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OPTIMAL ERROR ESTIMATES FOR CHEBYSHEV APPROXIMATIONS OF FUNCTIONS WITH LIMITED REGULARITY IN FRACTIONAL SOBOLEV-TYPE SPACES

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ABSTRACT. In this paper, we introduce a new theoretical framework built upon fractional Sobolev-type spaces involving Riemann-Liouville fractional integrals/derivatives for optimal error estimates of Chebyshev polynomial approximations to functions with limited regularity. It naturally arises from exact representations of Chebyshev expansion coefficients. Here, the essential pieces of the puzzle for the error analysis include (i) fractional integration by parts (under the weakest possible conditions), and (ii) generalised Gegenbauer functions of fractional degree (GGF-Fs): a new family of special functions with notable fractional calculus properties. Under this framework, we are able to estimate the optimal decay rate of Chebyshev expansion coefficients for a large class of functions with interior and endpoint singularities, which are deemed suboptimal or complicated to characterise in existing literature. Then we can derive optimal error estimates for spectral expansions and the related Chebyshev interpolation and quadrature measured in various norms, and also improve available results in usual Sobolev spaces with integer regularity exponentials in several senses. As a byproduct, this study results in some analytically perspicuous formulas particularly on GGF-Fs, which are potentially useful in spectral algorithms. The idea and analysis techniques can be extended to general Jacobi polynomial approximations.

1. Introduction

It is known that polynomial approximation theory is of fundamental importance in numerical analysis and algorithm development of many computational methods, e.g., p/hp finite elements or spectral/spectral-element methods (see, e.g., [9, 15, 18, 23, 35, 36] and the references therein). Typically, the documented approximation

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results take the form

$$\|Q_N u - u\|_{\mathcal{S}_\mathrm{l}} \le cN^{-\sigma}|u|_{\mathcal{B}_\mathrm{r}}, \quad \sigma \ge 0, \tag{1.1}$$

where Q_N is an orthogonal projection (or interpolation operator) upon the set of all polynomials of degree at most N, and c is a positive constant independent of N and u. In (1.1), $\mathcal{S}_\mathrm{l}$ is a certain Sobolev space, $\mathcal{B}_\mathrm{r}$ is a related Sobolev or Besov space, and σ depends on the regularity exponentials of both $\mathcal{B}_\mathrm{r}$ and $\mathcal{S}_\mathrm{l}$. In practice, one would expect **(a)** *the space $\mathcal{B}_\mathrm{r}$ should contain the classes of functions as broad as possible;* and **(b)** *the space $\mathcal{B}_\mathrm{r}$ can best characterise their regularity leading to optimal order of convergence.* In general, the space $\mathcal{B}_\mathrm{r}$ is of the following types:

(i) $\mathcal{B}_\mathrm{r}$ is the standard weighted Sobolev space $H^m_\omega(\Omega)$ with integer $m \ge 0$ and certain weight function $\omega(x)$ on $\Omega = (-1,1)$ (see, e.g., [9,15,23]). However, it could not lead to optimal order for functions with endpoint singularities (see, e.g., [9,22]) or with interior singularities, e.g., $|x|$ (see [38]).

(ii) $\mathcal{B}_\mathrm{r}$ is the non-uniformly Jacobi-weighted Sobolev space (see, e.g., [5,6,18,20,22,36]). For example, $\mathcal{B}_\mathrm{r} = H^{m,\beta}(\Omega)$ with integer $m \ge 0$ and $\beta > -1$, is defined as a closure of C^∞-functions endowed with the weighted norm

$$\|u\|_{H^{m,\beta}(\Omega)} = \left\{ \sum_{k=0}^{m} \int_{-1}^{1} |u^{(k)}(x)|^2 (1-x^2)^{\beta+k} dx \right\}^{1/2}. \tag{1.2}$$

Compared with the standard Sobolev space in (i), such spaces can better describe the endpoint singularities, but still produce suboptimal estimates for $(1+x)^\alpha$-type singular functions with non-integer $\alpha > 0$ (cf. [16, p. 474]). Indeed, for the Chebyshev approximation, we find that $u = (1+x)^\alpha \in H^{m,-1/2}(\Omega)$ with integer $m < 2\alpha + 1/2$, and

$$\|\pi^C_N u - u\|_{L^2_\omega(\Omega)} \le cN^{-m}|u|_{H^{m,-1/2}(\Omega)}, \tag{1.3}$$

where $\pi^C_N u$ is the L^2_ω-orthogonal projection of u (with $\omega = (1-x^2)^{-1/2}$). However, the expected optimal order is $O(N^{-2\alpha-1/2})$, so the loss of an order of the fractional part of $2\alpha + 1/2$ or one order (when $2\alpha = k + 1/2$ with integer $k \ge 0$), is inevitable under this framework. This is due to the space $H^{m,\beta}(\Omega)$ is only defined for integer $m \ge 0$.

(iii) In a series of works [5–7], Babuška and Guo introduced the Jacobi-weighted Besov space defined by space interpolation based on the so-called K-method. One commonly used Besov space for $(1+x)^\alpha$-type corner singularities is $\mathcal{B}^{s,\beta}_{2,2}(\Omega) = (H^{l,\beta}(\Omega), H^{m,\beta}(\Omega))_{\vartheta,2}$ with integers $l < m$ and $s = (1-\vartheta)l + \vartheta m, \vartheta \in (0,1)$, equipped with the norm

$$\begin{aligned} &\|u\|_{\mathcal{B}^{s,\beta}_{2,2}(\Omega)} = \left(\int_0^\infty t^{-2\vartheta} |K(t,u)|^2 \frac{dt}{t} \right)^{1/2}, \quad \text{where} \\ &K(t,u) = \inf_{u=v+w} \big(\|v\|_{H^{l,\beta}(\Omega)} + t\|w\|_{H^{m,\beta}(\Omega)} \big). \end{aligned} \tag{1.4}$$

However, to deal with $(1+x)^\alpha \log^\nu(1+x)$-type corner singularities, Babuška and Guo had to further modify the K-method by incorporating a log-factor into the norm.

The aforementioned framework might lead to suboptimal estimates for functions with interior singularities. For example, we consider $u(x) = |x|$ for $x \in (-1,1)$. Note that $u'(x) = 2H(x) - 1$ and $u''(x) = 2\delta(x)$ (where H, δ are, respectively, the

Heaviside function and the Dirac delta function). Since $u'' \notin L^2(\Omega)$, the Chebyshev approximation of $|x|$ has a convergence:

$$\|\pi_N^C u - u\|_{L^2_\omega(\Omega)} \le cN^{-1}|u|_{H^{1,-1/2}(\Omega)}, \tag{1.5}$$

but the expected optimal order is $O(N^{-3/2})$ (cf. [38,39]). In fact, as shown in [38, Thms 4.2-4.3] and [39, Thms 7.1-7.2] (also see Lemma 5.1 below), one should choose $\mathcal{B}_{\rm r} \subseteq {\rm BV}(\bar{\Omega})$ (the space of functions of bounded variation) to achieve optimality (see Section 5, and refer to [27,38,39,42] for more details). Unfortunately, the Sobolev spaces therein were defined through integer-order derivatives, so they could not best characterise the regularity of, e.g., $u(x) = |x|^\alpha$ with non-integer $\alpha > 0$. In other words, the order of convergence is suboptimal.

In this paper, we intend to introduce a new framework of fractional Sobolev-type spaces that can meet the two requirements **(a)-(b)** and overcome the deficiencies mentioned above. We focus on the Chebyshev approximation but the analysis techniques are extendable to general Jacobi approximations. Here, we put emphasis on estimating the decay rate of expansion coefficients for the reason that the errors of spectral expansions in various norms, and the related interpolation and quadratures, can be estimated directly from the sums of the coefficients (cf. [27,38]). The essential ideas and main contributions of this study are summarised as follows:

(i) We derive the exact representation of the Chebyshev expansion coefficients (see Theorem 4.1) by using the fractional calculus properties of GGF-Fs and fractional integration by parts (under the weakest possible conditions). This allows us to naturally define the fractional Sobolev spaces to characterise the regularity of a large class of singular functions, leading to optimal order of convergence.

(ii) As a byproduct, our estimates for the functions in this framework with integer regularity exponential can improve the existing bounds in usual Sobolev spaces (see, e.g., [27,38,39,42]).

(iii) We present some useful analytical formulas on fractional calculus of GGF-Fs, and the Chebyshev expansions of some specific singular functions. Some of them are new or difficult to be derived by other means (cf. [11,19,41]). They are also useful for the design of spectral algorithms.

The paper is organised as follows. In Sections 2-3, we introduce the GGF-Fs, and present their important properties, including the uniform bounds and Riemann-Liouville fractional integral/derivative formulas. We derive the main results in Section 4, and improve the existing estimates in Sobolev spaces with integer-order derivatives in Section 5. We discuss in Section 6 the extension of the main results to the analysis of interpolation, quadrature, and endpoint singularities.

2. Generalised Gegenbauer functions of fractional degree

In this section, we collect some relevant properties of the hypergeometric functions and Gegenbauer polynomials, upon which we define the GGF-Fs and derive their relevant properties. These pave the way for the forthcoming error analysis.

2.1. Hypergeometric functions and Gegenbauer polynomials.

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of all integers and real numbers, respectively, and denote

$$\mathbb{N} = \big\{k \in \mathbb{Z} : k \ge 1\big\},\ \ \mathbb{N}_0 := \{0\} \cup \mathbb{N},\ \ \mathbb{R}^+ := \big\{a \in \mathbb{R} : a > 0\big\},\ \ \mathbb{R}_0^+ := \{0\} \cup \mathbb{R}^+. \tag{2.1}$$

For $a \in \mathbb{R}$, the rising factorial in the Pochhammer symbol is defined by

$$(a)_0 = 1; \quad (a)_j = a(a+1)\cdots(a+j-1) \quad \forall\, j \in \mathbb{N}. \tag{2.2}$$

The hypergeometric function is a power series, defined by (cf. [4])

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!} \\ &= 1 + \sum_{j=1}^{\infty} \frac{a(a+1)\cdots(a+j-1)}{1\cdot 2\cdots j} \frac{b(b+1)\cdots(b+j-1)}{c(c+1)\cdots(c+j-1)} z^j, \end{aligned} \tag{2.3}$$

where $a, b, c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. The series converges absolutely for all $|z| < 1$. Clearly, we have

$${}_2F_1(a,b;c;0) = 1, \quad {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z). \tag{2.4}$$

If $a = -n$ with $n \in \mathbb{N}_0$, then $(a)_j = 0$ for all $j \geq n+1$, so ${}_2F_1(-n,b;c;x)$ reduces to a polynomial of degree not more than n.

The following properties can be found in [4, Ch. 2], if not stated otherwise.

- If $c - a - b > 0$, the series (2.3) converges absolutely at $z = \pm 1$, and

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.5}$$

  Here, the Gamma function with negative non-integer arguments should be understood by the Euler's reflection formula:

$$\Gamma(1-a)\Gamma(a) = \frac{\pi}{\sin(\pi a)}, \quad a \notin \mathbb{Z}. \tag{2.6}$$

  Note that $\Gamma(-a) = \infty$ if $a \in \mathbb{N}$.

- If $-1 < c - a - b \leq 0$, the series (2.3) converges conditionally at $z = -1$, but diverges at $z = 1$; while for $c - (a+b) \leq -1$, it diverges at $z = \pm 1$. In fact, it has the following singular behaviours at $z = 1$:

$$\lim_{z \to 1^-} \frac{{}_2F_1(a,b;c;z)}{-\ln(1-z)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \quad \text{if } c = a+b \tag{2.7}$$

  and

$$\lim_{z \to 1^-} \frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \quad \text{if } c < a+b. \tag{2.8}$$

Recall the transform identity: for $a, b, c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$,

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z), \quad |z| < 1. \tag{2.9}$$

The hypergeometric function satisfies the differential equation (cf. [4, p. 98]):

$$\big\{z^c(1-z)^{a+b-c+1}y'(z)\big\}' = abz^{c-1}(1-z)^{a+b-c}y(z), \quad y(z) = {}_2F_1(a,b;c;z). \tag{2.10}$$

We shall use the value at $z = 1/2$ (cf. [31, (15.4.28)]):

$${}_2F_1\Big(a,b;\frac{a+b+1}{2};\frac{1}{2}\Big) = \frac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}. \tag{2.11}$$

Many functions are associated with the hypergeometric function. For example, the Jacobi polynomial of degree $n \in \mathbb{N}_0$ with $\alpha, \beta > -1$ (cf. Szegő [37]) is defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\, {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big) \\ &= (-1)^n \frac{(\beta+1)_n}{n!}\, {}_2F_1\Big(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\Big) \end{aligned} \tag{2.12}$$

for $x \in (-1,1)$, which satisfies

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}. \tag{2.13}$$

For $\alpha, \beta > -1$, the Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$, namely,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_{n'}^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x)\, dx = \gamma_n^{(\alpha,\beta)} \delta_{nn'}, \tag{2.14}$$

where $\delta_{nn'}$ is the Kronecker Delta symbol, and

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}. \tag{2.15}$$

Remark 2.1. According to Szegő [37, pp. 63-67], the formula (2.12) furnishes the extension of the classical Jacobi polynomials to arbitrary real values of the parameters α and β. It is a polynomial in x. In fact, many properties of the classical Jacobi polynomials still hold, but the orthogonality is lacking in general.

Throughout this paper, the Gegenbauer polynomial with $\lambda > -1/2$ is defined by

$$\begin{aligned} G_n^{(\lambda)}(x) &= \frac{P_n^{(\lambda-1/2,\lambda-1/2)}(x)}{P_n^{(\lambda-1/2,\lambda-1/2)}(1)} = {}_2F_1\Big(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\Big) \\ &= (-1)^n\, {}_2F_1\Big(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1+x}{2}\Big), \quad x \in (-1,1), \end{aligned} \tag{2.16}$$

which has a normalisation different from that in Szegő [37]. If $\lambda = 0$, it reduces to the Chebyshev polynomial

$$T_n(x) = G_n^{(0)}(x) = {}_2F_1\Big(-n, n; \frac{1}{2}; \frac{1-x}{2}\Big) = \cos(n \arccos(x)). \tag{2.17}$$

Note that under the above normalisation, we derive from (2.14)-(2.15) the orthogonality:

$$\int_{-1}^{1} G_n^{(\lambda)}(x) G_m^{(\lambda)}(x)\, \omega_\lambda(x)\, dx = \gamma_n^{(\lambda)} \delta_{nm}; \quad \gamma_n^{(\lambda)} = \frac{2^{2\lambda-1}\Gamma^2(\lambda+1/2)\, n!}{(n+\lambda)\Gamma(n+2\lambda)}, \tag{2.18}$$

where $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$. In the analysis, we shall use the derivative relation derived from the generalised Rodrigues' formula (see [37, (4.10.1)] with $\alpha = \beta = \lambda - 1/2 > -1$ and $m = 1$):

$$\omega_\lambda(x) G_n^{(\lambda)}(x) = -\frac{1}{2\lambda+1} \frac{d}{dx}\big\{\omega_{\lambda+1}(x) G_{n-1}^{(\lambda+1)}(x)\big\}, \quad n \geq 1. \tag{2.19}$$

2.2. **Generalised Gegenbauer functions of fractional degree.** As an indispensable tool for the error analysis, we introduce the GGF-Fs by allowing the degree n of the Gegenbauer polynomials in (2.16) to be real.

Definition 2.1. For real $\lambda > -1/2$ and real $\nu \ge 0$, the right GGF-F of degree ν is defined by

$$
\begin{aligned}
{}^{r}G_\nu^{(\lambda)}(x) &= {}_2F_1\Big(-\nu, \nu+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\Big)\\
&= 1 + \sum_{j=1}^{\infty} \frac{(-\nu)_j(\nu+2\lambda)_j}{j!\,(\lambda+1/2)_j}\Big(\frac{1-x}{2}\Big)^j
\end{aligned} \tag{2.20}
$$

for $x \in (-1,1)$; while the left GGF-F of degree ν is defined by

$$
\begin{aligned}
{}^{l}G_\nu^{(\lambda)}(x) &= (-1)^{[\nu]}\,{}_2F_1\Big(-\nu, \nu+2\lambda; \lambda+\frac{1}{2}; \frac{1+x}{2}\Big)\\
&= (-1)^{[\nu]}\Big\{1 + \sum_{j=1}^{\infty} \frac{(-\nu)_j(\nu+2\lambda)_j}{j!\,(\lambda+1/2)_j}\Big(\frac{1+x}{2}\Big)^j\Big\},
\end{aligned} \tag{2.21}
$$

where $[\nu]$ is the largest integer $\le \nu$. □

Remark 2.2. For $\lambda = 1/2$, the right GGF-F turns out to be the Legendre function (cf. [4]): $P_\nu(x) = {}^{r}G_\nu^{(1/2)}(x)$. In [31, (15.9.15)], ${}^{r}G_\nu^{(\lambda)}(x)$ (with a different normalisation) is defined as the Gegenbauer function. However, there is nearly no discussion on its properties.

Observe from (2.16) and Definition 2.1 that the GGF-Fs reduce to the classical Gegenbauer polynomials when $\nu \in \mathbb{N}_0$, but they are non-polynomials when ν is not an integer.

Proposition 2.1. *The GGF-Fs defined in Definition* 2.1 *satisfy*

$$
{}^{r}G_n^{(\lambda)}(x) = {}^{l}G_n^{(\lambda)}(x) = G_n^{(\lambda)}(x), \quad n \in \mathbb{N}_0; \tag{2.22a}
$$

$$
{}^{r}G_\nu^{(\lambda)}(-x) = (-1)^{[\nu]}\,{}^{l}G_\nu^{(\lambda)}(x), \quad {}^{r}G_\nu^{(\lambda)}(1) = 1, \quad {}^{l}G_\nu^{(\lambda)}(-1) = (-1)^{[\nu]}. \tag{2.22b}
$$

The special GGF-Fs $\big\{{}^{r}G_{n-\alpha}^{(\alpha+1/2)}(x)\big\}$ and $\big\{{}^{l}G_{n-\alpha}^{(\alpha+1/2)}(x)\big\}$ are closely related to the Jacobi polynomials with the parameters ≤ -1 (cf. Remark 2.1).

Proposition 2.2. *For $\alpha > -1$ and $n \ge \alpha$ with $n \in \mathbb{N}_0$, we have*

$$
\begin{aligned}
\frac{P_n^{(\alpha,-\alpha)}(x)}{P_n^{(\alpha,-\alpha)}(1)} &= \Big(\frac{1+x}{2}\Big)^{\alpha}\,{}^{r}G_{n-\alpha}^{(\alpha+1/2)}(x);\\
\frac{P_n^{(-\alpha,\alpha)}(x)}{P_n^{(\alpha,-\alpha)}(1)} &= (-1)^{[\alpha]}\Big(\frac{1-x}{2}\Big)^{\alpha}\,{}^{l}G_{n-\alpha}^{(\alpha+1/2)}(x)\,.
\end{aligned} \tag{2.23}
$$

Proof. Taking $a = -n+\alpha, b = n+\alpha+1, c = \alpha+1$, and $z = (1-x)/2$ in (2.9), we obtain from (2.4) that

$$
{}_2F_1\Big(-n+\alpha, n+\alpha+1; \alpha+1; \frac{1-x}{2}\Big) = \Big(\frac{1+x}{2}\Big)^{-\alpha}\,{}_2F_1\Big(-n, n+1; \alpha+1; \frac{1-x}{2}\Big). \tag{2.24}
$$

By (2.12)-(2.13),

$$ {}_2F_1\Big(-n, n+1; \alpha+1; \frac{1-x}{2}\Big) = \frac{P_n^{(\alpha,-\alpha)}(x)}{P_n^{(\alpha,-\alpha)}(1)}, $$

and by (2.20) (taking $\nu = n-\alpha$), the hypergeometric function in the left-hand side of (2.24) equals ${}^{r}G_{n-\alpha}^{(\alpha+1/2)}(x)$. Thus, we derive the first identity in (2.23).

By (2.13) and (2.22b), the second identity in (2.23) follows from the first one immediately. □

Remark 2.3. If $-1<\alpha<1$, we rewrite (2.23) as

$$ \begin{aligned} {}^{r}G_{n-\alpha}^{(\alpha+1/2)}(x) &= d_{n,\alpha}(1+x)^{-\alpha}P_n^{(\alpha,-\alpha)}(x); \\ {}^{l}G_{n-\alpha}^{(\alpha+1/2)}(x) &= (-1)^{[\alpha]}d_{n,\alpha}(1-x)^{-\alpha}P_n^{(-\alpha,\alpha)}(x), \end{aligned} \tag{2.25} $$

where $d_{n,\alpha} = 2^{\alpha}/P_n^{(\alpha,-\alpha)}(1)$. From (2.14)-(2.15), we immediately obtain the orthogonality:

$$ \begin{aligned} &\int_{-1}^{1} {}^{r}G_{n-\alpha}^{(\alpha+1/2)}(x)\, {}^{r}G_{m-\alpha}^{(\alpha+1/2)}(x)\,(1-x^2)^{\alpha}\,dx \\ &\quad = d_{n,\alpha}d_{m,\alpha}\int_{-1}^{1} P_n^{(\alpha,-\alpha)}(x)P_m^{(\alpha,-\alpha)}(x)(1-x)^{\alpha}(1+x)^{-\alpha}\,dx = d_{n,\alpha}^2\gamma_n^{(\alpha,-\alpha)}\delta_{mm}, \end{aligned} \tag{2.26} $$

and likewise for $\{{}^{l}G_{n-\alpha}^{(\alpha+1/2)}(x)\}$. It is noteworthy that $\{(1+x)^{-\alpha}P_n^{(\alpha,-\alpha)}\}$ are defined as the Jacobi polyfractonomials in [43] and special generalised Jacobi functions in [17, 21], which serve as effective (singular) basis functions in accurate solutions of fractional differential equations (cf. [17, 43]). It is seen from (2.25) that they turn out to be special GGF-Fs.

It is important to point out that the GGF-Fs may be singular at $x=\pm1$, and they behave differently in different ranges of λ.

Proposition 2.3. *Let $\nu \in \mathbb{R}_0^+$.*

(i) *If $-1/2<\lambda<1/2$, then*

$$ {}^{r}G_{\nu}^{(\lambda)}(-1) = \frac{\cos((\nu+\lambda)\pi)}{\cos(\lambda\pi)} = (-1)^{[\nu]}\,{}^{l}G_{\nu}^{(\lambda)}(1). \tag{2.27} $$

(ii) *If $\lambda = 1/2$ and $\nu \notin \mathbb{N}_0$, then*

$$ \lim_{x\to-1^+}\frac{{}^{r}G_{\nu}^{(\lambda)}(x)}{\ln(1+x)} = \frac{\sin(\nu\pi)}{\pi} = \lim_{x\to1^-}\frac{(-1)^{[\nu]}\,{}^{l}G_{\nu}^{(\lambda)}(x)}{\ln(1-x)}. \tag{2.28} $$

(iii) *If $\lambda>1/2$ and $\nu \notin \mathbb{N}_0$, then*

$$ \begin{aligned} \lim_{x\to-1^+}\Big(\frac{1+x}{2}\Big)^{\lambda-1/2}\,{}^{r}G_{\nu}^{(\lambda)}(x) &= -\frac{\sin(\nu\pi)}{\pi}\frac{\Gamma(\lambda-1/2)\Gamma(\lambda+1/2)\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)} \\ &= (-1)^{[\nu]}\lim_{x\to1^-}\Big(\frac{1-x}{2}\Big)^{\lambda-1/2}\,{}^{l}G_{\nu}^{(\lambda)}(x). \end{aligned} \tag{2.29} $$

Proof. By virtue of (2.22b), it suffices to prove the results for the right GGF-F ${}^{r}G_{\nu}^{(\lambda)}(x)$.

(i) By (2.5), (2.6), and (2.20),

(2.30)
$$
\begin{aligned}
{}^{r}G_\nu^{(\lambda)}(-1) &= {}_2F_1(-\nu,\nu+2\lambda;\lambda+1/2;1) = \frac{\Gamma(\lambda+1/2)\Gamma(1/2-\lambda)}{\Gamma(\nu+\lambda+1/2)\Gamma(-\nu-\lambda+1/2)}\\
&= \frac{\pi}{\sin((\lambda+1/2)\pi)}\frac{\sin((\nu+\lambda+1/2)\pi)}{\pi} = \frac{\cos((\nu+\lambda)\pi)}{\cos(\lambda\pi)},
\end{aligned}
$$

which yields (2.27).

(ii) Using (2.6), (2.7), and (2.20), and noting that

$$
\frac{\ln((1+x)/2)}{\ln(1+x)} = \frac{\ln(1+x)-\ln 2}{\ln(1+x)} \to 1 \quad \text{as} \quad x\to -1^+,
$$

we obtain (2.28).

(iii) Next, taking $a=-\nu, b=\nu+2\lambda, c=\lambda+1/2$, and $z=(1-x)/2$ in (2.9), and using (2.4), we obtain

$$
\begin{aligned}
&{}_2F_1\Big(-\nu,\,\nu+2\lambda;\lambda+\frac12;\frac{1-x}{2}\Big)\\
&\quad=\Big(\frac{2}{1+x}\Big)^{\lambda-1/2}{}_2F_1\Big(\nu+\lambda+\frac12,-\nu-\lambda+\frac12;\lambda+\frac12;\frac{1-x}{2}\Big).
\end{aligned}
$$

For $\lambda>1/2$, we find from (2.5) and (2.6) that

$$
{}_2F_1\Big(\nu+\lambda+\frac12,-\nu-\lambda+\frac12;\lambda+\frac12;1\Big) = -\frac{\sin(\nu\pi)}{\pi}\frac{\Gamma(\lambda-1/2)\Gamma(\lambda+1/2)\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)},
$$

so we derive (2.29) from (2.20) and the above. □

To illustrate, we depict in Figure 2.1 the right generalised Chebyshev/Legendre functions, i.e., ${}^{r}G_\nu^{(\lambda)}(x)$ with $\lambda=0,1/2$ and for various ν. Note that the left counterparts ${}^{l}G_\nu^{(\lambda)}(x)=(-1)^{[\nu]}\,{}^{r}G_\nu^{(\lambda)}(-x)$ (cf. (2.22b)). Observe that in the Legendre case (the figure on the right), ${}^{r}G_\nu^{(1/2)}(x)$ with non-integer degree has a logarithmic singularity at $x=-1$ (cf. (2.28)), while the generalised Chebyshev functions (left) are well defined at $x=-1$.

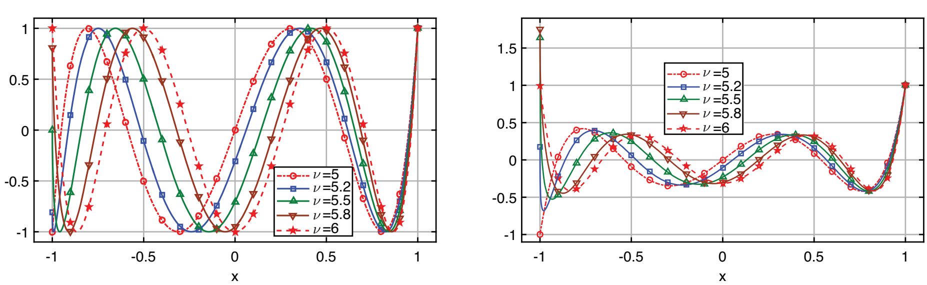

FIGURE 2.1. Graphs of ${}^{r}G_\nu^{(\lambda)}(x)$ with $\lambda=0$ (left) and $\lambda=1/2$ (right) for various ν.

2.3. **Uniform upper bounds.** The uniform bounds of the GGF-Fs stated in the following two theorems are of paramount importance in the forthcoming error analysis.

Theorem 2.1. *For $\lambda \ge 1$ and real $\nu \ge 0$, we have*

$$\max_{|x|\le 1} \big\{\omega_\lambda(x)|{}^rG_\nu^{(\lambda)}(x)|,\ \omega_\lambda(x)|{}^lG_\nu^{(\lambda)}(x)|\big\} \le \kappa_\nu^{(\lambda)}, \tag{2.31}$$

where $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$ *and*

(2.32)

$$\kappa_\nu^{(\lambda)} = \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}}\bigg(\frac{\cos^2(\pi\nu/2)\Gamma^2((\nu+1)/2)}{\Gamma^2((\nu+1)/2+\lambda)} + \frac{4\sin^2\big(\pi\nu/2\big)}{2\lambda-1+\nu(\nu+2\lambda)}\frac{\Gamma^2(\nu/2+1)}{\Gamma^2(\nu/2+\lambda)}\bigg)^{1/2}.$$

Proof. By virtue of (2.22b), it suffices to prove the result for ${}^rG_\nu^{(\lambda)}(x)$. For notational simplicity, we denote

$$\begin{aligned} &G(x) := {}^rG_\nu^{(\lambda)}(x);\quad M(x) := \omega_\lambda(x)G(x);\\ &H(x) := M^2(x) + \varrho^{-1}(1-x^2)\big(M'(x)\big)^2, \end{aligned} \tag{2.33}$$

where the constant $\varrho := 2\lambda - 1 + \nu(\nu+2\lambda)$.

We take three steps to complete the proof.

Step 1. Show that $H(x)$ is continuous on $[-1,1]$, that is, $H(\pm 1)$ are well defined. It is evident that by (2.22b), $M(1) = 0$; and from (2.29), we find that $M(-1)$ is a finite value when $\lambda \ge 1$. Next, from (3.13a) with $s = 1$, we derive

$$(1-x^2)^{1/2}M'(x) = (1-2\lambda)\,(1-x^2)^{\lambda-1}\,{}^rG_{\nu+1}^{(\lambda-1)}(x). \tag{2.34}$$

Similarly, by (2.22b), $(1-x^2)^{1/2}M'(x)|_{x=1} = 0$ for $\lambda > 1$, and it's finite for $\lambda = 1$. We now justify $(1-x^2)^{1/2}M'(x)|_{x=-1}$ is also well defined. We infer from Proposition 2.3 that (a) if $1 \le \lambda < 3/2$, ${}^rG_{\nu+1}^{(\lambda-1)}(x)$ is finite at $x = -1$; (b) if $\lambda = 3/2$, ${}^rG_{\nu+1}^{(\lambda-1)}(-1) = 0$; and (c) if $\lambda > 3/2$, ${}^rG_{\nu+1}^{(\lambda-1)}(x)$ tends to a finite value as $x \to -1$. Hence, by (2.33), $H(\pm 1)$ are well defined.

Step 2. Derive the identity:

$$H'(x) = -\frac{4(\lambda-1)x}{\varrho}\big(M'(x)\big)^2, \quad x \in (-1,1). \tag{2.35}$$

Indeed, taking $a = -\nu, b = \nu + 2\lambda, c = \lambda + 1/2$, and $z = (1 \pm x)/2$ in (2.10), we find that $G(x)$ satisfies the Sturm-Liouville problem

$$\big\{\omega_{\lambda+1}(x)G'(x)\big\}' + \nu(\nu+2\lambda)\omega_\lambda(x)G(x) = 0. \tag{2.36}$$

Substituting $G(x) = \omega_\lambda^{-1}(x)M(x)$ into (2.36), we obtain by direct calculations that

$$(1-x^2)M''(x) + (2\lambda-3)xM'(x) + \varrho M(x) = 0. \tag{2.37}$$

Differentiating $H(x)$ and using (2.37), leads to

$$\begin{aligned} H'(x) &= \frac{2}{\varrho}M'(x)\big\{(1-x^2)M''(x) + \varrho M(x)\big\} - \frac{2x}{\varrho}(M'(x))^2\\ &= -\frac{4(\lambda-1)x}{\varrho}\big(M'(x)\big)^2. \end{aligned} \tag{2.38}$$

Step 3. Prove the following bounds and calculate the values at $x = 0$:

$$M^2(x) \le H(x) \le H(0) = M^2(0) + \varrho^{-1}\big(M'(0)\big)^2 \quad \forall\, x \in [-1, 1]. \tag{2.39}$$

By (2.35), we have $H'(x) \equiv 0$ if $\lambda = 1$, so $H(x)$ is a constant and $H(x) = H(0)$. In other words, (2.39) is true for $\lambda = 1$.

If $\lambda > 1$, we deduce from (2.35) that the stationary points of $H(x)$ are $x = 0$ or zeros of $M'(x)$ (if any). Let $0 \ne \tilde{x} \in (-1, 1)$ be any zero of $M'(x)$ (note: $M(\tilde{x}) \ne 0$). Evidently, by (2.35), $H'(x)$ does not change sign in the neighbourhood of $\tilde{x}$, which means $\tilde{x}$ cannot be an extreme point of $H(x)$. In fact, $x = 0$ is the only extreme point in $(-1, 1)$. We also see from (2.35) that $H'(x) \ge 0$ (resp., $H'(x) \le 0$) for $x \in (0, 1)$ (resp., $x \in (-1, 0)$). Note that $H(x)$ attains its maximum at $x = 0$, since $H(x)$ is ascending when $x < 0$, and is descending when $x > 0$. Therefore, we obtain (2.39) from (2.33) and the above reasoning.

Now, we calculate $H(0)$. From (2.6) and (2.11), we obtain that for $\lambda \ge 0$,

$$\begin{aligned} M(0) &= {}^{r}G_\nu^{(\lambda)}(0) = {}_2F_1\Big(-\nu, \nu + 2\lambda; \lambda + \frac{1}{2}; \frac{1}{2}\Big) \\ &= \frac{\sqrt{\pi}\,\Gamma(\lambda + 1/2)}{\Gamma(-\nu/2 + 1/2)\Gamma(\nu/2 + \lambda + 1/2)} \\ &= \sin\big(\pi(\nu + 1)/2\big)\frac{\Gamma(\lambda + 1/2)\Gamma(\nu/2 + 1/2)}{\sqrt{\pi}\,\Gamma(\nu/2 + \lambda + 1/2)}, \end{aligned} \tag{2.40}$$

which, together with (3.13b), implies

$$\begin{aligned} \big\{(1 - x^2)^{1/2} M'(x)\big\}\big|_{x=0} &= (1 - 2\lambda)\,{}^{r}G_{\nu+1}^{(\lambda-1)}(0) \\ &= (1 - 2\lambda)\sin\big(\pi(\nu + 2)/2\big)\frac{\Gamma(\lambda - 1/2)\Gamma(\nu/2 + 1)}{\sqrt{\pi}\,\Gamma(\nu/2 + \lambda)} \\ &= 2\sin\big(\pi\nu/2\big)\frac{\Gamma(\lambda + 1/2)\Gamma(\nu/2 + 1)}{\sqrt{\pi}\,\Gamma(\nu/2 + \lambda)}. \end{aligned} \tag{2.41}$$

In the last step, we used the identity: $\Gamma(z + 1) = z\Gamma(z)$.

Substituting (2.40)-(2.41) into (2.39), we obtain the bound in (2.31). ☐

As a direct consequence of Theorem 2.1, we have the following bound for the Gegenbauer polynomials.

Corollary 2.1. *For real $\lambda \ge 1$ and integer $l \ge 0$, we have*

$$\max_{|x|\le 1}\big\{\omega_\lambda(x)|G_{2l}^{(\lambda)}(x)|\big\} \le \frac{\Gamma(\lambda + 1/2)\Gamma(l + 1/2)}{\sqrt{\pi}\,\Gamma(l + \lambda + 1/2)}; \tag{2.42a}$$

(2.42b)

$$\max_{|x|\le 1}\big\{\omega_\lambda(x)|G_{2l+1}^{(\lambda)}(x)|\big\} \le \frac{2l + 1}{\sqrt{2\lambda - 1 + (2l + 1)(2l + 2\lambda + 1)}}\,\frac{\Gamma(\lambda + 1/2)\Gamma(l + 1/2)}{\sqrt{\pi}\,\Gamma(l + \lambda + 1/2)}.$$

Remark 2.4. The bounds for Gegenbauer polynomials multiplied by a different weight function: $(1-x^2)^{\lambda/2-1/4}$ can be found in [28]. To the best of our knowledge, the bounds herein are new.

The upper bound in Theorem 2.1 is valid for $\lambda \ge 1$. In the analysis, we also need to use the upper bound for $0 < \lambda < 1$. In this case, we have to multiply the GGF-Fs by a different weight function, and conduct the analysis in a slightly different manner.

Theorem 2.2. *For real* $0<\lambda<1$ *and real* $\nu\geq 0$*, we have*

$$\max_{|x|\le 1}\big\{(1-x^2)^{\lambda/2}|{}^rG_\nu^{(\lambda)}(x)|,\ (1-x^2)^{\lambda/2}|{}^lG_\nu^{(\lambda)}(x)|\big\}\le \widehat{\kappa}_\nu^{(\lambda)}, \tag{2.43}$$

where

$$\begin{aligned}\widehat{\kappa}_\nu^{(\lambda)}=&\frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}}\bigg(\frac{\cos^2(\pi\nu/2)\Gamma^2(\nu/2+1/2)}{\Gamma^2((\nu+1)/2+\lambda)}\\&+\frac{4\sin^2\big(\pi\nu/2\big)}{\nu^2+2\lambda\nu+\lambda}\frac{\Gamma^2(\nu/2+1)}{\Gamma^2(\nu/2+\lambda)}\bigg)^{1/2}.\end{aligned} \tag{2.44}$$

Proof. Once again, by virtue of (2.22b), it suffices to prove the result for ${}^rG_\nu^{(\lambda)}(x)$.
Here, we denote

$$\begin{aligned}&\widehat{M}(x):=(1-x^2)^{\lambda/2}{}^rG_\nu^{(\lambda)}(x);\quad \widehat{H}(x):=\widehat{M}^2(x)+\frac{1}{\rho(x)}\big(\widehat{M}'(x)\big)^2,\\&\rho(x):=\big((\nu+\lambda)^2(1-x^2)-\lambda(\lambda-1)\big)(1-x^2)^{-2}.\end{aligned} \tag{2.45}$$

Using Proposition 2.3, we can justify that $\widehat{H}(x)$ is continuous on $[-1,1]$ in the same manner as Step 1 in the proof of Theorem 2.1. Indeed, direct calculations from (2.36) lead to

$$(1-x^2)\widehat{M}''(x)-x\widehat{M}'(x)+(1-x^2)\rho(x)\widehat{M}(x)=0,\quad x\in(-1,1). \tag{2.46}$$

Like (2.35), we can show that

$$\widehat{H}'(x)=\frac{2\lambda(\lambda-1)x}{(\lambda+\nu)^2(1-x^2)^2-\lambda(\lambda-1)(1-x^2)}\big(\widehat{M}'(x)\big)^2,\quad x\in(-1,1). \tag{2.47}$$

For $0<\lambda<1$, $\widehat{H}(x)$ is increasing for $x<0$, and decreasing for $x>0$, so $H(x)$ attains its maximum at $x=0$. Thus,

$$\widehat{M}^2(x)\le\widehat{H}(x)\le\widehat{H}(0)=\widehat{M}^2(0)+\rho^{-1}(0)\big(\widehat{M}'(0)\big)^2\quad\forall x\in[-1,1]. \tag{2.48}$$

By (2.40),

$$\widehat{M}(0)=\cos\big(\pi\nu/2\big)\frac{\Gamma(\lambda+1/2)\Gamma(\nu/2+1/2)}{\sqrt{\pi}\,\Gamma(\nu/2+\lambda+1/2)}. \tag{2.49}$$

Recall the identity (cf. [31, (15.5.1)]):

$$\frac{d}{dx}{}_2F_1(a,b;c;z)=\frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z). \tag{2.50}$$

From (2.20) and (2.50), we obtain

$$\begin{aligned}\frac{d}{dx}{}^rG_\nu^{(\lambda)}(x)=&\frac{d}{dx}{}_2F_1\Big(-\nu,\nu+2\lambda;\lambda+\frac12;\frac{1-x}{2}\Big)\\=&\frac{\nu(\nu+2\lambda)}{2\lambda+1}{}_2F_1\Big(-\nu+1,\nu+2\lambda+1;\lambda+\frac32;\frac{1-x}{2}\Big).\end{aligned} \tag{2.51}$$

In view of

$$\widehat{M}'(x)=-\lambda x(1-x^2)^{\lambda/2-1}\,{}^rG_\nu^{(\lambda)}(x)+(1-x^2)^{\lambda/2}\frac{d}{dx}{}^rG_\nu^{(\lambda)}(x),$$

we deduce from (2.6), (2.11), and (2.51) that

$$\{\rho^{-1/2}(x)\widehat{M}'(x)\}|_{x=0}=\frac{2\sin(\pi\nu/2)}{\sqrt{\pi}}\frac{\Gamma(\nu/2+1)\,\Gamma(\lambda+1/2)}{\sqrt{\nu^2+2\lambda\nu+\lambda}\,\Gamma(\lambda+\nu/2)}. \tag{2.52}$$

From (2.48)-(2.49) and (2.52), we derive (2.43)-(2.44). ∎

3. Fractional integral/derivative formulas of GGF-Fs

In this section, we show that GGF-Fs enjoy some remarkable fractional calculus properties, which are important for the error analysis.

3.1. **Fractional integrals/derivatives and related spaces of functions.**

Let $\Omega = (a,b) \subset \mathbb{R}$ be a finite open interval. For real $p \in [1,\infty]$, let $L^p(\Omega)$ (resp., $W^{m,p}(\Omega)$ with $m \in \mathbb{N}$) be the usual p-Lebesgue space (resp., Sobolev space), equipped with the norm $\|\cdot\|_{L^p(\Omega)}$ (resp., $\|\cdot\|_{W^{m,p}(\Omega)}$), as in Adams [1].

Let $C(\bar{\Omega})$ be the classical space of continuous functions on $[a,b]$. Denote by $\mathrm{AC}(\bar{\Omega})$ the space of absolutely continuous functions on $[a,b]$. According to [26, Ch. 3] (also see [13, p. 206] or [34, Ch. 1]), a real function $f(x) \in \mathrm{AC}(\bar{\Omega})$, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every finite sequence of disjoint intervals $(a_k, b_k) \subset \Omega$ such that $\sum_k |b_k - a_k| < \delta$, we have $\sum_k |f(b_k) - f(a_k)| < \varepsilon$. Recall that (cf. [34, Ch. 1] or [24, p. 285]): a real function $f(x) \in \mathrm{AC}(\bar{\Omega})$ if and only if $f(x) \in L^1(\Omega)$, $f(x)$ has a derivative $f'(x)$ almost everywhere on $[a,b]$ such that $f'(x) \in L^1(\Omega)$, and $f(x)$ has the integral representation:

$$f(x) = f(a) + \int_a^x f'(t)\, dt \quad \forall x \in [a,b]. \tag{3.1}$$

Let $\mathrm{BV}(\bar{\Omega})$ be the space of functions of bounded variation on $[a,b]$. According to [13, p. 207] (also see [32, Ch. 11] and [24, Ch. X]), a real function $f(x) \in \mathrm{BV}(\bar{\Omega})$, if there exists a constant $C > 0$ such that $V(P,f) := \sum_{i=0}^{k-1} |f(x_{i+1}) - f(x_i)| \le C$ for every partition $P : x_0 < x_1 < \ldots < x_k$ of $[a,b]$. Define the total variation of f on $[a,b]$ as $V_f[a,b] := \sup\{V(P,f)\}$, where the supreme is taken over all partitions. An important characterisation of functions of bounded variation is the Jordan decomposition (cf. [32, Thm. 11.19]): a function is of bounded variation if and only if it can be expressed as the difference of two increasing functions on $[a,b]$. As a result, every function in $\mathrm{BV}(\bar{\Omega})$ has at most a countable number of discontinuities, which are either jump or removable discontinuities, so it is differentiable almost everywhere. In view of this, the Riemann-Stieltjes (RS) integral can be defined on functions of bounded variation (see, e.g., [24, Ch. X]). In general, if $\int_a^b f dg < \infty$, we say f is $\mathrm{RS}(g)$-integrable. According to [24, Prop. 1.3], we have the following important property: if f is $\mathrm{RS}(g)$-integrable, and $g \in \mathrm{BV}(\bar{\Omega})$, then

$$\Big| \int_a^b f dg \Big| \le \|f\|_\infty V_g[a,b], \tag{3.2}$$

where $\|f\|_\infty$ is the L^∞-norm of f on $[a,b]$.

We recall the definitions of the Riemann-Liouville fractional integrals and derivatives (cf. [34, p. 33, p. 44]), and also follow the notation therein. Denote the ordinary derivatives by $\mathcal{D} = d/dx$ and $\mathcal{D}^k = d^k/dx^k$ with integer $k \ge 2$.

Definition 3.1. For any $u \in L^1(\Omega)$, the left-sided and right-sided Riemann-Liouville fractional integrals of order $s \in \mathbb{R}^+$ are defined by

$$\begin{aligned} (I_{a+}^s u)(x) &= \frac{1}{\Gamma(s)} \int_a^x \frac{u(y)}{(x-y)^{1-s}} dy; \\ (I_{b-}^s u)(x) &= \frac{1}{\Gamma(s)} \int_x^b \frac{u(y)}{(y-x)^{1-s}} dy, \quad x \in \Omega. \end{aligned} \tag{3.3}$$

A function $u \in L^1(\Omega)$ is said to possess a left-sided (resp., right-sided) Riemann-Liouville fractional derivative $\mathcal{D}^s_{a+}u$ (resp., $\mathcal{D}^s_{b-}u$) of order $s \in (0,1)$, if $I^{1-s}_{a+}\, u \in \mathrm{AC}(\bar\Omega)$ (resp., $I^{1-s}_{b-}\, u \in \mathrm{AC}(\bar\Omega)$). Moreover, we have

$$(3.4)\qquad (\mathcal{D}^s_{a+}u)(x) = \mathcal{D}\big\{I^{1-s}_{a+}\, u\big\}(x), \quad (\mathcal{D}^s_{b-}u)(x) = -\mathcal{D}\big\{I^{1-s}_{b-}\, u\big\}(x), \quad x \in \Omega.$$

Similarly, for $s \in [k-1,k)$ with $k \in \mathbb{N}$, the higher order left-sided and right-sided Riemann-Liouville fractional derivatives for $u \in L^1(\Omega)$ satisfying $I^{1-s}_{a+}\, u, I^{1-s}_{b-}\, u \in \mathrm{AC}^k(\bar\Omega)$ (i.e., the space of all $f(x)$ having continuous derivatives up to order $k-1$ on $\bar\Omega$, and $f^{(k-1)} \in \mathrm{AC}(\bar\Omega)$) are defined by

$$(3.5)\qquad (\mathcal{D}^s_{a+}u)(x) = \mathcal{D}^k\big\{\, I^{k-s}_{a+}\, u\big\}(x); \quad (\mathcal{D}^s_{b-}u)(x) = (-1)^k\mathcal{D}^k\big\{\, I^{k-s}_{b-}\, u\big\}(x).$$

As a generalisation of (3.1), we have the following fractional integral representation, which can also be regarded as the definition of Riemann-Liouville fractional derivatives alternative to Definition 3.1 (see [10, Prop. 5] and [34, p. 45]).

Proposition 3.1. *A function $u \in L^1(\Omega)$ possesses a left-sided Riemann-Liouville fractional derivative $\mathcal{D}^s_{a+}u$ of order $s \in (0,1)$ if and only if there exist $C_a \in \mathbb{R}$ and $\phi \in L^1(\Omega)$ such that*

$$(3.6)\qquad u(x) = \frac{C_a}{\Gamma(s)}(x-a)^{s-1} + (I^s_{a+}\, \phi)(x) \ \ a.e. \ on\ \ [a,b],$$

where $C_a = (I^{1-s}_{a+}u)(a)$ and $\phi(x) = (\mathcal{D}^s_{a+}u)(x)$ a.e. on $[a,b]$.

Similarly, a function $u \in L^1(\Omega)$ has a right-sided Riemann-Liouville fractional derivative $\mathcal{D}^s_{b-}u$ of order $s \in (0,1)$ if and only if there exist $C_b \in \mathbb{R}$ and $\psi \in L^1(\Omega)$ such that

$$(3.7)\qquad u(x) = \frac{C_b}{\Gamma(s)}(b-x)^{s-1} + (I^s_{b-}\, \psi)(x) \ \ a.e. \ on\ \ [a,b],$$

where $C_b = (I^{1-s}_{b-}u)(b)$ and $\psi(x) = (\mathcal{D}^s_{b-}u)(x)$ a.e. on $[a,b]$.

Remark 3.1. We infer from Proposition 3.1 the equivalence of these two fractional spaces:

$$(3.8)\qquad \begin{aligned} W^{s,1}_{\mathrm{RL},a+}(\Omega) &:= \big\{u \in L^1(\Omega) : I^{1-s}_{a+}u \in \mathrm{AC}(\bar\Omega)\big\} \\ &\equiv \big\{u \in L^1(\Omega) : \mathcal{D}^s_{a+}u \in L^1(\Omega)\big\} \end{aligned}$$

for $s \in (0,1)$. The inclusion "$\subseteq$" follows immediately from $u \in L^1(\Omega), I^{1-s}_{a+}u \in \mathrm{AC}(\bar\Omega)$ and Definition 3.1. To show the opposite inclusion "$\supseteq$", we find

$$\begin{aligned} &\int_a^b |(I^{1-s}_{a+}u)(x)|dx = \frac{1}{\Gamma(1-s)}\int_a^b \Big|\int_a^x (x-y)^{-s}u(y)dy\Big|dx \\ &\le \frac{1}{\Gamma(1-s)}\int_a^b\int_a^x (x-y)^{-s}|u(y)|dydx = \frac{1}{\Gamma(1-s)}\int_a^b \Big(\int_y^b (x-y)^{-s}dx\Big)|u(y)|dy \\ &= \frac{1}{\Gamma(2-s)}\int_a^b (b-y)^{1-s}|u(y)|dy \le \frac{(b-a)^{1-s}}{\Gamma(2-s)}\int_a^b |u(y)|dy\,. \end{aligned}$$

Since $u \in L^1(\Omega)$, we conclude $I^{1-s}_{a+}u \in L^1(\Omega)$. As $\mathcal{D}^s_{a+}u = \mathcal{D}\{I^{1-s}_{a+}u\} \in L^1(\Omega)$, we infer that $I^{1-s}_{a+}u \in W^{1,1}(\Omega)(= \mathrm{AC}(\bar\Omega))$. Therefore, the equivalence in (3.8) follows. The same property for $W^{s,1}_{\mathrm{RL},b-}(\Omega)$ with $I^{1-s}_{b-}u$, $\mathcal{D}^s_{b-}u$ in place of $I^{1-s}_{a+}u, \mathcal{D}^s_{a+}u$, respectively, holds. We refer to [8] for insightful discussions of the relation between $W^{s,1}_{\mathrm{RL},a+}(\Omega)$ and the fractional Sobolev space in the sense of Gagliardo [29].

Recall the explicit formulas (cf. [34]): for real $\eta > -1$ and $s > 0$,

$$
\begin{aligned}
I^s_{a+}\,(x-a)^\eta &= \frac{\Gamma(\eta+1)}{\Gamma(\eta+s+1)}(x-a)^{\eta+s};\\
\mathcal{D}^s_{a+}\,(x-a)^\eta &= \frac{\Gamma(\eta+1)}{\Gamma(\eta-s+1)}(x-a)^{\eta-s}.
\end{aligned}
\tag{3.9}
$$

We have similar formulas for right-sided Riemann-Liouville fractional integral/derivative of $(b-x)^\eta$. In particular,

$$
I^{1-s}_{a+}\,(x-a)^{s-1} = \Gamma(s); \quad I^{1-s}_{b-}\,(b-x)^{s-1} = \Gamma(s), \quad s \in (0,1), \tag{3.10}
$$

which implies the boundary values C_a and C_b in Proposition 3.1 are not always zero as $x \to a^+$ and $x \to b^-$, respectively. On the other hand, if $\eta - s + 1 = -n$ with $n \in \mathbb{N}_0$ in the second formula of (3.9) (note: $\Gamma(-n) = \infty$), then

$$
\mathcal{D}^s_{a+}\,(x-a)^{s-n-1} = \mathcal{D}^s_{b-}\,(b-x)^{s-n-1} = 0 \quad \text{for } s > n \in \mathbb{N}_0. \tag{3.11}
$$

We see that the first term in the integral representations in (3.6)-(3.7) actually plays the same role as a "constant" in (3.1).

3.2. **Important formulas.**

Theorem 3.1. *For real $\nu \ge s > 0$ and real $\lambda > -1/2$, the GGF-Fs on $(-1,1)$ satisfy the Riemann-Liouville fractional integral formulas:*

$$
I^s_{1-}\big\{\omega_\lambda(x)\,{}^rG^{(\lambda)}_\nu(x)\big\} = h^{(-s)}_\lambda\,\omega_{\lambda+s}(x)\,{}^rG^{(\lambda+s)}_{\nu-s}(x), \tag{3.12a}
$$

$$
I^s_{-1+}\big\{\omega_\lambda(x)\,{}^lG^{(\lambda)}_\nu(x)\big\} = (-1)^{[\nu]+[\nu-s]}\,h^{(-s)}_\lambda\,\omega_{\lambda+s}(x)\,{}^lG^{(\lambda+s)}_{\nu-s}(x). \tag{3.12b}
$$

For real $\lambda > s - 1/2$ and real $\nu \ge 0$, the GGF-Fs on $(-1,1)$ satisfy the Riemann-Liouville fractional derivative formulas:

$$
\mathcal{D}^s_{1-}\big\{\omega_\lambda(x)\,{}^rG^{(\lambda)}_\nu(x)\big\} = h^{(s)}_\lambda\,\omega_{\lambda-s}(x)\,{}^rG^{(\lambda-s)}_{\nu+s}(x), \tag{3.13a}
$$

$$
\mathcal{D}^s_{-1+}\big\{\omega_\lambda(x)\,{}^lG^{(\lambda)}_\nu(x)\big\} = (-1)^{[\nu]+[\nu+s]}\,h^{(s)}_\lambda\,\omega_{\lambda-s}(x)\,{}^lG^{(\lambda-s)}_{\nu+s}(x). \tag{3.13b}
$$

In the above, we denote

$$
\omega_\alpha(x) = (1-x^2)^{\alpha-\frac{1}{2}}, \quad h^{(\beta)}_\lambda = \frac{2^\beta\,\Gamma(\lambda+1/2)}{\Gamma(\lambda-\beta+1/2)}. \tag{3.14}
$$

Proof. Recall the Bateman's fractional integral formula (cf. [4, p. 313]): for $c, s > 0$ and $|z| < 1$,

$$
{}_2F_1(a,b;c+s;z) = z^{1-(c+s)}\frac{\Gamma(c+s)}{\Gamma(c)\Gamma(s)}\int_0^z t^{c-1}(z-t)^{s-1}{}_2F_1(a,b;c;t)\,dt, \tag{3.15}
$$

which, together with (2.9), yields

$$
\begin{aligned}
&z^{c+s-1}(1-z)^{c+s-a-b}{}_2F_1(c-a+s,c-b+s;c+s;z)\\
&\qquad = \frac{\Gamma(c+s)}{\Gamma(c)\Gamma(s)}\int_0^z t^{c-1}(z-t)^{s-1}(1-t)^{c-a-b}{}_2F_1(c-a,c-b;c;t)\,dt.
\end{aligned}
\tag{3.16}
$$

Applying the variable substitutions: $z = (1-x)/2$ and $t = (1-y)/2$ to (3.16), leads to

(3.17)
$$(1-x)^{c+s-1}(1+x)^{c+s-a-b}{}_2F_1\Big(c-a+s, c-b+s; c+s; \frac{1-x}{2}\Big)$$
$$= \frac{2^s\,\Gamma(c+s)}{\Gamma(c)\Gamma(s)} \int_x^1 (1-y)^{c-1}(1+y)^{c-a-b}(y-x)^{s-1}{}_2F_1\Big(c-a, c-b; c; \frac{1-y}{2}\Big)dy.$$

Taking $a = \nu+\lambda+1/2$, $b = -\nu-\lambda+1/2$, and $c = \lambda+1/2$ in (3.17), we obtain

$$(1-x^2)^{\lambda+s+1/2}\,{}_2F_1\Big(s-\nu, \nu+s+2\lambda; \lambda+s+\frac{1}{2}; \frac{1-x}{2}\Big)$$
$$= \frac{2^s\,\Gamma(\lambda+s+1/2)}{\Gamma(\lambda+1/2)\Gamma(s)} \int_x^1 (1-y^2)^{\lambda-1/2}(y-x)^{s-1}{}_2F_1\Big(-\nu, \nu+2\lambda; \lambda+\frac{1}{2}; \frac{1-y}{2}\Big)dy.$$

From (3.3) and (2.20), we derive (3.12a) immediately.

Similarly, performing the variable substitutions: $z = (1+x)/2$ and $t = (1+y)/2$ to (3.16), we can obtain (3.12b) in the same manner.

Applying $\mathcal{D}^s_{1-}$ to both sides of (3.12a) and noting that $\mathcal{D}^s_{1-}\, I^s_{1-}$ is an identity operator (cf. [34]), we obtain for real $\nu \ge s > 0$ and real $\lambda > -1/2$,

$$\omega_\lambda(x)\,{}^rG^{(\lambda)}_\nu(x) = h^{(-s)}_\lambda\,\mathcal{D}^s_{1-}\big\{\omega_{\lambda+s}(x)\,{}^rG^{(\lambda+s)}_{\nu-s}(x)\big\}. \tag{3.18}$$

Replacing λ, ν in the above equation by $\lambda - s, \nu + s$, and noting that

$$\big(h^{(-s)}_{\lambda-s}\big)^{-1} = \frac{2^s\,\Gamma(\lambda+1/2)}{\Gamma(\lambda-s+1/2)} = h^{(s)}_\lambda, \tag{3.19}$$

we obtain (3.13a). Similarly, applying $\mathcal{D}^s_{-1+}$ to both sides of (3.12b), we can derive (3.13b). ☐

4. Chebyshev approximations of functions in fractional Sobolev-type spaces

In this section, we introduce a new theoretical framework and present the main results on Chebyshev approximations. Here, we focus on the approximation of functions with interior singularities, and shall extend the estimates to deal with functions with endpoint singularities in Subsection 6.2.

4.1. **Fractional Sobolev-type spaces.**

For a fixed $\theta \in \Omega = (-1, 1)$, we denote $\Omega^-_\theta := (-1, \theta)$ and $\Omega^+_\theta := (\theta, 1)$. For $m \in \mathbb{N}_0$ and $s \in (0, 1)$, we define the fractional Sobolev-type space:

$$\begin{aligned}\mathbb{W}^{m+s}_\theta(\Omega) := \big\{u \in L^1(\Omega) \,:\, u, u', \cdots, u^{(m-1)} \in \mathrm{AC}(\bar\Omega) \ \text{ and} \\ I^{1-s}_{\theta-}u^{(m)} \in \mathrm{BV}(\bar\Omega^-_\theta), \quad I^{1-s}_{\theta+}u^{(m)} \in \mathrm{BV}(\bar\Omega^+_\theta)\big\},\end{aligned} \tag{4.1}$$

equipped with the norm:

$$\|u\|_{\mathbb{W}^{m+s}_\theta(\Omega)} = \sum_{k=0}^m \|u^{(k)}\|_{L^1(\Omega)} + U^{m,s}_\theta, \tag{4.2}$$

where the semi-norm is defined by

- for $m = 1, 2, \cdots$, and $s \in (0,1)$,

$$U_\theta^{m,s} := \int_{-1}^{\theta} \big|\mathcal{D}_{\theta-}^{s}\, u^{(m)}(x)\big|\, dx + \int_{\theta}^{1} |\mathcal{D}_{\theta+}^{s} u^{(m)}(x)| dx + \big|\big\{I_{\theta-}^{1-s} u^{(m)}\big\}(\theta+)\big| + \big|\big\{I_{\theta+}^{1-s} u^{(m)}\big\}(\theta-)\big|; \tag{4.3}$$

- for $m = 0$ and $s \in (1/2, 1)$,

$$U_\theta^{0,s} := \int_{-1}^{\theta} \big|\mathcal{D}_{\theta-}^{s}\, u(x)\big|\, \omega_{s/2}(x) dx + \int_{\theta}^{1} |\mathcal{D}_{\theta+}^{s} u(x)|\, \omega_{s/2}(x) dx + \big|\big\{\omega_{s/2}\, I_{\theta-}^{1-s} u\big\}(\theta+)\big| + \big|\big\{\omega_{s/2}\, I_{\theta+}^{1-s} u\big\}(\theta-)\big|. \tag{4.4}$$

Analogous to the integer derivative case (see [38, 39] or (5.1) below), the integrals in (4.3)-(4.4) are in the Riemann-Stieltjes sense. For example, for $I_{\theta-}^{1-s} u^{(m)} \in \mathrm{BV}(\bar{\Omega}_\theta^-)$, we understand

$$\int_{-1}^{\theta} \big|\mathcal{D}_{\theta-}^{s}\, u^{(m)}(x)\big|\, dx = \int_{-1}^{\theta} \big|d(I_{\theta-}^{1-s} u^{(m)})\big|.$$

By virtue of (3.2), it can be bounded by the total variation of $I_{\theta-}^{1-s} u^{(m)}$ on $\bar{\Omega}_\theta^-$.

Remark 4.1. The parameter θ is related to the location of the singular point of $u(x)$. For example, if $u = |x|$, then $\theta = 0$ (also see the examples in Subsection 4.4). For a function of multiple interior singular points, we partition $(-1, 1)$ into multiple subintervals and introduce the same number of parameters accordingly.

To deal with endpoint singularities, we define the fractional Sobolev-type spaces corresponding to $\theta = \pm 1$ by

(4.5)

$$\mathbb{W}_{1-}^{m+s}(\Omega) := \big\{u \in L^1(\Omega) \,:\, u, u', \cdots, u^{(m-1)} \in \mathrm{AC}(\bar{\Omega}),\ I_{1-}^{1-s} u^{(m)} \in \mathrm{BV}(\bar{\Omega})\big\},$$
$$\mathbb{W}_{-1+}^{m+s}(\Omega) := \big\{u \in L^1(\Omega) \,:\, u, u', \cdots, u^{(m-1)} \in \mathrm{AC}(\bar{\Omega}),\ I_{-1+}^{1-s} u^{(m)} \in \mathrm{BV}(\bar{\Omega})\big\}.$$

Accordingly, the semi-norm $U_+^{m,s}$ (resp., $U_-^{m,s}$) only involves the right (resp., left) Riemann-Liouville fractional integrals/derivatives. For example, when $\theta = -1$, the semi-norm corresponding to (4.3) becomes

$$U_{-1+}^{m,s} := \int_{-1}^{1} |\mathcal{D}_{-1+}^{s} u^{(m)}(x)| dx + \big|\big\{I_{-1+}^{1-s} u^{(m)}\big\}(-1+)\big|. \tag{4.6}$$

We remark that for $s \in (0,1)$, $\mathbb{W}_{-1+}^{s}(\Omega) \supsetneq W_{\mathrm{RL},-1+}^{s,1}(\Omega)$ defined in (3.8).

4.2. **Exact formulas and decay rate of Chebyshev expansion coefficients.** Let $\omega(x) = (1-x^2)^{-1/2} = \omega_0(x)$ be the Chebyshev weight function. For any $u \in L^2_\omega(\Omega)$, we expand it in Chebyshev series and denote the partial sum by

$$u(x) = {\sum_{n=0}^{\infty}}{}'\, \hat{u}_n^C\, T_n(x), \quad \pi_N^C u(x) = {\sum_{n=0}^{N}}{}'\, \hat{u}_n^C\, T_n(x), \tag{4.7}$$

where the prime denotes a sum whose first term is halved, and

$$\hat{u}_n^C = \frac{2}{\pi} \int_{-1}^{1} u(x) \frac{T_n(x)}{\sqrt{1-x^2}} dx = \frac{2}{\pi} \int_{0}^{\pi} u(\cos\theta) \cos(n\theta) d\theta. \tag{4.8}$$

Recall the formula of integration by parts involving the Riemann-Stieltjes integrals (cf. [25, (1.20)]).

Lemma 4.1. *For any $u, v \in \mathrm{BV}(\bar{\Omega})$, we have*

$$\int_a^b u(x)\,dv(x) = \{u(x)v(x)\}\big|_{a+}^{b-} - \int_a^b v(x)\,du(x), \tag{4.9}$$

where the notation $u(x\pm)$ stands for the right- and left-limit of u at x, respectively.

In particular, if $u, v \in \mathrm{AC}(\bar{\Omega})$, we have

$$\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx = \{u(x)v(x)\}\big|_a^b. \tag{4.10}$$

Remark 4.2. As highlighted in [27, 39], the error analysis of Chebyshev expansions in various norms, and the related interpolation and quadrature errors, essentially depends on estimating the decay rate of $|\hat{u}_n^C|$.

We present the main results below.

Theorem 4.1. *Given $\theta \in (-1, 1)$, if $u \in \mathbb{W}_\theta^{m+s}(\Omega)$ with $s \in (0, 1)$ and integer $m \ge 0$, then for $n > m + s > 1/2$,*

(4.11)

$$\begin{aligned}\hat{u}_n^C = &-\frac{1}{\sqrt{\pi}\,2^{m+s-1}\Gamma(m+s+1/2)}\Big\{\int_{-1}^{\theta} {}^{l}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\,d\big\{I_{\theta-}^{1-s}u^{(m)}(x)\big\}\\ &+\big\{I_{\theta-}^{1-s}u^{(m)}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\big\}\big|_{x=\theta-}\\ &-\int_{\theta}^{1} {}^{r}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\,d\big\{I_{\theta+}^{1-s}u^{(m)}(x)\big\}\\ &-\big\{I_{\theta+}^{1-s}u^{(m)}(x)\,{}^{r}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\big\}\big|_{x=\theta+}\Big\},\end{aligned}$$

where $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$. Moreover, we have the following bounds:

(i) *For $m = 0, s \in (1/2, 1)$ and $n > s$, we have*

$$|\hat{u}_n^C| \le \frac{U_\theta^{0,s}}{2^{s-1}\pi}\max\Big\{\frac{\Gamma((n-s+1)/2)}{\Gamma((n+s+1)/2)}, \frac{2}{\sqrt{n^2-s^2+s}}\frac{\Gamma((n-s)/2+1)}{\Gamma((n+s)/2)}\Big\}. \tag{4.12}$$

(ii) *For $m \ge 1, s \in (0, 1)$ and $n > m + s$, we have*

$$|\hat{u}_n^C| \le \frac{U_\theta^{m,s}}{2^{m+s-1}\pi}\,\frac{\Gamma((n-m-s+1)/2)}{\Gamma((n+m+s+1)/2)}. \tag{4.13}$$

Proof. Substituting $n \to n-k, \lambda \to k$ in (2.19), leads to

$$\omega_k(x)G_{n-k}^{(k)}(x) = -\frac{1}{2k+1}\big\{\omega_{k+1}(x)G_{n-k-1}^{(k+1)}(x)\big\}', \quad n \ge k+1. \tag{4.14}$$

For $u, u', \cdots, u^{(m-1)} \in \mathrm{AC}(\bar{\Omega})$, using (4.14) with $k = 0, 1, \cdots, m-1$, and the integration by parts in Lemma 4.1, we obtain that for $n \geq m$,

(4.15)
$$\begin{aligned}
\hat{u}_n^C &= \frac{2}{\pi}\int_{-1}^{1} u(x) G_n^{(0)}(x)\omega_0(x)\,dx = -\frac{2}{\pi}\int_{-1}^{1} u(x)\big\{G_{n-1}^{(1)}(x)\omega_1(x)\big\}'\,dx \\
&= \frac{2}{\pi}\int_{-1}^{1} u'(x) G_{n-1}^{(1)}(x)\omega_1(x)dx = -\frac{2}{3\pi}\int_{-1}^{1} u'(x)\big\{G_{n-2}^{(2)}(x)\omega_2(x)\big\}'\,dx \\
&= \frac{1}{3}\frac{2}{\pi}\int_{-1}^{1} u''(x) G_{n-2}^{(2)}(x)\omega_2(x)dx = -\frac{1}{3\cdot 5}\frac{2}{\pi}\int_{-1}^{1} u''(x)\big\{G_{n-3}^{(3)}(x)\omega_3(x)\big\}'\,dx \\
&= \cdots = \frac{1}{(2m-1)!!}\frac{2}{\pi}\int_{-1}^{1} u^{(m)}(x)\, G_{n-m}^{(m)}(x)\omega_m(x)\,dx.
\end{aligned}$$

Using the identity (cf. [31]):

$$\Gamma(k+1/2) = \frac{\sqrt{\pi}\,(2k-1)!!}{2^k}, \quad k \in \mathbb{N}_0, \tag{4.16}$$

we can rewrite the expansion coefficient as

$$\hat{u}_n^C = \frac{1}{\sqrt{\pi}\,2^{m-1}\Gamma(m+1/2)}\int_{-1}^{1} u^{(m)}(x)\, G_{n-m}^{(m)}(x)\omega_m(x)\,dx. \tag{4.17}$$

We proceed with the proof by fractional integration by parts. Then it is necessary to use the following identities: for $n > m+s > 1/2$,

$$\begin{aligned}
\omega_m(x)G_{n-m}^{(m)}(x) &= -\frac{\Gamma(m+1/2)}{2^s\,\Gamma(m+s+1/2)}\, I_{1-}^{1-s}\big\{\omega_{m+s}(x)\,{}^{r}G_{n-m-s}^{(m+s)}(x)\big\}' \\
&= -\frac{\Gamma(m+1/2)}{2^s\,\Gamma(m+s+1/2)}\, I_{-1+}^{1-s}\big\{\omega_{m+s}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x)\big\}'.
\end{aligned} \tag{4.18}$$

To derive (4.18), we substitute s, λ, ν in (3.12a)-(3.12b) by $1-s, m-s, n-m+s$, respectively, leading to

$$\begin{aligned}
\omega_m(x)G_{n-m}^{(m)}(x) &= \frac{2^{1-s}\Gamma(m+1/2)}{\Gamma(m+s-1/2)}\, I_{1-}^{1-s}\big\{\omega_{m+s-1}(x)\,{}^{r}G_{n-m-s+1}^{(m+s-1)}(x)\big\} \\
&= \frac{2^{1-s}\Gamma(m+1/2)}{\Gamma(m+s-1/2)}\, I_{-1+}^{1-s}\big\{\omega_{m+s-1}(x)\,{}^{l}G_{n-m-s+1}^{(m+s-1)}(x)\big\}.
\end{aligned} \tag{4.19}$$

Taking $s = 1, \lambda = m+s$ and $\nu = n-m-s$ in (3.13a)-(3.13b), we obtain that for $m+s > 1/2$,

$$\begin{aligned}
\omega_{m+s-1}(x)\,{}^{r}G_{n-m-s+1}^{(m+s-1)}(x) &= -\frac{\Gamma(m+s-1/2)}{2\,\Gamma(m+s+1/2)}\big\{\omega_{m+s}(x)\,{}^{r}G_{n-m-s}^{(m+s)}(x)\big\}', \\
\omega_{m+s-1}(x)\,{}^{l}G_{n-m-s+1}^{(m+s-1)}(x) &= -\frac{\Gamma(m+s-1/2)}{2\,\Gamma(m+s+1/2)}\big\{\omega_{m+s}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x)\big\}'.
\end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.19) leads to (4.18).

For notational convenience, we denote

$$\begin{aligned}
&f(x) = u^{(m)}(x), \quad g(x) = -\omega_{m+s}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x), \\
&h(x) = -\omega_{m+s}(x)\,{}^{r}G_{n-m-s}^{(m+s)}(x).
\end{aligned} \tag{4.21}$$

By (4.18), we can rewrite (4.17) as

$$\begin{aligned}\hat{u}_n^C &= \frac{1}{\sqrt{\pi}\,2^{m-1}\Gamma(m+1/2)}\bigg\{\int_{-1}^{\theta} u^{(m)}\,G_{n-m}^{(m)}\,\omega_m\,dx + \int_{\theta}^{1} u^{(m)}\,G_{n-m}^{(m)}\,\omega_m\,dx\bigg\} \\ &= \frac{1}{\sqrt{\pi}\,2^{m+s-1}\Gamma(m+s+1/2)}\bigg\{\int_{-1}^{\theta} f(x)\,I_{-1+}^{1-s}g'(x)\,dx + \int_{\theta}^{1} f(x)\,I_{1-}^{1-s}h'(x)\,dx\bigg\}.\end{aligned} \tag{4.22}$$

We find from (2.22b) and (4.20), $g'(x)$ (resp., $h'(x)$) is continuous on $(-1,\theta]$ (resp., $[\theta,1)$), and they are also integrable when $m+s>1/2$. Thus, for $f\in L^1(\Omega)$, changing the order of integration by Fubini's Theorem, we derive from (3.3) that

$$\begin{aligned}\int_{-1}^{\theta} f(x) I_{-1+}^{1-s}g'(x)\,dx &= \frac{1}{\Gamma(1-s)}\int_{-1}^{\theta}\bigg\{\int_{-1}^{x}\frac{g'(y)}{(x-y)^s}dy\bigg\}f(x)\,dx \\ &= \frac{1}{\Gamma(1-s)}\int_{-1}^{\theta}\bigg\{\int_{y}^{\theta}\frac{f(x)}{(x-y)^s}dx\bigg\}g'(y)\,dy \\ &= \frac{1}{\Gamma(1-s)}\int_{-1}^{\theta}\bigg\{\int_{x}^{\theta}\frac{f(y)}{(y-x)^s}dy\bigg\}g'(x)\,dx = \int_{-1}^{\theta} g'(x)\,I_{\theta-}^{1-s}f(x)\,dx.\end{aligned} \tag{4.23}$$

Similarly, we can show

$$\int_{\theta}^{1} f(x)\,I_{1-}^{1-s}h'(x)\,dx = \int_{\theta}^{1} h'(x)\,I_{\theta+}^{1-s}f(x)\,dx. \tag{4.24}$$

Thus, if $I_{\theta-}^{1-s}f(x)\in \mathrm{BV}(\bar{\Omega}_\theta^-)$ and $I_{\theta+}^{1-s}f(x)\in \mathrm{BV}(\bar{\Omega}_\theta^+)$, we use Lemma 4.1, and derive

$$\begin{aligned}\int_{-1}^{\theta} f(x)\,I_{-1+}^{1-s}g'(x)\,dx &= \int_{-1}^{\theta} g'(x)\,I_{\theta-}^{1-s}f(x)\,dx \\ &= \big\{g(x)\,I_{\theta-}^{1-s}f(x)\big\}\big|_{-1+}^{\theta-} - \int_{-1}^{\theta} g(x)\,d\big\{I_{\theta-}^{1-s}f(x)\big\} \\ &= \big\{g(x)\,I_{\theta-}^{1-s}f(x)\big\}\big|_{x=\theta-} + \int_{-1}^{\theta} g(x)\,d\big\{I_{\theta-}^{1-s}f(x)\big\},\end{aligned} \tag{4.25}$$

where we used the fact $g(-1)=0$ for $m+s>1/2$ due to (2.22b), and also used (3.5).

Similarly, we can show that for $m+s>1/2$,

$$\int_{\theta}^{1} f(x)\,I_{1-}^{1-s}h'(x)\,dx = -\big\{h(x)\,I_{\theta+}^{1-s}f(x)\big\}\big|_{x=\theta+} - \int_{\theta}^{1} h(x)\,d\big\{I_{\theta+}^{1-s}f(x)\big\}. \tag{4.26}$$

Substituting (4.21) and (4.25)-(4.26) into (4.22), we obtain (4.11).

We next derive the bounds in (4.12)-(4.13).

(i) For $m=0$ and $s\in(1/2,1)$, we take $\lambda=s$ and $\nu=n-s$ in Theorem 2.2, and then obtain from (4.11) and the bound (4.12) directly.

(ii) We now turn to the proof of (4.13). We first show the inequality:

$$\frac{2}{\sqrt{2\lambda-1+\nu(\nu+2\lambda)}}\frac{\Gamma(\nu/2+1)}{\Gamma(\nu/2+\lambda)} \le \frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+\lambda)}, \quad \nu\ge 0, \quad \lambda\ge 1. \tag{4.27}$$

To prove (4.27), we use the property in [14, Corollary 2], that is, the ratio

$$f(z) := \frac{1}{\sqrt{z}}\frac{\Gamma(z+1)}{\Gamma(z+1/2)}, \quad z>0,$$

is decreasing. Then using the facts:

$$(\nu-1)/2+\lambda>0, \quad (\nu-1)/2+\lambda>(\nu+1)/2,$$

we can derive that

$$\begin{aligned}\frac{1}{\sqrt{(\nu-1)/2+\lambda}}\frac{\Gamma((\nu+1)/2+\lambda)}{\Gamma(\nu/2+\lambda)} &\le \frac{1}{\sqrt{\nu/2+1/2}}\frac{\Gamma((\nu+3)/2)}{\Gamma(\nu/2+1)} \\ &= \sqrt{\frac{\nu+1}{2}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2+1)},\end{aligned} \tag{4.28}$$

where in the last step we used the identity: $\Gamma(z+1)=z\Gamma(z)$. Next, we rewrite (4.28) as

$$\frac{2}{\sqrt{(\nu+2\lambda-1)(\nu+1)}}\frac{\Gamma(\nu/2+1)}{\Gamma(\nu/2+\lambda)} \le \frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+\lambda)}. \tag{4.29}$$

Noting that

$$\frac{2}{\sqrt{2\lambda-1+\nu(\nu+2\lambda)}} = \frac{2}{\sqrt{(\nu+2\lambda-1)(\nu+1)}},$$

we obtain (4.27) from (4.29) immediately. Using (4.27), we derive from Theorem 2.1 that

$$\max_{|x|\le 1}\left\{\omega_\lambda(x)\left|{}^rG_\nu^{(\lambda)}(x)\right|,\ \omega_\lambda(x)\left|{}^lG_\nu^{(\lambda)}(x)\right|\right\} \le \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}}\frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+\lambda)}, \tag{4.30}$$

so the bound in (4.13) follows from (4.11) with $\lambda=m+s$ and $\nu=n-m-s$ in (4.30). $\square$

In Theorem 4.1, we only presented the formulas for functions with fractional regularity index $s\in(0,1)$. For the integer case with $s=1$, we stop at the step (4.17) and then estimate the bound of the Chebyshev expansion coefficients. Accordingly, we define the corresponding space as

$$\mathbb{W}^{m+1}(\Omega) := \left\{u\in L^1(\Omega) : u',\cdots,u^{(m-1)}\in \mathrm{AC}(\bar\Omega),\ u^{(m)}\in \mathrm{BV}(\bar\Omega)\right\}, \tag{4.31}$$

as in [27, 39]. We show in Section 5 that the analysis under this framework can improve the existing estimates in [27, 38, 39, 42].

4.3. **L^∞- and L^2-estimates of Chebyshev expansions.** With Theorem 4.1 at our disposal, we can analyse all related orthogonal projections, interpolations, and quadratures (cf. [27]). Here, we first estimate the Chebyshev expansion errors in the L^∞-norm and L^2_ω-norm for functions with interior singularities, i.e., $\theta\in(-1,1)$ and with $s\in(0,1)$. We consider the analysis of endpoint singularities with $\theta=\pm1$ in Section 6.

Theorem 4.2. *Given $\theta\in(-1,1)$, if $u\in\mathbb{W}_\theta^{m+s}(\Omega)$ with $s\in(0,1)$ and integer $m\ge 0$, we have the following estimates:*

(i) *For $1<m+s<N+1$,*

$$\|u-\pi_N^C u\|_{L^\infty(\Omega)} \le \frac{U_\theta^{m,s}}{2^{m+s-2}(m+s-1)\pi}\frac{\Gamma((N-m-s)/2+1)}{\Gamma((N+m+s)/2)}. \tag{4.32}$$

(ii) *For $1/2<m+s<N+1$,*

$$\|u-\pi_N^C u\|_{L^2_\omega(\Omega)} \le \left\{\frac{2^3}{(2m+2s-1)\pi}\frac{\Gamma(N-m-s+1)}{\Gamma(N+m+s)}\right\}^{1/2} U_\theta^{m,s}. \tag{4.33}$$

Proof. (i) We first prove (4.32). For simplicity, we denote

$$\mathcal{S}_n^\sigma := \frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma+1)/2)}, \quad \mathcal{T}_n^\sigma := \frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma-1)/2)}, \quad \sigma := m+s. \tag{4.34}$$

A direct calculation leads to the identity:

$$\begin{aligned}\mathcal{T}_n^\sigma - \mathcal{T}_{n+2}^\sigma =& \frac{n+\sigma-1}{2}\frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma+1)/2)} - \frac{n-\sigma+1}{2}\frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma+1)/2)} \\ =&(\sigma-1)\frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma+1)/2)} = (\sigma-1)\mathcal{S}_n^\sigma,\end{aligned} \tag{4.35}$$

where we used the identity $z\Gamma(z) = \Gamma(z+1)$. As $\sigma > 1$, we obtain from (4.13) and (4.34) that

$$\begin{aligned}\left|u(x) - \pi_N^C u(x)\right| &\le \sum_{n=N+1}^{\infty} |\hat{u}_n^C| \le \frac{U_\theta^{m,s}}{2^{\sigma-1}\pi}\sum_{n=N+1}^{\infty}\mathcal{S}_n^\sigma \\ &= \frac{U_\theta^{m,s}}{2^{\sigma-1}(\sigma-1)\pi}\lim_{K\to\infty}\sum_{n=N+1}^{K}\left\{\mathcal{T}_n^\sigma - \mathcal{T}_{n+2}^\sigma\right\} \\ &= \frac{U_\theta^{m,s}}{2^{\sigma-1}(\sigma-1)\pi}\lim_{K\to\infty}\left\{\mathcal{T}_{N+1}^\sigma + \mathcal{T}_{N+2}^\sigma - \left(\mathcal{T}_{K+1}^\sigma + \mathcal{T}_{K+2}^\sigma\right)\right\} \\ &= \frac{U_\theta^{m,s}}{2^{\sigma-1}(\sigma-1)\pi}\left\{\mathcal{T}_{N+1}^\sigma + \mathcal{T}_{N+2}^\sigma\right\}.\end{aligned} \tag{4.36}$$

We find from [2, (1.1) and Theorem 10] that for $0 \le a \le b$, the ratio

$$R_b^a(z) := \frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad z \ge 0, \tag{4.37}$$

is decreasing with respect to z. As $\sigma - 1 > 0$, we have

$$\mathcal{T}_{N+2}^\sigma = R_{\sigma-1}^0(1+(N-\sigma+1)/2) \le R_{\sigma-1}^0(1+(N-\sigma)/2) = \mathcal{T}_{N+1}^\sigma. \tag{4.38}$$

Therefore, the estimate (4.32) follows from (4.36) and (4.38).

(ii) We now turn to the estimate (4.33) with $1 < m+s < N+1$. Similar to (4.38), we can use (4.37) to show that $\mathcal{S}_n^\sigma \le \mathcal{S}_{n-1}^\sigma$. Thus, using the identity

$$\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+1/2), \tag{4.39}$$

we derive

$$\begin{aligned}(\mathcal{S}_n^\sigma)^2 \le \mathcal{S}_n^\sigma\mathcal{S}_{n-1}^\sigma &= \frac{\Gamma((n-\sigma+1)/2)}{\Gamma((n+\sigma+1)/2)}\frac{\Gamma((n-\sigma)/2)}{\Gamma((n+\sigma)/2)} = 2^{2\sigma}\frac{\Gamma(n-\sigma)}{\Gamma(n+\sigma)} \\ &= \frac{2^{2\sigma}}{2\sigma-1}\left(\frac{\Gamma(n-\sigma)}{\Gamma(n-1+\sigma)} - \frac{\Gamma(n+1-\sigma)}{\Gamma(n+\sigma)}\right).\end{aligned} \tag{4.40}$$

Then, for $\sigma > 1$,

$$\begin{aligned}\left\|u - \pi_N^C u\right\|_{L_\omega^2(\Omega)}^2 =& \frac{\pi}{2}\sum_{n=N+1}^{\infty}\left|\hat{u}_n^C\right|^2 \le \frac{(U_\theta^{m,s})^2}{2^{2\sigma-3}\pi}\sum_{n=N+1}^{\infty}(\mathcal{S}_n^\sigma)^2 \\ \le& \frac{2^3(U_\theta^{m,s})^2}{(2\sigma-1)\pi}\frac{\Gamma(N-\sigma+1)}{\Gamma(N+\sigma)}.\end{aligned} \tag{4.41}$$

Finally, we prove (4.33) with $m = 0$ and $s \in (1/2, 1)$ by using (4.12). Note that (4.40) is valid for $\sigma = s \in (1/2, 1)$, so we have

$$(\mathcal{S}_n^s)^2 \le \frac{2^{2s}}{2s-1}\left(\frac{\Gamma(n-s)}{\Gamma(n-1+s)} - \frac{\Gamma(n+1-s)}{\Gamma(n+s)}\right). \tag{4.42}$$

We now estimate the second factor in the upper bound (4.12). Using the property $z\Gamma(z) = \Gamma(z+1)$, we obtain from (4.37) that for $n \ge 1$ and $s \in (1/2, 1)$,

$$\begin{aligned}\frac{\Gamma((n-s)/2+1)}{\Gamma((n+s)/2)} &= \frac{n+s}{2}\frac{\Gamma((n-s)/2+1)}{\Gamma((n+s)/2+1)} = \frac{n+s}{2}\frac{\Gamma((n+s)/2+1-s)}{\Gamma((n+s)/2+1)}\\ &\le \frac{n+s}{2}\frac{\Gamma((n-1+s)/2+1-s)}{\Gamma((n-1+s)/2+1)}\\ &= \frac{n+s}{2}\frac{\Gamma((n-s)/2+1/2)}{(n-1+s)/2\,\Gamma((n-1+s)/2)}\\ &= \frac{n+s}{n+s-1}\frac{\Gamma((n-s)/2+1/2)}{\Gamma((n+s)/2-1/2)}.\end{aligned} \tag{4.43}$$

Then by (4.39) and (4.43),

$$\begin{aligned}&\frac{4}{n^2-s^2+s}\frac{\Gamma^2((n-s)/2+1)}{\Gamma^2((n+s)/2)}\\ &\quad\le \frac{4}{n^2-s^2+s}\frac{n+s}{n+s-1}\frac{\Gamma((n-s)/2+1/2)}{\Gamma((n+s)/2-1/2)}\frac{\Gamma((n-s)/2+1)}{\Gamma((n+s)/2)}\\ &\quad= \frac{2^{2s}}{n^2-s^2+s}\frac{n+s}{n-1+s}\frac{\Gamma(n-s+1)}{\Gamma(n+s-1)} = 2^{2s}\frac{n^2-s^2}{n^2-s^2+s}\frac{\Gamma(n-s)}{\Gamma(n+s)}\\ &\quad\le 2^{2s}\frac{\Gamma(n-s)}{\Gamma(n+s)} = \frac{2^{2s}}{2s-1}\left(\frac{\Gamma(n-s)}{\Gamma(n+s-1)} - \frac{\Gamma(n+1-s)}{\Gamma(n+s)}\right),\end{aligned} \tag{4.44}$$

where in the last step, we used the property: $z\Gamma(z) = \Gamma(z+1)$ again to show that

$$\frac{\Gamma(n-s)}{\Gamma(n+s-1)} = (n+s-1)\frac{\Gamma(n-s)}{\Gamma(n+s)}, \quad \frac{\Gamma(n+1-s)}{\Gamma(n+s)} = (n-s)\frac{\Gamma(n-s)}{\Gamma(n+s)}.$$

Thus, we obtain from (4.12), (4.34), (4.42), and (4.44) that

$$|\hat{u}_n^C|^2 \le \frac{4(U_\theta^{0,s})^2}{(2s-1)\pi^2}\left(\frac{\Gamma(n-s)}{\Gamma(n-1+s)} - \frac{\Gamma(n+1-s)}{\Gamma(n+s)}\right). \tag{4.45}$$

Consequently, we can derive

$$\|u - \pi_N^C u\|^2_{L^2_\omega(\Omega)} = \frac{\pi}{2}\sum_{n=N+1}^{\infty}|\hat{u}_n^C|^2 \le \frac{2^3(U_\theta^{0,s})^2}{(2s-1)\pi}\frac{\Gamma(N-s+1)}{\Gamma(N+s)}. \tag{4.46}$$

This completes the proof. □

Remark 4.3. Recall that (cf. [31, (5.11.13)]): for $a < b$,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} + \frac{1}{2}(a-b)(a+b-1)z^{a-b-1} + O(z^{a-b-2}), \quad z \gg 1. \tag{4.47}$$

Thus, under the conditions of Theorem 4.2 and for fixed m and large n or N, we have

$$
\begin{aligned}
&|\hat{u}_n^C| \le Cn^{-(m+s)}U_\theta^{m,s}, \qquad \|u-\pi_N^C u\|_{L^\infty(\Omega)} \le CN^{1-(m+s)}U_\theta^{m,s}, \\
&\|u-\pi_N^C u\|_{L^2_\omega(\Omega)} \le CN^{\frac{1}{2}-(m+s)}U_\theta^{m,s},
\end{aligned} \tag{4.48}
$$

where C is a positive constant independent of n, N, and u.

4.4. **Applications to functions with interior singularities.** In what follows, we apply the main results to two typical types of singular functions, and provide numerical illustrations of the optimal convergence order.

- Type-I: Consider

$$u(x) = |x-\theta|^\alpha, \quad \alpha > -1/2, \quad x,\theta \in (-1,1), \tag{4.49}$$

where α is not an even integer.

- Type-II: Consider

$$u(x) = |x-\theta|^\alpha \ln|x-\theta|, \quad \alpha > -1/2, \quad x,\theta \in (-1,1). \tag{4.50}$$

4.4.1. Type-I *singularity in* (4.49).

Theorem 4.3. *Given the function in* (4.49) (*where α is not an even integer*), *we have that* (i) *if α is an odd integer, then $u \in \mathbb{W}^{\alpha+1}(\Omega)$* (*defined in* (4.31)); *and* (ii) *if α is not an integer, then $u \in \mathbb{W}_\theta^{\alpha+1}(\Omega)$* (*defined in* (4.1)).

For all real $\alpha > -1/2$ and each $n \ge \alpha+1$, the Chebyshev expansion coefficient can be expressed as

$$\hat{u}_n^C = \frac{\Gamma(\alpha+1)}{2^\alpha \Gamma(\alpha+3/2)\sqrt{\pi}} \big\{ {}^r G_{n-\alpha-1}^{(\alpha+1)}(\theta) - (-1)^{n+[n-\alpha]}\, {}^l G_{n-\alpha-1}^{(\alpha+1)}(\theta) \big\} \omega_{\alpha+1}(\theta), \tag{4.51}$$

and we have

(a) *for $-1/2 < \alpha < 0$,*

$$
\begin{aligned}
|\hat{u}_n^C| \le & \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}(1-\theta^2)^{\alpha/2} \\
& \times \max\left\{ \frac{\Gamma((n-\alpha)/2)}{\Gamma((n+\alpha)/2+1)}, \frac{2}{\sqrt{n^2-\alpha(\alpha+1)}} \frac{\Gamma((n-\alpha+1)/2)}{\Gamma((n+\alpha+1)/2)} \right\};
\end{aligned} \tag{4.52}
$$

(b) *for $\alpha > 0$,*

$$|\hat{u}_n^C| \le \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi} \frac{\Gamma((n-\alpha)/2)}{\Gamma((n+\alpha)/2+1)}. \tag{4.53}$$

Proof. We first identify the space that can optimally characterise the regularity of u, and derive the formula (4.51).

(i) If α is an odd integer, we find

$$
\begin{aligned}
&u^{(k)} = d_\alpha^k\, |x-\theta|^{\alpha-k}\, (\mathrm{sgn}(x-\theta))^k \in \mathrm{AC}(\bar{\Omega}), \quad 0 \le k \le \alpha-1; \\
&u^{(\alpha)} = d_\alpha^\alpha\, (2H(x-\theta)-1) \in \mathrm{BV}(\bar{\Omega}),
\end{aligned} \tag{4.54}
$$

where $\mathrm{sign}(z), H(z), \delta(z)$ are the sign, Heaviside, and Dirac Delta functions, respectively, and

$$d_\alpha^k := \alpha(\alpha-1)\cdots(\alpha-k+1) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}. \tag{4.55}$$

From (4.31), we claim that $u \in \mathbb{W}^{\alpha+1}(\Omega)$.

Moreover, by (2.22a), (5.8) (with $m=\alpha$), (4.16), and (4.54),

$$\begin{aligned}\hat{u}_n^C &= \frac{1}{(2\alpha+1)!!}\frac{2}{\pi}\int_{-1}^{1} G_{n-\alpha-1}^{(\alpha+1)}(x)\omega_{\alpha+1}(x)\,d\{u^{(\alpha)}(x)\}\\ &= \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\Gamma(\alpha+3/2)\sqrt{\pi}}G_{n-\alpha-1}^{(\alpha+1)}(\theta)\omega_{\alpha+1}(\theta),\end{aligned}$$

which is identical to (4.51) with α being an odd integer.

(ii) Let $m=[\alpha]+1$ and $s=\{\alpha+1\}\in(0,1)$. Note that if $-1/2<\alpha<0$, we have $m=0$ and $s=\alpha+1$. If $\alpha>0$, then $m=[\alpha]+1$ and $s=\{\alpha\}$, so like (4.54), we can show $u,\cdots,u^{(m-1)}\in \mathrm{AC}(\bar{\Omega})$. If α is not an even integer, we infer from (3.9) and direct calculations that for $x\in(-1,\theta)$,

$$\begin{aligned}I_{\theta-}^{1-s}u^{(m)} &= (-1)^m\, d_\alpha^m\, I_{\theta-}^{m-\alpha}(\theta-x)^{\alpha-m} = (-1)^m\, d_\alpha^m\,\Gamma(s)\\ &= (-1)^{[\alpha]+1}\,\Gamma(\alpha+1),\end{aligned}\tag{4.56}$$

while for $x\in(\theta,1)$,

$$I_{\theta+}^{1-s}u^{(m)} = d_\alpha^m\, I_{\theta+}^{m-\alpha}(x-\theta)^{\alpha-m} = d_\alpha^m\,\Gamma(s) = \Gamma(\alpha+1).\tag{4.57}$$

Therefore, by the definition (4.1), we have $u\in\mathbb{W}_\theta^{\alpha+1}(\Omega)$.

It is clear that by (4.56)-(4.57),

$$\mathcal{D}_{\theta-}^s\, u^{(m)}(x) = \mathcal{D}_{\theta+}^s\, u^{(m)}(x) = 0,$$

so we can derive the exact formula (4.51) from (4.11) straightforwardly.

(a) For $-1/2<\alpha<0$, taking $s=\alpha+1$ in (4.12), leads to

$$\begin{aligned}|\hat{u}_n^C| \le& \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}(1-\theta^2)^{\alpha/2}\\ &\times\max\left\{\frac{\Gamma((n-\alpha)/2)}{\Gamma((n+\alpha)/2+1)},\ \frac{2}{\sqrt{n^2-\alpha(\alpha+1)}}\frac{\Gamma((n-\alpha+1)/2)}{\Gamma((n+\alpha+1)/2)}\right\},\end{aligned}$$

where we used the fact

$$U_{0,\alpha+1} = 2(1-\theta^2)^{\alpha/2}\Gamma(\alpha+1).$$

(b) Similarly, we can obtain (4.53) directly from (4.13). ☐

Remark 4.4. As a special case of (4.51) with $\theta=0$, we obtain from (2.22b) and (2.40) that the Chebyshev expansion coefficients of $|x|^\alpha$ have the exact representation for each integer $n\ge 0$,

$$\hat{u}_n^C = \big((-1)^n+1\big)\frac{\Gamma(\alpha+1)\Gamma((n-\alpha)/2)}{2^\alpha\pi\Gamma((n+\alpha)/2+1)}\sin\Big(\frac{(n-\alpha)\pi}{2}\Big),\tag{4.58}$$

which implies that for integer $k\ge 0$,

$$\hat{u}_{2k+1}^C = 0,\quad \hat{u}_{2k}^C = (-1)^k\sin\frac{\alpha\pi}{2}\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}\frac{\Gamma(k-\alpha/2)}{\Gamma(k+\alpha/2+1)}.\tag{4.59}$$

It is noteworthy that the following asymptotic estimate for large k was obtained in [30, Sec. 3.11]:

$$\hat{u}_{2k}^C \simeq (-1)^k\sin\frac{\alpha\pi}{2}\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}k^{-\alpha-1},\tag{4.60}$$

but by different means. Indeed, our approach leads to exact representations for all n.

Note that we can directly apply Theorem 4.2 (also see Remark 4.3) to bound the errors of the Chebyshev expansion of the above type of singular functions. For example, if α is not an integer, we know $u \in \mathbb{W}_\theta^{\alpha+1}(\Omega)$, so we have

$$\|u - \pi_N^C u\|_{L^\infty(\Omega)} \le CN^{-\alpha}, \quad \|u - \pi_N^C u\|_{L^2_\omega(\Omega)} \le CN^{-\alpha-1/2}. \tag{4.61}$$

We tabulate in Table 4.1 the errors and convergence order of Chebyshev approximations to $u(x) = |x-\theta|^\alpha$ with various α and $\theta = 0, 1/2$.

TABLE 4.1. Convergence order of $u = |x-\theta|^\alpha$ with $\theta = 0, 1/2$.

N	$u=\lvert x\rvert^\alpha$ (error in L^∞-norm)					
	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.6$	order
2^5	6.68e-1	–	8.37e-3	–	8.32e-5	–
2^6	6.24e-1	0.10	3.71e-3	1.17	1.42e-5	2.55
2^7	5.83e-1	0.10	1.63e-3	1.19	2.40e-6	2.57
2^8	5.44e-1	0.10	7.13e-4	1.19	3.99e-7	2.59
2^9	5.08e-1	0.10	3.11e-4	1.20	6.62e-8	2.59
2^{10}	4.74e-1	0.10	1.36e-4	1.20	1.09e-8	2.60
N	$u=\lvert x-1/2\rvert^\alpha$ (error in L^∞-norm)					
	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.6$	order
2^5	6.60e-1	–	7.31e-3	–	6.18e-5	–
2^6	6.16e-1	0.10	3.15e-3	1.21	1.00e-5	2.63
2^7	5.75e-1	0.10	1.38e-3	1.19	1.68e-6	2.57
2^8	5.36e-1	0.10	6.01e-4	1.20	2.76e-7	2.61
2^9	5.00e-1	0.10	2.62e-4	1.20	4.58e-8	2.59
2^{10}	4.67e-1	0.10	1.14e-4	1.20	7.54e-9	2.60
N	$u=\lvert x\rvert^\alpha$ (error in L^2_ω-norm)					
	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.6$	order
2^5	1.88e-2	–	1.68e-3	–	2.68e-5	–
2^6	1.25e-2	0.59	5.31e-4	1.66	3.27e-6	3.03
2^7	8.29e-3	0.59	1.66e-4	1.68	3.91e-7	3.07
2^8	5.48e-3	0.60	5.13e-5	1.69	4.61e-8	3.08
2^9	3.61e-3	0.60	1.58e-5	1.70	5.41e-9	3.09
2^{10}	2.37e-3	0.61	4.88e-6	1.70	6.33e-10	3.10
N	$u=\lvert x-1/2\rvert^\alpha$ (error in L^2_ω-norm)					
	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.6$	order
2^5	1.89e-2	–	1.49e-3	–	2.02e-5	–
2^6	1.24e-2	0.61	4.53e-4	1.72	2.31e-6	3.13
2^7	8.22e-3	0.59	1.41e-4	1.68	2.75e-7	3.07
2^8	5.42e-3	0.60	4.33e-5	1.70	3.19e-8	3.11
2^9	3.58e-3	0.60	1.34e-5	1.70	3.74e-9	3.09
2^{10}	2.36e-3	0.60	4.11e-6	1.70	4.36e-10	3.10

4.4.2. TYPE-II *singularity in* (4.50). We first present the following useful formulas.

Lemma 4.2. *For real* $\eta > -1, s \geq 0$ *and* $x > a$,

$$\begin{aligned} &I^s_{a+}\{(x-a)^\eta \ln(x-a)\} \\ &= \frac{\Gamma(\eta+1)}{\Gamma(\eta+s+1)}\big\{ \ln(x-a) + \psi(\eta+1) - \psi(\eta+s+1)\big\}(x-a)^{\eta+s}, \end{aligned} \tag{4.62}$$

and the same formula holds for $I^s_{b-}\{(b-x)^\eta \ln(b-x)\}$ (*for* $x < b$) *with* $b-x$ *in place of* $x-a$. *Here,*

$$\ln z - \frac{1}{2z} < \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} < \ln z - \frac{1}{z}, \quad z > 0. \tag{4.63}$$

Proof. The formula (4.62) is a direct consequence of [34, (2.50)]. The property of the ψ-function in (4.63) can be found in [2, (2.2)]. Note that we can derive the formula for $I^s_{b-}\{(b-x)^\eta \ln(b-x)\}$ in the same manner. □

Theorem 4.4. *For any* $\alpha \geq 0$ *and* $\theta \in (-1,1)$, *we have*

$$u(x) = |x-\theta|^\alpha \ln|x-\theta| \in \mathbb{W}^{\alpha+1-\epsilon}_\theta(\Omega) \quad \forall \epsilon \in (0,1). \tag{4.64}$$

Moreover, we have the following uniform bound of the Chebyshev expansion coefficients:

$$|\hat{u}^C_n| \leq \frac{U^{[\sigma],\{\sigma\}}_\theta}{2^{\sigma-1}\pi}\frac{\Gamma((n-\sigma-1)/2)}{\Gamma((n+\sigma+1)/2)}, \tag{4.65}$$

where $\sigma := \alpha + 1 - \epsilon$, *and* $|\hat{u}^C_n| \leq Cn^{-\sigma}$ *for large* n.

If $\theta = 0$, *then we have* $\hat{u}^C_{2k+1} = 0$, *and the exact formula:*

$$\begin{aligned} \hat{u}^C_{2k} =& \frac{\Gamma(\alpha+1)}{2^{\alpha-2}\pi}\frac{\Gamma(k-\alpha/2)}{\Gamma(k+\alpha/2+1)}\Big\{\pi\cos\frac{\alpha\pi}{2} + \sin\frac{\alpha\pi}{2}\big(2\psi(\alpha+1) \\ &- 2\ln 2 - \psi(k-\alpha/2) - \psi(k+\alpha/2+1)\big)\Big\} \quad \forall\, k \in \mathbb{N}_0, \end{aligned} \tag{4.66}$$

which enjoys the asymptotic behaviour

$$\begin{aligned} \hat{u}^C_{2k} =& \frac{\Gamma(\alpha+1)}{2^{\alpha-3}\pi}k^{-\alpha-1}\Big\{\frac{\pi}{2}\cos\frac{\alpha\pi}{2} + \sin\frac{\alpha\pi}{2}\big(\psi(\alpha+1) - \ln 2 - \ln k\big)\Big\} \\ &+ O(k^{-\alpha-3}\ln k)\,\sin\frac{\alpha\pi}{2} + O(k^{-\alpha-3}), \quad k \gg 1. \end{aligned} \tag{4.67}$$

Proof. Let $m = [\alpha]+1$ and $\nu = \alpha - m$. We derive from a direct calculation that

$$u^{(k)}(x) = (\mathrm{sgn}(x-\theta))^k |x-\theta|^{\alpha-k}\big(d^k_\alpha \ln|x-\theta| + f^k_\alpha\big), \quad k \geq 0, \tag{4.68}$$

where d^k_α is the same as in (4.55), and

$$f^k_\alpha := \sum_{j=1}^k \frac{(-1)^{j-1}\Gamma(k+1)\Gamma(\alpha+1)}{j\Gamma(k-j+1)\Gamma(\alpha-k+j+1)}.$$

We see that $u \in L^1(\Omega)$ and $u, \cdots, u^{(m-1)} \in \mathrm{AC}(\bar{\Omega})$. Next, using Lemma 4.2, we obtain that for $x \in (\theta, 1)$,

$$\begin{aligned} I_{\theta+}^{1-s} u^{(m)} =& d_\alpha^m I_{\theta+}^{1-s}\big\{(x-\theta)^{\alpha-m}\ln(x-\theta)\big\} + f_\alpha^m I_{\theta+}^{1-s}\big\{(x-\theta)^{\alpha-m}\big\} \\ =& d_\alpha^m \frac{\Gamma(\nu+1)}{\Gamma(\nu+2-s)} \ln(x-\theta)(x-\theta)^{\nu+1-s} \\ &+ \frac{\Gamma(\nu+1)}{\Gamma(\nu+2-s)}\big(\psi(\nu+1) - \psi(\nu+2-s) + f_\alpha^m\big)(x-\theta)^{\nu+1-s}. \end{aligned}$$

Thus, if $\nu + 1 - s > 0$, i.e., $s < \alpha + 1 - m$, then $I_{\theta+}^{1-s} u^{(m)} \in \mathrm{BV}(\bar{\Omega}_\theta^+)$. Similarly, under the same condition, we have $I_{\theta-}^{1-s} u^{(m)} \in \mathrm{BV}(\bar{\Omega}_\theta^-)$. By the definition (4.1), we obtain $u \in \mathbb{W}_\theta^\mu(\Omega)$, where $\mu = m + s < \alpha + 1$. This implies (4.64). The bound in (4.65) follows from (4.13) straightforwardly.

If $\theta = 0$, then $u(x)$ is an even function, so $\hat{u}_{2k+1}^C = 0$. It is known that

$$\ln z = \lim_{\varepsilon\to 0} \frac{z^\varepsilon - 1}{\varepsilon}, \quad z > 0. \tag{4.69}$$

Using (4.69), we derive from (4.59) that

$$\begin{aligned} \hat{u}_{2k}^C &= \frac{2}{\pi}\int_{-1}^{1}\Big\{\lim_{\varepsilon\to 0}\frac{|x|^{\varepsilon+\alpha} - |x|^\alpha}{\varepsilon}\Big\}\frac{T_{2k}(x)}{\sqrt{1-x^2}}dx \\ &= (-1)^k \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\sin\frac{(\varepsilon+\alpha)\pi}{2}\frac{\Gamma(\varepsilon+\alpha+1)}{2^{\varepsilon+\alpha-1}\pi}\frac{\Gamma(k-(\varepsilon+\alpha)/2)}{\Gamma(k+(\varepsilon+\alpha)/2+1)} \\ &\quad - \sin\frac{\alpha\pi}{2}\frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}\frac{\Gamma(k-\alpha/2)}{\Gamma(k+\alpha/2+1)}\Big\}. \end{aligned} \tag{4.70}$$

Noting that

$$\begin{aligned} &\frac{d}{d\varepsilon}\Big\{\sin\frac{(\varepsilon+\alpha)\pi}{2}\frac{\Gamma(\varepsilon+\alpha+1)}{2^\varepsilon}\frac{\Gamma(k-(\varepsilon+\alpha)/2)}{\Gamma(k+(\varepsilon+\alpha)/2+1)}\Big\} \\ &=\frac{\Gamma(\varepsilon+\alpha+1)}{2^\varepsilon}\frac{\Gamma(k-(\varepsilon+\alpha)/2)}{\Gamma(k+(\varepsilon+\alpha)/2+1)}\Big\{\frac{\pi}{2}\cos\frac{(\varepsilon+\alpha)\pi}{2} + \sin\frac{(\varepsilon+\alpha)\pi}{2} \\ &\quad\times\big(\psi(\varepsilon+\alpha+1) - \ln 2 - \psi(k-(\varepsilon+\alpha)/2)/2 - \psi(k+(\varepsilon+\alpha)/2+1)/2\big)\Big\}, \end{aligned}$$

we obtain (4.66) from (4.70) and l'Hôpital's rule immediately.

Taking $z = k - \alpha/2$ in (4.63), we obtain

$$\ln(k-\alpha/2) - \frac{1}{2k-\alpha} < \psi(k-\alpha/2) < \ln(k-\alpha/2) - \frac{1}{k-\alpha/2},$$

which implies that for $k \gg 1$,

$$\psi(k-\alpha/2) = \ln k + O(k^{-1}); \quad \psi(k+\alpha/2+1) = \ln k + O(k^{-1}). \tag{4.71}$$

Using (4.47) leads to

$$\frac{\Gamma(k-\alpha/2)}{\Gamma(k+\alpha/2+1)} = k^{-\alpha-1}\big(1 + O(k^{-2})\big), \quad k \gg 1. \tag{4.72}$$

From (4.66) and (4.71)-(4.72), we obtain (4.67). $\square$

Remark 4.5. Consider the Chebyshev expansion of $u = |x|^\alpha \ln|x|$, we observe from (4.67) that for $n \gg 1$, $|\hat{u}_n^C| \le C(\ln n) n^{-(\alpha+1)}$. Therefore, we obtain directly the optimal estimates:

$$\begin{aligned} \|u - \pi_N^C u\|_{L^\infty(\Omega)} &\le \sum_{n=N+1}^{\infty} |\hat{u}_n^C| \le C(\ln N) N^{-\alpha}; \\ \|u - \pi_N^C u\|_{L^2_\omega(\Omega)} &\le C(\ln N) N^{-\alpha-1/2}. \end{aligned} \tag{4.73}$$

However, we find from (4.64) that the space $\mathbb{W}_\theta^{\alpha+1-\epsilon}(\Omega)$ is suboptimal to characterise this type of singularity. Indeed, by Theorem 4.2, we only have $\|u - \pi_N^C u\|_{L^\infty(\Omega)} = O(N^{\epsilon-\alpha})$ and $\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} = O(N^{\epsilon-\alpha-1/2})$. The situation is reminiscent of the Besov framework in [5], where the spaces of Type-I and Type-II are defined through different space interpolation. The question of how to modify the fractional space to best characterise Type-II singularity in our setting appears non-trivial and is still open.

5. Improving existing results

In this section, we estimate the Chebyshev approximation of functions in the space $\mathbb{W}^{m+1}(\Omega)$ defined in (4.31), and derive sharper bounds than the existing results (see, e.g., [27, 38, 39, 42]).

5.1. **Existing estimates.**

As in [38], let $\|\cdot\|_T$ be the Chebyshev-weighted 1-norm:

$$\|u\|_T = \Big\| \frac{u'(x)}{\sqrt{1-x^2}} \Big\|_1, \tag{5.1}$$

which is defined via a Riemann-Stieltjes integral for any u of bounded variation.

Lemma 5.1 (See [38, Thms 4.2-4.3])**.** *If $u, u', \cdots, u^{(m-1)}$ are absolutely continuous on $[-1,1]$, and if $\|u^{(m)}\|_T = V_T < \infty$ with integer $m \ge 0$, then for each $n \ge m+1$,*

$$|\hat{u}_n^C| \le \frac{2\,V_T}{\pi n(n-1)\cdots(n-m)}, \tag{5.2}$$

and for integer $m \ge 1$, and integer $N \ge m+1$,

$$\big\|u - \pi_N^C u\big\|_{L^\infty(\Omega)} \le \frac{2\,V_T}{\pi m\,(N-m)^m}. \tag{5.3}$$

We remark that the Chebyshev weight is removed in Trefethen [39, Thms 7.1-7.2], i.e., V_T is replaced by the total variation of $u^{(m)}$.

Following the argument of summation by certain telescoping series in [42], Majidian (cf. [27, Thm 2.1]) derived sharper bounds. For comparison, we quote the estimates therein below.

Lemma 5.2 (See [27, Thm 2.1])**.** *If $u, u', \cdots, u^{(m-1)}$ are absolutely continuous on $[-1,1]$, and if $\|u^{(m)}\|_T = V_T < \infty$ with integer $m \ge 0$, then for each $n \ge m+1$,*

$$|\hat{u}_n^C| \le \frac{2\,V_T}{\pi} \prod_{j=0}^{m} \frac{1}{n-m+2j}\,. \tag{5.4}$$

5.2. **Improved estimates.**

Theorem 5.1. *Suppose that for integer $m \geq 0$, $u, u', \cdots, u^{(m-1)}$ are absolutely continuous on $[-1,1]$, and $u^{(m)}$ is of bounded variation with the total variation denoted by $V_L^{(m)}$.*

(i) *If $n \geq m+1$ and $n-m$ is odd, then*

$$\big|\hat{u}_n^C\big| \leq \frac{2\,V_L^{(m)}}{\pi} \prod_{j=0}^{m} \frac{1}{n-m+2j}\,. \tag{5.5}$$

(ii) *If $n \geq m+1$ and $n-m$ is even, then*

$$\big|\hat{u}_n^C\big| \leq \frac{2\,V_L^{(m)}}{\pi\sqrt{n^2-m^2}} \prod_{j=0}^{m-1} \frac{1}{n-m+2j-1}\,. \tag{5.6}$$

(iii) *If $0 \leq n \leq m+1$, then*

$$|\hat{u}_n^C| \leq \frac{2\,V_L^{(n)}}{\pi(2n-1)!!}\,. \tag{5.7}$$

Proof. We find from (4.11) (or (4.15) with one more step of integration by parts) that for $n \geq m+1$,

$$\hat{u}_n^C = \frac{1}{(2m+1)!!}\frac{2}{\pi}\int_{-1}^{1} G_{n-m-1}^{(m+1)}(x)\omega_{m+1}(x)\,d\{u^{(m)}(x)\}. \tag{5.8}$$

Thus, by (3.2) and (5.8),

$$\big|\hat{u}_n^C\big| \leq \frac{V_L^{(m)}}{(2m+1)!!}\frac{2}{\pi}\max_{|x|\leq 1}\big\{\omega_{m+1}(x)|G_{n-m-1}^{(m+1)}(x)|\big\}. \tag{5.9}$$

If $n = m+2p+1$ with $p \in \mathbb{N}_0$, we derive from (2.42a) with $l = p$ and $\lambda = m+1$ that

$$\begin{aligned}\max_{|x|\leq 1}\big\{\omega_{m+1}(x)|G_{n-m-1}^{(m+1)}(x)|\big\} &\leq \frac{\Gamma(m+3/2)\Gamma(p+1/2)}{\sqrt{\pi}\,\Gamma(m+p+3/2)}\\ &= \frac{(2m+1)!!\,(2p-1)!!}{(2m+2p+1)!!}.\end{aligned} \tag{5.10}$$

Consequently, for $n = m+2p+1$ with $p \in \mathbb{N}_n$, we obtain from (5.9)-(5.10) that

$$\begin{aligned}\big|\hat{u}_n^C\big| \leq \frac{2}{\pi}\frac{(2p-1)!!\,V_L^{(m)}}{(2p+2m+1)!!} &= \frac{2}{\pi}\frac{V_L^{(m)}}{(2p+1)\cdot(2p+3)\cdots(2p+2m+1)}\\ = \frac{2}{\pi}\frac{V_L^{(m)}}{(n-m)\cdot(n-m+2)\cdots(n+m)},&\end{aligned} \tag{5.11}$$

which implies (5.5).

Similarly, if $n = m+2p+2$ with $p \in \mathbb{N}_0$, we derive from (2.42b) with $l = p$ and $\lambda = m+1$ that

(5.12)

$$\max_{|x|\leq 1}\big\{\omega_{m+1}(x)|G_{n-m-1}^{(m+1)}(x)|\big\} \leq \frac{1}{\sqrt{(2p+2)(2m+2p+1)}}\frac{(2m+1)!!\,(2p+1)!!}{(2m+2p+1)!!},$$

so by (5.9), we have

$$
\begin{aligned}
|\hat{u}_n^C| &\le \frac{2}{\pi}\frac{1}{\sqrt{(2p+2)(2m+2p+1)}}\frac{V_L}{(2p+3)\cdot(2p+5)\cdots(2p+2m+1)} \\
&= \frac{2}{\pi}\frac{1}{\sqrt{n^2-m^2}}\frac{V_L}{(n-m+1)\cdot(n-m+3)\cdots(n+m-1)}.
\end{aligned} \tag{5.13}
$$

This leads to (5.6).

In case of $0 \le n \le m+1$, we derive from (5.8) (with $n = m+1$) and the factor $G_0^{(n)}(x) \equiv 1$ that

$$
\hat{u}_n^C = \frac{1}{(2n-1)!!}\frac{2}{\pi}\int_{-1}^{1}\omega_n(x)\, d\{u^{(n-1)}(x)\}. \tag{5.14}
$$

Then we obtain (5.7) immediately. □

Next, we unify the bounds in (i)-(ii) of Theorem 5.1 without loss of the rate of convergence. In fact, this relaxation leads to the estimate (5.4) in [27, Thm 2.1], but with $V_L^{(m)}$ in place of V_T. In other words, the bounds in Theorem 5.1 indeed improve the best available results.

Corollary 5.1. *Under the same conditions as in Theorem* 5.1*, we have that for all* $n \ge m+1$,

$$
|\hat{u}_n^C| \le \frac{2\,V_L^{(m)}}{\pi}\prod_{j=0}^{m}\frac{1}{n-m+2j}. \tag{5.15}
$$

Proof. It is evident that by (5.5)-(5.6), we only need to prove this bound for $n-m$ being even. One verifies readily the fundamental inequality:

$$
n^2-(p-1)^2 \ge \sqrt{(n^2-p^2)(n^2-(p-2)^2)} \quad \text{for} \quad 2 \le p \le n.
$$

If m is even, we can pair up the factors and use the above inequality with $p = m, m-2, \cdots, 2$ to derive

$$
\begin{aligned}
(n-&m+1)(n-m+3)\cdots(n-1)(n+1)\cdots(n+m-3)(n+m-1) \\
&= (n-(m-1)^2)(n^2-(m-3)^2)\cdots(n^2-1) \\
&\ge \sqrt{n^2-m^2}\,\sqrt{n^2-(m-2)^2}\,\sqrt{n^2-(m-2)^2}\,\sqrt{n^2-(m-4)^2}\cdots \\
&= \sqrt{n^2-m^2}\,(n-m+2)\cdots(n+m-2).
\end{aligned} \tag{5.16}
$$

Similarly, if m is odd, we remain the middle most factor intact and pair up the factors to derive the above. Therefore, multiplying both sides of (5.16) by $\sqrt{n^2-m^2}$, we obtain

$$
\frac{1}{\sqrt{n^2-m^2}}\prod_{j=0}^{m-1}\frac{1}{n-m+2j-1} \le \prod_{j=0}^{m}\frac{1}{n-m+2j}. \tag{5.17}
$$

Then (5.15) follows from (5.17) and (i)-(ii) of Theorem 5.1 directly. □

To show the sharpness of our improved bounds, we consider $u = |x-\theta|$, $\theta \in (-1,1)$ to compare upper bounds of $\hat{u}_n^C$. In this case, we have $m = 1$, $u'' = 2\delta(x-\theta)$, $V_L^{(1)} = 2$, and $V_T = 2(1-\theta^2)^{-1/2}$. Let Ratio_1 and Ratio_2 be the ratios of upper bounds in [27, 39] (cf. (5.2) with V_T being replaced by the bounded variation of u', and the bound in (5.4)) and our improved bound in Theorem 5.1, respectively.

In Figure 5.1, we depict two ratios against various n for two values of θ. We see that the improved bound is sharper than the existing ones, and the removal of the Chebyshev weight in V_T is also significant for the sharpness of the bounds.

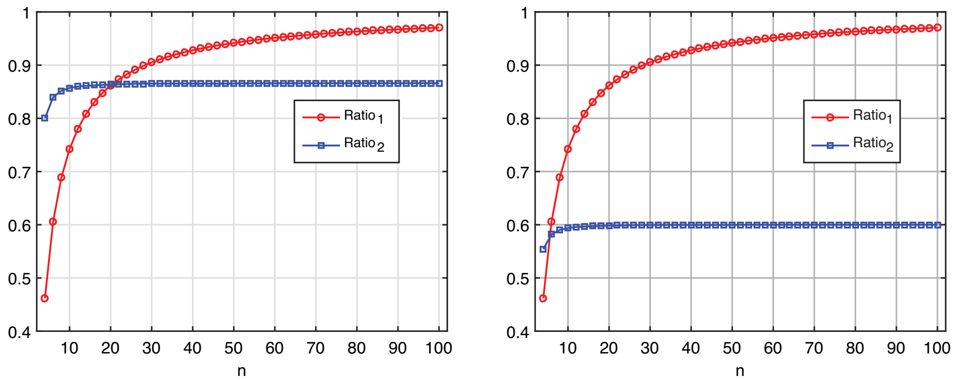

FIGURE 5.1. Ratios of the existing bounds and improved bound herein for $u = |x - \theta|$ and $\theta \in (-1, 1)$. Left: $\theta = 1/2$. Right: $\theta = 4/5$.

To conclude this section, we state below the improved L^∞-estimates, and remark on the improvements in Remark 5.1 below.

Theorem 5.2. *Let $u \in \mathbb{W}^{m+1}(\Omega)$ with integer $m \geq 0$, and denote by $V_L^{(m)}$ the total variation of $u^{(m)}(x)$.*

(i) *If $1 \leq m \leq N$, then*

$$\|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq \frac{2}{m\pi}\left(\prod_{j=1}^{m} \frac{1}{N - m + 2j - 1}\right) V_L^{(m)}. \tag{5.18}$$

(ii) *If $m = 0$, then for all integer $N \geq 1$,*

$$\big\|u - \pi_N^C u\big\|_{L^\infty(\Omega)} \leq V_L^{(0)}. \tag{5.19}$$

(iii) *If $m \geq N + 1$, then*

$$\|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq \frac{2}{(2N+1)!!\,\pi} \sum_{n=N}^{m} c_n \frac{(2N+1)!!}{(2n+1)!!} V_L^{(n)}, \tag{5.20}$$

where $c_n = 1$ for all $N \leq n \leq m - 1$ and $c_m = 2$.

Proof. From Theorem 4.2 with $s \to 1$ and (4.16), we obtain that for $1 \leq m \leq N+1$,

$$\begin{aligned} \|u - \pi_N^C u\|_{L^\infty(\Omega)} &\leq \frac{1}{2^{m-1} m\pi} \frac{\Gamma((N - m + 1)/2)}{\Gamma((N + m + 1)/2)} V_L^{(m)} \\ &= \frac{2}{m\pi} \frac{(N - m - 1)!!}{(N + m + 1)!!} V_L^{(m)} = \frac{2}{m\pi}\left(\prod_{j=1}^{m} \frac{1}{N - m + 2j - 1}\right) V_L^{(m)}. \end{aligned} \tag{5.21}$$

This gives (5.18). We now prove (5.19). Using integration by parts leads to

$$
\begin{aligned}
(u-\pi_N^C u)(x) &= \sum_{n=N+1}^{\infty} \hat{u}_n^C T_n(x) \\
&= \sum_{n=N+1}^{\infty} \Big(\int_0^{\pi} u(\cos\varphi)\cos(n\varphi)d\varphi \Big) \cos(n\theta) \\
&= \frac{2}{\pi} \sum_{n=N+1}^{\infty} \Big(\int_0^{\pi} \sin(n\varphi) d\{u(\cos\varphi)\} \Big) \frac{\cos(n\theta)}{n} \\
&= \frac{2}{\pi} \int_0^{\pi} \Psi_N^{\infty}(\varphi,\theta)\, d\{u(\cos\varphi)\},
\end{aligned} \tag{5.22}
$$

where

$$
\Psi_N^{\infty}(\varphi,\theta) = \sum_{n=N+1}^{\infty} \frac{\sin(n\varphi)\cos(n\theta)}{n} = \sum_{n=N+1}^{\infty} \frac{\sin(n(\varphi+\theta)) + \sin(n(\varphi-\theta))}{2n}. \tag{5.23}
$$

Thus, we have from (3.2) that

$$
\big|(u-\pi_N^C u)(x)\big| \le \frac{2}{\pi} \max_{\varphi\in[0,\pi]} \big|\Psi_N^{\infty}(\varphi,\theta)\big|\, V_L^{(0)}, \quad \theta\in[0,\pi]. \tag{5.24}
$$

We next show that for $\vartheta \in \mathbb{R}$,

$$
\Big| \sum_{n=N+1}^{\infty} \frac{\sin(n\vartheta)}{n} \Big| \le \frac{\pi}{2}. \tag{5.25}
$$

In fact, it suffices to derive this bound for $\vartheta \in (0,\pi)$, as the series defines an odd, 2π-periodic function which vanishes at $\vartheta = 0, \pi$. It is known that

$$
\sum_{n=1}^{\infty} \frac{\sin(n\vartheta)}{n} = \frac{\pi-\vartheta}{2}, \quad \vartheta\in(0,\pi). \tag{5.26}
$$

According to [3], we have that for $N \ge 2$,

$$
0 < \sum_{n=1}^{N} \frac{\sin(n\vartheta)}{n} \le \alpha(\pi-\vartheta), \quad \vartheta\in(0,\pi), \tag{5.27}
$$

with the best possible constant $\alpha = 0.66395\cdots$. Then by (5.26)-(5.27),

$$
\Big(\frac{1}{2}-\alpha\Big)(\pi-\vartheta) \le \sum_{n=N+1}^{\infty} \frac{\sin(n\vartheta)}{n} < \frac{\pi-\vartheta}{2}; \quad \Big| \sum_{n=N+1}^{\infty} \frac{\sin(n\vartheta)}{n} \Big| < \frac{\pi-\vartheta}{2} \tag{5.28}
$$

for $N \ge 2$ and $\vartheta \in (0,\pi)$. In fact, the bound (5.28) also holds for $N = 1$, since by (5.26),

$$
\sum_{n=2}^{\infty} \frac{\sin(n\vartheta)}{n} = \frac{\pi-\vartheta}{2} - \sin\vartheta < \frac{\pi-\vartheta}{2}.
$$

Hence, we complete the proof of (5.25). The estimate (5.19) is a direct consequence of (5.24)-(5.25).

Finally, we turn to the proof of the estimate (5.20). For $m \ge N+1$, we use (5.7) to bound $\{\hat{u}_n^C\}_{n=N+1}^m$, and use (5.18) (with $N \to m$) to derive

$$\begin{aligned}
\big|u(x) - \pi_N^C u(x)\big| &\le \big|\pi_m^C u(x) - \pi_N^C u(x)\big| + \big|u(x) - \pi_m^C u(x)\big| \\
&\le \sum_{n=N+1}^{m} \frac{2}{\pi(2n-1)!!} V_L^{(n-1)} + \frac{2}{m(2m-1)!!\pi} V_L^{(m)} \\
&\le \frac{2}{\pi(2N+1)!!}\bigg\{ \sum_{n=N+1}^{m} \frac{(2N+1)!!}{(2n-1)!!} V_L^{(n-1)} + \frac{2m+1}{m}\frac{(2N+1)!!}{(2m+1)!!} V_L^{(m)} \bigg\} \qquad (5.29) \\
&\le \frac{2}{\pi(2N+1)!!} \sum_{n=N}^{m} \frac{c_n(2N+1)!!}{(2n+1)!!} V_L^{(n)},
\end{aligned}$$

where $c_n = 1$ for all $N \le n \le m-1$ and $c_m = 2$. □

Remark 5.1. Taking a different route, we improve the existing bounds in the following senses:

(i) The Chebyshev-weighted 1-norm in Lemma 5.2 is replaced by the Legendre-weighted 1-norm.
(ii) Sharper bound is obtained than the best one in [27, Thm 2.1].
(iii) We obtain the "stability" result, that is, $m = 0$ in (5.3), and the estimate for the case $n \le m+1$ in (5.7), which are new.

6. Analysis of interpolation, quadrature, and endpoint singularities

In this section, we discuss the extension of our main results to error estimates of the related interpolation, quadratures, and also special types of functions with endpoint singularities. We then conclude the paper with some final remarks.

6.1. **Analysis of interpolations and quadrature.** As remarked in [27, 39, 42], the error analysis of several widely used interpolations and quadrature boils down to estimating the coefficients $\{\hat{u}_n^C\}$ and their partial sums. We refer to [27] for a list of more than six examples. Here, we just consider two cases and present sharp bounds by using our new estimates on the decay of expansion coefficients.

(i) Interpolation and quadrature at Chebyshev-Gauss (CG) points $\{x_j\}_{j=0}^N$, i.e., zeros of $T_{N+1}(x)$:

$$(\mathcal{I}_N^C u)(x) = {\sum_{n=0}^{N}}{}' b_n T_n(x), \quad b_n = \frac{2}{N+1}\sum_{j=0}^{N} u(x_j) T_n(x_j), \qquad (6.1)$$

and

$$\int_{-1}^{1} u(x)(1-x^2)^{-1/2} dx = \frac{\pi}{N+1}\sum_{j=0}^{N} u(x_j) + \mathcal{R}_N^C[u]. \qquad (6.2)$$

Then we have (cf. [42] and [33, (6)])

$$\|\mathcal{I}_N^C u - u\|_{L^\infty(\Omega)} \le 2 \sum_{n=N+1}^{\infty} |\hat{u}_n^C|; \quad \mathcal{R}_N^C[u] = \pi \sum_{k=1}^{\infty} (-1)^k \hat{u}_{2k(N+1)}^C. \qquad (6.3)$$

(ii) Legendre-Gauss quadrature rule at the zeros $\{x_j\}_{j=0}^N$ of the Legendre polynomial $P_{N+1}(x)$ and with quadrature weights $\{\omega_j\}_{j=0}^N$ (cf. [36, p. 96] and [27]):

$$\int_{-1}^{1} u(x)dx = \sum_{j=0}^{N} u(x_j)\omega_j + \mathcal{R}_N^L[u]. \tag{6.4}$$

Then we have (cf. [27, 38]):

$$\big|\mathcal{R}_N^L[u]\big| \le \frac{32}{15}\sum_{n=N+1}^{\infty} |\hat{u}_{2n}^C|. \tag{6.5}$$

Using Theorem 4.1 and the argument similar to Theorem 4.2 (also see Remark 4.3), we can obtain the following estimates.

Theorem 6.1. *Given $\theta \in (-1,1)$, if $u \in \mathbb{W}_\theta^{m+s}(\Omega)$ with $s \in (0,1)$ and integer $m \ge 0$, then for $m+s>1$, we have*

$$\|u - \mathcal{I}_N^C u\|_{L^\infty(\Omega)} \le CN^{1-m-s}U_\theta^{m,s}; \quad \|u - \mathcal{I}_N^C u\|_{L^2_\omega(\Omega)} \le CN^{\frac12-m-s}U_\theta^{m,s}, \tag{6.6}$$

and

$$|\mathcal{R}_N^C[u]| \le CN^{-(m+s)}U_\theta^{m,s}; \quad |\mathcal{R}_N^L[u]| \le CN^{-(m+s)}U_\theta^{m,s}, \tag{6.7}$$

where C is a positive constant independent of N and u.

Proof. We just provide the proof of the L^2_ω-error of the CG interpolation, since the others can be proved by summing up the bounds of $\{|\hat{u}_n^C|\}$ in Theorem 4.1 and Remark 4.3. Note that

$$\begin{aligned} \mathcal{I}_N^C u(x) - u(x) &= \mathcal{I}_N^C u(x) - \pi_N^C u + \pi_N^C u - u \\ &= \sum_{n=0}^{N}{}' (b_n - \hat{u}_n^C)T_n(x) + \pi_N^C u - u. \end{aligned} \tag{6.8}$$

Hence, we obtain

$$\|u - \mathcal{I}_N^C u\|^2_{L^2_\omega(\Omega)} \le \frac{\pi}{2}\sum_{n=0}^{N}{}' |b_n - \hat{u}_n^C|^2 + \|u - \pi_N^C u\|^2_{L^2_\omega(\Omega)}. \tag{6.9}$$

Recall that (cf. [12, (4.56)]):

$$b_n - \hat{u}_n^C = \sum_{k=1}^{\infty}(-1)^k(\hat{u}^C_{2k(N+1)-n} + \hat{u}^C_{2k(N+1)+n}), \quad n = 0,\cdots,N. \tag{6.10}$$

Using (4.13) and (4.47), we find that for $N \gg 1$, $\sigma = m+s > 1$, and $n = 0,\cdots,N$,

$$\begin{aligned} |b_n - \hat{u}_n^C| &\le \sum_{k=1}^{\infty}\big\{|\hat{u}^C_{2k(N+1)-n}| + |\hat{u}^C_{2k(N+1)+n}|\big\} \\ &\le \frac{U_\theta^{m,s}}{2^{\sigma-1}\pi} 2\sum_{k=1}^{\infty} \frac{\Gamma((2k(N+1)-n-\sigma+1)/2)}{\Gamma((2k(N+1)-n+\sigma+1)/2)} \\ &\le \frac{U_\theta^{m,s}}{2^{\sigma-1}\pi} 2\sum_{k=1}^{\infty} \frac{\Gamma((2k(N+1)-N-\sigma+1)/2)}{\Gamma((2k(N+1)-N+\sigma+1)/2)} \le CN^{-\sigma}U_\theta^{m,s}. \end{aligned} \tag{6.11}$$

By (6.9), we obtain from direct calculations and Remark 4.3 the L^2_ω-estimate. ☐

6.2. **Analysis of endpoint singularities.** The previous discussions were centred around the Chebyshev expansions and approximation of singular functions with interior singularities. In what follows, we extend the results to the cases with $\theta = \pm 1$, and study endpoint singularities. To fix the idea, we shall focus on the exact formulas and decay rate of the Chebyshev expansion coefficients, since it is the basis to derive many other related error bounds.

Let $\mathbb{W}^{m+s}_{1-}(\Omega)$ and $\mathbb{W}^{m+s}_{-1+}(\Omega)$ be the fractional Sobolev-type spaces defined in (4.5). The following representation of $\hat{u}^C_n$ is a direct consequence of Theorem 4.1.

Theorem 6.2. *If $u \in \mathbb{W}^{\sigma}_{1-}(\Omega)$ with $\sigma := m+s$, $s \in (0,1)$ and $m \in \mathbb{N}_0$, then for $n > \sigma > 1/2$,*

$$\begin{aligned}\hat{u}^C_n = -C_\sigma\Big\{ \int_{-1}^{1} {}^{l}G^{(\sigma)}_{n-\sigma}(x)\,\omega_\sigma(x)\,d\big\{I^{1-s}_{1-}u^{(m)}(x)\big\} \\ + \big\{I^{1-s}_{1-}u^{(m)}(x)\,{}^{l}G^{(\sigma)}_{n-\sigma}(x)\,\omega_\sigma(x)\big\}\big|_{x=1-}\Big\}.\end{aligned} \tag{6.12}$$

Similarly, if $u \in \mathbb{W}^{\sigma}_{-1+}(\Omega)$ with $\sigma := m+s$, $s \in (0,1)$ and $m \in \mathbb{N}_0$, then for $n > \sigma > 1/2$,

$$\begin{aligned}\hat{u}^C_n = C_\sigma\Big\{ \int_{-1}^{1} {}^{r}G^{(\sigma)}_{n-\sigma}(x)\,\omega_\sigma(x)\,d\big\{I^{1-s}_{-1+}u^{(m)}(x)\big\} \\ + \big\{I^{1-s}_{-1+}u^{(m)}(x)\,{}^{r}G^{(\sigma)}_{n-\sigma}(x)\,\omega_\sigma(x)\big\}\big|_{x=-1+}\Big\}.\end{aligned} \tag{6.13}$$

Here, $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$ and $C_\sigma := (\sqrt{\pi}\,2^{\sigma-1}\Gamma(\sigma+1/2))^{-1}$.

We next apply the formulas to several typical types of singular functions. We first consider $u(x) = (1+x)^\alpha$ with $\alpha > -1/2$ and $\alpha \notin \mathbb{N}_0$ (see, e.g., [19, 40]). Following the proof of Proposition 4.3, we have $u \in \mathbb{W}^{\alpha+1}_{-1+}(\Omega)$. Then using (6.13), one obtains the exact formula of the Chebyshev expansion coefficient. Equivalently, one can derive it by taking $\theta \to -1+$ in (4.51). More precisely, by (2.22b) and (4.51),

$$\begin{aligned}\hat{u}^C_n &= \frac{\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)\sqrt{\pi}} \lim_{\theta\to -1^+} \big\{{}^{r}G^{(\alpha+1)}_{n-\alpha-1}(\theta)\omega_{\alpha+1}(\theta) \\ &\quad - (-1)^{n+[n-\alpha]}\,{}^{l}G^{(\alpha+1)}_{n-\alpha-1}(\theta)\omega_{\alpha+1}(\theta)\big\} \\ &= \frac{\Gamma(\alpha+1)}{2^\alpha\Gamma(\alpha+3/2)\sqrt{\pi}} \lim_{\theta\to -1^+} {}^{r}G^{(\alpha+1)}_{n-\alpha-1}(\theta)\omega_{\alpha+1}(\theta).\end{aligned} \tag{6.14}$$

Using (2.29) leads to that for $\lambda > 1/2$,

$$\lim_{x\to -1^+} \omega_\lambda(x)\,{}^{r}G^{(\lambda)}_{\nu}(x) = -2^{2\lambda-1}\frac{\sin(\nu\pi)}{\pi}\frac{\Gamma(\lambda-1/2)\Gamma(\lambda+1/2)\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)}. \tag{6.15}$$

Therefore, from (4.39) and (6.14)-(6.15), we obtain the formula:

$$\hat{u}^C_n = \frac{(-1)^{n+1}\sin(\pi\alpha)\Gamma(2\alpha+1)}{2^{\alpha-1}\pi}\frac{\Gamma(n-\alpha)}{\Gamma(n+\alpha+1)}, \quad n \ge \alpha+1, \tag{6.16}$$

and for large n, we have $|\hat{u}^C_n| = O(n^{-2\alpha-1})$.

With the aid of (6.16), we next consider a more general case: $u(x) = (1+x)^\alpha g(x)$ with $g(x)$ being a sufficiently smooth function. Here, we need to use the formula of $\hat{u}_n^C$ for $(1+x)^\alpha$ with $n<\alpha+1$. Taking $m=n$ in (4.17) and using the property of the Beta function, yields

$$
\begin{aligned}
\hat{u}_n^C &= \frac{1}{\sqrt{\pi}\,2^{n-1}\Gamma(n+1/2)}\int_{-1}^{1} u^{(n)}(x)\,G_0^{(n)}(x)\omega_n(x)\,dx \\
&= \frac{1}{\sqrt{\pi}\,2^{n-1}\Gamma(n+1/2)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\int_{-1}^{1}(1+x)^{\alpha-n}(1-x^2)^{n-1/2}\,dx \\
&= \frac{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha-n+1)\Gamma(\alpha+n+1)}.
\end{aligned}
\tag{6.17}
$$

Using the Taylor expansion of $g(x)$ at $x=-1$, we obtain from (6.16)-(6.17) that

$$
\begin{aligned}
\hat{u}_n^C = &\sum_{l=0}^{[n-1-\alpha]}\frac{g^{(l)}(-1)}{l!}\frac{(-1)^{n+1}\sin(\pi(\alpha+l))\Gamma(2\alpha+2l+1)}{2^{\alpha+l-1}\pi}\frac{\Gamma(n-\alpha-l)}{\Gamma(n+\alpha+l+1)} \\
&+\sum_{l=[n-\alpha]}^{\infty}\frac{g^{(l)}(-1)}{l!}\frac{2^{\alpha+l+1}\Gamma(\alpha+l+1)\Gamma(\alpha+l+1/2)}{\sqrt{\pi}\Gamma(\alpha+l-n+1)\Gamma(\alpha+l+n+1)} \\
= &\frac{(-1)^{n+1}g(-1)\sin(\pi\alpha)\Gamma(2\alpha+1)}{2^{\alpha-1}\pi}n^{-2\alpha-1}+O(n^{-2\alpha-3}),
\end{aligned}
\tag{6.18}
$$

where we used (4.47).

Finally, we consider the singular function: $u(x)=(1+x)^\alpha\ln(1+x)$. Using (4.69), we derive from (4.59) and l'Hôpital's rule that

$$
\begin{aligned}
\hat{u}_n^C &= \frac{2}{\pi}\int_{-1}^{1}\Big\{\lim_{\varepsilon\to 0}\frac{(1+x)^{\varepsilon+\alpha}-(1+x)^\alpha}{\varepsilon}\Big\}\frac{T_n(x)}{\sqrt{1-x^2}}dx \\
&= (-1)^{n+1}\frac{2}{\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\frac{\sin(\pi(\alpha+\varepsilon))\Gamma(2\alpha+2\varepsilon+1)\Gamma(n-\alpha-\varepsilon)}{2^{\alpha+\varepsilon}\Gamma(n+\alpha+\varepsilon+1)} \\
&\quad -\frac{\sin(\pi\alpha)\Gamma(2\alpha+1)\Gamma(n-\alpha)}{2^\alpha\Gamma(n+\alpha+1)}\Big\} \\
&= \frac{(-1)^{n+1}\Gamma(2\alpha+1)\Gamma(n-\alpha)}{\pi 2^{\alpha-1}\Gamma(n+\alpha+1)}\big\{\pi\cos(\alpha\pi)+\sin(\alpha\pi)\big(2\psi(2\alpha+1) \\
&\quad -\ln 2-\psi(n+\alpha+1)-\psi(n-\alpha)\big)\big\}.
\end{aligned}
$$

In view of (4.63), we can obtain the asymptotic behaviour

$$
\begin{aligned}
\hat{u}_n^C = &\frac{(-1)^{n+1}\Gamma(2\alpha+1)}{\pi 2^{\alpha-1}}n^{-2\alpha-1}\big\{\pi\cos(\alpha\pi)+\sin(\alpha\pi)\big(2\psi(2\alpha+1)-\ln 2 \\
&-2\ln n\big)\big\}+O(n^{-2\alpha-3}\ln n)\,\sin(\alpha\pi)+O(n^{-2\alpha-3}).
\end{aligned}
$$

Remark 6.1. With the above analysis of the expansion coefficients, we can then obtain directly the optimal estimates for the Chebyshev approximation to these specific singular functions. More precisely, for $u(x)=(1+x)^\alpha g(x)$ with $g(x)$ being a sufficiently smooth function, we have

$$
\|u-\pi_N^C u\|_{L^\infty(\Omega)}\le CN^{-2\alpha},\quad \|u-\pi_N^C u\|_{L^2_\omega(\Omega)}\le CN^{-2\alpha-1/2},
\tag{6.19}
$$

and for $u(x)=(1+x)^\alpha\ln(1+x)$, we have

$$
\|u-\pi_N^C u\|_{L^\infty(\Omega)}\le C(\ln N)N^{-2\alpha},\quad \|u-\pi_N^C u\|_{L^2_\omega(\Omega)}\le C(\ln N)N^{-2\alpha-1/2}.
\tag{6.20}
$$

Compared with the interior singularities (see (4.61) and (4.73)), a higher convergence order $O(N^{-\alpha})$ is observed which is as expected.

6.3. **Concluding remarks.** Broadly speaking, we position this work as our first attempt to show how the Riemann-Liouville fractional calculus can alter the fundamental polynomial approximation theory. Some estimates and bounds herein are completely new, or significantly improve the existing results.

More precisely, we introduce a new theoretical framework of fractional Sobolev-type spaces for orthogonal polynomial approximations to functions with limited regularities (or interior/endpoint singularities). The proposed spaces are naturally arisen from the analytic representations of the expansion coefficients involving Riemann-Liouville fractional integrals/derivatives and GGF-Fs. We present a collection of notable properties of the new family of GGF-Fs, and derive optimal estimates of Chebyshev approximations in various norms for a wide class of singular functions. The analysis techniques can be extended to general Jacobi approximations. We are confident that this study, together with our follow-up works, will have far-reaching impact on numerical analysis of p-version and hp-version for singular problems.

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