}\int_0^t u(b,\tau)d\tau-\int_0^t\sigma_\nu(\tau)*u(b,\tau)d\tau\Big]\bar{u}(b,t)dt\\&=\mathrm{Re}\Big(\dot{u}(\cdot,0),\int_0^\xi u(\cdot,t)dt\Big)_\varpi+\mathrm{Re}\int_0^\xi\Big(f(\cdot,t),\int_t^\xi u(\cdot,\tau)d\tau\Big)_\varpi dt,\end{aligned}\tag{2.36}$$

where we used the property (2.24) to handle the integrals of the boundary terms. We now show the summation of three boundary terms is non-negative. By (2.14) and (2.19),

$$\begin{aligned}&cb^{d-1}\int_0^\xi|u(b,t)|^2dt+c^2b^{d-1}\mathrm{Re}\int_0^\xi\Big[\frac{d-1}{2b}\int_0^t u(b,\tau)d\tau-\int_0^t\sigma_\nu(\tau)*u(b,\tau)d\tau\Big]\bar{u}(b,t)dt\\&\ge c^2b^{d-1}\frac{d-2}{4b}\Big|\int_0^\xi u(b,t)dt\Big|^2\ge0.\end{aligned}$$

Using the Cauchy-Schwartz inequality leads to

$$\begin{aligned}&\mathrm{Re}\Big(\dot{u}(\cdot,0),\int_0^\xi u(\cdot,t)dt\Big)_\varpi=\mathrm{Re}\int_0^\xi\Big(u_1(\cdot),u(\cdot,t)\Big)_\varpi dt\\&\le\int_0^\xi\|u_1\|_{L^2_\varpi(I)}\|u(\cdot,t)\|_{L^2_\varpi(I)}dt\le T\|u_1\|_{L^2_\varpi(I)}\|u\|_{L^\infty(0,T;L^2_\varpi(I))},\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Re}\int_0^{\xi}\Big(f(\cdot,t),\int_t^{\xi}u(\cdot,\tau)d\tau\Big)_{\varpi}dt=\mathrm{Re}\int_0^{\xi}\Big(\int_0^{t}f(\cdot,\tau)d\tau,u(\cdot,t)\Big)_{\varpi}dt\\
\le&\int_0^{\xi}\int_0^{t}\|f(\cdot,\tau)\|_{L^2_{\varpi}(I)}\|u(\cdot,t)\|_{L^2_{\varpi}(I)}d\tau dt\\
\le&\Big(\int_0^{\xi}\|f(\cdot,\tau)\|_{L^2_{\varpi}(I)}d\tau\Big)\Big(\int_0^{\xi}\|u(\cdot,t)\|_{L^2_{\varpi}(I)}dt\Big)\\
\le&T\|f\|_{L^1(0,T;L^2_{\varpi}(I))}\|u\|_{L^\infty(0,T;L^2_{\varpi}(I))}.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\|u(\cdot,\xi)\|^2_{L^2_{\varpi}(I)}-\frac{1}{2}||u_0||^2_{L^2_{\varpi}(I)}\le T(\|u_1\|_{L^2_{\varpi}(I)}+\|f\|_{L^1(0,T;L^2_{\varpi}(I))})\|u\|_{L^\infty(0,T;L^2_{\varpi}(I))}. \tag{2.37}$$

This leads to the estimate (2.34). Moreover, integrating (2.37) from 0 to T, we obtain the L^2-estimate (2.35) like Theorem 2.1. □

Remark 2.3. The estimates in Theorem 2.2 are valid for each mode n, which cannot be derived from Theorem 2.1. However, the converse statement is true. Indeed, using the Parseval's identity of the Fourier/spherical harmonic series, we can claim Theorem 2.1 in the case of regular scatterers from Theorem 2.2 straightforwardly.

Remark 2.4. The stability results are essential for the analysis of numerical solvers for the reduced problem. We illustrate this in the forthcoming section.

3 Analysis of a waveguide problem

In this section, we apply the previous argument to analyze a waveguide problem considered in [18], which involves the exact planar non-reflecting boundary condition (cf. [8]). More precisely, let

$$\Omega_\infty:=\big\{(x,y):0<x<\infty,0<y<2\pi\big\},$$

and consider

$$\partial_t^2U=c^2\Delta U+F,\quad \text{in }\Omega_\infty,\ t>0, \tag{3.1}$$

$$U|_{t=0}=U_0,\quad \partial_tU|_{t=0}=U_1,\quad \text{in }\Omega_\infty;\quad U|_{x=0}=0,\quad t>0, \tag{3.2}$$

$$\partial_tU+c\partial_xU=o(x^{-1/2}),\ x\to\infty,\ t>0, \tag{3.3}$$

where $c>0$ is the wave speed. Here, we assume that the given data F,U_0 and U_1 are 2π-periodic in y, and are compactly supported (with respect to x), in an interval $(0,a)$ for some $a>0$.

We adopt the exact planar NRBC at the artificial boundary $x=a$. This leads to the reduced problem in $\Omega:=(0,a)\times[0,2\pi)$:

$$\partial_t^2 U = c^2\Delta U + F, \qquad \text{in } \Omega,\ t>0, \tag{3.4}$$

$$U|_{t=0}=U_0, \quad \partial_t U|_{t=0}=U_1, \quad \text{in } \Omega; \qquad U|_{x=0}=0, \quad t>0, \tag{3.5}$$

$$\partial_x U - \mathcal{T}_a U = 0, \quad \text{at } x=a,\ t>0. \tag{3.6}$$

Note that the time-domain DtN map is given by

$$\mathcal{T}_a(U) := -\frac{1}{c}\partial_t U - \frac{1}{c}\sum_{|m|=0}^{\infty} \rho_m(t) * \widehat{U}_m(a,t) e^{\mathrm{i}my}, \tag{3.7}$$

where the convolution kernel ρ_m and Fourier coefficients $\{\widehat{U}_m\}$ are given by (cf. [8]):

$$\rho_m(t) := \frac{mcJ_1(mct)}{t}, \qquad \widehat{U}_m(a,t) = \frac{1}{2\pi}\int_0^{2\pi} U(a,y,t)e^{-\mathrm{i}my}dy, \tag{3.8}$$

with $J_1(\cdot)$ being the Bessel function of the first kind of order 1 (cf. [17]). Alternatively, we have (cf. [9, Table 19.1, Page 90]):

$$\rho_m(t) = \mathcal{L}^{-1}\big[\sqrt{s^2+m^2c^2} - s\big]. \tag{3.9}$$

Since $\rho_0=0$ and $\rho_{-m}=\rho_m$, it suffices to consider $m>0$ below.

3.1 A priori estimates

To derive the L^2-*a priori* estimates for the reduced problem (3.4)-(3.6), we first recall the following properties (see [18]).

Lemma 3.1. *Let $s=s_1+\mathrm{i}s_2$ with $s_1,s_2\in\mathbb{R}$. Then for any integer m and $s_1>0$,*

$$\mathrm{Re}\big(\sqrt{s^2+m^2c^2}\big)\ge 0, \quad s_2\mathrm{Im}\big(\sqrt{s^2+m^2c^2}\big)\ge 0. \tag{3.10}$$

Like Lemma 2.3, the following result is very important for the analysis.

Lemma 3.2. *For any $v\in L^2(0,T)$, we have*

$$\mathrm{Re}\int_0^T \Big(\int_0^t [\rho_m * v](\tau)d\tau\Big)\bar{v}(t)dt \ge -\int_0^T |v(t)|^2 dt, \quad \forall T>0,\ m\ge 0. \tag{3.11}$$

Proof. Let $\tilde{v}=v\mathbf{1}_{[0,T]}$, where $\mathbf{1}_{[0,T]}$ is the characteristic function of $[0,T]$. Then we obtain from Lemma 2.1 that

$$\begin{aligned}
&\int_0^T e^{-2s_1t}\int_0^t [\rho_m * v](\tau)d\tau\bar{v}(t)dt = \int_0^\infty e^{-2s_1t}\int_0^t [\rho_m*\tilde{v}](\tau)d\tau\bar{\tilde{v}}(t)dt \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\frac{\sqrt{s^2+m^2c^2}}{s}-1\Big]\big|\mathcal{L}[\tilde{v}](s)\big|^2 ds_2 \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{s^2+m^2c^2}}{s}\big|\mathcal{L}[\tilde{v}](s)\big|^2 ds_2 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\big|\mathcal{L}[\tilde{v}](s)\big|^2 ds_2.
\end{aligned}$$

Taking the real part of the above equation, we get

$$\begin{aligned}&\mathrm{Re}\int_0^T e^{-2s_1t}\int_0^t[\rho_m*v](\tau)d\tau\bar{v}(t)dt\\ =&\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\Big(\frac{\sqrt{s^2+m^2c^2}}{s}\Big)\big|\mathcal{L}[\tilde{v}](s)\big|^2ds_2-\frac{1}{2\pi}\mathrm{Re}\int_{-\infty}^{\infty}\big|\mathcal{L}[\tilde{v}](s)\big|^2ds_2.\end{aligned} \tag{3.12}$$

It is clear that by (2.10) with $f=g=\tilde{v}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\big|\mathcal{L}[\tilde{v}](s)\big|^2ds_2=\int_0^{\infty}e^{-2s_1t}|\tilde{v}(t)|^2dt=\int_0^T e^{-2s_1t}|v(t)|^2dt. \tag{3.13}$$

By Lemma 3.1,

$$\mathrm{Re}\Big(\frac{\sqrt{s^2+m^2c^2}}{s}\Big)=\frac{1}{|s^2|}\Big[s_1\mathrm{Re}\big(\sqrt{s^2+m^2c^2}\big)+s_2\mathrm{Im}\big(\sqrt{s^2+m^2c^2}\big)\Big]\geq 0. \tag{3.14}$$

By letting $s_1\to 0^+$ in (3.12), a combination of (3.12)-(3.14) leads to (3.11). □

Define $X:=\{U\in H^1(\Omega):U|_{x=0}=0\}$, and denote by $\langle\cdot,\cdot\rangle_{L^2(\Gamma_a)}$ and $\|\cdot\|_{L^2(\Gamma_a)}$ the inner product and norm of $L^2(\Gamma_a)$, respectively, where $\Gamma_a=\{(a,y):0<y<2\pi\}$. The weak form of (3.4)-(3.6) is to find $U\in X$ for all $t>0$, such that

$$\int_\Omega\partial_{tt}U\overline{V}d\boldsymbol{x}=-c^2\int_\Omega\nabla U\cdot\nabla\overline{V}d\boldsymbol{x}+c^2\int_{\Gamma_a}\mathcal{T}_a(U)\overline{V}dy+\int_\Omega F\overline{V}d\boldsymbol{x},\quad\forall V\in X. \tag{3.15}$$

Theorem 3.1. *Let $U(\in X$ for $t>0)$ be the solution of* (3.4)-(3.6). *If $U_0\in L^2(\Omega),U_1\in L^2(\Omega)$, and $F\in L^1(0,T;L^2(\Omega))$ for any $T>0$, then we have $U\in L^\infty(0,T;L^2(\Omega))$, and there holds*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))}\leq C\big(\|U_0\|_{L^2(\Omega)}+T\|U_1\|_{L^2(\Omega)}+T\|F\|_{L^1(0,T;L^2(\Omega))}\big), \tag{3.16}$$

and

$$\|U\|_{L^2((0,T)\times\Omega))}\leq C\sqrt{T}\big(\|U_0\|_{L^2(\Omega)}+T\|U_1\|_{L^2(\Omega)}+T\|F\|_{L^1(0,T;L^2(\Omega))}\big), \tag{3.17}$$

where C is a positive constant independent of any functions and c.

Proof. Taking

$$V=\psi(x,y,t)=\int_t^\xi U(x,y,\tau)d\tau,\qquad\forall(x,y)\in\Omega,\quad 0\leq t\leq\xi\leq T,$$

in (3.15) and following the same lines as in the proof of Theorem 2.1, we have

$$\begin{aligned}&\frac{1}{2}||U(\cdot,\xi)||^2_{L^2(\Omega)}-\frac{1}{2}||U(\cdot,0)||^2_{L^2(\Omega)}+\frac{c^2}{2}\int_\Omega\Big|\int_0^\xi\nabla U(x,y,t)dt\Big|^2dxdy\\ =&\mathrm{Re}\int_\Omega\partial_tU(\cdot,0)\bar{\psi}(\cdot,0)dxdy+\mathrm{Re}\int_\Omega\int_0^\xi F\bar{\psi}dtdxdy+c^2\mathrm{Re}\int_{\Gamma_a}\int_0^\xi\mathcal{T}_a(U)\bar{\psi}dtdy.\end{aligned} \tag{3.18}$$

According to the definition of $\psi(x,y,t)$ and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\Big|\int_\Omega\int_0^\xi F\bar{\psi}dtdxdy\Big|&=\Big|\int_\Omega\int_0^\xi\int_0^t F(x,\tau)d\tau\overline{U}(x,t)dtdxdy\Big|\\&\le T\|F\|_{L^1(0,T;L^2(\Omega))}\|U\|_{L^\infty(0,T;L^2(\Omega))},\end{aligned}\tag{3.19}$$

and

$$\Big|\int_\Omega\partial_t U(x,0)\bar{\psi}(x,0)dx\Big|\le T\|U_1\|_{L^2(\Omega)}\|U\|_{L^\infty(0,T;L^2(\Omega))}.\tag{3.20}$$

We next show that for any $t>0$,

$$\mathrm{Re}\int_0^\xi\int_{\Gamma_a}\mathcal{T}_a(U)\bar{\psi}dydt\le 0.\tag{3.21}$$

It follows from (3.7), Theorem 3.2 and the orthogonality of $\{e^{\mathrm{i}my}\}$ that

$$\begin{aligned}&\mathrm{Re}\int_0^\xi\int_{\Gamma_a}\mathcal{T}_a(U)\bar{\psi}dydt\\&=-\int_0^\xi\|U\|^2_{L^2(\Gamma_a)}dt-2\pi\sum_{|m|=0}^{\infty}\mathrm{Re}\int_0^\xi\big[\rho_m*\widehat{U}_m(a,t)\big]\overline{\widehat{\psi}}_m(a,\tau)dt\\&=-\int_0^\xi\|U\|^2_{L^2(\Gamma_a)}dt-2\pi\sum_{|m|=0}^{\infty}\mathrm{Re}\int_0^\xi\int_0^t\big[\rho_m*\widehat{U}_m(a,\tau)\big]d\tau\overline{\widehat{U}}_m(a,t)dt\\&\le-\int_0^\xi\|U\|^2_{L^2(\Gamma_a)}d\tau+2\pi\sum_{|m|=0}^{\infty}\int_0^\xi\big|\widehat{U}_m(a,\tau)\big|^2d\tau=0.\end{aligned}$$

Thus, the estimate (3.16) follows from (3.18)-(3.21) and the Cauchy-Schwartz inequality. According to (3.18) and (3.21), we have

$$\frac{1}{2}\|U(\cdot,\xi)\|^2_{L^2(\Omega)}-\frac{1}{2}\|U(\cdot,0)\|^2_{L^2(\Omega)}\le\mathrm{Re}\int_\Omega\partial_t U(\cdot,0)\bar{\psi}(\cdot,0)dxdy+\mathrm{Re}\int_\Omega\int_0^\xi F\bar{\psi}dtdxdy.\tag{3.22}$$

Integrating this inequality from 0 to T w.r.t ξ and then using the Cauchy-Schwartz inequality we derive the L^2-estimate (3.17). □

3.2 Fourier-Legendre spectral-Galerkin approximation

We expand the solution U and given data U_0,U_1,F in Fourier series:

$$\{U,U_0,U_1,F\}=\sum_{|m|=0}^{\infty}\{\hat{u}_m,\hat{u}_{0,m},\hat{u}_{1,m},\hat{f}_m\}e^{\mathrm{i}my}.\tag{3.23}$$

With a little abuse of notation, we still denote by u the Fourier coefficient $\hat{u}_m$, and likewise, we use u_0, u_1 and f to denote $\hat{u}_{0,m}, \hat{u}_{1,m}$ and $\hat{f}_m$, respectively. Then the problem (3.4)-(3.6) reduces to a sequence of 1D problems:

$$\begin{aligned} &\partial_t^2 u = c^2(\partial_x^2 u - m^2 u) + f, \quad 0<x<a,\ t>0,\\ &u=u_0, \quad \partial_t u = u_1, \qquad 0<x<a,\ t=0; \qquad u|_{x=0}=0, \quad t>0,\\ &\partial_t u + c\partial_x u + \rho_m * u = 0, \qquad x=a,\ t>0. \end{aligned} \tag{3.24}$$

Now, we apply the Legendre spectral-Galerkin approximation to discretize (3.24) in space. For convenience of implementation, we transform the interval $(0,a)$ to the reference interval $I:=(-1,1)$ by $x=a(\tilde{x}+1)/2$, and denote

$$v(\tilde{x},t)=u(x,t), \quad g(\tilde{x},t)=f(x,t), \quad v_i(\tilde{x})=u_i(x),\ i=0,1.$$

Then (3.24) becomes

$$\begin{aligned} &\partial_t^2 v = \tilde{c}^2\partial_{\tilde{x}}^2 v - m^2c^2 v + g, \quad -1<\tilde{x}<1,\ t>0,\\ &v=v_0, \quad \partial_t v = v_1, \qquad -1<\tilde{x}<1,\ t=0; \qquad v|_{\tilde{x}=-1}=0,\ t>0,\\ &\partial_t v + \tilde{c}\partial_{\tilde{x}} v + \rho_m * v = 0, \qquad \tilde{x}=1,\ t>0, \end{aligned} \tag{3.25}$$

where the constant $\tilde{c}=2c/a$. Then the weak form of (3.25) is to find $v(\cdot,t)\in V:=\{v\in H^1(I): v(-1)=0\}$, such that for all $w\in V$ and $t>0$

$$\begin{aligned} \mathcal{A}_m(v,w) := (\ddot{v},w) + \tilde{c}\dot{v}(1,t)w(1) + \tilde{c}^2(\partial_{\tilde{x}} v, \partial_{\tilde{x}} w)&\\ + c^2m^2(v,w) + \tilde{c}(\rho_m * v)(1,t)\bar{w}(1) = (g,w),& \end{aligned} \tag{3.26}$$

$$v(\tilde{x},0)=v_0(\tilde{x}), \quad \partial_t v(\tilde{x},0) = v_1(\tilde{x}), \quad \tilde{x}\in I, \tag{3.27}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(I)$.

We can derive the following *a priori* estimates for each mode m.

Theorem 3.2. *Let $v(\in V$ for $t>0)$ be the solution of* (3.26)-(3.27). *If $v_0\in L^2(I), v_1\in L^2(I)$, and $f\in L^1(0,T;L^2(I))$ for any $T>0$, then we have $v\in L^\infty(0,T;L^2(I))$, and there holds*

$$\|v\|_{L^\infty(0,T;L^2(I))} \le C\big(\|v_0\|_{L^2(I)} + T\|v_1\|_{L^2(I)} + T\|g\|_{L^1(0,T;L^2(I))}\big), \tag{3.28}$$

and

$$\|v\|_{L^2((0,T)\times I)} \le C\sqrt{T}\big(\|v_0\|_{L^2(I)} + T\|v_1\|_{L^2(I)} + T\|g\|_{L^1(0,T;L^2(I))}\big), \tag{3.29}$$

where C is a positive constant independent of any functions and c.

Proof. Taking $w(\tilde{x})=\int_t^\xi v(\tilde{x},\tau)d\tau$ in (3.26), and integrating the resulted equation from 0 to ξ with respect to t, we use Lemma 3.2 and the argument similar to that for Theorem

3.1 to derive the estimates:

$$
\begin{aligned}
&\mathrm{Re}\Big\{\int_0^\xi \mathcal{A}_m\Big(v(\cdot,t),\int_t^\xi v(\cdot,\tau)d\tau\Big)dt\Big\}\\
\geq&\frac{1}{2}\|v(\cdot,\xi)\|^2_{L^2(I)}-\frac{1}{2}\|v(\cdot,0)\|^2_{L^2(I)}-\int_0^\xi(\dot{v}(\cdot,0),v(\cdot,t))dt\\
\geq&\frac{1}{2}\|v(\cdot,\xi)\|^2_{L^2(I)}-\frac{1}{2}\|v_0\|^2_{L^2(I)}-\mathrm{Re}\int_0^\xi(v_1,v(\cdot,t))dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Re}\Big\{\int_0^\xi\Big(g(\cdot,t),\int_t^\xi v(\cdot,\tau)d\tau\Big)dt\Big\}=\mathrm{Re}\Big\{\int_0^\xi\int_0^t\Big(g(\cdot,\tau),v(\cdot,t)\Big)d\tau dt\Big\}\\
\leq&\int_0^\xi\int_0^t\|f(\cdot,\tau)\|_{L^2(I)}\|v(\cdot,t)\|_{L^2(I)}d\tau dt\\
\leq&\Big(\int_0^\xi\|g(\cdot,\tau)\|_{L^2(I)}d\tau\Big)\Big(\int_0^\xi\|v(\cdot,t)\|_{L^2(I)}dt\Big)\\
\leq&T\|g\|_{L^1(0,T;L^2(I))}\|v\|_{L^\infty(0,T;L^2(I))}.
\end{aligned}
$$

Therefore, we derive the *a priori* estimates by using the Cauchy-Schwartz inequality. □

Let $V_N:=\{\psi\in P_N:\psi(-1)=0\}$, where P_N is the set of all polynomials of degree at most N. The semi-discretization Legendre spectral-Galerkin approximation of (3.25) is to find $v_N(\tilde{x},t)\in V_N$ for all $t>0$ such that

$$
\begin{aligned}
&\mathcal{A}_m(v_N,w_N)=(\mathcal{I}_Ng,w_N),\ \forall w_N\in V_N,\\
&v_N(\tilde{x},0)=v_{0,N}(\tilde{x}),\quad \dot{v}_N(\tilde{x},0)=v_{1,N}(\tilde{x}),\quad \tilde{x}\in I,
\end{aligned}
\tag{3.30}
$$

where $\mathcal{I}_N$ is the interpolation operator on $(N+1)$ Legendre-Gauss-Lobatto points, and $v_{0,N},v_{1,N}\in P_N$ are suitable approximations of the initial values.

In what follows, we perform the error estimates for the scheme (3.30). For this purpose, we make some preparations.

Lemma 3.3. *Let $\rho_m(t)$ be the kernel function defined in* (3.8). *Then we have*

$$
|\rho_m(t)|\leq\frac{\sqrt{2}}{2}m^2c^2,\quad \forall t>0,
\tag{3.31}
$$

for all integer m.

Proof. Recall the properties of the Bessel functions (see [1]):

$$
\frac{2n}{z}J_n(z)=J_{n+1}(z)+J_{n-1}(z),\quad n\geq 1;\qquad J_0^2(z)+2\sum_{n=1}^\infty J_n^2(z)=1,\ z>0.
\tag{3.32}
$$

By (3.8) and the above properties, we obtain that for $m\geq 1$,

$$\begin{aligned}\rho_m(t)&=\frac{mcJ_1(mct)}{t}=\frac{m^2c^2}{2}\big(J_0(mct)+J_2(mct)\big)\\&\leq\frac{m^2c^2}{2}\sqrt{2\big(J_0^2(mct)+J_2^2(mct)\big)}\leq\frac{\sqrt{2}}{2}m^2c^2.\end{aligned}\tag{3.33}$$

Since $\rho_0=0$, and $\rho_{-m}=\rho_m$, the upper bound is valid for all m and $t>0$. □

Consider the orthogonal projection: ${}_0\pi_N^1:{}_0H^1(I):=\{u\in H^1(I):u(-1)=0\}\to{}_0P_N:=P_N\cap{}_0H^1(I)$, such that

$$(({}_0\pi_N^1u-u)',w')+({}_0\pi_N^1u-u,w)=0,\quad\forall w\in{}_0P_N.\tag{3.34}$$

Recall the Legendre-approximation results (see e.g., [12]): for any $u\in{}_0H^1(I)\cap H^s(I)$ with $1\leq s\leq N+1$,

$$\|{}_0\pi_N^1u-u\|_{H^\mu(I)}\leq DN^{\mu-s}\|u^{(s)}\|_{L^2(I)},\quad\mu=0,1,\tag{3.35}$$

and

$$\|{}_0\pi_N^1u-u\|_{L^\infty(I)}\leq DN^{1/2-s}\|u^{(s)}\|_{L^2(I)},\tag{3.36}$$

where D is a positive constant independent of N,s and u.

We also recall the approximation result on Legendre-Gauss-Lobbatto interpolation: for any $u\in H^s(I)$ with $1\leq s\leq N+1$ (see e.g., [12]):

$$\|u-\mathcal{I}_Nu\|_{L^2(I)}\leq DN^{-s}\|u^{(s)}\|_{L^2(I)}.\tag{3.37}$$

Moreover, we shall use the trace inverse inequality (see e.g., [16]): for any $\phi\in P_N$,

$$|\phi(1)|\leq\frac{N+1}{\sqrt{2}}\|\phi\|_{L^2(I)}.\tag{3.38}$$

With the above preparations, we are now ready to carry out the error analysis. It is clear that by (3.26) and (3.30),

$$\mathcal{A}_m(v_N-v,w_N)=(\mathcal{I}_Ng-g,w_N),\quad\forall w_N\in V_N,\ t>0.\tag{3.39}$$

To this end, let

$$e_N=v_N-{}_0\pi_N^1v,\quad\hat{e}_N=v-{}_0\pi_N^1v,\qquad\text{so } v_N-v=e_N-\hat{e}_N.$$

Then we derive from (3.30) and (3.34) that for any $w_N\in V_N$,

$$\begin{aligned}\mathcal{A}_m(e_N,w_N)&=\mathcal{A}_m(\hat{e}_N,w_N)+(\mathcal{I}_Ng-g,w_N)\\&=(\partial_t^2\hat{e}_N,w_N)+(m^2c^2-\tilde{c}^2)(\hat{e}_N,w_N)+\tilde{c}\partial_t\hat{e}_N(1,t)\bar{w}_N(1)\\&\quad+\tilde{c}\rho_m(t)*\hat{e}_N(1,t)\bar{w}_N(1)+(\mathcal{I}_Ng-g,w_N),\end{aligned}\tag{3.40}$$

and

$$e_N(x,0)=v_{0,N}(x)-{}_0\pi_N^1 v_0(x),\ \dot{e}_N(x,0)=v_{1,N}(x)-{}_0\pi_N^1 v_1(x). \tag{3.41}$$

We apply the argument of taking $w_N=\int_t^\xi e_N(\cdot,\tau)d\tau$ in (3.39). Following the previous practice (see the proof of Theorem 3.2), leads to

$$\begin{aligned}&\mathrm{Re}\Big\{\int_0^\xi \mathcal{A}_m\Big(e_N,\int_t^\xi e_N(\cdot,\tau)d\tau\Big)dt\Big\}\\ \geq&\frac{1}{2}\|e_N(\cdot,\xi)\|_{L^2(I)}^2-\frac{1}{2}\|e_N(\cdot,0)\|_{L^2(I)}^2-\mathrm{Re}\int_0^\xi(\dot{e}_N(\cdot,0),e_N(\cdot,t))dt,\end{aligned}$$

and letting $f=(m^2c^2-\tilde{c}^2)\hat{e}_N+(\mathcal{I}_Ng-g)$, we have

$$\begin{aligned}&\mathrm{Re}\Big\{\int_0^\xi\Big(f,\int_t^\xi e_N(\cdot,\tau)d\tau\Big)dt\Big\}\\ \leq&T\big(|m^2c^2-\tilde{c}^2|\|\hat{e}_N\|_{L^1(0,T;L^2(I))}+\|\mathcal{I}_Ng-g\|_{L^1(0,T;L^2(I))}\big)\|e_N\|_{L^\infty(0,T;L^2(I))}.\end{aligned}$$

Thus it remains to deal with the other terms at the right-hand side of (3.40). We derive from the integration by parts and Cauchy-Schwartz inequality that

$$\begin{aligned}\mathrm{Re}\int_0^\xi(\partial_t^2\hat{e}_N,w_N)dt=&\mathrm{Re}\int_0^\xi\Big(\int_0^t\partial_t^2\hat{e}_N(\tilde{x},\tau)d\tau,e_N\Big)dt\\ =&\mathrm{Re}\int_0^\xi\int_{-1}^1(\partial_t\hat{e}_N(\tilde{x},t)-\partial_t\hat{e}_N(\tilde{x},0))\bar{e}_N(\tilde{x},t)d\tilde{x}dt\\ \leq&\big(\|\partial_t\hat{e}_N\|_{L^1(0,T;L^2(I))}+T\|\partial_t\hat{e}_N(\cdot,0)\|_{L^2(I)}\big)\|e_N\|_{L^\infty(0,T;L^2(I))}.\end{aligned}$$

Using the integration by parts, we then infer from Lemma 3.3 and the inverse inequality (3.38) that

$$\begin{aligned}&\tilde{c}\mathrm{Re}\int_0^\xi\{\partial_t\hat{e}_N(1,t)+\rho_m(t)*\hat{e}_N(1,t)\}\bar{w}_N(1)dt\\ =&\tilde{c}\mathrm{Re}\int_0^\xi\Big\{\hat{e}_N(1,t)-\hat{e}_N(1,0)+\Big(\int_0^t\rho_m(\tau)*\hat{e}_N(1,\tau)d\tau\Big)\Big\}\bar{e}_N(1,t)dt\\ \leq&\tilde{c}\Big[\int_0^\xi\Big(|\hat{e}_N(1,t)|+|\hat{e}_N(1,0)|+\int_0^t|\rho_m(\tau)*\hat{e}_N(1,\tau)|d\tau\Big)dt\Big]\|e_N(1,\cdot)\|_{L^\infty(0,T)}\\ \leq&\tilde{c}DTN\Big\{\|\hat{e}_N(1,\cdot)\|_{L^\infty(0,T)}+\frac{\sqrt{2}m^2c^2}{2}\|\hat{e}_N(1,\cdot)\|_{L^1(0,T)}\Big\}\|e_N\|_{L^\infty(0,T;L^2(I))}.\end{aligned}$$

Choosing $v_{0,N}={}_0\pi_N^1v_0$ and $v_{1,N}={}_0\pi_N^1v_1$, so we have $e_N(\cdot,0)=\partial_te_N(\cdot,0)=0$. Consequently, we obtain that

$$\begin{aligned}\|e_N\|_{L^\infty(0,T;L^2(I))}\leq&D\Big\{\|\partial_t\hat{e}_N\|_{L^1(0,T;L^2(I))}+|m^2c^2-\tilde{c}^2|T\|\hat{e}_N\|_{L^1(0,T;L^2(I))}\\ &+\tilde{c}TN\|\hat{e}_N(1,\cdot)\|_{L^\infty(0,T)}+m^2c^2\tilde{c}TN\|\hat{e}_N(1,\cdot)\|_{L^1(0,T)}+T\|g-\mathcal{I}_Ng\|_{L^1(0,T;L^2(I))}\Big\}.\end{aligned} \tag{3.42}$$

Finally, using the fact $v_N - v = e_N - \hat{e}_N$, the triangle inequality and (3.41)-(3.42), we obtain from the approximation results (3.35)-(3.37) the following error bound for each mode m.

Theorem 3.3. *Let v and v_N be respectively the solution of* (3.26) *and* (3.30)*. If $v_0, v_1 \in {}_0H^1(I) \cap H^s(I), g \in L^1(0,T;H^s(I))$ $v \in L^\infty(0,T;{}_0H^1(I) \cap H^s(I))$ and $\partial_t v \in L^1(0,T;H^s(I))$ with $1 \le s \le N+1$, then*

$$\begin{aligned} \|v - v_N\|_{L^\infty(0,T;L^2(I))} \le & DN^{-s}\big\{\|\partial_x^s v_0\|_{L^2(I)} + T\|\partial_x^s v_1\|_{L^2(I)} + T\|\partial_x^s g\|_{L^1(0,T;L^2(I))} \\ & + \|\partial_t \partial_x^s v\|_{L^1(0,T;L^2(I))} + |m^2c^2 - \tilde{c}^2| T \|\partial_x^s v\|_{L^1(0,T;L^2(I))}\big\} \\ & + \tilde{c} D T N^{3/2-s}\big\{\|\partial_x^s v\|_{L^\infty(0,T;L^2(I))} + m^2c^2 \|\partial_x^s v\|_{L^1(0,T;L^2(I))}\big\}, \end{aligned} \quad (3.43)$$

where D is a positive constant independent of N, T and any function. A similar error bound holds for $\|v - v_N\|_{L^2(0,T;L^2(I))}$.

Remark 3.1. The presence of the NRBC brings about significantly subtle issues for the analysis compared with the standard setting in [4,7]. Moreover, the error bounds appear suboptimal.

Remark 3.2. We can further assemble the Fourier approximation and derive the error estimates for the full Fourier-Legendre spectral approximation with the aid of Theorem 3.3. This follows a standard procedure (cf. [12]), so we omit the details.

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