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ON EXPONENTIAL CONVERGENCE OF GEGENBAUER INTERPOLATION AND SPECTRAL DIFFERENTIATION

ZIQING XIE, LI-LIAN WANG, AND XIAODAN ZHAO

Abstract. This paper is devoted to a rigorous analysis of exponential convergence of polynomial interpolation and spectral differentiation based on the Gegenbauer-Gauss and Gegenbauer-Gauss-Lobatto points, when the underlying function is analytic on and within an ellipse. Sharp error estimates in the maximum norm are derived.

1. Introduction

Perhaps the most significant advantage of the spectral method is its high-order of accuracy. The typical convergence rate of the spectral method is $O(n^{-m})$ for every m, provided that the underlying function is sufficiently smooth [21, 4, 6, 30, 8]. If the function is suitably analytic, the expected rate is $O(q^n)$ with $0 < q < 1$. This is the so-called *exponential convergence,* which is well accepted among the community.

There has been much investigation on exponential decay of spectral expansions of analytic functions. For instance, the justification or description of the results for Fourier or Chebyshev series can be found in [40, 10, 38, 5, 35, 43]. In the seminal work of Gottlieb and Shu on the resolution of the Gibbs phenomenon (the interested readers are referred to Gustafsson [31] for a review of this significant contribution attributed to a series of papers [26, 23, 24, 22, 25]), the exponential convergence, in the maximum norm (termed as the regularization error), of Gegenbauer polynomial expansions was derived, when the index (denoted by λ below) grows linearly with the degree n. Boyd [7] provided an insightful study of the "diagonal limit" (i.e., $\lambda = \beta n$ for some constant $\beta > 0$) convergence of the Gegenbauer reconstruction algorithm in [26]. Indeed, under the assumption of analyticity in [26], the exponential accuracy of Gegenbauer expansions is actually valid for fixed λ, which can be justified by a direct application of the Stirling's formula (see (2.11) below) to Theorem 4.3 in [26]. We can also find the estimates under the analytic assumption on and within the Bernstein ellipse $\mathcal{E}_\rho$ (as defined in (2.15) below) from various sources. We refer to [10, 38, 35] for the results on Chebyshev expansions. Davis [10] stated an estimate (with proof, see Page 312 of [10]) due to K. Neumann, that is, if

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u is analytic on and inside $\mathcal{E}_\rho$ with $\rho > 1$, then the Legendre expansion coefficient satisfies

$$\limsup_{n\to\infty} |\hat{u}_n|^{1/n} = \rho^{-1}. \tag{1.1}$$

A similar result for the general Jacobi expansion was presented in Theorem 9.1.1 (without proof) on Page 245 of [40]. A more informative and sharper estimate with explicit dependence on n for Gegenbauer expansions was obtained in Theorem 4.2 of [25]. Based on an argument different from that of [25], the very recent paper [44] showed the exponential convergence of the Legendre expansion.

It is known that the heart of a collocation/pseudospectral method is the spectral differentiation process. That is, given a set of Gauss-type points $\{x_j\}_{j=0}^n$, e.g., on $[-1, 1]$, the derivative values $\{u'(x_j)\}$ can be approximated by an exact differentiation of the polynomial interpolant $\{(I_n u)'(x_j)\}$. Such a direct differentiation technique is also called a differencing method in the literature (see, e.g., [41, 18, 37]). For the first time, Tadmor [41] showed the exponential accuracy of differencing analytic functions on Chebyshev-Gauss points, where the main argument was based on analyzing the (continuous) Chebyshev coefficients and the aliasing error, and where the intimate relation between Fourier and Chebyshev basis functions played an essential role in the analysis. Reddy and Weideman [37] took a different approach and improved the estimate in [41] for Chebyshev differencing of functions analytic on and within $\mathcal{E}_\rho$ with $\rho > 1$. As pointed out in [37], although the exponential convergence of spectral differentiation of analytic functions is appreciated and mentioned in the literature (see, e.g., [18, 42, 6]), the rigorous proofs (merely for the Fourier and Chebyshev methods) can only be found in [41, 37]. Indeed, to the best of our knowledge, the theoretical justification even for the Legendre method is lacking. It is worthwhile to point out that under the regularity condition (M): $\|u^{(k)}\|_{L^\infty} \le cM^k$, the super-geometric convergence of Legendre spectral differentiation was proved by Zhang [47]:

$$\max_{0\le j\le n} |(u - I_n u)'(x_j)| \le C\Big(\frac{eM}{2(n+1)}\Big)^{n+2}, \tag{1.2}$$

where $\{x_j\}_{j=0}^n$ are the Legendre-Gauss points. A similar estimate was nontrivially extended to the Chebyshev collocation method in [48]. The condition (M) covers a large class of functions, but it is even more restrictive than analyticity. On the other hand, the regularity index k could be infinite, while the dependence of (1.2) on k is not clear.

The main concern of this paper is to derive sharp estimates of exponential convergence, in the maximum norm, of interpolation and spectral differentiation on Gegenbauer-Gauss and Gegenbauer-Gauss-Lobatto points, provided that the underlying function is analytic on and within Bernstein's ellipse. The essential argument is based on the classical Hermite's contour integral (see (2.19) below), and a delicate estimate of the asymptotics of the Gegenbauer polynomial on the ellipse of interest. It is important to remark that the Chebyshev polynomial takes a very simple explicit form (see, e.g., [10] or (3.5) below), but the Gegenbauer polynomial has a complicated expression. Accordingly, compared with the Chebyshev case in [37], the analysis in this paper is much more involved. The Chebyshev and Legendre methods are commonly used in spectral approximations, but we have also witnessed renewed applications of the Gegenbauer (or more general Jacobi) polynomial based methods in, e.g., defeating Gibbs phenomenon (see, e.g., [26, 20]),

hp-elements (see, e.g., [16, 2, 33, 27]), and numerical solutions of differential equations (see, e.g., [3, 28, 29, 12, 13, 14, 15, 45, 39]) and integral equations (see, e.g., [9]). The results in this paper might be useful for a better understanding of the methods and have implications in other applications.

The rest of the paper is organized as follows. As some preliminaries, we briefly review basic properties of Gegenbauer polynomials, Gamma functions and analytic functions in Section 2. We study the asymptotics of the Gegenbauer polynomials in Section 3, and present the main results on exponential convergence of interpolation and spectral differentiation, together with some numerical results and extensions in the last section.

2. Preliminaries

In this section, we collect some relevant properties of Gegenbauer polynomials and assorted facts to be used throughout the paper.

2.1. Gegenbauer polynomials.

The analysis heavily relies on the normalization of [40], so we define the Gegenbauer polynomials[1] by the three-term recurrence:

$$\begin{aligned} &nC_n^\lambda(x) = 2\,(n+\lambda-1)\,xC_{n-1}^\lambda(x) - (n+2\lambda-2)C_{n-2}^\lambda(x), \quad n \ge 2,\\ &C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x, \quad \lambda > -1/2, \;\; x \in [-1,1]. \end{aligned} \tag{2.1}$$

Notice that if $\lambda = 0$, $C_n^\lambda(x)$ vanishes identically for $n \ge 1$. This corresponds to the Chebyshev polynomial, and there holds

$$\lim_{\lambda \to 0} \lambda^{-1} C_n^\lambda(x) = \frac{2}{n} T_n(x) = \frac{2}{n} \cos(n \arccos(x)), \quad n \ge 1. \tag{2.2}$$

Hereafter, if not specified explicitly, we assume $\lambda \ne 0$, and refer to [37] for the analysis of the Chebyshev case. Notice that for $\lambda = 1/2$, $C_n^\lambda(x) = L_n(x)$, i.e., the usual Legendre polynomial of degree n.

The Gegenbauer polynomials are orthogonal with respect to the weight function $(1-x^2)^{\lambda-1/2}$, namely,

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x)(1-x^2)^{\lambda-1/2} dx = h_n^\lambda \delta_{mn}, \tag{2.3}$$

where δ_{mn} is the Kronecker symbol, and

$$h_n^\lambda = \frac{2^{1-2\lambda}\pi}{\Gamma^2(\lambda)} \frac{\Gamma(n+2\lambda)}{n!(n+\lambda)}. \tag{2.4}$$

Moreover, we have

$$C_n^\lambda(-x) = (-1)^n C_n^\lambda(x), \quad C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} \tag{2.5}$$

and

$$\frac{d}{dx} C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}(x). \tag{2.6}$$

[1] Historically, they are sometimes called “ultraspherical polynomials” (see, e.g., the footnote on Page 80 of [40] and Page 302 of [1]).

By Formula (4.7.1) and Theorems 7.32.1 and 7.33.1 of Szegö [40], we have

$$
\begin{aligned}
&|C_n^\lambda(x)| \le C_n^\lambda(1) \quad \text{if } \lambda > 0;\\
&|C_n^\lambda(x)| \le D_\lambda\, n^{\lambda-1} \quad \text{if } -\frac{1}{2} < \lambda < 0 \text{ and } n \gg 1,
\end{aligned}
\tag{2.7}
$$

where D_λ is a positive constant independent of n. A tight bound can be found in [36] (also see [34]):

$$
\max_{|x|\le 1}\Big\{(1-x^2)^\lambda\big(C_n^\lambda(x)\big)^2\Big\} \le \frac{2e(2+\sqrt{2}\lambda)}{\pi} h_n^\lambda, \quad \lambda > 0,\ n \ge 0. \tag{2.8}
$$

2.2. **Gamma and incomplete Gamma functions.** The following properties of the Gamma and incomplete Gamma functions (cf. [46]) are found to be useful. The Gamma function satisfies

$$
\Gamma(x)\Gamma(x+1/2) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x), \quad \forall x \ge 0 \tag{2.9}
$$

and

$$
\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}, \quad \forall x > 0. \tag{2.10}
$$

We would like to quote Stirling's formula (see, e.g., [25]):

$$
\sqrt{2\pi}x^{x+1/2}e^{-x} \le \Gamma(x+1) \le \sqrt{2\pi}x^{x+1/2}e^{-x}e^{\frac{1}{12x}}, \quad \forall x \ge 1. \tag{2.11}
$$

We also need to use the incomplete Gamma function defined by

$$
\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt, \quad \alpha > 0,\ \ x \ge 0, \tag{2.12}
$$

which satisfies (see P. 899 of [32])

$$
\Gamma(n+1, x) = n!e^{-x}\sum_{k=0}^n \frac{x^k}{k!}, \quad n = 0, 1, \ldots . \tag{2.13}
$$

2.3. **Basics of analytic functions.** Suppose that $u(x)$ is analytic on $[-1,1]$. Based on the notion of analytic continuation, there always exists a simple connected region R in the complex plane containing $[-1,1]$ into which $f(x)$ can be continued analytically. The analyticity may be characterized by the growth of the derivatives of u. More precisely, let $\mathcal{C}$ be a simple positively oriented closed contour surrounding $[-1,1]$ and lying in R. Then we have (see, e.g., [10]):

$$
\frac{|u^{(m)}(x)|}{m!} \le \frac{\max_{z\in\mathcal{C}}|u(z)|L(\mathcal{C})}{2\pi\delta^{m+1}}, \quad \forall x \in [-1,1], \tag{2.14}
$$

where $L(\mathcal{C})$ is the length of $\mathcal{C}$, and δ is the distance from $\mathcal{C}$ to $[-1,1]$ (can be viewed as the distance from $[-1,1]$ to the nearest singularity of u in the complex plane). Mathematically, an appropriate contour to characterize the analyticity is the so-called Bernstein ellipse:

$$
\mathcal{E}_\rho := \Big\{z \in \mathbb{C}\ :\ z = \frac{1}{2}(w + w^{-1}) \ \text{ with }\ w = \rho e^{\mathrm{i}\theta},\ \theta \in [0, 2\pi]\Big\}, \quad \rho > 1, \tag{2.15}
$$

where $\mathbb{C}$ is the set of all complex numbers, and $\mathrm{i} = \sqrt{-1}$ is the complex unit. The ellipse $\mathcal{E}_\rho$ has the foci at ± 1 and the major and minor semi-axes are, respectively,

$$
a = \frac{1}{2}\big(\rho + \rho^{-1}\big), \quad b = \frac{1}{2}(\rho - \rho^{-1}), \tag{2.16}
$$

so the sum of two axes is ρ. As illustrated in Figure 2.1, ρ is the radius of the circle $w = \rho e^{\mathrm{i}\theta}$ that is mapped to the ellipse $\mathcal{E}_\rho$ under the conformal mapping: $z = \frac{1}{2}(w + w^{-1})$.

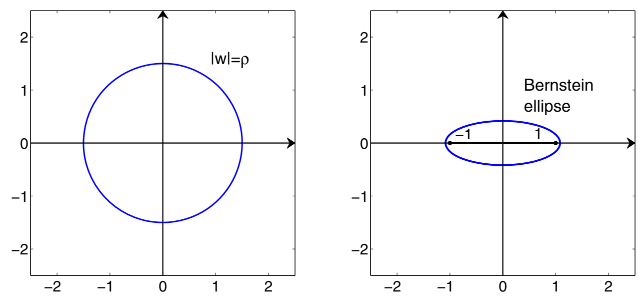


FIGURE 2.1. Circle (left): $|w| = \rho = 1.5$, and Bernstein ellipse (right): $\mathcal{E}_\rho$ with foci ± 1 linked by the conformal mapping: $z = \frac{1}{2}(w + w^{-1})$.

According to [37], the perimeter of $\mathcal{E}_\rho$ satisfies

$$L(\mathcal{E}_\rho) \le \pi\sqrt{\rho^2 + \rho^{-2}}, \tag{2.17}$$

which overestimates the perimeter by less than 12 percent. The distance from $\mathcal{E}_\rho$ to the interval $[-1, 1]$ is

$$\delta_\rho = \frac{1}{2}(\rho + \rho^{-1}) - 1. \tag{2.18}$$

We are concerned with the interpolation and spectral differentiation of analytic functions on the Gegenbauer-Gauss-type points. Let $\{x_j := x_j(\lambda, n)\}_{j=0}^{n}$ be the Gegenbauer-Gauss points (i.e., the zeros of $C_{n+1}^{\lambda}(x)$) or the Gegenbauer-Gauss-Lobatto points (i.e., the zeros of $(1-x^2)C_{n-1}^{\lambda+1}(x)$). The associated Lagrange interpolation polynomial of u is given by $I_n u \in \mathbb{P}_n$ (the set of all polynomials of degree $\le n$) such that $(I_n u)(x_j) = u(x_j)$ for $0 \le j \le n$. Our starting point is the Hermite's contour integral (see, e.g., [11]):

$$(u - I_n u)(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{E}_\rho} \frac{Q_{n+1}(x)}{Q_{n+1}(z)} \frac{u(z)}{z - x} dz, \quad \forall x \in [-1, 1], \tag{2.19}$$

where $Q_{n+1}(x) = C_{n+1}^{\lambda}(x)$ or $(1-x^2)C_{n-1}^{\lambda+1}(x)$. Consequently, we have

$$(u - I_n u)'(x_j) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{E}_\rho} \frac{Q_{n+1}'(x_j)}{Q_{n+1}(z)} \frac{u(z)}{z - x_j} dz, \quad 0 \le j \le n. \tag{2.20}$$

A crucial component of the error analysis is to obtain a sharp asymptotic estimate of $Q_{n+1}(z)$ on $\mathcal{E}_\rho$ with large n. This will be the main concern of the forthcoming section.

3. Asymptotic estimate of Gegenbauer polynomials on $\mathcal{E}_\rho$

Much of our analysis relies on the following representation of the Gegenbauer polynomial.

Lemma 3.1. *Let $z = \frac{1}{2}(w + w^{-1})$. We have*

$$C_n^\lambda(z) = \sum_{k=0}^{n} g_k^\lambda g_{n-k}^\lambda w^{n-2k}, \quad n \geq 0, \ \ \lambda > -1/2, \tag{3.1}$$

where

$$g_0^\lambda = 1, \quad g_k^\lambda = \binom{k+\lambda-1}{k} = \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}, \quad 1 \leq k \leq n. \tag{3.2}$$

This formula is derived from the three-term recurrence formula (2.1) and the mathematical induction. Its proof is provided in Appendix A.

Remark 3.1. Some consequences of Lemma 3.1 are in order.

(a) Comparing the coefficients w^n on both sides of (3.1), we find that the leading coefficient of C_n^λ is $2^n g_n^\lambda$. This can be also verified from (2.1) by the mathematical induction.

(b) If $\lambda > 0$, then $g_k^\lambda > 0$ for all $0 \leq k \leq n$. On the other hand, if $\lambda < 0$, we find from (2.10) and (3.2) that

$$g_k^\lambda = \frac{\sin(\pi\lambda)}{\pi}\frac{\Gamma(k+\lambda)\Gamma(1-\lambda)}{k!} < 0, \quad 1 \leq k \leq n. \tag{3.3}$$

(c) If $\lambda = 1/2$, it follows from (2.9) and (3.2) that

$$g_k^\lambda = \frac{(2k)!}{(k!)^2 2^{2k}}, \quad 0 \leq k \leq n. \tag{3.4}$$

Such a representation for the Legendre polynomial can be found in, e.g., [11] and [40], but the derivation is different.

(d) If $\lambda = 0$, then by (2.2),

$$T_n(z) = \frac{1}{2}\big(w^n + w^{-n}\big), \quad n \geq 1. \tag{3.5}$$

(e) If $\lambda = 1$, then $g_k^\lambda \equiv 1$ for $0 \leq k \leq n$. Therefore, the Chebyshev polynomial of the second kind has the representation

$$U_n(z) = \frac{w^{n+1} - w^{-(n+1)}}{w - w^{-1}} = w^n \sum_{k=0}^{n} w^{-2k} = C_n^1(z), \quad n \geq 0, \tag{3.6}$$

which can be found in [35]. □

It is interesting to observe from (3.6) that for $\lambda = 1$, $C_n^\lambda(z)/w^n$ converges to $(1 - w^{-2})^{-\lambda}$ uniformly for all $|w| > 1$, that is,

$$\sum_{k=0}^{\infty} w^{-2k} = \frac{1}{1 - w^{-2}}, \quad |w| > 1.$$

In what follows, we show a similar property holds for general $\lambda > -1/2$ and $\lambda \neq 0$. More precisely, we estimate the upper bound of remainder:

$$\left|\big(1 - w^{-2}\big)^{-\lambda} - \frac{C_n^\lambda(z)}{g_n^\lambda w^n}\right| \leq \sum_{k=1}^{n} |d_{n,k}^\lambda||g_k^\lambda|\rho^{-2k} + \sum_{k=n+1}^{\infty} |g_k^\lambda|\rho^{-2k} := R_n(\rho, \lambda), \tag{3.7}$$

where $z \in \mathcal{E}_\rho$ with $|w| = \rho > 1$, and

$$d_{n,k}^\lambda = 1 - \frac{g_{n-k}^\lambda}{g_n^\lambda}, \quad 1 \leq k \leq n. \tag{3.8}$$

To obtain a sharp estimate of $R_n(\rho, \lambda)$, it is necessary to understand the behavior of the coefficients $\{g_k^\lambda\}_{k=1}^n$ and $\{d_{n,k}^\lambda\}_{k=1}^n$, which are summarized in the following two lemmas.

Lemma 3.2. *For $\lambda > -1/2$, $k \geq 1$ and $k + \lambda \geq 1$,*

$$c_1 \left(1 + \frac{\lambda}{k}\right)^{k+1/2} e^{-\lambda} \leq \frac{\Gamma(\lambda)\, g_k^\lambda}{(k+\lambda)^{\lambda-1}} \leq c_2 \left(1 + \frac{\lambda}{k}\right)^{k+1/2} e^{-\lambda}, \tag{3.9}$$

where $c_1 = e^{-\frac{1}{12k}}$ and $c_2 = e^{\frac{1}{12(k+\lambda)}}$.

Proof. Applying Stirling's formula (2.11) to

$$(k+\lambda)\Gamma(\lambda)\, g_k^\lambda = \frac{\Gamma(k+\lambda+1)}{\Gamma(k+1)}$$

leads to (3.9). □

Lemma 3.3. *Let $\{g_k^\lambda\}_{k=1}^n$ and $\{d_{n,k}^\lambda\}_{k=1}^n$ be the sequences as defined in* (3.2) *and* (3.8)*, respectively.*

(i) *If $\lambda > 1$, then there holds*

$$0 < d_{n,1}^\lambda < d_{n,2}^\lambda < \cdots < d_{n,n}^\lambda < 1. \tag{3.10}$$

(ii) *If $-1/2 < \lambda < 1$ and $\lambda \neq 0$, then*

$$\cdots < |g_{k+1}^\lambda| < |g_k^\lambda| < \cdots < |g_1^\lambda| < g_0^\lambda = 1, \tag{3.11}$$

and we have

$$0 < -d_{n,1}^\lambda < -d_{n,2}^\lambda < \cdots < -d_{n,n-1}^\lambda, \tag{3.12}$$

and

$$|d_{n,k}^\lambda g_k^\lambda| < 1 \quad \text{for} \quad 1 \leq k \leq n-1,\ n \geq 3. \tag{3.13}$$

Proof. By (3.2),

$$\frac{g_{k+1}^\lambda}{g_k^\lambda} = \frac{k+\lambda}{k+1} \quad \text{for} \ \ \lambda \neq 0. \tag{3.14}$$

Thus for $\lambda > 1$, $\{g_k^\lambda\}$ is strictly increasing with respect to k, which, together with the fact $g_k^\lambda > 0$, implies

$$0 < d_{n,k}^\lambda = 1 - \frac{g_{n-k}^\lambda}{g_n^\lambda} < 1$$

and

$$d_{n,k+1}^\lambda - d_{n,k}^\lambda = \frac{g_{n-k}^\lambda - g_{n-k-1}^\lambda}{g_n^\lambda} > 0.$$

This completes the proof of (i).

The property (3.11) is a direct consequence of (3.14), and (3.12) can be proved in a fashion similar to (3.10). It remains to verify (3.13). If $k = 1$, a direct calculation by using (3.2) yields

$$|d_{n,1}^\lambda g_1^\lambda| = |\lambda| \frac{1-\lambda}{n-(1-\lambda)} < 1, \quad \forall n \geq 3.$$

For $2 \le k \le n-1$, it follows from (3.2) and (3.12) that

$$|d^\lambda_{n,k} g^\lambda_k| = \Big(\frac{g^\lambda_{n-k}}{g^\lambda_n} - 1\Big)|g^\lambda_k| < \frac{g^\lambda_{n-k}}{g^\lambda_n}|g^\lambda_k| = \left(\prod_{j=0}^{k-2} \frac{1-\frac{1-\lambda}{k-j}}{1-\frac{1-\lambda}{n-j}}\right) \frac{|\lambda|}{1-\frac{1-\lambda}{n-k+1}} < 1.$$

This ends the proof. □

With the above preparation, we are ready to present the main result on the upper bound of $R_n(\rho,\lambda)$ in (3.7).

Theorem 3.1. *Let $R_n(\rho,\lambda)$ with $\rho > 1$ be the remainder as defined in* (3.7).

(i) *If $\lambda > 1$, then for all $n \ge m \ge 1$ and*

$$m + 2 \ge (\lambda - 1)\Big(\frac{1}{2\ln\rho} - 1\Big), \tag{3.15}$$

we have

$$R_n(\rho,\lambda) \le d^\lambda_{n,m}\Big((1-\rho^{-2})^{-\lambda} - 1\Big) + A\frac{[\lambda]!}{(2\ln\rho)^\lambda}\frac{(m+\lambda)^{[\lambda]}}{\rho^{2(m-1)}}, \tag{3.16}$$

where $[\lambda]$ is the largest integer $\le \lambda$, and

$$A = \frac{1}{\Gamma(\lambda)}\exp\Big(\frac{1}{12(m+1+\lambda)} + \frac{\lambda}{2(m+1)}\Big). \tag{3.17}$$

(ii) *If $-1/2 < \lambda < 1$ and $\lambda \ne 0$, then for all $n \ge m \ge 1$ and $n \ge 3$,*

$$R_n(\rho,\lambda) \le |d^\lambda_{n,m}|\big|(1-\rho^{-2})^{-\lambda} - 1\big| + \frac{\rho^{-2m}}{\rho^2 - 1} + 2\rho^{-2n}. \tag{3.18}$$

Here, the factor $d^\lambda_{n,m}$ is given by (3.8).

Proof. (i) For $\lambda > 1$, we obtain from (3.10) that

$$\begin{aligned} R_n(\rho,\lambda) &= \sum_{k=1}^{m} d^\lambda_{n,k} g^\lambda_k \rho^{-2k} + \sum_{k=m+1}^{n} d^\lambda_{n,k} g^\lambda_k \rho^{-2k} + \sum_{k=n+1}^{\infty} g^\lambda_k \rho^{-2k} \\ &\overset{(3.10)}{\le} d^\lambda_{n,m}\sum_{k=1}^{m} g^\lambda_k \rho^{-2k} + \sum_{k=m+1}^{n} g^\lambda_k \rho^{-2k} + \sum_{k=n+1}^{\infty} g^\lambda_k \rho^{-2k} \\ &\le d^\lambda_{n,m}\Big((1-\rho^{-2})^{-\lambda} - 1\Big) + \sum_{k=m+1}^{\infty} g^\lambda_k \rho^{-2k}. \end{aligned} \tag{3.19}$$

By Lemma 3.2,

$$\begin{aligned} \sum_{k=m+1}^{\infty} g^\lambda_k \rho^{-2k} &\le (\Gamma(\lambda))^{-1} e^{-\lambda} e^{\frac{1}{12(m+1+\lambda)}} \sum_{k=m+1}^{\infty} \frac{(k+\lambda)^{\lambda-1}}{\rho^{2k}}\Big(1 + \frac{\lambda}{k}\Big)^{k+1/2} \\ &\le (\Gamma(\lambda))^{-1} e^{-\lambda} e^{\frac{1}{12(m+1+\lambda)}} \sum_{k=m+1}^{\infty} \frac{(k+\lambda)^{\lambda-1}}{\rho^{2k}} e^{\lambda + \frac{\lambda}{2k}}, \end{aligned}$$

where we used the inequality $1 + x < e^x$ for $x > 0$. Hence,

$$\sum_{k=m+1}^{\infty} g^\lambda_k \rho^{-2k} \le A \sum_{k=m+1}^{\infty} \frac{(k+\lambda)^{\lambda-1}}{\rho^{2k}}, \tag{3.20}$$

where A is given by (3.17). One verifies that under condition (3.15), $(k+\lambda)^{\lambda-1}/\rho^{2k}$ is decreasing with respect to k. Therefore, by (2.12) and (2.13),

$$\begin{aligned}\sum_{k=m+1}^{\infty}\frac{(k+\lambda)^{\lambda-1}}{\rho^{2k}} &\le \int_m^{\infty}(x+\lambda)^{\lambda-1}\rho^{-2x}dx = \frac{\rho^{2\lambda}}{(2\ln\rho)^{\lambda}}\int_{2(m+\lambda)\ln\rho}^{\infty}x^{\lambda-1}e^{-x}dx\\ &= \frac{\rho^{2\lambda}}{(2\ln\rho)^{\lambda}}\Gamma\big(\lambda,2(m+\lambda)\ln\rho\big) \le \frac{\rho^{2\lambda}}{(2\ln\rho)^{\lambda}}\Gamma\big([\lambda]+1,2(m+\lambda)\ln\rho\big)\\ &= \frac{[\lambda]!\rho^{-2m}}{(2\ln\rho)^{\lambda}}\sum_{k=0}^{[\lambda]}\frac{(m+\lambda)^k(2\ln\rho)^k}{k!} \le \frac{[\lambda]!}{(2\ln\rho)^{\lambda}}\frac{(m+\lambda)^{[\lambda]}}{\rho^{2m}}\sum_{k=0}^{\infty}\frac{(2\ln\rho)^k}{k!}\\ &= \frac{[\lambda]!}{(2\ln\rho)^{\lambda}}\frac{(m+\lambda)^{[\lambda]}}{\rho^{2(m-1)}}.\end{aligned}$$

A combination of the above estimates leads to (3.16).

(ii) Now, we turn to the proof of the second case: $-1/2<\lambda<1$ and $\lambda\ne 0$. By Lemma 3.3,

$$\begin{aligned}R_n(\rho,\lambda) &= \sum_{k=1}^{m}|d_{n,k}^{\lambda}||g_k^{\lambda}|\rho^{-2k} + \sum_{k=m+1}^{n}|d_{n,k}^{\lambda}||g_k^{\lambda}|\rho^{-2k} + \sum_{k=n+1}^{\infty}|g_k^{\lambda}|\rho^{-2k}\\ &\overset{(3.12)}{\le} |d_{n,m}^{\lambda}|\sum_{k=1}^{m}|g_k^{\lambda}|\rho^{-2k} \overset{(3.13)}{+} \sum_{k=m+1}^{n-1}\rho^{-2k} + |d_{n,n}^{\lambda}g_n^{\lambda}|\rho^{-2n} \overset{(3.11)}{+} \sum_{k=n+1}^{\infty}\rho^{-2k}\\ &\le |d_{n,m}^{\lambda}|\big|(1-\rho^{-2})^{-\lambda}-1\big| + \frac{\rho^{-2m}}{\rho^2-1} + 2\rho^{-2n},\end{aligned}$$

where in the last step, we used the following facts:

$$\sum_{k=1}^{m}|g_k^{\lambda}|\rho^{-2k} \le \mathrm{sign}(\lambda)\sum_{k=1}^{\infty}g_k^{\lambda}\rho^{-2k} = \mathrm{sign}(\lambda)\big((1-\rho^{-2})^{-\lambda}-1\big),\quad \rho>1,$$

(note: $\mathrm{sign}(\lambda)$ is the sign of λ), and $|d_{n,n}^{\lambda}g_n^{\lambda}| = |g_n^{\lambda}-1|<2$, thanks to (3.11). $\square$

The estimate in Theorem 3.1 is quite tight and is valid even for small n. By choosing a suitable m to balance the two error terms in the upper bound, we are able to derive the anticipated asymptotic estimate.

Theorem 3.2. *For any $z\in\mathcal{E}_\rho$ with $|w|=\rho>1$, and any $\lambda>-1/2$ and $\lambda\ne 0$, there exists $0<\varepsilon\le 1/2$ such that*

$$\left|\big(1-w^{-2}\big)^{-\lambda} - \frac{C_n^{\lambda}(z)}{g_n^{\lambda}w^n}\right| \le A(\rho,\lambda)n^{\varepsilon-1} + O(n^{-1}), \tag{3.21}$$

where

$$A(\rho,\lambda) = |1-\lambda|\big|(1-\rho^{-2})^{-\lambda}-1\big|. \tag{3.22}$$

Proof. We first estimate $|d^\lambda_{n,m}|$ in Theorem 3.1, when $n-m$ is large. Using Stirling's formula (2.11) and (3.2) we get

$$\begin{aligned}\frac{g^\lambda_{n-m}}{g^\lambda_n} &= \Big(1+\frac{1-\lambda}{n+\lambda-1}\Big)^{n+\frac12}\Big(1-\frac{1-\lambda}{n-m}\Big)^{n-m+\frac12}\Big(1-\frac{m}{n+\lambda-1}\Big)^{\lambda-1}\Big\{1+O\Big(\frac{1}{n-m}\Big)\Big\}\\ &= \Big(1-\frac{m}{n+\lambda-1}\Big)^{\lambda-1}\Big\{1+O\Big(\frac{1}{n-m}\Big)\Big\}\\ &= \Big\{1+\frac{(1-\lambda)m}{n+\lambda-1}+O\Big(\frac{m^2}{n^2}\Big)\Big\}\Big\{1+O\Big(\frac{1}{n-m}\Big)\Big\}.\end{aligned}$$

Hereafter, taking $m=[n^\varepsilon]$ with $0<\varepsilon\le 1/2$ yields

$$\begin{aligned}\frac{g^\lambda_{n-m}}{g^\lambda_n} &= 1+(1-\lambda)n^{\varepsilon-1}+O\Big(\frac{1}{n-n^\varepsilon}\Big)\\ &\Longrightarrow\ d^\lambda_{n,m}=(\lambda-1)n^{\varepsilon-1}+O(n^{-1}).\end{aligned}\tag{3.23}$$

One verifies readily that for $\lambda>1$ and any $0<\varepsilon\le 1/2$,

$$\frac{m^{[\lambda]}}{\rho^{2m}}\le\frac{1}{n}\iff\frac{\ln n}{n^\varepsilon}\le\frac{2\ln\rho}{1+\varepsilon[\lambda]},\tag{3.24}$$

which, together with (3.16) and (3.23), implies (3.21) with $\lambda>1$.

If $-1/2<\lambda<1$ and $\lambda\neq 0$, it follows from (3.24) that $\rho^{-2m}\le n^{-1}$ for any $0<\varepsilon\le 1/2$. This validates the desired estimate. □

A direct consequence of Theorem 3.2 is that

$$\lim_{n\to\infty}\frac{C^\lambda_n(z)}{g^\lambda_n}=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{g^\lambda_{n-k}}{g^\lambda_n}g^\lambda_k w^{-2k}=(1-w^{-2})^{-\lambda},\tag{3.25}$$

for all $z\in\mathcal{E}_\rho$ with $|w|=\rho>1$, and any $\lambda>-1/2$ and $\lambda\neq 0$.

Remark 3.2. Based on a completely different argument, Elliott [17] derived an asymptotic expansion for large n near $z=1$ (but not near $z=-1$): $C^\lambda_n(z)\sim\frac{B(n,\lambda)}{(z^2-1)^{\lambda/2}}$, where B is a series involving modified Bessel functions, and some other asymptotic expansions for $|z|$ large and n fixed. Although they are valid for general z off the interval $[-1,1]$, our results in Theorems 3.1 and 3.2 provide tighter and sharper bounds when z is sitting on $\mathcal{E}_\rho$. □

At the end of this section, we provide some numerical results to illustrate the tightness of the upper bound in (3.21). Denote by

$$E_n(\rho;\lambda):=\frac{1}{A(\rho,\lambda)}\max_{z\in\mathcal{E}_\rho}\Big|\big(1-w^{-2}\big)^{-\lambda}-\frac{C^\lambda_n(z)}{g^\lambda_n w^n}\Big|.\tag{3.26}$$

To approximate the maximum value, we sample a set of points dense on the ellipse $\mathcal{E}_\rho$ based on the conformal mapping $z=\frac12(w+w^{-1})$ of the Fourier points on the circle $w=\rho e^{\mathrm{i}\theta}$. We plot in Figure 3.1 in (Matlab) log-log scale of $E_n(\rho;\lambda), n^{-1}$ and $n^{\varepsilon-1}$ (with $\varepsilon=0.1$) for several sets of parameters λ and ρ, and for large n. According to Theorem 3.2, E_n should be bounded by $n^{\varepsilon-1}$ from above, and it is anticipated to be bounded below by n^{-1}, if the estimate is tight. Indeed, we observe from Figure 3.1 such a behavior when n is large.

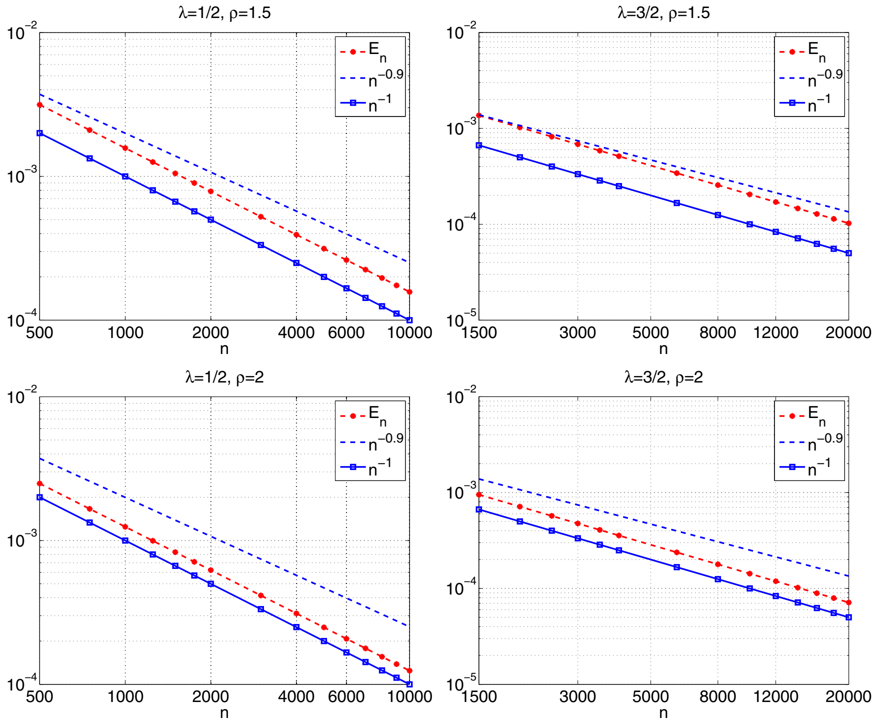


FIGURE 3.1. E_n against $n^{\varepsilon-1}$ (with $\varepsilon = 0.1$) and n^{-1} for large n.

4. ERROR ESTIMATES OF INTERPOLATION AND SPECTRAL DIFFERENTIATION

After collecting all the necessary results, we are ready to estimate exponential convergence of interpolation and spectral differentiation of analytic functions.

Hereafter, the notation $a_n \cong b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, for any two sequences $\{a_n\}$ and $\{b_n\}$ (with $b_n \neq 0$) of complex numbers.

4.1. Gegenbauer-Gauss interpolation and differentiation.

We start with the analysis of interpolation and spectral differentiation on zeros of the Gegenbauer polynomial $C^{\lambda}_{n+1}(x)$.

Theorem 4.1. *Let u be analytic on and within the ellipse $\mathcal{E}_\rho$ with foci ± 1 and $\rho > 1$ as defined in (2.15), and let $(I_n u)(x)$ be the interpolant of $u(x)$ at the set of $(n+1)$ Gegenbauer-Gauss points.*

(i) *If $\lambda > 0$, we have*

$$\max_{|x|\le 1} \big|(u - I_n u)(x)\big| \le \frac{c\Gamma(\lambda) M_\rho \sqrt{\rho^2+\rho^{-2}}}{\Gamma(2\lambda)(\rho-1)^2(1+\rho^{-2})^{-\lambda}} \frac{n^\lambda}{\rho^n}. \tag{4.1}$$

(ii) *If $-1/2 < \lambda < 0$, we have*

$$\max_{|x|\le 1} \big|(u - I_n u)(x)\big| \le \frac{cD_\lambda |\Gamma(\lambda)| M_\rho \sqrt{\rho^2+\rho^{-2}}}{(\rho-1)^2(1-\rho^{-2})^{-\lambda}} \frac{1}{\rho^n}. \tag{4.2}$$

Here, $M_\rho = \max_{z\in\mathcal{E}_\rho} |u(z)|$, D_λ is defined in (2.7), and $c \cong 1$ is a generic positive constant.

Proof. By the formula (2.19) with $Q_{n+1} = C^{\lambda}_{n+1}$ and (2.17)-(2.18), we have the bound of the pointwise error:

$$\begin{aligned} \big|(u - I_n u)(x)\big| &\le \frac{|C^{\lambda}_{n+1}(x)|}{2\pi} \frac{\max_{z\in\mathcal{E}_\rho} |u(z)|}{\min_{z\in\mathcal{E}_\rho} |C^{\lambda}_{n+1}(z)|} \oint_{\mathcal{E}_\rho} \frac{|dz|}{|z-x|} \\ &\le \frac{M_\rho L(\mathcal{E}_\rho)}{2\pi\delta_\rho} \frac{|C^{\lambda}_{n+1}(x)|}{\min_{z\in\mathcal{E}_\rho} |C^{\lambda}_{n+1}(z)|} \\ &\le \frac{M_\rho\sqrt{\rho^2+\rho^{-2}}}{\rho+\rho^{-1}-2} \frac{|C^{\lambda}_{n+1}(x)|}{\min_{z\in\mathcal{E}_\rho} |C^{\lambda}_{n+1}(z)|}, \quad x\in[-1,1],\ n\ge 0. \end{aligned} \tag{4.3}$$

Therefore, it is essential to obtain the lower bound of $|C^{\lambda}_{n+1}(z)|$. Recall that for any two complex numbers z_1 and z_2, we have $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$. It follows from Theorem 3.2 that

$$\left| |1-w^{-2}|^{-\lambda} - \frac{|C^{\lambda}_{n+1}(z)|}{|g^{\lambda}_{n+1}|\rho^{n+1}} \right| \le A(\rho,\lambda) n^{\varepsilon-1} + O(n^{-1}),$$

which implies

$$\begin{aligned} \big|1-w^{-2}\big|^{-\lambda} - A(\rho,\lambda)n^{\varepsilon-1} - O(n^{-1}) &\le \frac{|C^{\lambda}_{n+1}(z)|}{|g^{\lambda}_{n+1}|\rho^{n+1}} \\ &\le \big|1-w^{-2}\big|^{-\lambda} + A(\rho,\lambda)n^{\varepsilon-1} + O(n^{-1}). \end{aligned} \tag{4.4}$$

Notice that

$$1-\rho^{-2} \le |1-w^{-2}| \le 1+\rho^{-2}. \tag{4.5}$$

Consequently,

$$|C^{\lambda}_{n+1}(z)| \ge c\frac{n^{\lambda-1}\rho^{n+1}}{|\Gamma(\lambda)|} \begin{cases} (1+\rho^{-2})^{-\lambda}, & \text{if } \lambda > 0, \\ (1-\rho^{-2})^{-\lambda}, & \text{if } \lambda < 0, \end{cases} \tag{4.6}$$

where we used (3.9), and the constant $c \cong 1$.

On the other hand, we derive from (2.5), (2.7) and (2.11) that if $\lambda > 0$,

$$\max_{|x|\le 1} |C^{\lambda}_{n+1}(x)| = C^{\lambda}_{n+1}(1) \cong \frac{n^{2\lambda-1}}{\Gamma(2\lambda)}. \tag{4.7}$$

Hence, a combination of (4.3), (4.6) and (4.7) leads to (4.1). Similarly, for $-1/2 < \lambda < 0$, we use (2.7) to derive (4.2). □

Remark 4.1. For $\lambda > 0$, we obtain from (2.4), (2.8) and (2.11) that

$$|C^{\lambda}_{n+1}(x)| \le \frac{c2^{1-\lambda}\sqrt{e(2+\sqrt{2}\lambda)}}{\Gamma(\lambda)} n^{\lambda-1}(1-x^2)^{-\lambda/2}, \ \ |x|<1. \tag{4.8}$$

Replacing (4.7) by this bound in the above proof, we can derive the pointwise estimate for $\lambda > 0$:

$$\big|(u - I_n u)(x)\big| \le D(\rho,\lambda)\frac{(1-x^2)^{-\lambda/2}}{\rho^n}, \quad |x|<1, \tag{4.9}$$

where the positive constant $D(\rho,\lambda)$ can be worked out as well. It appears to be sharper than (4.1) at the points which are not too close to the endpoints $x = \pm 1$. A similar remark also applies to the Gegenbauer-Gauss-Lobatto interpolation to be addressed in a minute. □

Now, we turn to the estimate of spectral differentiation.

Theorem 4.2. *Let u be analytic on and within the ellipse $\mathcal{E}_\rho$ with foci ± 1 and $\rho > 1$ as defined in (2.15), and let $(I_n u)(x)$ be the interpolant of $u(x)$ at $(n+1)$ Gegenbauer-Gauss points $\{x_j\}_{j=0}^n$. Then we have*

$$\max_{0\le j\le n} \big|(u - I_n u)'(x_j)\big| \le \Lambda(\rho,\lambda)\frac{n^{\lambda+2}}{\rho^n}, \tag{4.10}$$

where the constant

$$\Lambda(\rho,\lambda) = \frac{2c\Gamma(\lambda+1)M_\rho\sqrt{\rho^2+\rho^{-2}}}{\Gamma(2\lambda+2)(\rho-1)^2}\begin{cases}(1+\rho^{-2})^\lambda, & \text{if } \lambda > 0,\\ (1-\rho^{-2})^\lambda, & \text{if } \lambda < 0,\end{cases} \tag{4.11}$$

and c, M_ρ are the same as in Theorem 4.1.

Proof. In view of (2.19) and (2.20), it is enough to replace x and $C_{n+1}^\lambda(x)$ by x_j and $\frac{d}{dx}C_{n+1}^\lambda(x)$, respectively, in (4.3). Thus, we have

$$\begin{aligned}\big|(u - I_n u)'(x_j)\big| &\le \frac{M_\rho}{2\pi}\frac{|(C_{n+1}^\lambda)'(x_j)|}{\min_{z\in\mathcal{E}_\rho}|C_{n+1}^\lambda(z)|}\oint_{\mathcal{E}_\rho}\frac{|dz|}{|z-x_j|}\\ &\le \frac{M_\rho\sqrt{\rho^2+\rho^{-2}}}{\rho+\rho^{-1}-2}\frac{|(C_{n+1}^\lambda)'(x_j)|}{\min_{z\in\mathcal{E}_\rho}|C_{n+1}^\lambda(z)|}, \quad 0\le j\le n.\end{aligned} \tag{4.12}$$

By (2.6) and (4.7),

$$\max_{|x|\le 1}\big|(C_{n+1}^\lambda)'(x)\big| = 2|\lambda||C_n^{\lambda+1}(1)| \cong \frac{2|\lambda|}{\Gamma(2\lambda+2)}n^{2\lambda+1}, \tag{4.13}$$

which, together with (4.6) and (4.12), leads to the desired estimate. ☐

Remark 4.2. Obviously, by (2.19),

$$(u - I_n u)'(x) = \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{E}_\rho}\Big(\frac{(C_{n+1}^\lambda)'(x)}{z-x} + \frac{C_{n+1}^\lambda(x)}{(z-x)^2}\Big)\frac{u(z)}{C_{n+1}^\lambda(z)}dz. \tag{4.14}$$

If $x \ne x_j$, we need to estimate the second term in the summation, which can be done in the same fashion in the proof of Theorem 4.1. The first term is actually estimated above. Consequently, we have

$$\max_{|x|\le 1}\big|(u - I_n u)'(x)\big| \le \Lambda(\rho,\lambda)\frac{n^{\lambda+2}}{\rho^n} + \frac{1}{\delta_\rho}\max_{|x|\le 1}\big|(u - I_n u)(x)\big|,$$

where δ_ρ is given by (2.18). Hence, by Theorem 4.1,

$$\max_{|x|\le 1}\big|(u - I_n u)'(x)\big| = O\Big(\frac{n^{\lambda+2}}{\rho^n}\Big). \tag{4.15}$$

In fact, the results for higher-order derivatives can be derived recursively, and it is anticipated that

$$\max_{|x|\le 1}\big|(u - I_n u)^{(k)}(x)\big| = O\Big(\frac{n^{\lambda+2k}}{\rho^n}\Big), \quad k \ge 1. \tag{4.16}$$

A similar remark applies to the Gegenbauer-Gauss-Lobatto case below. ☐

4.2. **Gegenbauer-Gauss-Lobatto interpolation and differentiation.** We are now in a position to estimate the Gegenbauer-Gauss-Lobatto interpolation and spectral differentiation. In this case, $Q_{n+1}(x) = (1-x^2)C_{n-1}^{\lambda+1}(x)$ in (2.19)-(2.20). The main result is stated as follows.

Theorem 4.3. *Let u be analytic on and within the ellipse $\mathcal{E}_\rho$ with foci ± 1 and $\rho > 1$ as defined in* (2.15)*, and let $(I_n u)(x)$ be the interpolant of $u(x)$ at the set of $(n+1)$ Gegenbauer-Gauss-Lobatto points.*

(a) *We have the interpolation error:*

$$\max_{|x|\le 1} \big|(u - I_n u)(x)\big| \le \frac{4cM_\rho\sqrt{\rho^2+\rho^{-2}}(1+\rho^{-2})^{\lambda+1}}{(1-\rho^{-1})^2(\rho-\rho^{-1})^2} \frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+2)} \frac{n^{\lambda+1}}{\rho^n}. \tag{4.17}$$

(b) *We have the estimate:*

$$\max_{0\le j\le n} \big|(u - I_n u)'(x_j)\big| \le \frac{8cM_\rho\sqrt{\rho^2+\rho^{-2}}(1+\rho^{-2})^{\lambda+1}}{(1-\rho^{-1})^2(\rho-\rho^{-1})^2} \frac{\Gamma(\lambda+2)}{\Gamma(2\lambda+4)} \frac{n^{\lambda+3}}{\rho^n}. \tag{4.18}$$

Here, $c \cong 1$ and $M_\rho = \max_{z\in\mathcal{E}_\rho} |u(z)|$.

Proof. For any $z \in \mathcal{E}_\rho$, one verifies that

$$\frac{1}{4}(\rho-\rho^{-1})^2 \le |z^2 - 1| \le \frac{1}{4}(\rho+\rho^{-1})^2$$

and

$$\min_{z\in\mathcal{E}_\rho} \big|(1-z^2)C_{n-1}^{\lambda+1}(z)\big| \ge \frac{1}{4}(\rho-\rho^{-1})^2 \min_{z\in\mathcal{E}_\rho} \big|C_{n-1}^{\lambda+1}(z)\big|. \tag{4.19}$$

(a) By (2.19) with $Q_{n+1}(x) = (1-x^2)C_{n-1}^{\lambda+1}(x)$ and (2.17)-(2.18), we have the bound of the pointwise error:

$$\begin{aligned} \big|(u - I_n u)(x)\big| &\le \frac{\big|(1-x^2)C_{n-1}^{\lambda+1}(x)\big|}{2\pi} \frac{\max_{z\in\mathcal{E}_\rho}|u(z)|}{\min_{z\in\mathcal{E}_\rho}\big|(1-z^2)C_{n-1}^{\lambda+1}(z)\big|} \oint_{\mathcal{E}_\rho} \frac{|dz|}{|z-x|} \\ &\overset{(4.19)}{\le} \frac{M_\rho\sqrt{\rho^2+\rho^{-2}}}{\rho+\rho^{-1}-2} \frac{4(\rho-\rho^{-1})^{-2}|C_{n-1}^{\lambda+1}(x)|}{\min_{z\in\mathcal{E}_\rho}|C_{n-1}^{\lambda+1}(z)|}, \quad x\in[-1,1],\ n\ge 0. \end{aligned} \tag{4.20}$$

Thus, the estimate (4.17) follows from (4.6) and (4.7).

(b) Similarly, we have

$$\begin{aligned} \big|(u - I_n u)'(x_j)\big| &\le \frac{M_\rho}{2\pi} \frac{\big|[(1-x^2)C_{n-1}^{\lambda+1}(x)]'(x_j)\big|}{\min_{z\in\mathcal{E}_\rho}\big|(1-z^2)C_{n-1}^{\lambda+1}(z)\big|} \oint_{\mathcal{E}_\rho} \frac{|dz|}{|z-x_j|} \\ &\le \frac{4M_\rho\sqrt{\rho^2+\rho^{-2}}}{\rho+\rho^{-1}-2} \frac{\big|[(1-x^2)C_{n-1}^{\lambda+1}(x)]'(x_j)\big|}{(\rho-\rho^{-1})^2\min_{z\in\mathcal{E}_\rho}\big|C_{n-1}^{\lambda+1}(z)\big|}, \quad 0\le j\le n. \end{aligned} \tag{4.21}$$

A direct calculation leads to

$$\big[(1-x^2)C_{n-1}^{\lambda+1}(x)\big]' = -2xC_{n-1}^{\lambda+1}(x) + (1-x^2)\big[C_{n-1}^{\lambda+1}(x)\big]', \tag{4.22}$$

which, together with (4.7) and (4.13), gives

$$\max_{|x|\le 1} \big|[(1-x^2)C_{n-1}^{\lambda+1}(x)]'\big| \cong \frac{2(\lambda+1)}{\Gamma(2\lambda+4)} n^{2\lambda+3}. \tag{4.23}$$

A combination of (4.6), (4.21) and (4.23) yields the desired estimate. ◻

Remark 4.3. Similar to Remarks 4.1 and 4.2, we can derive a sharper pointwise estimate and analyze higher-order derivatives of interpolation errors. □

4.3. **Analysis of quadrature errors.** Recall the interpolatory Gegenbauer-Gauss-type quadrature formula:

$$\int_{-1}^{1} u(x)(1-x^2)^{\lambda-1/2}dx \approx \sum_{j=0}^{n} u(x_j)\omega_j = \int_{-1}^{1} (I_n u)(x)(1-x^2)^{\lambda-1/2}dx, \tag{4.24}$$

where the quadrature weights $\{\omega_j\}_{j=0}^{n}$ are expressed by the Lagrange basis polynomials (see, e.g., [40, 19]). Observe that

$$\Big|\int_{-1}^{1} (u-I_n u)(x)(1-x^2)^{\lambda-1/2}dx\Big| \le h_0^\lambda \max_{|x|\le 1} |(u-I_n u)(x)|, \tag{4.25}$$

where h_0^λ is given by (2.4). With the aid of interpolation error estimates in Theorem 4.1 and Theorem 4.3, we are able to derive the quadrature errors immediately.

4.4. **Numerical results.** In what follows, we provide two numerical examples to demonstrate the sharpness of the estimates established in Theorems 4.2 and 4.3.

4.4.1. *Example 1.* We take

$$u(x) = \frac{1}{x^2+1}, \tag{4.26}$$

which has two simple poles at $\pm$i. By (2.16), we are free to choose $\mathcal{E}_\rho$ with ρ in the range

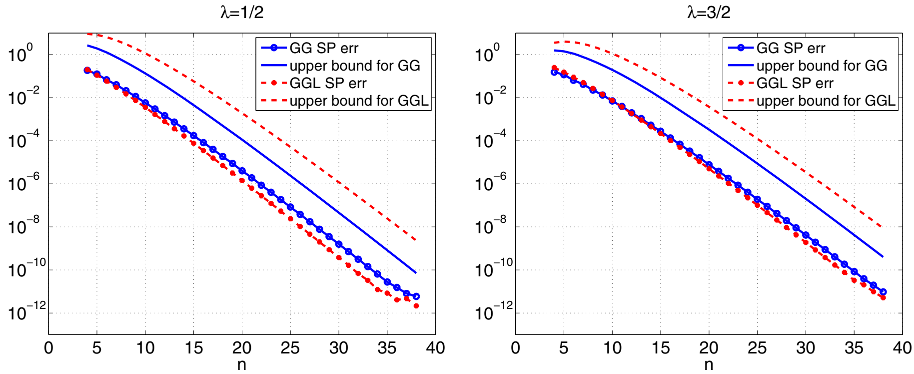
$$1 < \rho < 1+\sqrt{2} \approx 2.414, \tag{4.27}$$

such that u is analytic on and within $\mathcal{E}_\rho$. To compare the (discrete) maximum error of spectral differentiation with the upper bound, we sample about 2000 values of ρ equally from $(1, 1+\sqrt{2})$, and find a tighter upper bound (which is usually attained when ρ is close to $1+\sqrt{2}$). In Figure 4.1 (a)-(b), we plot the (discrete) maximum errors of spectral differentiation against the upper bounds. We visualize the exponential decay of the errors, and the upper bounds and the errors decay at almost the same rate. Moreover, it seems that the bounds are slightly sharper in the Gegenbauer-Gauss case.

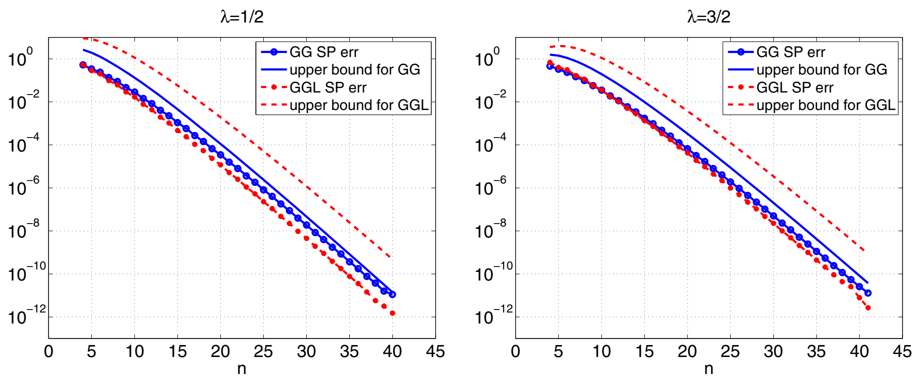
4.4.2. *Example 2.* The estimates indicate that the errors essentially depend on the location of singularity (although it affects the constant M_ρ) rather than the behavior of the singularity. To show this, we test the function with poles at $\pm$i of order 2:

$$u(x) = \frac{1}{(x^2+1)^2}. \tag{4.28}$$

We plot in Figure 4.1 (c)-(d) the errors and upper bounds as in (a)-(b). Indeed, a similar convergence behavior is observed. Indeed, Boyd [6] pointed out that the type of singularity might change the rate of convergence by a power of n, but not an exponential function of n.



(a) Upper bound vs Max.-error (Example 1) (b) Upper bound vs Max.-error (Example 1)



(c) Upper bound vs Max.-error (Example 2) (d) Upper bound vs Max.-error (Example 2)

FIGURE 4.1. The (discrete) maximum errors of Gegenbauer-Gauss and Gegenbauer-Gauss-Lobatto spectral differentiation against the upper bounds in Theorems 4.2 and 4.3 with $\lambda = 1/2, 3/2$. Example 1 (a)-(b), and Example 2 (c)-(d). In the legend, GG SP err (resp. GGL SP err) represents the Gegenbauer-Gauss (resp. Gegenbauer-Gauss-Lobatto) spectral differentiation error.

Concluding remarks

In this paper, the exponential convergence of Gegenbauer interpolation, spectral differentiation and quadrature of functions analytic on and within a sizable ellipse is analyzed. Sharp estimates in the maximum norm with explicit dependence on the important parameters are obtained. Illustrative numerical results are provided to support the analysis. For clarity of presentation, it is assumed that λ is fixed in our analysis, but the dependence of the error on this parameter can also be tracked if it is necessary.

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Appendix A. Proof of Lemma 3.1

We carry out the proof by induction.

Apparently, by (2.1), $C_0^\lambda(z) = 1$, so (3.1)-(3.2) holds for $n = 0$. Similarly, we can verify the case with $n = 1$.

Assume that the formula holds for $C_{n-2}^\lambda(z)$ and $C_{n-1}^\lambda(z)$ with $n \geq 2$. It follows from the three-term recurrence relation (2.1) that

$$\begin{aligned} nC_n^\lambda(z) &= 2(n+\lambda-1)z\sum_{k=0}^{n-1} g_k^\lambda g_{n-1-k}^\lambda w^{n-2k-1} \\ &\quad - (n+2\lambda-2)\sum_{k=0}^{n-2} g_k^\lambda g_{n-2-k}^\lambda w^{n-2k-2} \\ &= (n+\lambda-1)\sum_{k=0}^{n-1} g_k^\lambda g_{n-1-k}^\lambda w^{n-2k} + (n+\lambda-1)\sum_{k=0}^{n-1} g_k^\lambda g_{n-1-k}^\lambda w^{n-2k-2} \\ &\quad - (n+2\lambda-2)\sum_{k=0}^{n-2} g_k^\lambda g_{n-2-k}^\lambda w^{n-2k-2} \\ &= (n+\lambda-1)g_{n-1}^\lambda(w^n + w^{-n}) + \sum_{k=1}^{n-1} D_{n,k}^\lambda\, g_k^\lambda g_{n-k}^\lambda w^{n-2k}, \end{aligned}$$

where

$$D_{n,k}^\lambda = (n+\lambda-1)\left(\frac{g_{n-k-1}^\lambda}{g_{n-k}^\lambda} + \frac{g_{k-1}^\lambda}{g_k^\lambda}\right) - (n+2\lambda-2)\frac{g_{n-k-1}^\lambda}{g_{n-k}^\lambda}\frac{g_{k-1}^\lambda}{g_k^\lambda}.$$

One verifies from (3.2) that

$$g_n^\lambda = \frac{n+\lambda-1}{n} g_{n-1}^\lambda, \quad D_{n,k}^\lambda = n. \tag{A.1}$$

This completes the induction. □

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School of Mathematics and Computer Science, Guizhou Normal University, Guiyang, Guizhou 550001, China — and — Key Laboratory of High Performance Computing and Stochastic Information Processing (Ministry of Education of China), College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China

E-mail address: ziqingxie@yahoo.com.cn

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore

E-mail address: lilian@ntu.edu.sg

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore

E-mail address: zhao0122@e.ntu.edu.sg