NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **x**, No. x, pp. 1-22 (20xx)

On Spectral Approximations by Generalized Slepian Functions

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Abstract. We introduce a family of orthogonal functions, termed as generalized Slepian functions (GSFs), closely related to the time-frequency concentration problem on a unit disk in D. Slepian [19]. These functions form a complete orthogonal system in $L^2_{\varpi_\alpha}(-1,1)$ with $\varpi_\alpha(x)=(1-x)^\alpha$, $\alpha>-1$, and can be viewed as a generalization of the Jacobi polynomials with parameter $(\alpha,0)$. We present various analytic and asymptotic properties of GSFs, and study spectral approximations by such functions.

AMS subject classifications: 33E30, 33C47, 41A30, 65N35

Key words: Generalized Slepian functions, orthogonal systems, approximation errors, spectral accuracy

1. Introduction

The investigation of time-frequency concentration problem back to 1960s gives rises to some interesting special functions with attractive properties. The most significant ones (see a series of papers by Slepian et al. [16, 17, 20]) are known as the prolate spheriodal wave functions (PSWFs) or Slepian functions, which are bandlimited and mostly time-concentrated within a finite interval. This discovery has motivated many subsequential research works in various directions (see, e.g., [4, 5, 7, 10, 12, 14, 15, 22–24, 27]).

In a very recent work [25], we introduced a family of generalized PSWFs as the eigenfunctions of a singular Sturm-Liouville problem, and interestingly, they are also the eigenfunctions of an integral operator. This orthogonal system is complete in $L^2_{w_\alpha}(-1,1)$ with $w_\alpha(x)=(1-x^2)^\alpha, \alpha>-1$, and generalizes both the PSWFs (from order zero to order α) and the Gegenbauer polynomials (to a system with a bandwidth tuning parameter). However, this study could not cover the case when the weight function is nonsymmetric (i.e., the Jacobi weight function $w_{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ with $\alpha\neq\beta$). Indeed, it seems implausible to generate an orthogonal system of functions which is simultaneously the eigenfunctions of a second-order differential operator and an integral operator with complex exponential kernel, based on the argument in [25].

In this paper, we explore such a generalization, but restrict our discussion to the nonsymmetric Jacobi weight $\varpi_\alpha(x)=(1-x)^\alpha$ with $\alpha>-1$. More precisely, we define the

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orthogonal system as the eigenfunctions of a Sturm-Liouville problem (see (2.9) below), and show that they satisfy an integral equation which has a close relation with the time-frequency concentration problem over a unit disk studied in D. Slepian [19] (so we term this new family of orthogonal functions as generalized Slepian functions (GSFs)). We derive some analytic and asymptotic properties of the GSFs and their associated eigenvalues, and study spectral approximations of functions in $L^2_{\varpi_\alpha}(-1,1)$ using the GSFs as basis functions.

The paper is organized as follows. In section 2, we define the GSFs and describe the algorithm for their evaluation. We present various properties in section 3, and derive the spectral approximation results using the GSFs in section 4, together with some numerical experiments to support the analysis.

2. Generalized Slepian functions

In this section, we define the generalized Slepian functions, and introduce an efficient algorithm for their numerical evaluation.

2.1. Jacobi polynomials

We first review some properties of the Jacobi polynomials (cf. [21]). For $\alpha,\beta > -1$, the Jacobi polynomials, denoted by $J_n^{(\alpha,\beta)}(x), x\in I := (-1,1)$, are orthogonal with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$; namely,

$$\int_I J_m^{(\alpha,\beta)}(x) J_n^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\,dx = 0, \quad \text{if } m \neq n.$$

In this paper, we mainly use the Jacobi polynomials with $\beta = 0$, and particularly, denote $J_n^{(\alpha)}(x) := J_n^{(\alpha,0)}(x)$ and $\varpi_\alpha(x) = w_{\alpha,0}(x) = (1-x)^\alpha$. We further assume that they are normalized so that

$$\int_I J_m^{(\alpha)}(x) J_n^{(\alpha)}(x)\varpi_\alpha(x)\,dx = \delta_{mn}, \tag{2.1}$$

where δ_{mn} is the Kronecker delta symbol. The Jacobi polynomials $\{J_n^{(\alpha)}\}$ are the eigenfunctions of the Sturm-Liouville problem

$$\mathscr{L}_x^{(\alpha)}\big[J_n^{(\alpha)}\big] := -\varpi_{-\alpha}\partial_x\big((1-x^2)\varpi_\alpha\partial_x J_n^{(\alpha)}\big) = \gamma_n^{(\alpha)} J_n^{(\alpha)}, \quad x\in I, \tag{2.2}$$

with the corresponding eigenvalues $\gamma_n^{(\alpha)} = n(n+\alpha+1)$. Hereafter, we use ∂_x to denote the ordinary derivative $\frac{d}{dx}$, and likewise for higher-order ordinary derivatives.

Recall that $\{J_n^{(\alpha)}\}$ satisfy the three-term recurrence relation:

$$\begin{aligned} &xJ_n^{(\alpha)}(x) = a_n J_{n-1}^{(\alpha)}(x) + b_n J_n^{(\alpha)}(x) + c_n J_{n+1}^{(\alpha)}(x), \quad n\geq 1,\\ &J_0^{(\alpha)}(x) = \sqrt{\frac{\alpha+1}{2^{\alpha+1}}}, \quad J_1^{(\alpha)}(x) = \sqrt{\frac{\alpha+3}{2^{\alpha+3}}}\,(\alpha+(\alpha+2)x). \end{aligned} \tag{2.3}$$

where

$$a_n = \frac{2n(n+\alpha)}{(2n+\alpha)\sqrt{(2n+\alpha-1)(2n+\alpha+1)}},$$
$$b_n = \frac{-\alpha^2}{(2n+\alpha)(2n+\alpha+2)}, \quad c_n = a_{n+1}. \tag{2.4}$$

The leading coefficient of $J_n^{(\alpha)}(x)$ (i.e., the coefficient of x^n) is

$$k_n^{(\alpha)} = \frac{\sqrt{2n+\alpha+1}\,\Gamma(2n+\alpha+1)}{2^{n+(\alpha+1)/2} n!\,\Gamma(n+\alpha+1)}. \tag{2.5}$$

Moreover, we have

$$J_n^{(\alpha)}(1) = \frac{\sqrt{2n+\alpha+1}\,\Gamma(n+\alpha+1)}{2^{(\alpha+1)/2} n!\,\Gamma(\alpha+1)}. \tag{2.6}$$

2.2. Definition of GSFs

Define the second-order differential operator

$$\begin{aligned}\mathscr{D}_x &:= \mathscr{D}_x(\alpha, c) = \mathscr{L}_x^{(\alpha)} + \frac{c^2}{2}(1-x)\\ &= -(1-x^2)\partial_x^2 + (\alpha + (\alpha+2)x)\partial_x + \frac{c^2}{2}(1-x), \quad x \in I,\end{aligned} \tag{2.7}$$

where $\alpha > -1$, $c > 0$ and $\mathscr{L}_x^{(\alpha)}$ is defined in (2.2). It is clear that $\mathscr{D}_x$ is a strictly positive self-adjoint operator in the sense that for any u and v in the domain of $\mathscr{D}_x$,

$$\left(\mathscr{D}_x u, v\right)_{\varpi_\alpha} = \left(u, \mathscr{D}_x v\right)_{\varpi_\alpha}, \ \left(\mathscr{D}_x u, u\right)_{\varpi_\alpha} = \left\|\sqrt{1+x}\,\partial_x u\right\|^2_{\varpi_{\alpha+1}} + \frac{c^2}{2}\|u\|^2_{\varpi_{\alpha+1}} \geq 0, \tag{2.8}$$

where for a generic weight function w, $(\cdot,\cdot)_w$ and $\|\cdot\|_w$ denote the inner product and the norm of the weighted space $L^2_w(I)$, respectively. Hence, by the Sturm-Louville theory (cf. [3, 11]), the operator $\mathscr{D}_x$ admits a countable and infinite set of bounded, analytical eigenfunctions, denoted by $\{\varphi_n^{(\alpha)}(x;c)\}_{n=0}^{\infty}$, which forms a complete orthogonal system of $L^2_{\varpi_\alpha}(I)$. Thus, we have

$$\mathscr{D}_x \varphi_n^{(\alpha)} = \chi_n^{(\alpha)} \varphi_n^{(\alpha)}, \quad n \geq 0, \ x \in I, \ c \geq 0, \tag{2.9}$$

where $\{\chi_n^{(\alpha)} := \chi_n^{(\alpha)}(c)\}_{n=0}^{\infty}$ are the corresponding eigenvalues. We define $\varphi_n^{(\alpha)}(x;c)$ as the *generalized Slepian function of order α and of degree n*. Moreover, if $c = 0$, the eigen-problem (2.9) is reduced to (2.2), and therefore we have

$$\varphi_n^{(\alpha)}(x;0) = J_n^{(\alpha)}(x), \quad \chi_n^{(\alpha)}(0) = \gamma_n^{(\alpha)} = n(n+\alpha+1). \tag{2.10}$$

In view of this, the GSFs can be viewed as a generalization of $\{J_n^{(\alpha)}\}$, but equipped with a tuning parameter c. It's worthwhile to point out that when $\alpha = 0$, $\varphi_n^{(\alpha)}(x;c)$ is different from the prolate spheroidal wave functions (cf. [20]), since the latter family is the eigenfunctions of the operator $\mathscr{D}_x$ $(\alpha = 0)$ with c^2x^2 in place of $c^2(1-x)/2$.

The following properties can be derived from the general theory of Sturm-Louville problems (cf. [3, 11]). More precisely, for any $c > 0$ and $\alpha > -1$, we have

(i) $\{\varphi_n^{(\alpha)}\}_{n=0}^{\infty}$ are all real, smooth, and form a complete orthonormal system of $L^2_{\varpi_\alpha}(I)$. Thus,

$$\int_{-1}^{1} \varphi_m^{(\alpha)}(x;c)\varphi_n^{(\alpha)}(x;c)\varpi_\alpha(x)\,dx = \delta_{mn}. \tag{2.11}$$

(ii) $\{\chi_n^{(\alpha)}\}_{n=0}^{\infty}$ are all real, positive, simple and ordered as

$$0 < \chi_0^{(\alpha)}(c) < \chi_1^{(\alpha)}(c) < \cdots < \chi_n^{(\alpha)}(c) < \cdots. \tag{2.12}$$

(iii) $\varphi_n^{(\alpha)}$ has exactly n distinct zeros on the interval $[-1,1]$, which lie in $(-1,1)$.

For $\alpha > -1$ and $c > 0$, we define the integral operator $Q_c^{(\alpha)}$:

$$Q_c^{(\alpha)}[\phi](x) = \int_{-1}^{1} \mathrm{K}_c^{(\alpha)}(x,t)\phi(t)\varpi_\alpha(t)\,dt, \quad \forall \phi \in L^2_{\varpi_\alpha}(I), \;\; x \in I, \tag{2.13}$$

where

$$\mathrm{K}_c^{(\alpha)}(x,t) = \frac{\mathscr{J}_\alpha\big(c\sqrt{(1-x)(1-t)}\big)}{\big(c\sqrt{(1-x)(1-t)}\big)^{\alpha}}, \tag{2.14}$$

and $\mathscr{J}_\alpha(\cdot)$ is the Bessel function of the first kind. We find from the asymptotic properties (cf. [26]):

$$\mathscr{J}_\alpha(z) = O(z^\alpha), \quad z \to 0^+; \qquad \mathscr{J}_\alpha(z) = O(z^{-1/2}), \quad z \to +\infty.$$

The kernel function $\mathrm{K}_c^{(\alpha)}(x,t)$ in (2.14) is well-defined for $x = t$. Moreover, one verifies that $Q_c^{(\alpha)} : L^2_{\varpi_\alpha}(I) \to L^2_{\varpi_\alpha}(I)$ is compact.

A remarkable property of the GSFs is that they are the eigenfunctions of $Q_c^{(\alpha)}$.

Theorem 2.1. *For any $c > 0$, the GSFs $\{\varphi_n^{(\alpha)}\}_{n=0}^{\infty}$ are the eigenfunctions of $Q_c^{(\alpha)}$:*

$$Q_c^{(\alpha)}[\varphi_n^{(\alpha)}] = \nu_n^{(\alpha)}\varphi_n^{(\alpha)}, \tag{2.15}$$

where $\{\nu_n^{(\alpha)} := \nu_n^{(\alpha)}(c)\}$ are the corresponding eigenvalues.

Proof. An essential step is to show that

$$\mathscr{D}_x \mathrm{K}_c^{(\alpha)}(x,t) = \frac{\mathscr{J}'_\alpha(z)}{2z^{\alpha-1}} + \Big(\frac{c^2}{4}(3-x-t-xt) - \frac{\alpha}{2}\Big)\frac{\mathscr{J}_\alpha(z)}{z^\alpha} = \mathscr{D}_t \mathrm{K}_c^{(\alpha)}(x,t), \tag{2.16}$$

where $\mathscr{D}_x$ is the differential operator defined in (2.7), and $z = c\sqrt{(1-x)(1-t)}$. In this proof, $\mathscr{J}'_\alpha$ (resp. $\partial_x \mathscr{J}_\alpha$) means the derivative (resp. partial derivative) with respect to z (resp. x). A direct calculation yields

$$\begin{aligned} z^\alpha \mathscr{D}_x \mathrm{K}_c^{(\alpha)}(x,t) &= -(1-x^2)\partial_x^2 \mathscr{J}_\alpha(z) + 2x\partial_x \mathscr{J}_\alpha(z) \\ &\quad + \Big(\frac{\alpha^2(1+x)}{4(1-x)} + \frac{c^2}{2}(1-x) - \frac{\alpha}{2}\Big)\mathscr{J}_\alpha(z). \end{aligned} \tag{2.17}$$

Using the fact

$$\partial_x \mathscr{J}_\alpha(z) = -\frac{c}{2}\sqrt{\frac{1-t}{1-x}}\mathscr{J}'_\alpha(z), \tag{2.18}$$

and the property of the Bessel functions (cf. [26]):

$$\mathscr{J}''_\alpha(z) + \frac{1}{z}\mathscr{J}'_\alpha(z) + \Big(1 - \frac{\alpha^2}{z^2}\Big)\mathscr{J}_\alpha(z) = 0, \quad z > 0, \ \ \alpha > -1, \tag{2.19}$$

we find

$$\begin{aligned}\partial_x^2 \mathscr{J}_\alpha(z) &= \frac{c^2(1-t)}{4(1-x)}\Big(\mathscr{J}''_\alpha(z) - \frac{\mathscr{J}'_\alpha(z)}{z}\Big)\\ &= -\frac{c^2(1-t)}{4(1-x)}\Big(\frac{2}{z}\mathscr{J}'_\alpha(z) + \Big(1-\frac{\alpha^2}{z^2}\Big)\mathscr{J}_\alpha(z)\Big).\end{aligned} \tag{2.20}$$

Inserting (2.18) and (2.20) into (2.17) leads to the formula for $\mathscr{D}_x \mathrm{K}_c^{(\alpha)}$ in (2.16). Since $\mathrm{K}_c^{(\alpha)}(x,t) = \mathrm{K}_c^{(\alpha)}(t,x)$, we interchange x and t, and derive $\mathscr{D}_x\mathrm{K}_c^{(\alpha)} = \mathscr{D}_t\mathrm{K}_c^{(\alpha)}$ in (2.16).

We obtain from (2.8)-(2.9) and (2.16) that

$$\begin{aligned}\chi_n^{(\alpha)}\int_{-1}^1 \mathrm{K}_c^{(\alpha)}(x,t)\varphi_n^{(\alpha)}(t;c)\varpi_\alpha(t)dt &\overset{(2.9)}{=} \int_{-1}^1 \mathrm{K}_c^{(\alpha)}(x,t)\mathscr{D}_t\varphi_n^{(\alpha)}(t;c)\varpi_\alpha(t)\,dt\\ &\overset{(2.8)}{=} \int_{-1}^1 \varphi_n^{(\alpha)}(t;c)\mathscr{D}_t\mathrm{K}_c^{(\alpha)}(x,t)\varpi_\alpha(t)\,dt\\ &\overset{(2.16)}{=} \int_{-1}^1 \varphi_n^{(\alpha)}(t;c)\mathscr{D}_x\mathrm{K}_c^{(\alpha)}(x,t)\varpi_\alpha(t)\,dt\\ &= \mathscr{D}_x\int_{-1}^1 \mathrm{K}_c^{(\alpha)}(x,t)\varphi_n^{(\alpha)}(t;c)\varpi_\alpha(t)\,dt,\end{aligned}$$

or equivalently,

$$\mathscr{D}_x\Big(\mathrm{Q}_c^{(\alpha)}[\varphi_n^{(\alpha)}]\Big) = \chi_n^{(\alpha)}\mathrm{Q}_c^{(\alpha)}[\varphi_n^{(\alpha)}].$$

This means $Q_c^{(\alpha)}[\varphi_n^{(\alpha)}]$ is an eigenfunction of $\mathscr{D}_x$ with the corresponding eigenvalue $\chi_n^{(\alpha)}$, so it must be proportional to $\varphi_n^{(\alpha)}$. We denote the proportional constant by $\nu_n^{(\alpha)}$, so (2.15) follows. □

Remark 2.1. Slepian [19] discussed the time-frequency concentration problem on a unit disk:

$$\gamma\psi(x,y) = \int_D e^{\mathrm{i}c(x\xi+y\eta)}\psi(\xi,\eta)d\xi d\eta, \quad c>0,$$

where $D = \{(x,y) \ : \ x^2+y^2 \le 1\}$. This induces a family of orthogonal functions which are eigenfunctions of the integral operator:

$$\mu\phi(r) = \int_0^1 \mathscr{J}_\alpha(crs)\sqrt{crs}\phi(s)\,ds, \quad 0 \le r \le 1. \tag{2.21}$$

Such a family is closely related to the Jacobi polynomials $\{x^{\alpha+1/2}J_n^{(\alpha,0)}(1-2x^2)\}$ (where $x \in (-1,1)$). Indeed, by using a suitable transform, we realize that the problem (2.21) can be converted to an eigen-problem similar to (2.13). □

2.3. Computation of the GSFs and the eigenvalues

Next, we introduce an efficient algorithm for numerical evaluation of the GSFs and the corresponding eigenvalues.

Since the GSFs are analytic, an efficient approach to use the Jacobi spectral-Galerkin methods as with the Bouwkamp-type algorithm (cf. [6,8,27]). More precisely, for any fixed $n \geq 0$, we write

$$\varphi_n^{(\alpha)}(x;c) = \sum_{k=0}^{\infty} \beta_k^n J_k^{(\alpha)}(x), \tag{2.22}$$

where

$$\beta_k^n := \beta_k^n(c) = \int_{-1}^{1} \varphi_n^{(\alpha)}(x;c) J_k^{(\alpha)}(x) \varpi_\alpha(x)\, dx. \tag{2.23}$$

Substituting it into (2.9) and using the properties (2.2) and (2.3), we obtain the following equivalent eigen-problem:

$$(\boldsymbol{A} - \chi_n^{(\alpha)} \cdot \boldsymbol{I})\vec{\beta}^n = 0, \tag{2.24}$$

where $\vec{\beta}^n = (\beta_0^n, \beta_1^n, \beta_2^n, \cdots)^t \in l^2$ and $\boldsymbol{A}$ is a symmetric tri-diagonal matrix. The non-zero entries of $\boldsymbol{A}$ are given by

$$A_{k,k} = k(k+\alpha+1) + c^2(1-b_k)/2; \quad A_{k,k+1} = A_{k+1,k} = -c^2 a_{k+1}/2. \tag{2.25}$$

The linear system (2.24) involves infinitely many unknowns, so an appropriate truncation is necessary. Following the rule for the PSWFs in [8], we suggest a cutoff $m = 2n + 2[\alpha] + 30$ for the computation of $\{\varphi_l^{(\alpha)}(x;c), \chi_l^{(\alpha)}(c)\}_{l=0}^n$. Notice that $\varphi_n^{(\alpha)}$ is sufficiently smooth, so this could lead to a very accurate evaluation for c (feasible for approximating general functions in $L^2_{\varpi_\alpha}(I)$).

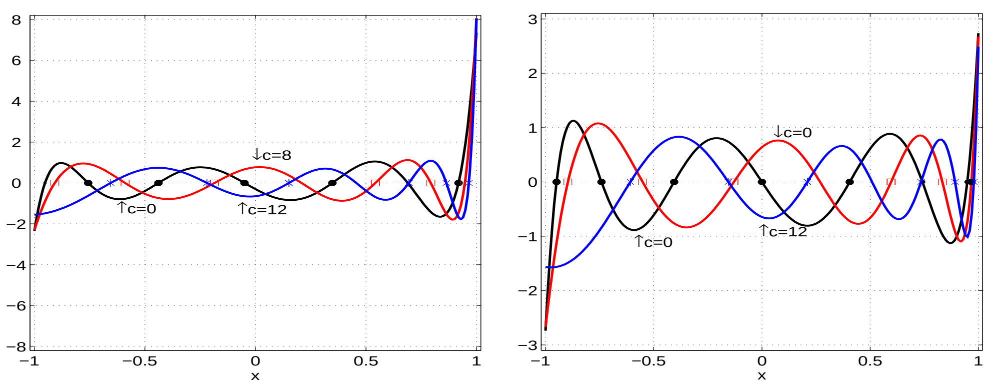


Figure 2.1: Graph of $\varphi_7^{(\alpha)}(x;c)$ with $c = 0, 8, 12$. Left: $\alpha = 0.5$. Right: $\alpha = 0$.

In Figure 2.1, we plot several samples of GSFs, and see that as c increases the GSFs oscillate more and more uniformly. Such a behavior is analogous to the PSWFs (see, e.g., [7]).

Now, we turn to the evaluation of the eigenvalues $\{\nu_n^{(\alpha)}(c)\}$ of the integral operator (2.13). The computation is based on the following explicit formula.

Theorem 2.2. *For any $\alpha > -1$ and $c > 0$,*

$$\nu_n^{(\alpha)}(c) = \frac{2^{\frac{1-\alpha}{2}} \beta_0^n}{\sqrt{\alpha+1}\,\Gamma(\alpha+1)\varphi_n^{(\alpha)}(1;c)}, \tag{2.26}$$

where β_0^n is the first expansion coefficient given by (2.23).

Proof. By Formula (9.1.10) of [1],

$$\mathscr{J}_\alpha(z) = \frac{z^\alpha}{2^\alpha}\left\{\frac{1}{\Gamma(\alpha+1)} - \frac{z^2}{4\Gamma(\alpha+2)} + O(z^4)\right\}, \quad \alpha > -1.$$

Thus, by (2.13)-(2.15),

$$\begin{aligned}\nu_n^{(\alpha)}\varphi_n^{(\alpha)}(x;c) = & \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_{-1}^{1}\varphi_n^{(\alpha)}(t;c)\varpi_\alpha(t)\,dt \\ & - \frac{c^2(1-x)}{2^{\alpha+2}\Gamma(\alpha+2)}\int_{-1}^{1}\varphi_n^{(\alpha)}(t;c)\varpi_{\alpha+1}(t)\,dt + O((1-x)^2).\end{aligned}$$

Letting $x \to 1^-$ leads to

$$\begin{aligned}\nu_n^{(\alpha)}\varphi_n^{(\alpha)}(1;c) = & \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_{-1}^{1}\varphi_n^{(\alpha)}(t;c)\varpi_\alpha(t)\,dt \\ \overset{(2.23)}{=} & \frac{1}{2^\alpha\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\beta_k^n\int_{-1}^{1}J_k^{(\alpha)}(t)\varpi_\alpha(t)\,dt \\ \overset{(2.1)}{=} & \frac{\beta_0^n}{2^\alpha\Gamma(\alpha+1)}\int_{-1}^{1}J_0^{(\alpha)}(t)\varpi_\alpha(t)\,dt \overset{(2.3)}{=} \frac{2^{\frac{1-\alpha}{2}}\beta_0^n}{\sqrt{\alpha+1}\,\Gamma(\alpha+1)}.\end{aligned}$$

Notice that $\varphi_n^{(\alpha)}(1;c) \neq 0$ (refer to Property (iii) below (2.12)), so we have (2.26). □

In Figure 3.1 (left), we depict the distribution of $\nu_n^{(\alpha)}(c)$ with $\alpha = 0.5$ and for $n \in [0, 100]$, and $c \in [0, 60]$, which shows that for fixed c, the eigenvalues decay exponentially with respect to n. Intuitively, $|\beta_0^n|$ must be sufficiently small for large n, since $\varphi_n^{(\alpha)}$ is analytic. A quantitative analysis will be conducted in the forthcoming section.

3. Properties of the GSFs and the eigenvalues

In this section, we derive more properties of the GSFs and the associated eigenvalues $\{\chi_n^{(\alpha)}\}$ and $\{\nu_n^{(\alpha)}\}$.

Theorem 3.1. *For any $\alpha > -1$ and $c > 0$,*

$$n(n+\alpha+1) < \chi_n^{(\alpha)}(c) < n(n+\alpha+1) + c^2, \quad n \geq 0. \tag{3.1}$$

Proof. Differentiating (2.9) with respect to c yields

$$\partial_x\big((1-x)^{\alpha+1}(1+x)\partial_x\partial_c\varphi_n^{(\alpha)}\big)+\Big(\chi_n^{(\alpha)}(c)-\frac{c^2}{2}(1-x)\Big)\varpi_\alpha\partial_c\varphi_n^{(\alpha)} \\ =\big(c(1-x)-\partial_c\chi_n^{(\alpha)}(c)\big)\varpi_\alpha\varphi_n^{(\alpha)}.$$

Multiplying the above equation by $\varphi_n^{(\alpha)}$, and integrating the resulting equation over $(-1,1)$, we derive from (2.9) and integration by parts that

$$\begin{aligned} c\int_{-1}^{1}&(1-x)[\varphi_n^{(\alpha)}(x;c)]^2\varpi_\alpha(x)dx-\frac{\partial\chi_n^{(\alpha)}(c)}{\partial c} \\ &=\int_{-1}^{1}\Big\{\partial_x\big((1-x)^{\alpha+1}(1+x)\partial_x\partial_c\varphi_n^{(\alpha)}(x;c)\big) \\ &\quad+\Big(\chi_n^{(\alpha)}(c)-\frac{c^2}{2}(1-x)\Big)\varpi_\alpha\partial_c\varphi_n^{(\alpha)}(x;c)\Big\}\varphi_n^{(\alpha)}(x;c)dx \\ &=\int_{-1}^{1}\Big\{\partial_x\big((1-x)^{\alpha+1}(1+x)\partial_x\varphi_n^{(\alpha)}(x;c)\big) \\ &\quad+\Big(\chi_n^{(\alpha)}(c)-\frac{c^2}{2}(1-x)\Big)\varpi_\alpha\varphi_n^{(\alpha)}(x;c)\Big\}\partial_c\varphi_n^{(\alpha)}(x;c)dx=0, \end{aligned}$$

which, together with (2.11), implies

$$\begin{aligned} 0<\frac{\partial\chi_n^{(\alpha)}}{\partial c}&=c\int_{-1}^{1}(1-x)[\varphi_n^{(\alpha)}(x;c)]^2\varpi_\alpha(x)dx<2c \\ &\Rightarrow\ 0<\chi_n^{(\alpha)}(c)-\chi_n^{(\alpha)}(0)<c^2. \end{aligned}$$

Since $\chi_n^{(\alpha)}(0)=n(n+\alpha+1)$, the desired result follows. □

For $0<c\ll 1$, the GSF $\varphi_n^{(\alpha)}(x;c)$ turns out to be a small perturbation of the Jacobi polynomial $J_n^{(\alpha)}(x)$, so is the eignvalue $\chi_n^{(\alpha)}(c)$ (a direct consequence of (3.1)). The following estimate follows from a perturbation method as described in [19], and a sketch of the proof is given in Appendix A for the readers' reference.

Lemma 3.1. *For any $\alpha>-1$ and $0<c\ll 1$,*

$$\varphi_n^{(\alpha)}(x;c)=J_n^{(\alpha)}(x)+O(c^2);\quad \chi_n^{(\alpha)}(c)=\gamma_n^{(\alpha)}+O(c^2),\quad n\ge 0. \tag{3.2}$$

With the aid the above lemma, we can show that the sequence $\{\nu_n^{(\alpha)}(c)\}$ is strictly decreasing with respect to n.

Theorem 3.2. *For any $\alpha>-1$ and $c>0$,*

$$\nu_n^{(\alpha)}(c)>\nu_{n+1}^{(\alpha)}(c)>0,\quad n=0,1,2,\cdots. \tag{3.3}$$

Proof. It is enough to verify this ordering for sufficiently small c. Indeed, if (3.3) is true for $0 < c \ll 1$, and if there exists a positive constant $\tilde{c} > c$ that violates this ordering, then we can find $c < c_* < \tilde{c}$ such that $\nu_n^{(\alpha)}(c_*) = \nu_{n+1}^{(\alpha)}(c_*)$. This contradicts to the fact that $\{\nu_n^{(\alpha)}\}$ are distinct.

To this end, we assume that $0 < c \ll 1$, and carry out the proof by using Lemma 3.1. Differentiating (2.15) with respect to x gives

$$\nu_n^{(\alpha)} \partial_x \varphi_n^{(\alpha)}(x;c) = \int_{-1}^{1} \varphi_n^{(\alpha)}(t;c) \varpi_\alpha(t) \partial_x \mathrm{K}_c^{(\alpha)}(x,t)\, dt, \tag{3.4}$$

and

$$\nu_{n+1}^{(\alpha)} \partial_x \varphi_{n+1}^{(\alpha)}(x;c) = \int_{-1}^{1} \varphi_{n+1}^{(\alpha)}(t;c) \varpi_\alpha(t) \partial_x \mathrm{K}_c^{(\alpha)}(x,t) dt. \tag{3.5}$$

It's clear that

$$\partial_x \mathrm{K}_c^{(\alpha)}(x,t) = \partial_t \mathrm{K}_c^{(\alpha)}(x,t). \tag{3.6}$$

Multiplying (3.4) by $\varphi_{n+1}^{(\alpha)} \varpi_\alpha$ and integrating the resulting equation over $(-1,1)$, we derive from (3.6) that

$$\begin{aligned}
&\nu_n^{(\alpha)} \int_{-1}^{1} \partial_x \varphi_n^{(\alpha)} \varphi_{n+1}^{(\alpha)} \varpi_\alpha dx \\
&= \int_{-1}^{1} \Big(\int_{-1}^{1} \varphi_n^{(\alpha)}(t;c) \varpi_\alpha(t) \partial_x \mathrm{K}_c^{(\alpha)}(x,t) dt \Big) \varphi_{n+1}^{(\alpha)}(x;c) \varpi_\alpha(x) dx \\
&\overset{(3.6)}{=} \int_{-1}^{1} \Big(\int_{-1}^{1} \varphi_n^{(\alpha)}(t;c) \varpi_\alpha(t) \partial_t \mathrm{K}_c^{(\alpha)}(x,t) dt \Big) \varphi_{n+1}^{(\alpha)}(x;c) \varpi_\alpha(x) dx \\
&= \int_{-1}^{1} \Big(\int_{-1}^{1} \varphi_{n+1}^{(\alpha)}(x;c) \varpi_\alpha(x) \partial_t \mathrm{K}_c^{(\alpha)}(x,t) dx \Big) \varphi_n^{(\alpha)}(t;c) \varpi_\alpha(t) dt \\
&= \nu_{n+1}^{(\alpha)} \int_{-1}^{1} \partial_x \varphi_{n+1}^{(\alpha)} \varphi_n^{(\alpha)} \varpi_\alpha dx,
\end{aligned} \tag{3.7}$$

where the last equality is obtained by multiplying (3.5) by $\varphi_n^{(\alpha)} \varpi_\alpha$, and integrating the resulting equation over $(-1,1)$. For $0 < c \ll 1$, we find from (2.1), (2.5) and (3.2) that

$$\int_{-1}^{1} \partial_x \varphi_n^{(\alpha)} \varphi_{n+1}^{(\alpha)} \varpi_\alpha dx = \int_{-1}^{1} \partial_x J_n^{(\alpha)} J_{n+1}^{(\alpha)} \varpi_\alpha dx + O(c^2) = O(c^2),$$

and

$$\begin{aligned}
\int_{-1}^{1} \varphi_n^{(\alpha)} \partial_x \varphi_{n+1}^{(\alpha)} \varpi_\alpha dx &= \int_{-1}^{1} \partial_x J_{n+1}^{(\alpha)} J_n^{(\alpha)} \varpi_\alpha dx + O(c^2) = \frac{(n+1) k_{n+1}^{(\alpha)}}{k_n^{(\alpha)}} + O(c^2) \\
&= \frac{(2n+\alpha+2)\sqrt{(2n+\alpha+1)(2n+\alpha+3)}}{2(n+\alpha+1)} + O(c^2) \\
&> n + 1 + \frac{\alpha}{2} + O(c^2).
\end{aligned}$$

Thus, we can rewrite (3.7) as

$$\nu_n^{(\alpha)}(c)-\nu_{n+1}^{(\alpha)}(c)=\nu_n^{(\alpha)}(c)\left(1-\frac{\int_{-1}^{1}\partial_x\varphi_n^{(\alpha)}\varphi_{n+1}^{(\alpha)}\varpi_\alpha dx}{\int_{-1}^{1}\varphi_n^{(\alpha)}\partial_x\varphi_{n+1}^{(\alpha)}\varpi_\alpha dx}\right)=\nu_n^{(\alpha)}(c)\big(1-O(c^2)\big)>0.$$

This completes the proof. □

The rest of this section is to analyze the asymptotic properties of $\varphi_n^{(\alpha)}$ and $\nu_n^{(\alpha)}$. In view of (2.23) and (2.26), it is necessary to study the Jacobi expansion coefficients $\{\beta_k^n\}$. We first derive an explicit upper bound for β_0^n by using an argument in [10].

Lemma 3.2. *Denote*

$$q_n:=q(n;\alpha,c)=\frac{c^2}{\chi_n^{(\alpha)}}, \tag{3.8}$$

and let m be a nonnegative integer such that

$$m(m+\alpha+1)<\frac{\ln 2}{2}\chi_n^{(\alpha)}-\frac{c^2}{2}. \tag{3.9}$$

Then for any $\alpha\ge 0$ and $c>0$, we have

$$|\beta_0^n|\le\Big(\frac{q_n}{2}\Big)^m\exp\Big(\frac{2m^3+3\alpha m^2+3(c^2-\alpha-2/3)m}{3\chi_n^{(\alpha)}}\Big)\sqrt{\frac{\alpha+1}{2m+1}}. \tag{3.10}$$

Proof. Define

$$A_k^n=\int_{-1}^{1}x^k\varphi_n^{(\alpha)}(x;c)\varpi_\alpha(x)\,dx, \tag{3.11}$$

and $\eta_n:=\frac{q_n}{2}$. To establish (3.10), we first show that

$$|A_0^n|\le\eta_n^m|A_m^n|\prod_{l=0}^{m-1}\left(1-\frac{c^2+2l(l+\alpha+1)}{2\chi_n^{(\alpha)}}\right)^{-1}. \tag{3.12}$$

Rewrite (2.9) as

$$\partial_x\left((1-x^2)\varpi_\alpha\partial_x\varphi_n^{(\alpha)}\right)+\chi_n^{(\alpha)}\left(1-\eta_n(1-x)\right)\varpi_\alpha\varphi_n^{(\alpha)}=0.$$

Multiplying the above equation by x^l and integrating the resulting equation over $(-1,1)$, leads to

$$l(l-1)A_{l-2}^n-\alpha lA_{l-1}^n+\Big\{\chi_n^{(\alpha)}-\frac{c^2}{2}-l(l+\alpha+1)\Big\}A_l^n+\frac{c^2}{2}A_{l+1}^n=0,\quad l\ge 0, \tag{3.13}$$

where $A_{-2}^n=A_{-1}^n=0$. In particular, we have

$$\Big\{\chi_n^{(\alpha)}-\frac{c^2}{2}\Big\}A_0^n+\frac{c^2}{2}A_1^n=0. \tag{3.14}$$

Without loss of generality, we assume that $A_0^n>0$ (if $A_0^n<0$, we just replace A_k^n by $-A_k^n$ in (3.13)-(3.14)). Observe from (3.9) that for $0\le l\le m$, the coefficient of A_l^n in (3.13) is positive. Hence, (3.14) implies $A_1^n<0$. For $\alpha\ge 0$, we find from (3.13) that

- if l is even, then $A^n_{l-2}, A^n_l > 0$ and $A^n_{l-1}, A^n_{l+1} < 0$. Thus, by (3.13),

$$\Big\{\chi_n^{(\alpha)} - \frac{c^2}{2} - l(l+\alpha+1)\Big\}A^n_l + \frac{c^2}{2}A^n_{l+1} \le 0, \quad 0 \le l \le m,$$

  which implies

$$0 < A^n_l \le \eta_n |A^n_{l+1}| \Big(1 - \frac{c^2 + 2l(l+\alpha+1)}{2\chi_n^{(\alpha)}}\Big)^{-1}.$$

- if l is odd, then $A^n_{l-2}, A^n_l < 0$ and $A^n_{l-1}, A^n_{l+1} > 0$. By (3.13),

$$\Big\{\chi_n^{(\alpha)} - \frac{c^2}{2} - l(l+\alpha+1)\Big\}A^n_l + \frac{c^2}{2}A^n_{l+1} \ge 0,$$

  which gives

$$0 < -A^n_l \le \eta_n A^n_{l+1} \Big(1 - \frac{c^2 + 2l(l+\alpha+1)}{2\chi_n^{(\alpha)}}\Big)^{-1}.$$

Consequently, for $\alpha \ge 0$,

$$|A^n_l| \le \eta_n |A^n_{l+1}| \Big(1 - \frac{c^2 + 2l(l+\alpha+1)}{2\chi_n^{(\alpha)}}\Big)^{-1}, \quad 0 \le l \le m, \tag{3.15}$$

which leads to (3.12).

Notice that

$$1 - x \ge \exp(-2x), \quad \text{for} \ \ 0 \le x \le \frac{\ln 2}{2}.$$

Therefore, under the condition (3.9),

$$1 - \frac{c^2 + 2l(l+\alpha+1)}{2\chi_n^{(\alpha)}} \ge \exp\Big(-\frac{c^2 + 2l(l+\alpha+1)}{\chi_n^{(\alpha)}}\Big), \quad 0 \le l \le m.$$

This yields

$$\begin{aligned} \prod_{l=0}^{m-1}\left(1 - \frac{c^2 + 2l(l+\alpha+1)}{2\chi_n^{(\alpha)}}\right)^{-1} &\le \exp\Big(\frac{\sum_{l=0}^{m-1}(c^2 + 2l(l+\alpha+1))}{\chi_n^{(\alpha)}}\Big) \\ &= \exp\Big(\frac{2m^3 + 3\alpha m^2 + 3(c^2 - \alpha - \frac{2}{3})m}{3\chi_n^{(\alpha)}}\Big). \end{aligned} \tag{3.16}$$

Next, we obtain from (2.11) and (3.11) that for $\alpha \ge 0$,

$$|A^n_m| \le \|x^m\|_{\varpi_\alpha} \|\varphi_n^{(\alpha)}\|_{\varpi_\alpha} \le \sqrt{\frac{2^{\alpha+1}}{2m+1}}. \tag{3.17}$$

Moreover, by (2.11), (2.23) and (3.11),

$$\beta^n_0 = J_0^{(\alpha)} A^n_0 = \sqrt{\frac{\alpha+1}{2^{\alpha+1}}}A^n_0. \tag{3.18}$$

A combination of (3.12) and (3.16)-(3.18) leads to the desired result (3.10). □

Remark 3.1. Suppose that

$$0<q_n<1 \quad \text{and} \quad m=O\big((\chi_n^{(\alpha)})^{1/3}\big)=O(n^{2/3}).$$

Then we deduce from (3.10) that $|\beta_0^n|$ decays exponentially with respect to n.

We have the following upper bound for $|\beta_k^n|$ involving $|\beta_0^k|$, which together with Lemma 3.2, will play an essential role for the analysis of GSF approximation in the proceeding section.

Lemma 3.3. *Let β_k^n be the Jacobi expansion coefficient defined in* (2.23)*, and let q_n and m be the same as in Lemma 3.2. If $0<q_n<1$, then for $\alpha>-1, c>0$ and $0\le k\le m$,*

$$|\beta_k^n|\le\Big(\frac{C_\alpha}{q_n}\Big)^k|\beta_0^n| \quad \text{with} \quad C_\alpha=\frac{2(\alpha+2)^2}{\alpha+1}. \tag{3.19}$$

Proof. The $(k+1)$th equation of the system (2.24) can be written as

$$\beta_{k+1}^n=\frac{2}{a_{k+1}}\Big(-\frac{1}{q_n}\Big(1-\frac{k(k+\alpha+1)}{\chi_n^{(\alpha)}}\Big)+\frac12-\frac{b_k}{2}\Big)\beta_k^n-\frac{a_k}{a_{k+1}}\beta_{k-1}^n, \tag{3.20}$$

for $k\ge 0$ (with $\beta_{-1}^n=0$).

Using (2.4), one verifies that for $\alpha>-1$ and $k\ge 1$,

$$\frac{2(\alpha+1)}{(\alpha+2)^2}\le\frac{2k(k+\alpha)}{(2k+\alpha)^2}<a_k<\frac{2k(k+\alpha)}{(2k+\alpha)(2k+\alpha-1)}, \tag{3.21}$$

and

$$0\le -b_k\le -b_1=\frac{\alpha^2}{(\alpha+2)(\alpha+4)}<1. \tag{3.22}$$

Next, we prove (3.19) by induction. Using (3.20) with $k=0$, we derive from (3.21) and the fact $0<q_n<1$ that

$$|\beta_1^n|=\frac{2}{a_1q_n}\frac{|(\alpha+1)q_n-2(\alpha+2)|}{2(\alpha+2)}|\beta_0^n|\le\frac{4}{a_1q_n}|\beta_0^n|\le\frac{C_\alpha}{q_n}|\beta_0^n|.$$

Recalling that $q_n<1$, we find from (3.9) that for $k\le m$

$$\frac{q_n}{2}-\Big(1-\frac{k(k+\alpha+1)}{\chi_n^{(\alpha)}}\Big)<-\frac12+\frac{\ln 2}{2}<0. \tag{3.23}$$

Assuming that (3.19) is true for k and $k-1$ (with $k\ge1$), we deduce from (3.20), (3.21) and (3.23) that for $k\le m$

$$\begin{aligned}|\beta_{k+1}^n|&\le\frac{2}{a_{k+1}q_n}\Big|\frac{q_n}{2}-\Big(1-\frac{k(k+\alpha+1)}{\chi_n^{(\alpha)}}\Big)-\frac{b_kq_n}{2}\Big||\beta_k^n|+\frac{a_k}{a_{k+1}}|\beta_{k-1}^n|\\&=\frac{2}{a_{k+1}q_n}\Big(1-\frac{k(k+\alpha+1)}{\chi_n^{(\alpha)}}-\frac{q_n}{2}-\frac{b_kq_n}{2}\Big)|\beta_k^n|+\frac{a_k}{a_{k+1}}|\beta_{k-1}^n|\\&\le\frac{(\alpha+2)^2}{(\alpha+1)q_n}\Big(1-\frac{k(k+\alpha+1)}{\chi_n^{(\alpha)}}\Big)\Big(\frac{C_\alpha}{q_n}\Big)^k|\beta_0^n|+\frac{a_k}{a_{k+1}}\Big(\frac{C_\alpha}{q_n}\Big)^{k-1}|\beta_0^n|\\&=\Big(\frac{C_\alpha}{q_n}\Big)^{k+1}|\beta_0^n|\left\{\frac12-\frac{k(k+\alpha+1)}{2\chi_n^{(\alpha)}}+\frac{a_kq_n^2}{2a_{k+1}C_\alpha^2}\right\}.\end{aligned} \tag{3.24}$$

In view of (3.1), the summation in the curly brackets is bigger than 0 for all $1 \le k \le m$, so it suffices to show it is less than 1, that is, to prove

$$\frac{a_k}{a_{k+1}} \le \frac{(\alpha+2)^4}{2(\alpha+1)^2} \le \frac{C_\alpha^2}{q_n^2}, \quad 1 \le k \le m, \ \ \alpha > -1. \tag{3.25}$$

We first consider the case with $k=1$. A direct calculation by using (2.4) yields

$$\frac{a_1}{a_2} = \frac{(\alpha+1)(\alpha+4)\sqrt{(\alpha+3)(\alpha+5)}}{2(\alpha+2)^2\sqrt{(\alpha+1)(\alpha+3)}} < \frac{(\alpha+4)^2}{2(\alpha+2)^2}.$$

Hence, it is enough to verify

$$f(\alpha) := (\alpha+2)^3 - (\alpha+1)(\alpha+4) > 0, \quad \alpha > -1,$$

which holds, since $f(-1) = 1$ and $f'(\alpha) > 0$ for all $\alpha > -1$. We next turn to the verification of (3.25) with $k \ge 2$. Indeed, by (3.21),

$$\frac{a_k}{a_{k+1}} < \frac{k(k+\alpha)(2k+\alpha+2)^2}{(k+1)(k+\alpha+1)(2k+\alpha-1)(2k+\alpha)} < \Big(\frac{2k+\alpha+2}{2k+\alpha-1}\Big)^2 \le \Big(\frac{\alpha+6}{\alpha+3}\Big)^2.$$

Thus, it suffices to show

$$(\alpha+3)(\alpha+2)^2 - \sqrt{2}(\alpha+1)(\alpha+6) > 0, \quad \alpha > -1.$$

Once again, it can be verified by simple calculus.

The induction is then completed by (3.24) and (3.25). □

In the sequel, we study the asymptotic behavior of the GSFs and the eigenvalues $\nu_n^{(\alpha)}(c)$ with large n. For this purpose, we first establish the explicit asymptotic formulas for the expansion coefficients β_{n+k}^n in (2.23) with $k = 0, \pm 1$ by using the inverse power method (cf. [18]). Basically, we solve the eigen-problem (2.24) with $\boldsymbol{A}$ being the 3×3 symmetric tri-diagonal matrix, whose main diagonal is

$$\big(A_{n-1,n-1}, A_{n,n}, A_{n+1,n+1}\big),$$

and the upper off-diagonal is

$$\big(A_{n-1,n},\, A_{n,n+1}\big),$$

where the entries $A_{i,j}$ are given by (2.25). Using the standard inverse power method, we can obtain a good approximation of β_{n+k}^n $(k = 0, \pm 1)$ and $\chi_n^{(\alpha)}$ for large n :

$$\big(\tilde\beta_{n-1}^n, \tilde\beta_n^n, \tilde\beta_{n+1}^n\big)^t \simeq \big(\beta_{n-1}^n, \beta_n^n, \beta_{n+1}^n\big)^t, \quad \tilde\chi_n^{(\alpha)}(c) \simeq \chi_n^{(\alpha)}(c).$$

Hereafter, the notation $A \simeq B$ means that for $B \ne 0$, the ratio $A/B \to 1$ in certain limiting process.

Proposition 3.1. *For fixed $c > 0, \alpha > -1$ and large n,*

$$\beta^n_{n\pm k} = \tilde{\beta}^n_{n\pm k} + O\Big(\frac{c^2}{n^3}\Big), \quad k = 0, 1; \quad \chi^{(\alpha)}_n(c) = \tilde{\chi}^{(\alpha)}_n(c) + O\Big(\frac{c^2}{n^3}\Big), \tag{3.26}$$

where

$$\tilde{\beta}^n_n = 1 - \frac{c^4}{64n^2}, \quad \tilde{\beta}^n_{n-1} = -\frac{c^2}{8n} + \alpha\frac{c^2}{16n^2}, \quad \tilde{\beta}^n_{n+1} = \frac{c^2}{8n} - (\alpha+2)\frac{c^2}{16n^2}, \tag{3.27}$$

and

$$\tilde{\chi}^{(\alpha)}_n(c) = n(n+\alpha+1) + \frac{c^2}{2} + \frac{c^2(c^2+4\alpha^2)}{32n^2}. \tag{3.28}$$

Consequently, we have

$$\varphi^{(\alpha)}_n(x;c) \simeq \beta^n_{n-1}J^{(\alpha)}_{n-1}(x) + \beta^n_n J^{(\alpha)}_n(x) + \beta^n_{n+1}J^{(\alpha)}_{n+1}(x). \tag{3.29}$$

Remark 3.2. In theory, this approach can be applied to find the asymptotic formulas for β^n_k with $|k-n| > 1$, but the symbolic computation is very tedious. Moreover, as pointed out in [18], such a procedure is somehow heuristic and the rigorous proof is lengthy and elementary.

As a consequence of Lemma 3.2 and Proposition 3.1, we have the following asymptotic bound for $\nu^{(\alpha)}_n(c)$ with large n.

Corollary 3.1. *If $0 < q_n = c^2/\chi^{(\alpha)}_n < 1$, then for $\alpha \geq 0$, $c > 0$ and $n \gg 1$,*

$$\nu^{(\alpha)}_n(c) \leq Cm^{-1/2}n^{-(\alpha+1/2)}(q_n/2)^m, \tag{3.30}$$

where $m = O(n^{2/3})$ and C is a positive constant independent of m and n.

Proof. Using (2.6), (3.29) and the Stirling's formula

$$\Gamma(x) \simeq \sqrt{2\pi}x^{x-1/2}e^{-x}, \quad \forall x \gg 1, \tag{3.31}$$

we obtain that

$$\varphi^{(\alpha)}_n(1;c) \simeq J^{(\alpha)}_n(1) \simeq \frac{n^{\alpha+1/2}}{2^{\alpha/2}\Gamma(\alpha+1)}, \quad n \gg 1. \tag{3.32}$$

Taking $m = O(n^{2/3})$ such that the condition (3.9) holds, we obtain (3.30) from (2.26), (3.10) and (3.32). ☐

As a numerical illustration of the asymptotic formula (3.26)-(3.27), we plot in Figure 3.1 (right) the approximation error $\log_{10}\big(\max_{|k|\leq 1}|\beta^n_{n+k} - \tilde{\beta}^n_{n+k}|/c^2\big)$ against $\log_{10} n$ for $\alpha = -0.5, 0$ and various c, which verifies the expected rate of convergence $O(n^{-3})$.

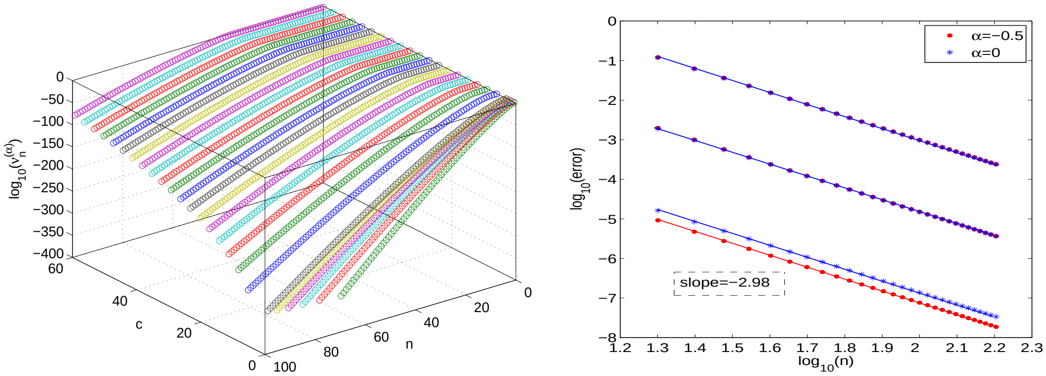


Figure 3.1: Left: Profiles of $\log_{10}(\nu_n^{(\alpha)})$ for various $n \in [0,100]$ and $c \in (0,60]$ with $\alpha = 0.5$. Right: Decay of the error: $\log_{10}\big(\max_{|k|\le 1}|\beta_{n+k}^n - \tilde{\beta}_{n+k}^n|/c^2\big)$ with $n \in [20,160]$, $c = 1,5,10$ and $\alpha = -0.5, 0$.

4. Approximation by GSFs

In this section, we study the approximability of GSFs, and derive the approximation errors with explicit dependence on the tuning parameter c.

Hereafter, let $\mathbb{N}$ be the set of all non-negative integers. For $N \in \mathbb{N}$, we consider the approximation of function $u \in L^2_{\varpi_\alpha}(I)$ by the truncated GSF series:

$$(\pi_{N,c}^{(\alpha)}u)(x) = \sum_{n=0}^{N} \hat{u}_n^{(\alpha)}\varphi_n^{(\alpha)}(x;c), \tag{4.1}$$

where

$$\hat{u}_n^{(\alpha)} := \hat{u}_n^{(\alpha)}(c) = \int_{-1}^{1} u(x)\varphi_n^{(\alpha)}(x;c)\varpi_\alpha(x)\,dx. \tag{4.2}$$

To describe the approximation errors, we introduce the weighted Sobolev space $H^r_{\varpi_\alpha}(I)$ with $r \in \mathbb{N}$, defined as in [2], whose norm and semi-norm are denoted by $\|\cdot\|_{r,\varpi_\alpha}$ and $|\cdot|_{r,\varpi_\alpha}$, respectively. In particular, $L^2_{\varpi_\alpha}(I) = H^0_{\varpi_\alpha}(I)$ with the norm $\|\cdot\|_{\varpi_\alpha}$. We also use the non-uniformly weighted Sobolev space:

$$B^r_{\varpi_\alpha}(I) := \Big\{u : \partial_x^k u \in L^2_{\varpi_{\alpha+k}}(I),\ \ 0 \le k \le r\Big\}, \tag{4.3}$$

equipped with the norm and semi-norm:

$$\|u\|_{B^r_{\varpi_\alpha}} = \Big(\sum_{k=0}^{r} \|\partial_x^k u\|^2_{\varpi_{\alpha+k}}\Big)^{1/2}, \quad |u|_{B^r_{\varpi_\alpha}} = \|\partial_x^r u\|_{\varpi_{\alpha+r}}.$$

Roughly speaking, the truncation error $\|\pi_N^{(\alpha)}u - u\|_{\varpi_\alpha}$ can be characterized by the decay rate of $|\hat{u}_N^{(\alpha)}|$, as stated below. It should be pointed out that the argument is similar to that for the PSWFs in [10], but the following analysis is subtler.

Theorem 4.1. *For $\alpha \ge 0$ and $c > 0$, let $q_N = c^2/\chi_N^{(\alpha)}$ and assume that $u \in B^r_{\varpi_\alpha}(I)$ with $r \ge 0$.*

- *If $0 < q_N < 1$, then we have*

$$\left|\hat{u}_N^{(\alpha)}\right| \le D\Big(N^{-2r/3}\|\partial_x^r u\|_{\varpi_{\alpha+r}} + \Big(\frac{q_N}{\sqrt{2}}\Big)^{\delta N^{2/3}}\|u\|_{\varpi_\alpha}\Big). \tag{4.4}$$

- *Given $\ln 2 < c_0 < 1$, let q_* be the root of $x = 2c_0 e^{-2x}$. If $0 < q_N < q_*$, then there exists $0 < p_N \le c_0 < 1$ such that*

$$\left|\hat{u}_N^{(\alpha)}\right| \le D\big(N^{-r}\|\partial_x^r u\|_{\varpi_{\alpha+r}} + (p_N)^{\delta N}\|u\|_{\varpi_\alpha}\big). \tag{4.5}$$

Here, D and δ are positive generic constants independent of N and u.

Proof. Let M be a positive integer to be specified later, and let u_M be the truncated Jacobi series:

$$u_M(x) = \sum_{k=0}^{M} g_k^{(\alpha)} J_k^{(\alpha)}(x), \quad g_k^{(\alpha)} = \int_{-1}^{1} u(x) J_k^{(\alpha)}(x)\varpi_\alpha(x)dx$$

Rewrite $\hat{u}_N^{(\alpha)}$ as

$$\hat{u}_N^{(\alpha)} = \int_{-1}^{1}(u(x) - u_M(x))\varphi_N^{(\alpha)}(x;c)\varpi_\alpha(x)dx + \int_{-1}^{1} u_M(x)\varphi_N^{(\alpha)}(x;c)\varpi_\alpha(x)dx. \tag{4.6}$$

Next, we estimate the two terms separately. Firstly, using the Cauchy-Schwartz inequality and (2.11), we derive from the fundamental Jacobi approximation result (see, e.g., [9] or Theorem 2.1 in [13]) that

$$\left|\int_{-1}^{1}(u - u_M)\varphi_N^{(\alpha)}\varpi_\alpha dx\right| \le \|u - u_M\|_{\varpi_\alpha}\|\varphi_N^{(\alpha)}\|_{\varpi_\alpha} \le DM^{-r}\|(1-x^2)^{r/2}\partial_x^r u\|_{\varpi_\alpha}. \tag{4.7}$$

On the other hand, using the orthogonality and Lemma 3.3, we treat the second term as

$$\begin{aligned}
\left|\int_{-1}^{1} u_M \varphi_N^{(\alpha)}\varpi_\alpha dx\right| &\le \left|\sum_{k=0}^{M} g_k^{(\alpha)}\int_{-1}^{1} J_k^{(\alpha)}\varphi_N^{(\alpha)}\varpi_\alpha dx\right| \\
&\le \Big(\sum_{k=0}^{M}(g_k^{(\alpha)})^2\Big)^{\frac{1}{2}}\Big(\sum_{k=0}^{M}\Big(\int_{-1}^{1} J_k^{(\alpha)}\varphi_N^{(\alpha)}\varpi_\alpha dx\Big)^2\Big)^{\frac{1}{2}} \\
&\le \|u\|_{\varpi_\alpha}\Big(\sum_{k=0}^{M}(\beta_k^N)^2\Big)^{\frac{1}{2}} \le \Big(\sum_{k=0}^{M}\Big(\frac{C_\alpha}{q_N}\Big)^{2k}\Big)^{\frac{1}{2}}|\beta_0^N|\|u\|_{\varpi_\alpha} \\
&\le \sqrt{M+1}\Big(\frac{C_\alpha}{q_N}\Big)^{M}|\beta_0^N|\|u\|_{\varpi_\alpha}.
\end{aligned} \tag{4.8}$$

To prove (4.4), we take $m = O\big((\chi_N^{(\alpha)})^{1/3}\big) = O(N^{2/3})$ in Lemma 3.2 (note: this m verifies the condition (3.9), and the exponential factor in (3.10) is uniformly bounded), and derive from (3.10) that for $\alpha \ge 0$ and $c > 0$,

$$|\beta_0^N| \le \frac{D}{\sqrt{m}}\Big(\frac{q_N}{2}\Big)^m. \tag{4.9}$$

We choose $M = O(m)$ and find a constant $0 < \gamma < 1$ such that

$$\frac{M}{m} = \frac{\ln\sqrt{2} + \gamma \ln\frac{\sqrt{2}}{q_N}}{\ln\frac{C_\alpha}{q_N}} \quad \Leftrightarrow \quad \frac{1}{\sqrt{2}}\Big(\frac{C_\alpha}{q_N}\Big)^{\frac{M}{m}} = \Big(\frac{q_N}{\sqrt{2}}\Big)^{-\gamma}. \tag{4.10}$$

To illustrate the existence of γ, we take $\gamma = \frac{1}{4}$ for simplicity and define

$$f(y_N) := \frac{\ln\sqrt{2} + \gamma \ln y_N}{\ln\frac{C_\alpha}{\sqrt{2}} + \ln y_N}, \tag{4.11}$$

where $y_N := \frac{\sqrt{2}}{q_N} \in (\sqrt{2}, +\infty)$. By using simple calculus, we can verified that

$$\frac{5\ln\sqrt{2}}{4\ln C_\alpha} = f(\sqrt{2}) < f(y_N) < f(+\infty) = \frac{1}{4}, \tag{4.12}$$

since $f'(y_N) > 0$. In view of (4.12), $\frac{M}{m}$ is uniformly bounded. We take $M = O(m) = O(N^{2/3})$. Hence, a combination of (4.8), (4.9) and (4.10) leads to

$$\begin{aligned}\Big|\int_{-1}^{1} u_M \varphi_N^{(\alpha)} \varpi_\alpha dx\Big| &\le D\Big(\frac{C_\alpha}{q_N}\Big)^M \Big(\frac{q_N}{2}\Big)^m \|u\|_{\varpi_\alpha} = D\Big(\frac{q_N}{2}\Big(\frac{C_\alpha}{q_N}\Big)^{\frac{M}{m}}\Big)^m \|u\|_{\varpi_\alpha} \\ &\overset{(4.10)}{=} D\Big(\frac{q_N}{\sqrt{2}}\Big)^{\frac{3}{4}m} \|u\|_{\varpi_\alpha} \le D\Big(\frac{q_N}{\sqrt{2}}\Big)^{\delta N^{2/3}} \|u\|_{\varpi_\alpha},\end{aligned}$$

where $\delta = \frac{3}{4}\tau$ with $\tau = N^{2/3}/m$. Thus, (4.4) follows from (4.6), (4.7) and the above estimate.

Now, we turn to the proof of (4.5). We derive from (3.10) that for every m satisfying (3.9),

$$|\beta_0^N| \le \frac{D}{\sqrt{m}}\Big(\frac{q_N}{2}\Big)^m \exp\Big(\frac{2m^3 + 3\alpha m^2 + 3c^2 m}{3\chi_N^{(\alpha)}}\Big) = D\frac{p_N^m}{\sqrt{m}}, \tag{4.13}$$

where we denoted by

$$p_N := \frac{q_N}{2}\exp\Big(\frac{2m^2 + 3\alpha m + 3c^2}{3\chi_N^{(\alpha)}}\Big). \tag{4.14}$$

It is clear that $0 < p_N \le c_0 < 1$, if and only if

$$m\Big(m + \frac{3}{2}\alpha\Big) < \frac{3}{2}\chi_N^{(\alpha)} \ln\frac{2c_0}{q_N} - \frac{3}{2}c^2 \quad \text{and} \quad 0 < q_N < q_*, \tag{4.15}$$

where $q_* = 2c_0e^{-q_*}$ with $\ln 2 < c_0 < 1$. Moreover, one verifies readily that when $0 < q_N < q_*$, we have

$$q_N^3 e^{2q_N} < 4c_0^3 \quad \Leftrightarrow \quad \frac{\ln 2}{2}\chi_N^{(\alpha)} - \frac{c^2}{2} < \frac{3}{2}\chi_N^{(\alpha)} \ln\frac{2c_0}{q_N} - \frac{3}{2}c^2. \tag{4.16}$$

We choose m to be the largest positive integer such that

$$m\Big(m + \frac{3}{2}(\alpha+1)\Big) \le \frac{\ln 2}{2}\chi_N^{(\alpha)} - \frac{c^2}{2}, \tag{4.17}$$

which guarantees (3.9) and (4.15). Similar to (4.10)-(4.12), we can find a constant $0 < \bar{\gamma} < 1$, and choose M such that when $0 < q_N < q_*$,

$$\frac{M}{m} = \bar{\gamma}\frac{\ln\frac{1}{p_N}}{\ln\frac{C_\alpha}{q_N}} \quad \Leftrightarrow \quad \Big(\frac{C_\alpha}{q_N}\Big)^{\frac{M}{m}} = {p_N}^{-\bar{\gamma}}. \tag{4.18}$$

Once again, by the same argument in (4.12), we can take $M = O(m) = O(N)$. Thus, by (4.8), (4.13) and (4.18),

$$\begin{aligned} \Big|\int_{-1}^{1} u_M \varphi_N^{(\alpha)} \varpi_\alpha dx\Big| &\le D\Big(\frac{C_\alpha}{q_N}\Big)^M p_N^m \|u\|_{\varpi_\alpha} = D\Big(p_N\Big(\frac{C_\alpha}{q_N}\Big)^{\frac{M}{m}}\Big)^m \|u\|_{\varpi_\alpha} \\ &= Dp_N^{(1-\bar{\gamma})m}\|u\|_{\varpi_\alpha} = Dp_N^{\delta N}\|u\|_{\varpi_\alpha}. \end{aligned} \tag{4.19}$$

where $\delta = (1-\bar{\gamma})\bar{\tau}$ with $\bar{\tau} = m/N$. Finally, the estimate (4.5) follows from (4.6), (4.7) and (4.19). □

Remark 4.1. Notice that the root of $x = 2e^{-x}$ is approximately $\tilde{q} \approx 0.8524$. Roughly, if $q_N < \tilde{q}$, the estimate (4.5) holds.

Notice that the conditions in Theorem 4.1 involve $\chi_N^{(\alpha)}$ and c, while it is more desirable to express them in terms of the ratio $\kappa = c/N$. By (3.1),

$$1 + \frac{\alpha+1}{N} < \frac{\kappa^2}{q_N} < 1 + \frac{\alpha+1}{N} + \kappa^2, \quad \forall N \ge 1, \tag{4.20}$$

which implies the for $N \gg 1$,

$$\underline{\kappa} := \frac{\kappa^2}{1+\kappa^2} \le q_N < \kappa^2. \tag{4.21}$$

Hence, q_N sits in the interval $[\underline{\kappa}, \kappa)$, but it seems implausible to obtain an explicit relation between q_N and κ. Here, we just provide in Table 4.1 some samples of κ and the corresponding numerical approximations of q_N with $\alpha = 0.5$ and $N = 128$.

In what follows, we provide some numerical examples to demonstrate the convergence behavior of approximation by GSFs.

Table 4.1: Samples of κ and q_N.

$\kappa = 0.5$		$\kappa = 1$		$\kappa = 1.2$		$\kappa = 1.268$		$\kappa = 1.5$	
$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N
0.200	0.222	0.500	0.654	0.590	0.807	0.615	0.8524	0.692	0.976

In the first example, we consider the GSF approximation of $u(x) = \sin(3\pi x)\exp(5x)$. In Figure 4.1(left), we plot $\log_{10}(|\hat{u}_N^{(\alpha)}|)$ (with $\alpha = 0$) against N, and observe an exponential decay of $|\hat{u}_N^{(\alpha)}|$, when q_N meets the condition for (4.5). However, $|\hat{u}_N^{(\alpha)}|$ grows fast as q_N gets close to 1. We also find that in general, $|\hat{u}_N^{(\alpha)}|$ decays faster when $c \approx N$.

In the second example, we test $u(x) = (x-a)^{7/5} e^{\sin x}$ with $a = 0, 1$, which has a finite regularity in the space $B^r_{\varpi_\alpha}(I)$ (cf. (4.3)). It follows from Theorem 4.1 that if $a = 1$ and q_N satisfies the condition for (4.5), then we have $|\hat{u}_N^{(\alpha)}| = O\big(N^{\varepsilon-(14/5+\alpha+1)}\big)$ for any $0 < \varepsilon \ll 1$. We plot in in Figure 4.1(right), $\log_{10}(|\hat{u}_N^{(\alpha)}|)$ (with $\alpha = 0.5$) against $\log_{10}(N)$. The slopes are slightly smaller than the theoretical prediction when $q_N \le 0.8524$ (cf. Remark 4.1) or $c \approx 1.268N$ (cf. Table 4.1). Once again, we find that $c \approx N$ is a good choice.

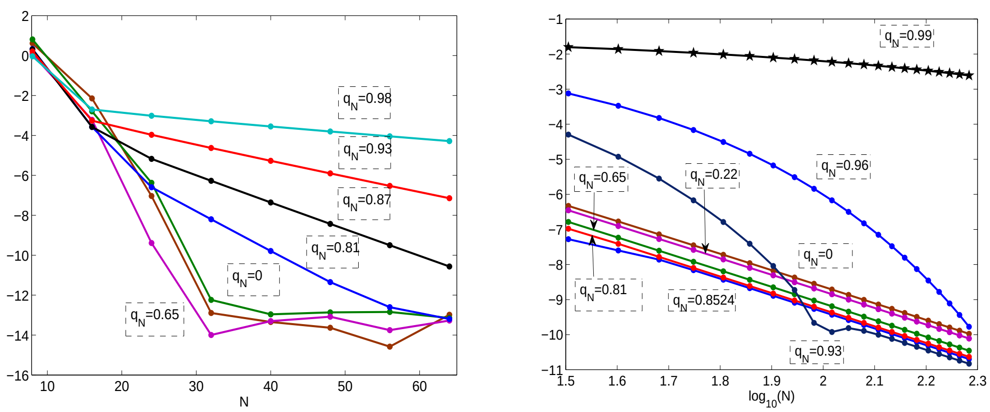
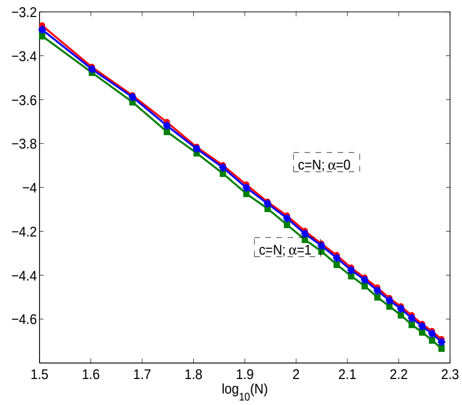


Figure 4.1: Left (Example 1): $\log_{10}(|\hat{u}_N^{(\alpha)}|)$ against $N \in [8, 64]$ with $\alpha = 0$ and $q_N = 0, 0.22, 0.65, 0.81, 0.87, 0.93, 0.98$. Correspondingly, we have $c = \kappa N$ with $\kappa = 0, 0.5, 1, 1.2, 1.3, 1.4, 1.5$. Right (Example 2 with $a = 1$): $\log_{10}(|\hat{u}_N^{(\alpha)}|)$ against $\log_{10}(N)$ with $\alpha = 0.5$ and $q_N = 0, 0.22, 0.65, 0.81, 0.8524, 0.93, 0.96, 0.99$. Correspondingly, $c \approx \kappa N$ with $\kappa = 0, 0.5, 1, 1.2, 1.268, 1.4, 1.47, 1.56$.

We still test the second example with $a = 0, 1$, but fix $c = N$ and choose $\alpha = 0, 0.5, 1$. In Figure 4.2 (left), we take $a = 0$ in the second example. It is predicted by Theorem 4.1 that $|\hat{u}_N^{(\alpha)}|$ behaves like $O\big(N^{\varepsilon-19/10}\big)$. As expected, we observe almost the same decay rate for different α. In contrast, if $a = 1$, the decay rate $O\big(N^{\varepsilon-(19/5+\alpha)}\big)$ varies with α, as depicted in Figure 4.2 (right).



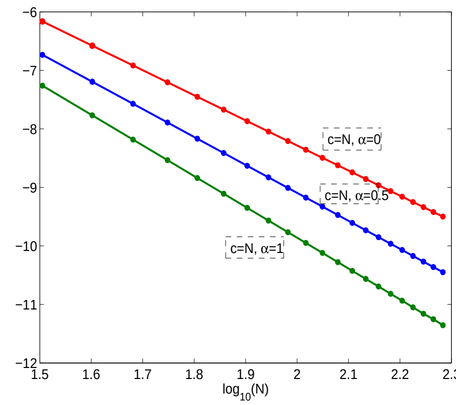


Figure 4.2: $\log_{10}(|\hat{u}_N^{(\alpha)}|)$ against $\log_{10}(N)$ with $N \in [32, 192]$ and $c = N$. Left: Example 2 with $a = 0$. Right: Example 2 with $a = 1$.

A. Proof of Lemma 3.1

Following the general perturbation scheme in [19], we expand the eigen-pair in series of c^2 :

$$\varphi_n^{(\alpha)}(x;c) = J_n^{(\alpha)}(x) + \sum_{j=1}^{\infty} c^{2j} Q_{n,j}(x,\alpha); \quad \chi_n^{(\alpha)}(c) = \gamma_n^{(\alpha)} + \sum_{j=1}^{\infty} c^{2j} a_{n,j}(\alpha), \tag{A.1}$$

where $\gamma_n^{(\alpha)} = \chi_n^{(\alpha)}(0)$ (cf. (2.10)), and

$$Q_{n,j}(x,\alpha) = \sum_{k=-j}^{j} B_{k,n}(j,\alpha) J_{n+k}^{(\alpha)}(x), \tag{A.2}$$

with the convectional choice $B_{0,n} = 0$. Let $\mathscr{L}_x^{(\alpha)}$ and $\mathscr{D}_x$ be the Sturm-Liouville operators associated with $J_n^{(\alpha)}$ and $\varphi_n^{(\alpha)}$, respectively, as defined in Section 2:

$$\mathscr{L}_x^{(\alpha)} = -\varpi_{-\alpha}\partial_x\left((1-x^2)\varpi_\alpha\partial_x\right), \quad \mathscr{D}_x = \mathscr{L}_x^{(\alpha)} + \frac{c^2}{2}(1-x). \tag{A.3}$$

Hence, substituting the expansion (A.1) into

$$\mathscr{D}_x \varphi_n^{(\alpha)}(x;c) = \chi_n^{(\alpha)}(c)\varphi_n^{(\alpha)}(x;c), \quad n \ge 1, \tag{A.4}$$

equating to zero the coefficients of distinct powers of c^2, we find the equation corresponding to the coefficient of c^2 is

$$(\mathscr{L}_x^{(\alpha)} - \gamma_n^{(\alpha)})Q_{n,1} + \frac{1}{2}(1-x)J_n^{(\alpha)} - a_{n,1}J_n^{(\alpha)} = 0. \tag{A.5}$$

Hence, using $\mathscr{L}_x^{(\alpha)} J_n^{(\alpha)} = \gamma_n^{(\alpha)} J_n^{(\alpha)}$, the property (2.3) and (A.2), we find that

$$\begin{aligned} B_{1,n} &= \frac{e_n}{\gamma_{n+1}^{(\alpha)} - \gamma_n^{(\alpha)}} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+2)^2\sqrt{(2n+\alpha+1)(2n+\alpha+3)}}, \\ a_{n,1} &= d_n = \frac{-\alpha^2}{2(2n+\alpha)(2n+\alpha+2)} + \frac{1}{2}, \\ \text{and} \quad B_{-1,n} &= \frac{e_n c_n}{\gamma_{n-1}^{(\alpha)} - \gamma_n^{(\alpha)}} = \frac{-n(n+\alpha)}{(2n+\alpha)^2\sqrt{(2n+\alpha-1)(2n+\alpha+1)}} = -B_{1,n-1}, \end{aligned} \tag{A.6}$$

where e_n and d_n are defined in (2.25). In view of $B_{0,n} = 0$, we obtain that

$$\varphi_n^{(\alpha)}(x;c) = J_n^{(\alpha)}(x) + c^2\big(B_{-1,n} J_{n-1}^{(\alpha)}(x) + B_{1,n} J_{n+1}^{(\alpha)}(x)\big) + O(c^4), \tag{A.7}$$

and

$$\chi_n^{(\alpha)}(c) = \gamma_n^{(\alpha)} + c^2 a_{n,1} + O(c^4). \tag{A.8}$$

This ends the proof. □

Acknowledgement. This work is partially supported by Singapore AcRF Tier 1 Grant RG58/08, Singapore MOE Grant T207B2202, and Singapore NRF2007IDM-IDM002-010.

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