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A Spectral Collocation Method for Nonlinear Fractional Boundary Value Problems with a Caputo Derivative

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Abstract In this paper, we consider the nonlinear boundary value problems involving the Caputo fractional derivatives of order $\alpha \in (1, 2)$ on the interval $(0, T)$. We present a Legendre spectral collocation method for the Caputo fractional boundary value problems. We derive the error bounds of the Legendre collocation method under the L^2- and L^∞-norms. Numerical experiments are included to illustrate the theoretical results.

Keywords Spectral collocation method · Caputo fractional derivative · Fredholm integral equations · Convergence analysis

Mathematics Subject Classification 65N35 · 45D05 · 41A05 · 41A10 · 41A25

1 Introduction

Fractional-order derivatives have recently emerged in the modelling of various processes, see, e.g., [8,15] for several applications. The fractional calculus and fractional differential equations (FDEs) have also attracted much attention (see, e.g., the survey paper [14]). In this paper, we contribute to these developments by describing and analyzing a numerical method for the following nonlinear two-point boundary value problem involving a Caputo fractional derivative,

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$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in (0, T), \\ y(0) = 0, \quad y(T) = 0, \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and ${}_0^C D_t^{\alpha}$ is the left-sided Caputo derivative of order $\alpha \in (1, 2)$. In case of $\alpha = 2$, ${}_0^C D_t^{\alpha}$ coincides with the usual second order derivative $y''(t)$, and the model (1.1) recovers the classical two point boundary value problem. For any positive non-integer real number β with $n - 1 < \beta < n$, $n \in \mathbb{N}$, the (formal) left-sided Caputo fractional derivative of order β is defined by (see, e.g., [10, pp. 70])

$${}_0^C D_t^{\beta} \phi = {}_0 I_t^{n-\beta} \Big(\frac{d^n \phi}{dt^n} \Big), \quad t \in (0, T). \tag{1.2}$$

here, ${}_0 I_t^{\gamma}$ for $\gamma > 0$ is the left-sided Riemann–Liouville integral operator of order γ defined by

$${}_0 I_t^{\gamma} \phi(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \phi(s) ds, \tag{1.3}$$

which satisfies a semigroup property: for $\gamma, \delta > 0$ (cf. [10, pp. 73]),

$${}_0 I_t^{\gamma+\delta} \phi = {}_0 I_t^{\gamma} {}_0 I_t^{\delta} \phi, \qquad \forall \phi \in L^2(0, T). \tag{1.4}$$

The Eq. (1.1) stem from the mathematical modeling of anomalous diffusion, especially super-diffusion, in which the mean squares variance grows faster than that in a Gaussian process. The space fractional derivative admits a micro interpretation as asymmetric Levy flights [7]. Such phenomena were observed in applications, e.g., geophysical flows and magnetized plasmas [3]. Hence, the accurate simulation of the model (1.1) has great significance in the field of science and engineering.

During the last decades, there appears a growing interest in developing numerical methods for solving FDEs, and plenty of literature is available on this subject. Amongst the existing methods, finite difference methods [18,20] and finite element methods [5,8,9,21] are predominant. However, these methods are based on local operations and lack the capability to effectively deal with the problems with non-locality and weakly singularities. To overcome these challenges, the spectral method is often a good candidate, which appears to be a global approach and very suitable for non-local problems.

Several spectral methods for FDEs have been proposed recently. For instance, Li and Xu [13] developed a space-time spectral Galerkin method for the diffusion equation with a Riemann–Liouville derivatives in time, which is exponentially convergent for smooth solutions; Esmaeili et al. [6] described a pseudo-spectral method for solving FDEs with initial conditions and derived the pseudo-spectral differentiation matrix of fractional order; Mokhtary and Ghoreishi [17] presented a spectral tau method for initial value problems with a Caputo derivative, which converges exponentially provided that the data in the given fractional integro-differential equations are smooth; Li et al. [12] derived recursive formulae based on Legendre, Chebyshev and Jacobi polynomials for approximating the Caputo derivative, and proposed a collocation method for solving initial/boundary value problems; Chen et al. [2] established spectral approximation results for a new class of generalized Jacobi functions (GJFs) in weighted Sobolev spaces involving fractional derivatives and constructed efficient GJF–Petrov–Galerkin methods for a class of prototypical fractional initial value problems and fractional boundary value problems of general order.

Since the Caputo fractional boundary value problem can be reformulated as a Volterra or Fredholm integral equation [4], the numerical methods for integral equations can also be used in solving fractional differential equations, e.g., Kopteva and Stynes [11] reformulated the

Caputo two-point boundary value problem as a Volterra integral equation of the second kind, and proposed a multi-step collocation method for the Volterra integral equation. Recently, Wang et al. [22,23] proposed and analyzed spectral collocation methods for Volterra integral equations and Volterra functional integro-differential equations. On the basis of the works, this paper shall further introduce and analyze a Legendre spectral collocation method for the nonlinear Caputo fractional boundary value problem (1.1). Due to the influence of the nonlinear term, the convergence analysis of the spectral collocation method becomes very difficult. To this end, we employ two kinds of polynomial interpolations, i.e., the Legendre–Gauss and Jacobi–Gauss interpolations. Accordingly, we construct the Legendre spectral collocation scheme and design the algorithm. We also carry out a rigorous error analysis of the proposed method and present some numerical experiments to verify the theoretical results. The main contributions of this paper are summarized as follows:

(i) We convert the nonlinear Caputo boundary value problem (1.1) into a Volterra–Fredholm integral equation (2.6), and present a Legendre collocation method. Numerical results show that our method converges quickly for problems with very smooth solutions and reasonably well for problems with singular solutions.

(ii) We establish a priori error estimate for the numerical scheme in the function spaces $L^2(0, T)$ and $L^\infty(0, T)$, in the case that the original problem has a smooth solution. It shows that the spectral accuracy can be obtained for the proposed approximation.

The remainder of the paper is organized as follows. In Sect. 2, we reformulate the original equation as an equivalent Volterra–Fredholm equation and further translate it into an equation defined on the interval $(-1, 1)$. The existence and uniqueness theorems of the reformulation are also introduced there. In Sect. 3, we introduce some basic properties of the Legendre/Jacobi polynomial interpolations and propose the Legendre spectral collocation method for the reformulated nonlinear Volterra–Fredholm equation (2.6). In Sect. 4, we derive the error bounds of the Legendre collocation method for smooth solutions in the function spaces $L^2(0, T)$ and $L^\infty(0, T)$, respectively. Our theoretical results are verified by the numerical experiments in Sect. 5.

2 Reformulation of the Boundary Value Problem

We can convert (1.1) into an equivalent Fredholm equation, stated below.

Lemma 2.1 *Let $1 < \alpha < 2$. Assume that $y(t)$ is a function with an absolutely continuous first derivative, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then we have that $y \in C^1[0, T]$ is a solution of the boundary value problem (1.1) if and only if it is a solution of the Fredholm integral equation:*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s))ds. \quad (2.1)$$

Proof This is a direct result of Lemma 6.43 and Theorem 3.1 in [4]. □

The existence and uniqueness of the solution to (1.1) read as follows.

Lemma 2.2 (see [4, Theorem 6.44]) *Assume the hypotheses of Lemma 2.1. Moreover let f be uniformly bounded by an absolute constant. Then the boundary value problem (1.1) has a solution $y \in C^1[0, T]$.*

Lemma 2.3 *Assume the hypothesis of Lemma 2.1. Moreover let f satisfy a Lipschitz condition with respect to the second variable with the Lipschitz constant $L < \frac{\Gamma(\alpha+1)}{2T^\alpha}$. Then the boundary value problem (1.1) has a unique solution $y \in C^1[0, T]$.*

Proof The uniqueness of the solution for $y \in C^1[0, T]$ can be found in Theorem 6.45 of [4]. Here, we mainly derive the upper bound of the Lipschitz constant L. Define the operator $\mathcal{A}$ by

$$\mathcal{A}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s))ds.$$

It is clear that the operator maps $C^1[0, T]$ into itself and that

$$\begin{aligned}
|\mathcal{A}y(t) - \mathcal{A}\widehat{y}(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y(s)) - f(s, \widehat{y}(s))| ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, y(s)) - f(s, \widehat{y}(s))| ds. \\
&\le \frac{L}{\Gamma(\alpha)} \|y - \widehat{y}\|_{L^\infty(0,T)} \Big(\int_0^t (t-s)^{\alpha-1} ds + \frac{t}{T} \int_0^T (T-s)^{\alpha-1} ds \Big) \\
&\le \frac{2LT^\alpha}{\Gamma(\alpha+1)} \|y - \widehat{y}\|_{L^\infty(0,T)}.
\end{aligned}$$

Since $C^1[0, T] \subset L^\infty(0, T)$, this implies, under our assumption, that $\mathcal{A}$ is a contraction. Thus, by Banach's fixed point theorem, we obtained that $\mathcal{A}$ has a unique fixed point. □

Next, let $\Lambda = (-1, 1)$. For ease of analysis, we transfer the problem (2.1) to an equivalent problem defined in Λ. More specifically, we use the change of variable

$$t = \frac{1}{2} T(x+1), \quad x \in \Lambda$$

to rewrite the Eq. (2.1) as follows,

$$\begin{aligned}
y\Big(\frac{1}{2}T(x+1)\Big) &= \frac{1}{\Gamma(\alpha)} \int_0^{\frac{1}{2}T(x+1)} \Big(\frac{1}{2}T(x+1) - s\Big)^{\alpha-1} f(s, y(s))ds \\
&\quad - \frac{x+1}{2\Gamma(\alpha)} \int_0^T (T-\hat{s})^{\alpha-1} f(\hat{s}, y(\hat{s}))d\hat{s}.
\end{aligned} \tag{2.2}$$

Moreover, to transfer the integral intervals $(0, \frac{1}{2}T(x+1))$ to $(-1, x)$ and $(0, T)$ to $(-1, 1)$, we make the transformation $s = \frac{1}{2}T(\xi+1)$ and $\hat{s} = \frac{1}{2}T(\lambda+1)$. Then, (2.2) reads

$$\begin{aligned}
y\left(\frac{1}{2}T(x+1)\right) &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \int_{-1}^x (x-\xi)^{\alpha-1} f\left(\frac{1}{2}T(\xi+1), y\left(\frac{1}{2}T(\xi+1)\right)\right) d\xi \\
&\quad - \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^1 (1-\lambda)^{\alpha-1} f\left(\frac{1}{2}T(\lambda+1), y\left(\frac{1}{2}T(\lambda+1)\right)\right) d\lambda.
\end{aligned} \tag{2.3}$$

Further, let

$$Y(x) = y\Big(\frac{1}{2}T(x+1)\Big), \qquad F(\xi, Y(\xi)) = f\Big(\frac{1}{2}T(\xi+1), y\Big(\frac{1}{2}T(\xi+1)\Big)\Big).$$

Then, (2.3) can be reduced to

$$Y(x)=\frac{T^\alpha}{2^\alpha\Gamma(\alpha)}\int_{-1}^{x}\left(x-\xi\right)^{\alpha-1}F(\xi,Y(\xi))d\xi-\frac{T^\alpha(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\left(1-\lambda\right)^{\alpha-1}F(\lambda,Y(\lambda))d\lambda. \tag{2.4}$$

Finally, under the linear transformation:

$$\xi=\xi(x,\theta):=\frac{x+1}{2}\theta+\frac{x-1}{2},\quad \theta\in\Lambda, \tag{2.5}$$

the Eq. (2.4) becomes

$$\begin{aligned}Y(x)=&\frac{T^\alpha(x+1)^\alpha}{4^\alpha\Gamma(\alpha)}\int_{-1}^{1}\left(1-\theta\right)^{\alpha-1}F\Big(\xi(x,\theta),Y\big(\xi(x,\theta)\big)\Big)d\theta\\&-\frac{T^\alpha(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\left(1-\lambda\right)^{\alpha-1}F(\lambda,Y(\lambda))d\lambda.\end{aligned} \tag{2.6}$$

3 The Spectral Collocation Method

In this section, we shall propose a Legendre spectral collocation method for solving the Eq. (2.6).

3.1 The Jacobi–Gauss Interpolation

For $\alpha,\beta>-1$, let $J_k^{\alpha,\beta}(x)$, $x\in\Lambda$ be the standard Jacobi polynomial of degree k, and denote the weight function $\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$. The set of Jacobi polynomials is a complete $L^2_{\omega^{\alpha,\beta}}(\Lambda)$-orthogonal system, i.e.,

$$\int_{-1}^{1}J_k^{\alpha,\beta}(x)J_j^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx=\gamma_k^{\alpha,\beta}\delta_{k,j}, \tag{3.1}$$

where $\delta_{k,j}$ is the Kronecker function, and

$$\gamma_k^{\alpha,\beta}=\begin{cases}\dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & k=0,\\ \dfrac{2^{\alpha+\beta+1}}{(2k+\alpha+\beta+1)}\dfrac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!\Gamma(k+\alpha+\beta+1)}, & k\geq 1.\end{cases}$$

In particular,

$$J_0^{\alpha,\beta}(x)=1,\quad J_1^{\alpha,\beta}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta). \tag{3.2}$$

For a given integer $N\geq 0$, let $\mathcal{P}_N$ be the space of all polynomials of degree at most N. We denote by $\{x_j^{\alpha,\beta},\omega_j^{\alpha,\beta}\}_{j=0}^N$ the nodes and the corresponding Christoffel numbers of the standard Jacobi–Gauss interpolation on the interval Λ. Then, the standard Jacobi–Gauss quadrature formula is stated as

$$\int_\Lambda\phi(x)\omega^{\alpha,\beta}(x)dx\approx\sum_{j=0}^{N}\phi(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta}, \tag{3.3}$$

which is exact for any $\phi(x) \in \mathcal{P}_{2N+1}$. Particularly,

$$\sum_{j=0}^{N} J_p^{\alpha,\beta}(x_j^{\alpha,\beta}) J_q^{\alpha,\beta}(x_j^{\alpha,\beta}) \omega_j^{\alpha,\beta} = \gamma_p^{\alpha,\beta} \delta_{p,q}, \quad \forall\, 0 \le p+q \le 2N+1. \tag{3.4}$$

For any $v \in C(\Lambda)$, we denote by $\mathcal{I}_{x,N}^{\alpha,\beta} : C(\Lambda) \to \mathcal{P}_N$ the Jacobi–Gauss interpolation operator in the x-direction, such that

$$\mathcal{I}_{x,N}^{\alpha,\beta} v(x_j^{\alpha,\beta}) = v(x_j^{\alpha,\beta}), \qquad 0 \le j \le N. \tag{3.5}$$

Clearly

$$\mathcal{I}_{x,N}^{\alpha,\beta} v(x) = \sum_{p=0}^{N} v_p^{\alpha,\beta} J_p^{\alpha,\beta}(x), \quad \text{where} \quad v_p^{\alpha,\beta} = \frac{1}{\gamma_p^{\alpha,\beta}} \sum_{j=0}^{N} v(x_j) J_p^{\alpha,\beta}(x_j) \omega_j^{\alpha,\beta}. \tag{3.6}$$

In the special case where $\alpha = \beta = 0$, the Jacobi polynomial is reduced to the Legendre polynomial $L_k(x)$. Accordingly, we write $x_j = x_j^{0,0}$, $\omega_j = \omega_j^{0,0}$ and $\mathcal{I}_{x,N} = \mathcal{I}_{x,N}^{0,0}$ for the purpose of convenience.

3.2 The Legendre Spectral Collocation Scheme

The Legendre spectral collocation scheme for (2.6) is to seek $U(x) \in \mathcal{P}_N(\Lambda)$ with $N \ge 1$, such that

$$\begin{aligned} U(x) = &\frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N}\Big[(x+1)^\alpha \int_{-1}^{1} \big(1-\theta\big)^{\alpha-1} \mathcal{I}_{\theta,N}^{\alpha-1,0} F\Big(\xi(x,\theta), U\big(\xi(x,\theta)\big)\Big) d\theta\Big] \\ &- \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \big(1-\lambda\big)^{\alpha-1} \mathcal{I}_{\lambda,N}^{\alpha-1,0} F(\lambda, U(\lambda)) d\lambda. \end{aligned} \tag{3.7}$$

This is an implicit scheme. If F satisfies the Lipschitz condition with the Lipschitz constant $L < \frac{\Gamma(\alpha+1)}{2T^\alpha}$, then (3.7) has a unique solution, see “Appendix” of this paper.

We now describe the numerical implementations of scheme (3.7). To this end, we set

$$\begin{aligned} &U(x) = \sum_{p=0}^{N} u_p L_p(x), \\ &\mathcal{I}_{x,N} \mathcal{I}_{\theta,N}^{\alpha-1,0} \Big(\big(x+1\big)^\alpha F\big(\xi(x,\theta), U\big(\xi(x,\theta)\big)\big)\Big) = \sum_{p=0}^{N} \sum_{p'=0}^{N} d_{p,p'} L_p(x) J_{p'}^{\alpha-1,0}(\theta). \end{aligned} \tag{3.8}$$

Then by (3.8) and (3.1), a direct computation leads to

$$\begin{aligned} &\frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \int_{-1}^{1} \big(1-\theta\big)^{\alpha-1} \mathcal{I}_{x,N} \mathcal{I}_{\theta,N}^{\alpha-1,0} \Big(\big(x+1\big)^\alpha F\big(\xi(x,\theta), U\big(\xi(x,\theta)\big)\big)\Big) d\theta \\ &\quad = \frac{T^\alpha}{4^\alpha \Gamma(\alpha)} \sum_{p=0}^{N} \sum_{p'=0}^{N} d_{p,p'} L_p(x) \int_{-1}^{1} (1-\theta)^{\alpha-1} J_{p'}^{\alpha-1,0}(\theta) d\theta \\ &\quad = \frac{T^\alpha}{2^\alpha \Gamma(\alpha+1)} \sum_{p=0}^{N} d_{p,0} L_p(x). \end{aligned} \tag{3.9}$$

Applying (3.4) to (3.8), one can verify readily that

$$d_{p,0}=\frac{\alpha(2p+1)}{2^{1+\alpha}}\sum_{i=0}^{N}\sum_{j=0}^{N}(x_i+1)^{\alpha}F\Big(\xi(x_i,\theta_j^{\alpha-1,0}),U\big(\xi(x_i,\theta_j^{\alpha-1,0})\big)\Big)L_p(x_i)\omega_i\omega_j^{\alpha-1,0}. \tag{3.10}$$

Moreover, by (3.3) we have

$$\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}F(\lambda,U(\lambda))d\lambda=\sum_{j=0}^{N}F(x_j^{\alpha-1,0},U(x_j^{\alpha-1,0}))\omega_j^{\alpha-1,0}. \tag{3.11}$$

Hence, by using (3.7)–(3.11) we deduce that

$$\sum_{p=0}^{N}u_pL_p(x)=\frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\sum_{p=0}^{N}d_{p,0}L_p(x)-\frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\sum_{j=0}^{N}F(x_j^{\alpha-1,0},U(x_j^{\alpha-1,0}))\omega_j^{\alpha-1,0}. \tag{3.12}$$

Comparing the expansion coefficients of (3.12) yields

$$\begin{cases} u_p=\dfrac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}d_{p,0}-\dfrac{T^{\alpha}}{2^{\alpha+1}\Gamma(\alpha)}\displaystyle\sum_{j=0}^{N}F(x_j^{\alpha-1,0},U(x_j^{\alpha-1,0}))\omega_j^{\alpha-1,0}, & \text{for } p=0,1,\\ u_p=\dfrac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}d_{p,0}, \quad \text{for } 2\le p\le N. \end{cases} \tag{3.13}$$

The system (3.13) can be solved by an iterative process (e.g., the Newton–Raphson iteration method or the successive substitution method).

4 Error Analysis

In this section, we will carry out the error analysis for the numerical scheme (3.7) under $L^2(\Lambda)$ and $L^{\infty}(\Lambda)$, respectively. To this end, we need some preparations.

We first recall some lemmas which will be used later. For any integer $m\ge 0$, we introduce the Jacobi-weighted Sobolev space

$$H^m_{\omega^{\alpha,\beta}}(\Lambda)=\Big\{v:\|v\|_{H^m_{\omega^{\alpha,\beta}}}<\infty\Big\}$$

with the norm and semi-norm

$$\|v\|_{H^m_{\omega^{\alpha,\beta}}}=\left(\sum_{k=0}^{m}|v|_{H^m_{\omega^{\alpha,\beta}}}\right)^{\frac{1}{2}},\qquad |v|_{H^k_{\omega^{\alpha,\beta}}}=\|\partial_x^k v\|_{\omega^{\alpha+k,\beta+k}},$$

where $\|\cdot\|_{\omega^{\alpha,\beta}}$ denotes the weighted $L^2_{\omega^{\alpha,\beta}}(\Lambda)$-norm. In particular, $L^2(\Lambda)=H^0_{\omega^{0,0}}(\Lambda)$, $\|\cdot\|=\|\cdot\|_{L^2(\Lambda)}$ and $\|\cdot\|_{\infty}=\|\cdot\|_{L^{\infty}(\Lambda)}$.

Lemma 4.1 (cf. [19, pp. 133]) *For any $v\in H^m_{\omega^{\alpha,\beta}}(\Lambda)$ with $\alpha,\beta>-1$, $m\ge 1$ and integers $0\le k\le m\le N+1$,*

$$\|\partial_x^k(v-\mathcal{I}_{x,N}^{\alpha,\beta}v)\|_{\omega^{\alpha+k,\beta+k}}\le cN^{k-m}\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}.$$

Moreover, for any $v \in H^m(\Lambda)$ with $1 \le m \le N+1$, we have the following result (cf. [1], pp. 289]),

$$\|v - \mathcal{I}_{x,N} v\|_{H^1(\Lambda)} \le c N^{\frac{3}{2}-m} \|\partial_x^m v\|.$$

Lemma 4.2 (cf., [16, pp. 330]) *Assume that $\{l_j(x)\}_{j=0}^N$ are the Lagrange basis polynomials associated with $N+1$ Jacobi–Gauss points. Then*

$$\|\mathcal{I}_N^{\alpha,\beta}\|_\infty = \begin{cases} \mathcal{O}(\log N), & -1 < \alpha, \beta \le -\dfrac{1}{2}, \\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), & \gamma = \max(\alpha,\beta), \quad \text{otherwise}. \end{cases} \tag{4.1}$$

Next, let $\theta \in \Lambda$ and $\mathcal{I}_{\theta,N}^{\alpha-1,0} : C(\Lambda) \to \mathcal{P}_N$ be the Jacobi–Gauss interpolation operator in the θ-direction with the parameter $(\alpha-1, 0)$. As in (2.5), we set

$$\xi = \xi(x,\theta) := \frac{x+1}{2}\theta + \frac{x-1}{2}, \quad \theta \in \Lambda.$$

It is clear that $\xi \in (-1, x)$. Let $\{\theta_j^{\alpha-1,0}\}_{j=0}^N$ be the Jacobi–Gauss points in Λ and $\xi_j^{\alpha-1,0} = \xi(x, \theta_j^{\alpha-1,0})$. We define a new Jacobi–Gauss interpolation operator in the ξ-direction ${}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} : C(-1,x) \to \mathcal{P}_N(-1,x)$ as follows:

$${}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v\left(\xi_j^{\alpha-1,0}\right) = v\left(\xi_j^{\alpha-1,0}\right), \qquad 0 \le j \le N.$$

Obviously,

$${}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v\left(\xi_j^{\alpha-1,0}\right) = v\left(\xi_j^{\alpha-1,0}\right) = v\left(\xi(x, \theta_j^{\alpha-1,0})\right) = \mathcal{I}_{\theta,N}^{\alpha-1,0} v\left(\xi(x, \theta_j^{\alpha-1,0})\right).$$

Moreover, ${}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v(\xi)$ and $\mathcal{I}_{\theta,N}^{\alpha-1,0} v(\xi(x,\theta))\big|_{\theta=\frac{2\xi}{x+1}-\frac{x-1}{x+1}}$ belong to $\mathcal{P}_N(-1,x)$ in the variable ξ. Hence

$${}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v(\xi) = \mathcal{I}_{\theta,N}^{\alpha-1,0} v(\xi(x,\theta))\big|_{\theta=\frac{2\xi}{x+1}-\frac{x-1}{x+1}}. \tag{4.2}$$

Thus, by (4.2) and (3.3) we obtain

$$\begin{aligned} \int_{-1}^x (x-\xi)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v(\xi) d\xi &= \left(\frac{1+x}{2}\right)^\alpha \int_{-1}^1 (1-\theta)^{\alpha-1} \mathcal{I}_{\theta,N}^{\alpha-1,0} v(\xi(x,\theta)) d\theta \\ &= \left(\frac{1+x}{2}\right)^\alpha \sum_{j=0}^N v\left(\xi\left(x, \theta_j^{\alpha-1,0}\right)\right) \omega_j^{\alpha-1,0} \\ &= \left(\frac{1+x}{2}\right)^\alpha \sum_{j=0}^N v\left(\xi_j^{\alpha-1,0}\right) \omega_j^{\alpha-1,0}. \end{aligned} \tag{4.3}$$

Similarly,

$$\int_{-1}^x (x-\xi)^{\alpha-1} \left({}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} v(\xi)\right)^2 d\xi = \left(\frac{1+x}{2}\right)^\alpha \sum_{j=0}^N v^2\left(\xi_j^{\alpha-1,0}\right) \omega_j^{\alpha-1,0}. \tag{4.4}$$

Then, according to (4.2) and Lemma 4.1, we get that for integer $1 \leq m \leq N+1$,

$$\begin{aligned}
&\int_{-1}^{x}(x-\xi)^{\alpha-1}\left|v(\xi)-{}_{x}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}v(\xi)\right|^{2}d\xi \\
&=\Big(\frac{1+x}{2}\Big)^{\alpha}\int_{-1}^{1}(1-\theta)^{\alpha-1}\left|v(\xi(x,\theta))-\mathcal{I}_{\theta,N}^{\alpha-1,0}v(\xi(x,\theta))\right|^{2}d\theta \\
&\leq cN^{-2m}\Big(\frac{1+x}{2}\Big)^{\alpha}\int_{-1}^{1}\left|\partial_{\theta}^{m}v(\xi(x,\theta))\right|^{2}(1-\theta)^{\alpha+m-1}(1+\theta)^{m}d\theta \\
&=cN^{-2m}\int_{-1}^{x}\left|\partial_{\xi}^{m}v(\xi)\right|^{2}(x-\xi)^{\alpha+m-1}(1+\xi)^{m}d\xi.
\end{aligned} \tag{4.5}$$

We now analyze the numerical errors of scheme (3.7). Let $e(x) = Y(x) - U(x)$ and denote by $\mathcal{I}$ the identity operator. Clearly,

$$\|e\| \leq \|Y - \mathcal{I}_{x,N}Y\| + \|\mathcal{I}_{x,N}Y - U\|. \tag{4.6}$$

Lemma 4.3 *For $N \geq 1$, the following inequality holds*

$$\|e\| \leq \sum_{j=1}^{5}\|B_j\|,$$

where

$$\begin{aligned}
B_1(x) &= Y(x) - \mathcal{I}_{x,N}Y(x), \\
B_2(x) &= \frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha)}\mathcal{I}_{x,N}\int_{-1}^{x}(x-\xi)^{\alpha-1}\big(\mathcal{I}-{}_{x}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big)F(\xi,Y(\xi))d\xi, \\
B_3(x) &= \frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha)}\mathcal{I}_{x,N}\int_{-1}^{x}\big(x-\xi\big)^{\alpha-1}{}_{x}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\Big(F(\xi,Y(\xi))-F(\xi,U(\xi))\Big)d\xi, \\
B_4(x) &= \frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\Big)d\lambda, \\
B_5(x) &= \frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\big(\mathcal{I}_{\lambda,N}^{\alpha-1,0}-\mathcal{I}\big)F(\lambda,Y(\lambda))d\lambda.
\end{aligned}$$

Proof Consider the term $\mathcal{I}_{x,N}Y(x) - U(x)$. By (2.4) we have that for $N \geq 1$,

$$\begin{aligned}
\mathcal{I}_{x,N}Y(x) = {} & \frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha)}\mathcal{I}_{x,N}\int_{-1}^{x}\big(x-\xi\big)^{\alpha-1}F(\xi,Y(\xi))d\xi \\
& - \frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}F(\lambda,Y(\lambda))d\lambda.
\end{aligned} \tag{4.7}$$

Moreover, by (4.2) we deduce that

$$\begin{aligned}
\int_{-1}^{x}(x-\xi)^{\alpha-1}{}_{x}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}F(\xi,U(\xi))d\xi = {} & \Big(\frac{x+1}{2}\Big)^{\alpha}\int_{-1}^{1}\big(1-\theta\big)^{\alpha-1}\mathcal{I}_{\theta,N}^{\alpha-1,0} \\
& F\Big(\xi(x,\theta),U\big(\xi(x,\theta)\big)\Big)d\theta.
\end{aligned} \tag{4.8}$$

This, along with (3.7), gives that

$$
\begin{aligned}
U(x) &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} F(\xi, U(\xi)) d\xi \\
&\quad - \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \big(1-\lambda\big)^{\alpha-1} \mathcal{I}_{\lambda,N}^{\alpha-1,0} F(\lambda, U(\lambda)) d\lambda.
\end{aligned} \tag{4.9}
$$

By subtracting (4.9) from (4.7), we derive that

$$
\begin{aligned}
\mathcal{I}_{x,N} Y(x) - U(x) = &\frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N} \int_{-1}^{x} \big(x-\xi\big)^{\alpha-1} \Big(F(\xi, Y(\xi)) - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} F(\xi, U(\xi))\Big) d\xi \\
&+ \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \big(1-\lambda\big)^{\alpha-1} \Big(\mathcal{I}_{\lambda,N}^{\alpha-1,0} F(\lambda, U(\lambda)) - F(\lambda, Y(\lambda))\Big) d\lambda.
\end{aligned}
$$

The previous formula can be rewritten as

$$
\begin{aligned}
\mathcal{I}_{x,N} Y(x) - U(x) = &\frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} \big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \\
&+ \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N} \int_{-1}^{x} \big(x-\xi\big)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \Big(F(\xi, Y(\xi)) - F(\xi, U(\xi))\Big) d\xi \\
&+ \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \big(1-\lambda\big)^{\alpha-1} \mathcal{I}_{\lambda,N}^{\alpha-1,0} \Big(F(\lambda, U(\lambda)) - F(\lambda, Y(\lambda))\Big) d\lambda \\
&+ \frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \big(1-\lambda\big)^{\alpha-1} \big(\mathcal{I}_{\lambda,N}^{\alpha-1,0} - \mathcal{I}\big) F(\lambda, Y(\lambda)) d\lambda.
\end{aligned} \tag{4.10}
$$

This, together with (4.6), leads to the desired result. □

We define the Nemytskii operator $\mathbb{F}(Y)(x) := F(x, Y(x))$. The following lemma establishes the convergence of our spectral collocation method in the function space $L^2(\Lambda)$.

Lemma 4.4 *Let $Y(x)$ and $U(x)$ be the the solutions to the Eqs. (2.6) and (3.7), respectively. Assume that $\alpha \in (1, 2)$, $Y \in H^m_{\omega^{m,m}}(\Lambda)$, $\mathbb{F} : H^m_{\omega^{m,m}}(\Lambda) \to H^m_{\omega^{\alpha+m-1,m}}(\Lambda)$ with integer $1 \le m \le N+1$ and $N \ge 1$. Moreover, F fulfills the Lipschitz condition with respect to the second variable with the Lipschitz constant $L < \frac{\Gamma(\alpha+1)}{2T^\alpha}$. Then we have*

$$
\|Y - U\| \le cN^{-m} (\|\partial_x^m Y\|_{\omega^{m,m}} + \|\partial_x^m F(\cdot, Y(\cdot))\|_{\omega^{\alpha+m-1,m}}).
$$

Proof According to Lemma 4.1, we get that for any integer $1 \le m \le N+1$,

$$
\|B_1\| = \|Y - \mathcal{I}_{x,N} Y\| \le cN^{-m} \|\partial_x^m Y\|_{\omega^{m,m}}. \tag{4.11}
$$

Next, by (3.3) we obtain

$$
\begin{aligned}
\|B_2\| &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\| \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} \big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \right\| \\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left[\sum_{j=0}^{N} \omega_j \Big(\int_{-1}^{x_j} (x_j - \xi)^{\alpha-1} \big(\mathcal{I} - {}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \Big)^2 \right]^{1/2}.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality and (4.5), we further get

$$
\begin{aligned}
&\|B_2\| \\
&\leq \frac{T^\alpha}{2^\alpha \Gamma(\alpha)}\left[\sum_{j=0}^{N}\omega_j \int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}d\xi \int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|(\mathcal{I}-{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0})F(\xi,Y(\xi))\right|^2 d\xi\right]^{1/2}\\
&\leq c\left[\sum_{j=0}^{N}\omega_j (x_j+1)^\alpha \int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|(\mathcal{I}-{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0})F(\xi,Y(\xi))\right|^2 d\xi\right]^{1/2}\\
&\leq cN^{-m}\left[\sum_{j=0}^{N}\omega_j (x_j+1)^\alpha \int_{-1}^{x_j}\left|\partial_\xi^m F(\xi,Y(\xi))\right|^2 (x_j-\xi)^{\alpha+m-1}(1+\xi)^m d\xi\right]^{1/2}\\
&\leq cN^{-m}\left\|\partial_\xi^m F(\cdot,Y(\cdot))\right\|_{\omega^{\alpha+m-1,m}}.
\end{aligned}
\tag{4.12}
$$

Similarly, by (3.3), (4.4) and the Cauchy–Schwarz inequality, we derive that

$$
\begin{aligned}
&\|B_3\| \\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)}\left\|\mathcal{I}_{x,N}\int_{-1}^{x}(x-\xi)^{\alpha-1}{}_{x}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big(F(\xi,Y(\xi))-F(\xi,U(\xi))\big)d\xi\right\|\\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)}\left[\sum_{j=0}^{N}\omega_j\Big(\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big(F(\xi,Y(\xi))-F(\xi,U(\xi))\big)d\xi\Big)^2\right]^{1/2}\\
&\leq \frac{T^\alpha}{2^\alpha \Gamma(\alpha)}\left[\sum_{j=0}^{N}\omega_j \int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}d\xi \int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big(F(\xi,Y(\xi))\right.\right.\\
&\quad \left.\left.-F(\xi,U(\xi))\big)\right|^2 d\xi\right]^{1/2}\\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)}\left[\sum_{j=0}^{N}\omega_j \frac{(x_j+1)^{2\alpha}}{2^\alpha \alpha}\sum_{k=0}^{N}\left|F(\xi_k^{\alpha-1,0},Y(\xi_k^{\alpha-1,0}))\right.\right.\\
&\quad \left.\left.-F(\xi_k^{\alpha-1,0},U(\xi_k^{\alpha-1,0}))\right|^2\omega_k^{\alpha-1,0}\right]^{1/2}.
\end{aligned}
$$

Next, for any given $x_j \in (-1,1)$, let $g(t)=(x_j+1)^t$. Since

$$
\frac{d^2}{dt^2}g(t)=(x_j+1)^t \ln^2(x_j+1)>0,
$$

and hence $g(t)$ is a convex function of t. Thus by Jensen's inequality we know that for $t\in[1,2]$,

$$
g(t)=g(2-t+2(t-1))\leq(2-t)g(1)+(t-1)g(2).
$$

Applying the previous inequality yields

$$\begin{aligned}\sum_{j=0}^{N}\omega_j(x_j+1)^\alpha &\le \sum_{j=0}^{N}\omega_j\Big[(2-\alpha)(x_j+1)+(\alpha-1)(x_j+1)^2\Big]\\ &=(2-\alpha)\int_\Lambda (x+1)dx+(\alpha-1)\int_\Lambda (x+1)^2dx\\ &=\frac{4}{3}+\frac{2\alpha}{3}\le\frac{8}{3},\qquad \forall\alpha\in[1,2].\end{aligned}\tag{4.13}$$

Therefore, by (4.4), (4.5), (4.13), the Lipschitz condition and the triangle inequality, we further deduce that for $L<\frac{\Gamma(\alpha+1)}{2T^\alpha}$,

$$\begin{aligned}\|B_3\| &\le \frac{LT^\alpha}{2^\alpha\Gamma(\alpha)}\left[\sum_{j=0}^{N}\omega_j\frac{(x_j+1)^{2\alpha}}{2^\alpha\alpha}\sum_{k=0}^{N}\left|Y(\xi_k^{\alpha-1,0})-U(\xi_k^{\alpha-1,0})\right|^2\omega_k^{\alpha-1,0}\right]^{1/2}\\ &\le\frac{\alpha}{2^{\alpha+1}}\left[\sum_{j=0}^{N}\omega_j\frac{(x_j+1)^\alpha}{\alpha}\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big(Y(\xi)-U(\xi)\big)\right|^2d\xi\right]^{1/2}\\ &\le\frac{\alpha}{2^{\alpha+1}}\left(\sum_{j=0}^{N}\omega_j\frac{(x_j+1)^\alpha}{\alpha}\right)^{1/2}\max_{0\le j\le N}\left(\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}(Y(\xi)-U(\xi))\right|^2d\xi\right)^{1/2}\\ &\le\frac{\alpha}{2^{\alpha+1}}\sqrt{\frac{8}{3\alpha}}\max_{0\le j\le N}\left[\left(\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|{}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}Y(\xi)-Y(\xi)\right|^2d\xi\right)^{1/2}\right.\\ &\quad\left.+\left(\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|Y(\xi)-U(\xi)\right|^2d\xi\right)^{1/2}\right]\\ &\le cN^{-m}\max_{0\le j\le N}\left(\int_{-1}^{x_j}(x_j-\xi)^{\alpha+m-1}(1+\xi)^m\left|\partial_\xi^mY(\xi)\right|^2d\xi\right)^{1/2}\\ &\quad+\frac{\alpha}{2^{\alpha+1}}\sqrt{\frac{8}{3\alpha}}\max_{0\le j\le N}\left(\int_{-1}^{x_j}(x_j-\xi)^{\alpha-1}\left|Y(\xi)-U(\xi)\right|^2d\xi\right)^{1/2}\\ &\le cN^{-m}\|\partial_x^mY\|_{\omega^{\alpha+m-1,m}}+\frac{\alpha}{2^{\alpha+1}}\sqrt{\frac{2^{\alpha+2}}{3\alpha}}\left(\int_{-1}^{1}\left|Y(\xi)-U(\xi)\right|^2d\xi\right)^{1/2}\\ &\le cN^{-m}\|\partial_x^mY\|_{\omega^{\alpha+m-1,m}}+\sqrt{\frac{\alpha}{3\times2^\alpha}}\|e\|.\end{aligned}\tag{4.14}$$

We next estimate the term $\|B_4\|$. By the Cauchy–Schwarz inequality, we know that

$$\begin{aligned}\|B_4\| &=\frac{T^\alpha\|x+1\|}{2^{\alpha+1}\Gamma(\alpha)}\left|\int_{-1}^{1}(1-\lambda)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\Big)d\lambda\right|\\ &\le\frac{T^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left[\int_{-1}^{1}(1-\lambda)^{\alpha-1}\left|\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\Big)\right|^2d\lambda\right]^{1/2}.\end{aligned}$$

The previous result, along with (3.3), Lemma 4.1 and the Lipschitz condition, yields

$$\begin{aligned}\|B_4\| &\le\frac{T^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\left|F\left(x_j^{\alpha-1,0},U\left(x_j^{\alpha-1,0}\right)\right)-F\left(x_j^{\alpha-1,0},Y\left(x_j^{\alpha-1,0}\right)\right)\right|^2\right)^{1/2}\\ &\le\frac{LT^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\left|U\left(x_j^{\alpha-1,0}\right)-Y\left(x_j^{\alpha-1,0}\right)\right|^2\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{LT^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\int_{-1}^{1}(1-\lambda)^{\alpha-1}\left|U(\lambda)-\mathcal{I}_{\lambda,N}^{\alpha-1,0}Y(\lambda)\right|^2 d\lambda\right)^{1/2}\\
&\le \sqrt{\frac{\alpha}{3\times 2^{\alpha+1}}}\left(\|U-Y\|_{\omega^{\alpha-1,0}}+\|Y-\mathcal{I}_{\lambda,N}^{\alpha-1,0}Y\|_{\omega^{\alpha-1,0}}\right)\\
&\le \sqrt{\frac{\alpha}{12}}\|e\|+cN^{-m}\|\partial_x^m Y\|_{\omega^{\alpha+m-1,m}}.
\end{aligned} \tag{4.15}
$$

It remains to estimate the term $\|B_5\|$. By Lemma 4.1 we have

$$
\begin{aligned}
\|B_5\| &= \frac{T^\alpha\|x+1\|}{2^{\alpha+1}\Gamma(\alpha)}\left|\int_{-1}^{1}(1-\lambda)^{\alpha-1}(\mathcal{I}_{\lambda,N}^{\alpha-1,0}-\mathcal{I})F(\lambda,Y(\lambda))d\lambda\right|\\
&\le c\left(\int_{-1}^{1}(1-\lambda)^{\alpha-1}d\lambda\right)^{1/2}\left(\int_{-1}^{1}(1-\lambda)^{\alpha-1}\left|(\mathcal{I}_{\lambda,N}^{\alpha-1,0}-\mathcal{I})F(\lambda,Y(\lambda))\right|^2 d\lambda\right)^{1/2}\\
&\le cN^{-m}\|\partial_x^m F(\cdot,Y(\cdot))\|_{\omega^{\alpha+m-1,m}}.
\end{aligned} \tag{4.16}
$$

Obviously,

$$
\sqrt{\frac{\alpha}{3\times 2^\alpha}}+\sqrt{\frac{\alpha}{12}}<1, \qquad \forall\alpha\in(1,2).
$$

Hence, a combination of (4.11), (4.12), (4.14), (4.15) and (4.16) leads to the desired result. □

Let $u(t):=U(\frac{2t}{T}-1)$ be the numerical solution of $y(t)$ with $t\in(0,T)$ and $\chi^{\alpha,\beta}(t):=(T-t)^\alpha t^\beta$ be the weight function. Define the Nemytskii operator $\mathbb{K}(y)(t):=f(t,y(t))$. Then, by the previous result, we obtain the following theorem.

Theorem 4.1 *Let $y(t)$ be the exact solution to the Eq. (1.1), and $u(t)$ be the numerical solution defined above. Assume that $\alpha\in(1,2)$, $y\in H^m_{\chi^{m,m}}(0,T)$, $\mathbb{K}:\ H^m_{\chi^{m,m}}(0,T)\to H^m_{\chi^{\alpha+m-1,m}}(0,T)$ with integer $1\le m\le N+1$ and $N\ge 1$. Moreover, f fulfills the Lipschitz condition with respect to the second variable with the Lipschitz constant $L<\frac{\Gamma(\alpha+1)}{2T^\alpha}$. Then we have*

$$
\|y-u\|_{L^2(0,T)}\le cN^{-m}\left(\|\partial_t^m y\|_{L^2_{\chi^{m,m}}(0,T)}+\|\partial_t^m f(\cdot,y(\cdot))\|_{L^2_{\chi^{\alpha+m-1,m}}(0,T)}\right).
$$

We now derive the error estimation in the function space $L^\infty(\Lambda)$.

Lemma 4.5 *Let $Y(x)$ and $U(x)$ be the the solutions to the Eqs. (2.6) and (3.7), respectively. Assume that $\alpha\in(1,2)$, $Y\in L^\infty(\Lambda)\cap H^m(\Lambda)$, $\mathbb{F}:\ H^m(\Lambda)\to H^m_{\omega^{\alpha+m-1,m}}(\Lambda)$ with integer $1\le m\le N+1$ and $N\ge 1$. Moreover, F fulfills the Lipschitz condition with respect to the second variable with the Lipschitz constant $L<\frac{\Gamma(\alpha+1)}{2T^\alpha}$. Then we have*

$$
\|Y-U\|_\infty\le cN^{\frac{3}{4}-m}\|\partial_x^m Y\|+cN^{\frac{1}{2}-m}\|\partial_x^m F(\cdot,Y(\cdot))\|_{\omega^{\alpha+m-1,m}}.
$$

Proof Clearly, by (4.10) we deduce readily that

$$
\|e\|_\infty\le\|Y-\mathcal{I}_{x,N}Y\|_\infty+\|\mathcal{I}_{x,N}Y-U\|_\infty\le\sum_{j=1}^{5}\|B_j\|_\infty. \tag{4.17}
$$

Moreover, according to the Sobolev inequality and Lemma 4.1, we get that for any integer $1\le m\le N+1$,

$$
\|B_1\|_\infty\le c\|Y-\mathcal{I}_{x,N}Y\|^{\frac{1}{2}}\|Y-\mathcal{I}_{x,N}Y\|^{\frac{1}{2}}_{H^1(\Lambda)}\le cN^{\frac{3}{4}-m}\|\partial_x^m Y\|. \tag{4.18}
$$

Next, by Lemma 4.2 we obtain

$$
\begin{aligned}
|B_2| &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \Big| \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} \big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \Big| \\
&\le \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \|\mathcal{I}_{x,N}\|_\infty \max_{-1\le x\le 1} \Big| \int_{-1}^{x} (x-\xi)^{\alpha-1} \big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \Big| \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \Big| \int_{-1}^{x} (x-\xi)^{\alpha-1} \big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\big) F(\xi, Y(\xi)) d\xi \Big|.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality and (4.5), we further get

$$
\begin{aligned}
|B_2| &\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\int_{-1}^{x} (x-\xi)^{\alpha-1} d\xi \int_{-1}^{x} (x-\xi)^{\alpha-1} \Big| (\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}) F(\xi, Y(\xi)) \Big|^2 d\xi \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\int_{-1}^{x} (x-\xi)^{\alpha-1} \Big| (\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}) F(\xi, Y(\xi)) \Big|^2 d\xi \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}-m} \max_{-1\le x\le 1} \left[\int_{-1}^{x} \Big| \partial_\xi^m F(\xi, Y(\xi)) \Big|^2 (x-\xi)^{\alpha+m-1} (1+\xi)^m d\xi \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}-m} \Big\| \partial_\xi^m F(\cdot, Y(\cdot)) \Big\|_{\omega^{\alpha+m-1,m}}.
\end{aligned} \tag{4.19}
$$

Similarly, by Lemma 4.2, (4.4) and the Cauchy–Schwarz inequality, we derive that

$$
\begin{aligned}
|B_3| &= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \Big| \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \Big(F(\xi, Y(\xi)) - F(\xi, U(\xi)) \Big) d\xi \Big| \\
&\le c\|\mathcal{I}_{x,N}\|_\infty \max_{-1\le x\le 1} \Big| \int_{-1}^{x} (x-\xi)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \Big(F(\xi, Y(\xi)) - F(\xi, U(\xi)) \Big) d\xi \Big| \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \Bigg[\int_{-1}^{x} (x-\xi)^{\alpha-1} d\xi \int_{-1}^{x} (x-\xi)^{\alpha-1} \Big| {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \Big(F(\xi, Y(\xi)) \\
&\quad - F(\xi, U(\xi)) \Big) \Big|^2 d\xi \Bigg]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\left(\frac{1+x}{2}\right)^\alpha \sum_{k=0}^{N} \Big| F(\xi_k^{\alpha-1,0}, Y(\xi_k^{\alpha-1,0})) - F(\xi_k^{\alpha-1,0}, U(\xi_k^{\alpha-1,0})) \Big|^2 \omega_k^{\alpha-1,0} \right]^{\frac{1}{2}}.
\end{aligned}
$$

Further, by using (4.4), (4.5) and a similar argument as in Theorem 4.4, we deduce that for $L < \frac{\Gamma(\alpha+1)}{2T^\alpha}$,

$$
\begin{aligned}
|B_3| &\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\left(\frac{x+1}{2}\right)^\alpha \sum_{k=0}^{N} \Big| Y(\xi_k^{\alpha-1,0}) - U(\xi_k^{\alpha-1,0}) \Big|^2 \omega_k^{\alpha-1,0} \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\int_{-1}^{x} (x-\xi)^{\alpha-1} \Big| {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} Y(\xi) - U(\xi) \Big|^2 d\xi \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}} \max_{-1\le x\le 1} \left[\int_{-1}^{x} (x-\xi)^{\alpha-1} \left(\Big| {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} Y(\xi) - Y(\xi) \Big|^2 + \Big| Y(\xi) - U(\xi) \Big|^2 \right) d\xi \right]^{\frac{1}{2}} \\
&\le cN^{\frac{1}{2}-m} \|\partial_x^m Y\|_{\omega^{\alpha+m-1,m}} + cN^{\frac{1}{2}} \|e\|.
\end{aligned} \tag{4.20}
$$

We next estimate the term $\|B_4\|_\infty$. By the Cauchy–Schwarz inequality we know that

$$|B_4| = \Big|\frac{T^\alpha(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\Big)d\lambda\Big|$$
$$\le \frac{T^\alpha}{2^\alpha\Gamma(\alpha)}\Big|\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\big)d\lambda\Big|$$
$$\le \frac{T^\alpha}{2^\alpha\Gamma(\alpha)}\Big(\frac{2^\alpha}{\alpha}\Big)^{\frac12}\Big(\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\Big|\mathcal{I}_{\lambda,N}^{\alpha-1,0}\big(F(\lambda,U(\lambda))-F(\lambda,Y(\lambda))\big)\Big|^2d\lambda\Big)^{\frac12}.$$

The previous result, along with (3.3), Lemma 4.1 and the Lipschitz condition, yields

$$
\begin{aligned}
|B_4| &\le \frac{T^\alpha}{2^\alpha\Gamma(\alpha)}\left(\frac{2^\alpha}{\alpha}\right)^{\frac12}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\Big|F(x_j^{\alpha-1,0},U(x_j^{\alpha-1,0}))-F(x_j^{\alpha-1,0},Y(x_j^{\alpha-1,0}))\Big|^2\right)^{\frac12}\\
&\le \frac{LT^\alpha}{2^\alpha\Gamma(\alpha)}\left(\frac{2^\alpha}{\alpha}\right)^{\frac12}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\Big|U(x_j^{\alpha-1,0})-Y(x_j^{\alpha-1,0})\Big|^2\right)^{\frac12}\\
&= \frac{LT^\alpha}{2^\alpha\Gamma(\alpha)}\left(\frac{2^\alpha}{\alpha}\right)^{\frac12}\left(\int_{-1}^{1}(1-\lambda)^{\alpha-1}\Big|U(\lambda)-\mathcal{I}_{\lambda,N}^{\alpha-1,0}Y(\lambda)\Big|^2d\lambda\right)^{\frac12}\\
&\le \frac{LT^\alpha}{2^\alpha\Gamma(\alpha)}\left(\frac{2^\alpha}{\alpha}\right)^{\frac12}\left(\|U-Y\|_{\omega^{\alpha-1,0}}+\|Y-\mathcal{I}_{\lambda,N}^{\alpha-1,0}Y\|_{\omega^{\alpha-1,0}}\right)\\
&\le \frac12\|U-Y\|_\infty + cN^{-m}\|\partial_x^m Y\|_{\omega^{\alpha+m-1,m}}.
\end{aligned}
\tag{4.21}
$$

Furthermore, by the Cauchy–Schwarz inequality and Lemma 4.1 we have

$$
\begin{aligned}
|B_5| &\le \frac{T^\alpha}{2^\alpha\Gamma(\alpha)}\Big|\int_{-1}^{1}(1-\lambda)^{\alpha-1}\,(\mathcal{I}_{\lambda,N}^{\alpha-1,0}-\mathcal{I})F(\lambda,Y(\lambda))d\lambda\Big|\\
&\le c\left(\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}d\lambda\int_{-1}^{1}\big(1-\lambda\big)^{\alpha-1}\big|(\mathcal{I}_{\lambda,N}^{\alpha-1,0}-\mathcal{I})F(\lambda,Y(\lambda))\big|^2d\lambda\right)^{\frac12}\\
&\le cN^{-m}\|\partial_x^m F(\cdot,Y(\cdot))\|_{\omega^{\alpha+m-1,m}}.
\end{aligned}
\tag{4.22}
$$

A combination of (4.17), (4.18), (4.19), (4.20), (4.21), (4.22) and Theorem 4.4 leads to the desired result. □

By Lemma 4.5, we derive the following theorem.

Theorem 4.2 *Let $y(t)$ and $u(t)$ be the exact solution and the numerical solution, respectively. Assume that $\alpha\in(1,2)$, $y\in L^\infty(0,T)\cap H^m(0,T)$, $\mathbb{K}:\ H^m(0,T)\to H^m_{\chi^{\alpha+m-1,m}}(0,T)$ with integer $1\le m\le N+1$ and $N\ge1$. Moreover, f fulfills the Lipschitz condition with respect to the second variable with the Lipschitz constant $L<\frac{\Gamma(\alpha+1)}{2T^\alpha}$. Then we have*

$$\|y-u\|_{L^\infty(0,T)}\le cN^{\frac34-m}\|\partial_t^m y\|_{L^2(0,T)}+cN^{\frac12-m}\|\partial_t^m f(\cdot,y(\cdot))\|_{L^2_{\chi^{\alpha+m-1,m}}(0,T)}.$$

5 Numerical Experiments

In this section, we present some numerical results to illustrate the efficiency of the Legendre spectral collocation method.

5.1 Linear Problems with Smooth Solutions

We start by considering the linear boundary problems with a Caputo derivative as follows (cf. [7]):

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = q(t)y(t) + g(t), & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{5.1}$$

where $q(t) = 20t^3(1-t)e^{-t}$, $g(t) = -\dfrac{\Gamma(128/17)}{\Gamma(128/17-\alpha)} t^{111/17-\alpha} - q(t)(t - t^{111/17})$. The solution of this problem can be written as $y(t) = t - t^{111/17}$, which is smooth on the interval $[0, 1]$.

In Figs. 1 and 2, we show the discrete L^{∞}- and L^2-errors of the Legendre spectral collocation method, respectively. The numerical results demonstrate that the method converges

Fig. 1 L^{∞}-errors of (5.1)

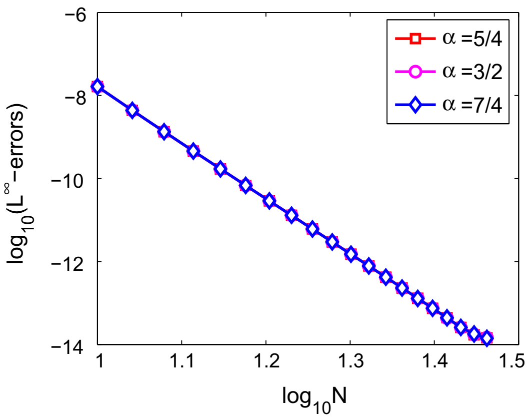
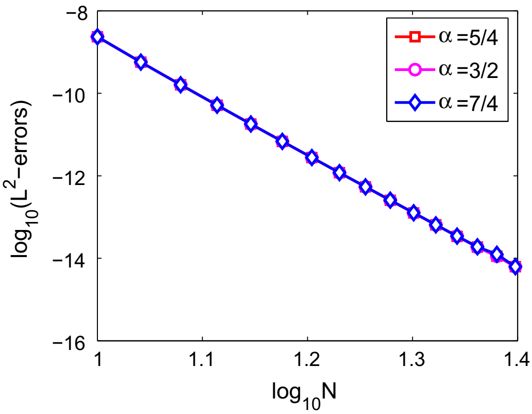


Fig. 2 L^2-errors of (5.1)



rapidly for this case, irrespective of α, since the true solution y is very smooth. They also indicate clearly the efficiency and accuracy of the suggested method for smooth solutions.

5.2 Linear Problems with Weakly Singular Solutions

Usually for fractional elliptic problems, the solution cannot be arbitrarily smooth, even if the source term is very smooth [8]: it contains a leading term $c_\alpha t^{1+\alpha}$ for t close to the origin. Next, we use our method to solve a kind of linear problem with weakly singular solutions:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = q(t)y(t) + g(t), & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{5.2}$$

where $q(t) = 20t^3(1-t)e^{-t}$, $g(t) = -\Gamma(\alpha+2)t - q(t)(t - t^{1+\alpha})$. The solution of this problem can be written as $y(t) = t - t^{1+\alpha}$, which is weakly singular at the endpoint $t = 0$.

In Figs. 3 and 4, we show the discrete L^∞- and L^2-errors of the Legendre spectral collocation method, respectively. The method converges slower due to limited regularity of the solution. Nonetheless, for all three α and two kinds of errors, algebraic rates of convergence are observed. This example also examines the influence of the fractional order α: the solution y becomes smoother as the order α increases.

Fig. 3 L^∞-errors of (5.2)

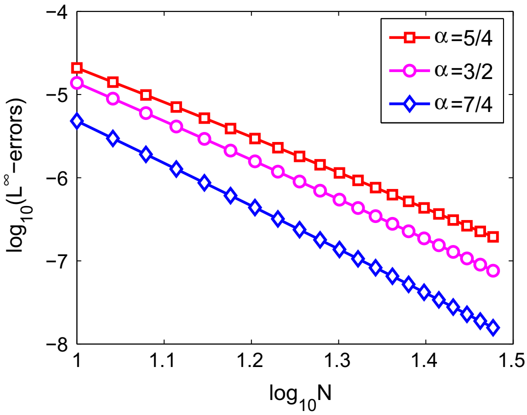
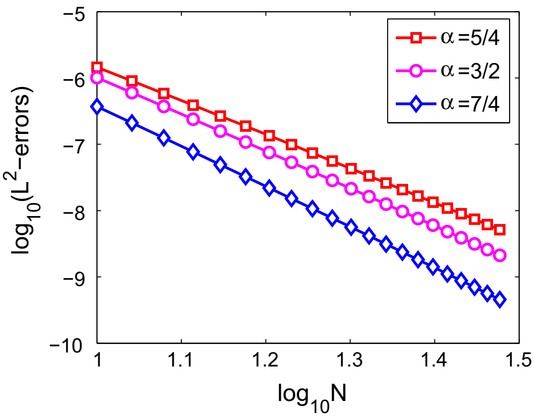


Fig. 4 L^2-errors of (5.2)



5.3 Nonlinear Problem with Smooth Solutions

We next consider the nonlinear boundary problem with a Caputo derivative as follows:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = y^2(t) + g(t), \quad t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{5.3}$$

where $g(t) = -\frac{\Gamma(128/17)}{\Gamma(128/17-\alpha)} t^{111/17-\alpha} - (t - t^{111/17})^2$. The solution of this problem can be written as $y(t) = t - t^{111/17}$, which is smooth on the interval [0, 1].

In Figs. 5 and 6, we show the discrete L^∞- and L^2-errors of the Legendre spectral collocation method, respectively. The numerical results show again that the method converges rapidly for this case, irrespective of α. They also indicate clearly the efficiency and accuracy.

Fig. 5 L^∞-errors of (5.3)

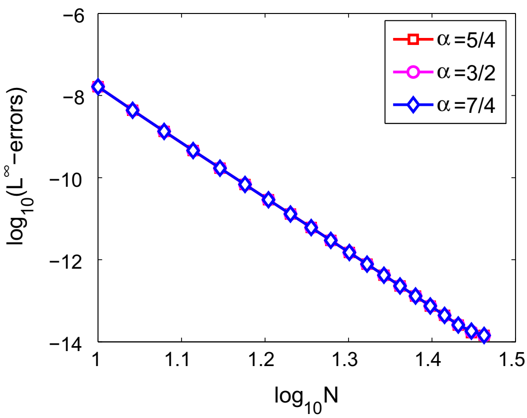
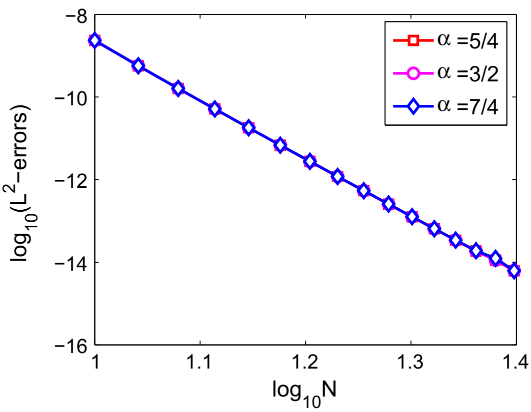


Fig. 6 L^2-errors of (5.3)



5.4 Nonlinear Problems with Weakly Singular Solutions

We also apply our method to solve a kind of nonlinear problem with weakly singular solutions as follows:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = y^2(t) + g(t), \quad t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{5.4}$$

where $g(t) = -\Gamma(2+\alpha)t - (t - t^{1+\alpha})^2$. The solution of this problem can be written as $y(t) = t - t^{1+\alpha}$, which is weakly singular at the endpoint $t = 0$.

In Figs. 7 and 8, we show the discrete L^∞- and L^2- errors of the Legendre spectral collocation method, respectively. Analogously, for all three α and two kinds of errors, algebraic rates of convergence are observed. They also indicate that, as α increases from $\alpha = 5/4$ to $\alpha = 7/4$, the convergence rate improves accordingly. The numerical results show that the method converges reasonably well for nonlinear problems with weakly singular solution.

To exhibit the numerical stability of the proposed method for the nonlinear problem (5.4), we present the L^∞- and L^2- errors for $\alpha = \frac{5}{4}$ and large N in Figs. 9 and 10. We find that this method is numerically stable even if the polynomial degree N is extremely large.

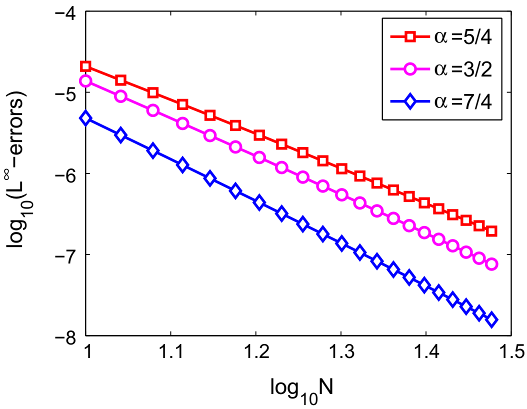


Fig. 7 L^∞-errors of (5.4)

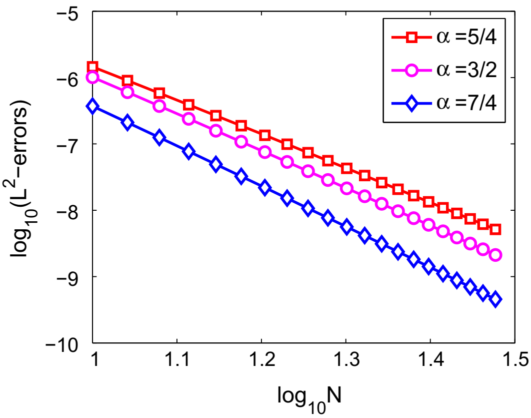


Fig. 8 L^2-errors of (5.4)

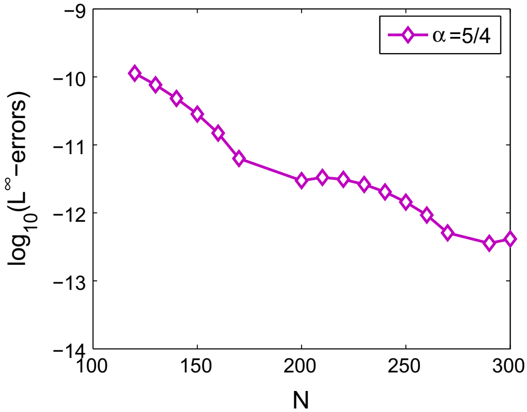


Fig. 9 L^{∞}-errors of (5.4) for large N

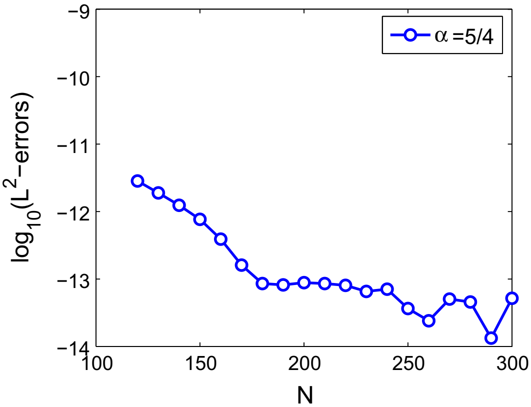


Fig. 10 L^2-errors of (5.4) for large N

Appendix

In this appendix, we verify the existence and uniqueness of scheme (3.7) with the Lipschitz constant $L < \frac{\Gamma(\alpha+1)}{2T^{\alpha}}$. We consider the following iteration process:

$$\begin{aligned} U^m(x) = &\frac{T^{\alpha}}{4^{\alpha}\Gamma(\alpha)}\mathcal{I}_{x,N}\left[(x+1)^{\alpha}\int_{-1}^{1}(1-\theta)^{\alpha-1}\mathcal{I}_{\theta,N}^{\alpha-1,0}F\Big(\xi(x,\theta),U^{m-1}\big(\xi(x,\theta)\big)\Big)d\theta\right] \\ &-\frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}(1-\lambda)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}F(\lambda,U^{m-1}(\lambda))d\lambda. \end{aligned} \tag{5.5}$$

Let $\widetilde{U}^m(x) = U^m(x) - U^{m-1}(x)$. Then we have from (5.5) and (4.8) that

$$\begin{aligned} \widetilde{U}^m(x) = &\frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha)}\mathcal{I}_{x,N}\left[\int_{-1}^{x}(x-\xi)^{\alpha-1}{}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0}\Big(F\big(\xi,U^{m-1}(\xi)\big)-F\big(\xi,U^{m-2}(\xi)\big)\Big)d\xi\right] \\ &-\frac{T^{\alpha}(x+1)}{2^{\alpha+1}\Gamma(\alpha)}\int_{-1}^{1}(1-\lambda)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U^{m-1}(\lambda))-F(\lambda,U^{m-2}(\lambda))\Big)d\lambda \\ =: &\, A_1 + A_2, \end{aligned} \tag{5.6}$$

where

$$A_1 = \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \mathcal{I}_{x,N} \left[\int_{-1}^{x} \left(x-\xi\right)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \Big(F\big(\xi, U^{m-1}(\xi)\big) - F\big(\xi, U^{m-2}(\xi)\big) \Big) d\xi \right],$$

$$A_2 = -\frac{T^\alpha (x+1)}{2^{\alpha+1}\Gamma(\alpha)} \int_{-1}^{1} \left(1-\lambda\right)^{\alpha-1} \mathcal{I}_{\lambda,N}^{\alpha-1,0} \Big(F(\lambda, U^{m-1}(\lambda)) - F(\lambda, U^{m-2}(\lambda)) \Big) d\lambda.$$

By (3.3), (4.4) and the Cauchy–Schwarz inequality, we derive that

$$\begin{aligned}
&\|A_1\| \\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left\| \mathcal{I}_{x,N} \int_{-1}^{x} (x-\xi)^{\alpha-1} {}_x\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \big(F(\xi, U^{m-1}(\xi)) - F(\xi, U^{m-2}(\xi))\big) d\xi \right\| \\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left[\sum_{j=0}^{N} \omega_j \left(\int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} {}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \big(F(\xi, U^{m-1}(\xi)) - F(\xi, U^{m-2}(\xi))\big) d\xi \right)^2 \right]^{\frac{1}{2}} \\
&\le \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left[\sum_{j=0}^{N} \omega_j \int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} d\xi \int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} \Big| {}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \big(F(\xi, U^{m-1}(\xi)) \right. \\
&\quad \left. - F(\xi, U^{m-2}(\xi))\big) \Big|^2 d\xi \right]^{\frac{1}{2}} \\
&= \frac{T^\alpha}{2^\alpha \Gamma(\alpha)} \left[\sum_{j=0}^{N} \omega_j \frac{(x_j+1)^{2\alpha}}{2^\alpha \alpha} \sum_{k=0}^{N} \Big| F(\xi_k^{\alpha-1,0}, U^{m-1}(\xi_k^{\alpha-1,0})) \right. \\
&\quad \left. - F(\xi_k^{\alpha-1,0}, U^{m-2}(\xi_k^{\alpha-1,0})) \Big|^2 \omega_k^{\alpha-1,0} \right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by (4.4), (4.13) and the Lipschitz condition, we further deduce that for $L < \frac{\Gamma(\alpha+1)}{2T^\alpha}$,

$$\begin{aligned}
\|A_1\| &\le \frac{L T^\alpha}{2^\alpha \Gamma(\alpha)} \left[\sum_{j=0}^{N} \omega_j \frac{(x_j+1)^{2\alpha}}{2^\alpha \alpha} \sum_{k=0}^{N} \Big| U^{m-1}(\xi_k^{\alpha-1,0}) - U^{m-2}(\xi_k^{\alpha-1,0}) \Big|^2 \omega_k^{\alpha-1,0} \right]^{\frac{1}{2}} \\
&\le \frac{\alpha}{2^{\alpha+1}} \left[\sum_{j=0}^{N} \omega_j \frac{(x_j+1)^{\alpha}}{\alpha} \int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} \Big| {}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \widetilde{U}^{m-1}(\xi) \Big|^2 d\xi \right]^{\frac{1}{2}} \\
&\le \frac{\alpha}{2^{\alpha+1}} \left(\sum_{j=0}^{N} \omega_j \frac{(x_j+1)^{\alpha}}{\alpha} \right)^{\frac{1}{2}} \max_{0\le j\le N} \left(\int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} \Big| {}_{x_j}\widetilde{\mathcal{I}}_{\xi,N}^{\alpha-1,0} \widetilde{U}^{m-1}(\xi) \Big|^2 d\xi \right)^{\frac{1}{2}} \\
&= \frac{\alpha}{2^{\alpha+1}} \left(\sum_{j=0}^{N} \omega_j \frac{(x_j+1)^{\alpha}}{\alpha} \right)^{\frac{1}{2}} \max_{0\le j\le N} \left(\int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} \Big| \widetilde{U}^{m-1}(\xi) \Big|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \frac{\alpha}{2^{\alpha+1}} \sqrt{\frac{8}{3\alpha}} \max_{0\le j\le N} \left(\int_{-1}^{x_j} (x_j-\xi)^{\alpha-1} \Big| \widetilde{U}^{m-1}(\xi) \Big|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \frac{\alpha}{2^{\alpha+1}} \sqrt{\frac{2^{\alpha+2}}{3\alpha}} \left(\int_{-1}^{1} \Big| \widetilde{U}^{m-1}(\xi) \Big|^2 d\xi \right)^{\frac{1}{2}} \\
&= \sqrt{\frac{\alpha}{3 \times 2^\alpha}} \|\widetilde{U}^{m-1}\|.
\end{aligned}$$

We next estimate the term $\|A_2\|$. By the Cauchy–Schwarz inequality, we know that

$$\|A_2\| = \frac{T^\alpha \|x+1\|}{2^{\alpha+1}\Gamma(\alpha)}\Big|\int_{-1}^{1}(1-\lambda)^{\alpha-1}\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U^{m-1}(\lambda)) - F(\lambda,U^{m-2}(\lambda))\Big)d\lambda\Big|$$
$$\leq \frac{T^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\Big[\int_{-1}^{1}(1-\lambda)^{\alpha-1}\Big|\mathcal{I}_{\lambda,N}^{\alpha-1,0}\Big(F(\lambda,U^{m-1}(\lambda)) - F(\lambda,U^{m-2}(\lambda))\Big)\Big|^2 d\lambda\Big]^{\frac{1}{2}}.$$

The previous result, along with (3.3) and the Lipschitz condition, yields

$$\begin{aligned}
\|A_2\| &\leq \frac{T^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\Big|F\left(x_j^{\alpha-1,0},U^{m-1}(x_j^{\alpha-1,0})\right)\right.\\
&\quad \left.-F\left(x_j^{\alpha-1,0},U^{m-2}(x_j^{\alpha-1,0})\right)\Big|^2\right)^{\frac{1}{2}}\\
&\leq \frac{LT^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\sum_{j=0}^{N}\omega_j^{\alpha-1,0}\Big|\widetilde{U}^{m-1}(x_j^{\alpha-1,0})\Big|^2\right)^{\frac{1}{2}}\\
&= \frac{LT^\alpha}{2^{\alpha+1}\Gamma(\alpha)}\sqrt{\frac{2^{\alpha+3}}{3\alpha}}\left(\int_{-1}^{1}(1-\lambda)^{\alpha-1}\Big|\widetilde{U}^{m-1}(\lambda)\Big|^2 d\lambda\right)^{\frac{1}{2}}\\
&\leq \sqrt{\frac{\alpha}{12}}\|\widetilde{U}^{m-1}\|, \qquad \forall\alpha\in(1,2).
\end{aligned} \tag{5.7}$$

Hence

$$\|\widetilde{U}^m\| \leq \left(\sqrt{\frac{\alpha}{3\times 2^\alpha}}+\sqrt{\frac{\alpha}{12}}\right)\|\widetilde{U}^{m-1}\|.$$

Since

$$\sqrt{\frac{\alpha}{3\times 2^\alpha}}+\sqrt{\frac{\alpha}{12}} < 1, \qquad \forall\alpha\in(1,2),$$

we have $\|\widetilde{U}^m\| \to 0$ as $m\to\infty$. This implies the existence of solution of (3.7). It is easy to prove the uniqueness of solution of (3.7).

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