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Full Length Article

Bernstein-type constants for approximation of $|x|^\alpha$ by partial Fourier–Legendre and Fourier–Chebyshev sums[☆]

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Abstract

In this paper, we study the approximation of $f_\alpha(x) = |x|^\alpha, \alpha > 0$ in $L_\infty[-1, 1]$ by its Fourier–Legendre partial sum $S_n^{(\alpha)}(x)$. We derive the upper and lower bounds of the approximation error in the L^∞-norm that are valid uniformly for all $n \ge n_0$ for some $n_0 \ge 1$. Such an optimal L^∞-estimate requires a judicious summation rule that can recover the lost half order if one uses a naive summation. Consequently, we can obtain the explicit Bernstein-type constant

$$B_\infty^{(\alpha)} := \lim_{n\to\infty} n^\alpha \|f_\alpha - S_n^{(\alpha)}\|_{L^\infty} = \frac{2\Gamma(\alpha)}{\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|.$$

Interestingly, using a similar argument, we can show that the Fourier–Chebyshev sum has the same Bernstein-type constant $B_\infty^{(\alpha)}$ as the Legendre case.

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Keywords: Bernstein constants; Legendre polynomial expansion; Chebyshev polynomial expansion; Lower and upper bounds; Optimal estimates

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1. Introduction

The approximation of $f_\alpha(x) = |x|^\alpha$, $\alpha > 0$ by polynomials has long been known as a fundamental problem of much relevance in approximation theory and numerical analysis (see, e.g., [6,25,29]). The foundational Weierstrass' theorem on polynomial approximation can be approached via approximating $|x|$ on $[-1, 1]$ (see [22,28]). The study of its best uniform polynomial approximation dates back to Bernstein in 1913, but to date, there are still some unsolved problems (see the excellent survey [25]: "*Towards best approximations for* $|x|^\alpha$"). The Chebyshev polynomial approximation of the prototype function $|x|$ has motivated the introduction of the Chebyshev-weighted 1-norm in Trefethen [29] that can best characterise the regularity of f_α with integer α. Moreover, the investigation of Legendre polynomial approximation of such singular functions played a fundamental role in the theory of *hp* finite element since the seminal work of Gui and Babuška [13]. It is also worth noting the important role of this benchmark function $f_\alpha(x)$ in the introduction of approximation framework and results for numerical analysis (see, e.g., [2,14–16,33] and the references therein).

Remarkably, Bernstein [4,5] proved the existence of the limit (which is dubbed as the Bernstein constant):

$$B^*_{\infty,\alpha} := \lim_{n\to\infty} n^\alpha \big\| f_\alpha - p^*_n \big\|_{L^\infty} \ \text{ with } \ \big\| f_\alpha - p^*_n \big\|_{L^\infty} = \inf_{p\in\mathbb{P}_n} \big\| f_\alpha - p \big\|_{L^\infty}, \tag{1.1}$$

where $\mathbb{P}_n$ is the set of all polynomials of degree at most n, and $p^*_n \in \mathbb{P}_n$ is the best uniform approximation polynomial. Moreover, Bernstein [5] derived the bounds

$$\frac{1}{\pi}\left|\sin\frac{\pi\alpha}{2}\right| \Gamma(\alpha)\Big(1 - \frac{1}{\alpha-1}\Big) \le B^*_{\infty,\alpha} \le \frac{1}{\pi}\left|\sin\frac{\pi\alpha}{2}\right| \Gamma(\alpha), \quad \alpha > 2, \tag{1.2}$$

but the exact value of $B^*_{\infty,\alpha}$ is still unknown (cf. [9]). The estimate (1.2) implies that for $\alpha \gg 1$,

$$B^*_{\infty,\alpha} \cong \frac{\Gamma(\alpha)}{\pi}\left|\sin\frac{\pi\alpha}{2}\right|. \tag{1.3}$$

In fact, Bernstein [5] speculated that for $\alpha = 1$, $B^*_{\infty,1} = \frac{1}{2\sqrt{\pi}} \approx 0.2820947917\ldots$. High precision numerical verification of Bernstein's speculation was carried out in [21,31] and $B^*_{\infty,1} \approx 0.2801694990\ldots$ was reported in [31]. We also remark that Ganzburg and Lubinsky [11] and Lubinsky [17] studied the Bernstein constant from the angle of best approximating entire functions to $|x|^\alpha$.

Ganzburg [9] and Lubinsky [18] showed the existence of the Bernstein constant in the L^p-sense

$$B^*_{p,\alpha} := \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} \big\| f_\alpha - p^*_n \big\|_{L^p} \ \text{ where } \ \big\| f_\alpha - p^*_n \big\|_{L^p} = \inf_{p\in\mathbb{P}_n} \big\| f_\alpha - p \big\|_{L^p}, \tag{1.4}$$

for all $1 \le p \le \infty$, but its exact value is known only for $p = 1, 2$ (see [19,23]). In particular, Raitsin [23] obtained that for $p = 2$,

$$B^*_{2,\alpha} = \frac{2\Gamma(\alpha+1)}{\sqrt{(2\alpha+1)\pi}}\left|\sin\frac{\alpha\pi}{2}\right|, \quad \alpha > -\frac{1}{2}. \tag{1.5}$$

In fact, Ganzburg [9] considered the analogues of the Bernstein's problem involving the Lagrange interpolation at the Chebyshev nodes of the first and second kind, from which the existence result in (1.4) was followed. One important result therein is

$$B^C_{\infty,\alpha} := \lim_{n\to\infty} (2n)^\alpha \| f_\alpha - \mathcal{I}^{(1)}_{2n}[f_\alpha]\|_{L^\infty} = \frac{4}{\pi}\left|\sin\frac{\pi\alpha}{2}\right| \int_0^\infty \frac{t^{\alpha-1}}{e^t + e^{-t}} \mathrm{d}t, \tag{1.6}$$

for $\alpha > 0$, where $\mathcal{I}_{2n}^{(1)}[f_\alpha]$ is the Lagrange interpolation of f_α at the nodes: $x_0 = 0$ and $\{x_j\}_{j=1}^{2n}$ being the zeros of the Chebyshev polynomials $T_{2n}(x)$. Revers [24] (also see some earlier works cited therein) studied an alternative interpolation and obtained

$$\limsup_{n\to\infty}(2n)^\alpha \|f_\alpha - \mathcal{I}_{2n}^{(2)}[f_\alpha]\|_{L^\infty} \le \frac{4}{\pi}\Big|\sin\frac{\pi\alpha}{2}\Big|\int_0^\infty \frac{t^\alpha}{e^t - e^{-t}}\mathrm{d}t, \quad \alpha > 0, \tag{1.7}$$

where $\mathcal{I}_{2n}^{(2)}[f_\alpha]$ is the Lagrange interpolant at the $(2n+1)$ zeros of $T_{2n+1}(x)$. We refer to Revers [26] for an up-to-date review and other interesting developments, and also to the dissertation [10] for many other explorations in various aspects.

In this paper, we aim to study the approximation of $f_\alpha(x)$ by its Fourier–Legendre partial sum

$$f_\alpha(x) = \sum_{k=0}^\infty \hat{f}_k^{(\alpha)} P_k(x), \quad S_n^{(\alpha)}(x) := \sum_{k=0}^n \hat{f}_k^{(\alpha)} P_k(x), \tag{1.8}$$

where $P_k(x)$ is the Legendre polynomial of degree k and the expansion coefficients are explicitly given by (see e.g., [23, P.60]):

$$\hat{f}_k^{(\alpha)} = \big\{1 + (-1)^k\big\}\frac{\Gamma(\alpha+1)}{2^{\alpha+2}\sqrt{\pi}}\frac{(2k+1)\Gamma((k-\alpha)/2)}{\Gamma((k+\alpha+3)/2)}\sin\frac{(k-\alpha)\pi}{2}. \tag{1.9}$$

We first derive the upper and lower bounds of the truncation errors in the L^∞-norm that are valid for all $n \ge n_0$ with some $n_0 \ge 1$. Consequently, we are able to obtain the explicit Bernstein-type constant $B_\infty^{(\alpha)}$. We can recover the lost half order (see [32]) or remove the extra $\log n$-factor (see [3]) in the existing estimates. The analysis is accomplished by subtle estimations of the involving Gamma functions and a proper summation rule.

2. Main results

We present the first main result on the L^∞-estimate of approximating f_α by its the Fourier–Legendre partial sum, and the related Bernstein-type constant.

Theorem 2.1. *For real $\alpha > 0$ and integer $n > \alpha + 1$, we have*

$$C_\alpha\,\widehat{\Upsilon}_{\hat{n}}^{(\alpha)}\frac{\Gamma(\hat{n}-\alpha/2)}{\Gamma(\hat{n}+\alpha/2)} \le \big\|f_\alpha - S_n^{(\alpha)}\big\|_{L^\infty} \le C_\alpha\,\Upsilon_{\hat{n}}^{(\alpha)}\frac{\Gamma(\hat{n}-\alpha/2-1)}{\Gamma(\hat{n}+\alpha/2-1)}, \tag{2.1}$$

where $\hat{n} := \lceil \frac{n+1}{2} \rceil$ is the smallest integer $\ge \frac{n+1}{2}$,

$$\begin{aligned} \Upsilon_{\hat{n}}^{(\alpha)} &:= 1 + \frac{(2\alpha+1)\sqrt{\pi}}{4}\frac{\Gamma(\hat{n}+\alpha/2+1)}{\Gamma(\hat{n}+(\alpha+3)/2)}, \\ \widehat{\Upsilon}_{\hat{n}}^{(\alpha)} &:= \frac{(\hat{n}+1/4)\Gamma(\hat{n}+1/2)\Gamma(\hat{n}+\alpha/2+1)}{\Gamma(\hat{n}+1)\Gamma(\hat{n}+(\alpha+3)/2)}, \end{aligned} \tag{2.2}$$

and

$$C_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-1}\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|. \tag{2.3}$$

Then the Bernstein-type constant is

$$B_\infty^{(\alpha)} = \lim_{n\to\infty} n^\alpha \big\|f_\alpha - S_n^{(\alpha)}\big\|_{L^\infty} = \frac{2\Gamma(\alpha)}{\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|. \tag{2.4}$$

Proof. We take three steps to prove the above results. (i) We first derive the upper bound in (2.1). Observe from (1.9) that $\hat{f}_k^{(\alpha)} = 0$ if k is odd. We group the terms in the summation as

$$\begin{aligned}\left|f_\alpha(x) - S_n^{(\alpha)}(x)\right| &= \Big|\sum_{k=n+1}^{\infty} \hat{f}_k^{(\alpha)} P_k(x)\Big| = \Big|\sum_{j=\hat{n}}^{\infty} \hat{f}_{2j}^{(\alpha)} P_{2j}(x)\Big| \\ &\le \mathbb{S}_{\hat{n}} + \mathbb{S}_{\hat{n}+2} + \mathbb{S}_{\hat{n}+4} + \cdots = \sum_{i=0}^{\infty} \mathbb{S}_{\hat{n}+2i},\end{aligned} \tag{2.5}$$

where

$$\mathbb{S}_j := \left|\hat{f}_{2j}^{(\alpha)} P_{2j} + \hat{f}_{2j+2}^{(\alpha)} P_{2j+2}\right| \le |\hat{f}_{2j}^{(\alpha)}||P_{2j} - P_{2j+2}| + |\hat{f}_{2j}^{(\alpha)} + \hat{f}_{2j+2}^{(\alpha)}||P_{2j+2}|. \tag{2.6}$$

We next show that for $j \ge \hat{n} \ge (\alpha+1)/2$,

$$\mathbb{S}_j \le \left|\sin\frac{\alpha\pi}{2}\right| \frac{\Gamma(\alpha+1)}{2^{\alpha-2}\pi} \left\{1 + \frac{(2\alpha+1)\sqrt{\pi}}{4} \frac{\Gamma(\hat{n}+\alpha/2+1)}{\Gamma(\hat{n}+\alpha/2+3/2)}\right\} \frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)}. \tag{2.7}$$

To do this, we infer from [16, (3.17)] with $\mu \to 0$ that

$$\max_{|x|\le 1}\left\{\Big|\int_{-1}^{x} P_k(y)\mathrm{d}y\Big|\right\} \le \frac{1}{2\sqrt{\pi}} \frac{\Gamma(k/2)}{\Gamma((k+3)/2)}.$$

Using the properties (cf. [27, (3.175) and (3.176a)]):

$$P_k(\pm 1) = (\pm 1)^k, \quad k \ge 0; \quad (2k+1)P_k(x) = P'_{k+1}(x) - P'_{k-1}(x), \quad k \ge 1,$$

we have that for $k \ge 1$,

$$\left|P_{k+1}(x) - P_{k-1}(x)\right| = (2k+1)\Big|\int_{-1}^{x} P_k(y)\mathrm{d}y\Big| \le \frac{1}{\sqrt{\pi}} \frac{(k+1/2)\Gamma(k/2)}{\Gamma((k+3)/2)}. \tag{2.8}$$

On the other hand, we obtain from (1.9) that

$$|\hat{f}_{2j}^{(\alpha)} + \hat{f}_{2j+2}^{(\alpha)}| \le \left|\sin\frac{\alpha\pi}{2}\right| \frac{\Gamma(\alpha+1)}{2^\alpha\sqrt{\pi}} \frac{2(2\alpha+1)j + 3\alpha + 3/2}{2j+\alpha+3} \frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+3/2)}. \tag{2.9}$$

A combination of (1.9), (2.6), (2.8), (2.9) and the fact $|P_k(x)| \le 1$, leads to

$$\begin{aligned}\mathbb{S}_j \le \left|\sin\frac{\alpha\pi}{2}\right| \frac{\Gamma(\alpha+1)}{2^\alpha\sqrt{\pi}} \frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)} &\left\{ \frac{4(j+1/4)(j+3/4)}{\sqrt{\pi}} \frac{\Gamma(j+1/2)}{\Gamma(j+2)}\right. \\ &\left. + \frac{2(2\alpha+1)j + 3\alpha + 3/2}{2j+\alpha+3}\right\} \frac{\Gamma(j+\alpha/2+1)}{\Gamma(j+\alpha/2+3/2)}.\end{aligned} \tag{2.10}$$

From [1, (1.1) and Thm. 10], we know that for $c \ge b \ge 0$, the ratio

$$\mathcal{R}_c^b(z) := \frac{\Gamma(z+b)}{\Gamma(z+c)}, \quad z \ge 0, \tag{2.11}$$

is decreasing with respect to z. Thus for $j \ge \hat{n}$,

$$\frac{2(2\alpha+1)j + 3\alpha + 3/2}{2j+\alpha+3} \frac{\Gamma(j+\alpha/2+1)}{\Gamma(j+\alpha/2+3/2)} \le (2\alpha+1)\frac{\Gamma(\hat{n}+\alpha/2+1)}{\Gamma(\hat{n}+\alpha/2+3/2)}. \tag{2.12}$$

Recall [20, (5.11.13)]: for $a < b$,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} + \frac{1}{2}(a-b)(a+b-1)z^{a-b-1} + \mathcal{O}(z^{a-b-2}), \quad z \gg 1. \tag{2.13}$$

Therefore

$$\lim_{j\to\infty}\frac{(j+1/4)^{1/2}\Gamma(j+1/2)}{\Gamma(j+1)}=1,\qquad \lim_{j\to\infty}\frac{(j+\alpha/2+3/4)^{1/2}\Gamma(j+\alpha/2+1)}{\Gamma(j+\alpha/2+3/2)}=1. \tag{2.14}$$

Moreover, the ratio (cf. [7, Corollary 2])

$$\widehat{\mathcal{R}}_c(z):=\frac{1}{\sqrt{z+c}}\frac{\Gamma(z+1)}{\Gamma(z+1/2)}, \tag{2.15}$$

is decreasing $(-1/4,\infty)$ (increasing on $(-1/2,\infty)$) if $c=1/4$ (if $c=1/2$), which, together with (2.14), implies

$$\frac{1}{\widehat{\mathcal{R}}_{1/4}(j)}=\frac{(j+1/4)^{1/2}\Gamma(j+1/2)}{\Gamma(j+1)}\le\lim_{j\to\infty}\frac{1}{\widehat{\mathcal{R}}_{1/4}(j)}=1,$$

and

$$\begin{aligned}\frac{1}{\widehat{\mathcal{R}}_{1/4}(j+\alpha/2+1/2)}&=\frac{(j+\alpha/2+3/4)^{1/2}\Gamma(j+\alpha/2+1)}{\Gamma(j+\alpha/2+3/2)}\\&\le\lim_{j\to\infty}\frac{1}{\widehat{\mathcal{R}}_{1/4}(j+\alpha/2+1/2)}=1.\end{aligned}$$

Thus

$$(j+1/4)(j+3/4)\frac{\Gamma(j+1/2)}{(j+1)\Gamma(j+1)}\frac{\Gamma(j+\alpha/2+1)}{\Gamma(j+\alpha/2+3/2)}\le\frac{(j+1/4)^{1/2}}{(j+\alpha/2+3/4)^{1/2}}\frac{j+3/4}{j+1}\le 1. \tag{2.16}$$

From (2.10), (2.12), (2.16) and $\Gamma(z+1)=z\Gamma(z)$, we get (2.7).

With (2.7), we can now estimate the summation in (2.5). We obtain from (2.11) that

$$\begin{aligned}\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)}&\le\frac{1}{2}\frac{\Gamma(j-\alpha/2-1)}{\Gamma(j+\alpha/2)}+\frac{1}{2}\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)}\\&=\frac{1}{2\alpha}\Big(\frac{\Gamma(j-\alpha/2-1)}{\Gamma(j+\alpha/2-1)}-\frac{\Gamma(j-\alpha/2+1)}{\Gamma(j+\alpha/2+1)}\Big),\end{aligned} \tag{2.17}$$

where we observed

$$\frac{\Gamma(j-\alpha/2-1)}{\Gamma(j+\alpha/2)}=\frac{1}{\alpha}\Big(\frac{\Gamma(j-\alpha/2-1)}{\Gamma(j+\alpha/2-1)}-\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2)}\Big),$$

and

$$\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)}=\frac{1}{\alpha}\Big(\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2)}-\frac{\Gamma(j-\alpha/2+1)}{\Gamma(j+\alpha/2+1)}\Big).$$

As a direct consequence of (2.17), we have

$$\sum_{i=0}^{\infty}\frac{\Gamma(\hat{n}+2i-\alpha/2)}{\Gamma(\hat{n}+2i+\alpha/2+1)}\le\frac{1}{2\alpha}\frac{\Gamma(\hat{n}-\alpha/2-1)}{\Gamma(\hat{n}+\alpha/2-1)}. \tag{2.18}$$

Then the upper bound in (2.1) is a direct consequence of (2.5), (2.7) and (2.18).

(ii) We next derive the lower bound in (2.1). We recall (see [12, 8.911(5) & 8.339(2)])

$$P_{2j}(0)=(-1)^j\frac{\Gamma(j+1/2)}{\sqrt{\pi}\,j!},$$

and obtain from (1.9) that

$$\begin{aligned}\big\| f_\alpha - S_n^{(\alpha)}\big\|_{L^\infty} \ge \big|(f_\alpha - S_n^{(\alpha)})(0)\big| &= \Big|\sum_{j=\hat n}^{\infty} \hat f_{2j}^{(\alpha)} P_{2j}(0)\Big| \\ &= \Big|\sin\frac{\alpha\pi}{2}\Big| \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi} \sum_{j=\hat n}^{\infty} \frac{(j+1/4)\Gamma(j+1/2)\Gamma(j-\alpha/2)}{\Gamma(j+1)\Gamma(j+(\alpha+3)/2)} \\ &= \Big|\sin\frac{\alpha\pi}{2}\Big| \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi} \sum_{j=\hat n}^{\infty} \widehat{\Upsilon}_j^{(\alpha)} \frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)},\end{aligned} \tag{2.19}$$

where $\widehat{\Upsilon}_j^{(\alpha)}$ is given in (2.2). Thus direct calculation leads to

$$\frac{\widehat{\Upsilon}_j^{(\alpha)}}{\widehat{\Upsilon}_{j+1}^{(\alpha)}} = \frac{(j+1/4)(j+1)(j+(\alpha+3)/2)}{(j+1/2)(j+5/4)(j+\alpha/2+1)} = 1 - \frac{(\alpha+1)j+3\alpha/4+1}{4(j+1/2)(j+5/4)(j+\alpha/2+1)} < 1,$$

so the sequence $\{\Upsilon_j^{(\alpha)}\}_{j\ge\hat n}$ is increasing. Hence by (2.11),

$$\begin{aligned}\sum_{j=\hat n}^{\infty} \widehat{\Upsilon}_j^{(\alpha)} \frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)} &\ge \widehat{\Upsilon}_{\hat n}^{(\alpha)} \sum_{j=\hat n}^{\infty} \frac{1}{\alpha}\Big(\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2)} - \frac{\Gamma(j-\alpha/2+1)}{\Gamma(j+(\alpha+2)/2)}\Big) \\ &= \widehat{\Upsilon}_{\hat n}^{(\alpha)} \frac{\Gamma(\hat n-\alpha/2)}{\alpha\Gamma(\hat n+\alpha/2)}.\end{aligned}$$

Thus, we obtain from (2.19) the lower bound (2.1).

(iii) We now calculate the Bernstein-type constant. From (2.13), we get $\lim_{n\to\infty} \Upsilon_{\hat n}^{(\alpha)} = \lim_{n\to\infty} \widehat{\Upsilon}_{\hat n}^{(\alpha)} = 1$, and

$$\lim_{n\to\infty} n^\alpha \frac{\Gamma(\hat n-\alpha/2)}{\Gamma(\hat n+\alpha/2)} = \lim_{n\to\infty} n^\alpha \frac{\Gamma(\hat n-\alpha/2-1)}{\Gamma(\hat n+\alpha/2-1)} = 2^\alpha.$$

Then we obtain the desired value from the bounds in (2.1) and the above facts. □

Remark 2.1. Recently, Wang [32, Corollary2.5] obtained the estimate for $\alpha = 1$,

$$\| f_1 - S_n^{(1)}\|_{L^\infty} \le \frac{8}{\sqrt{\pi(2n-5)}}, \tag{2.20}$$

but the numerical evidence showed the order $O(n^{-1})$. As a by-product of Theorem 2.1, we have

$$\frac{1}{\pi(\hat n-1/2)}\Big(1-\frac{3}{4(\hat n+1)}\Big) \le \big\| f_1 - S_n^{(1)}\big\|_{L^\infty} \le \frac{1}{\pi(\hat n-3/2)}\Big(1+\frac{3\sqrt{\pi}}{4\sqrt{\hat n+5/4}}\Big), \tag{2.21}$$

which follows from (2.2) and (2.15). The key is the summation rule where (2.20) was obtained by a naive summation using $|P_k(x)| \le 1$ as the Chebyshev case, while the optimal estimate (2.21) is derived from a proper grouping the terms in (2.5)–(2.6).

Coincidentally, we can show that the Bernstein-type constant of the Fourier–Chebyshev partial sum is identical to that of the Legendre case.

Theorem 2.2. *Consider the Fourier–Chebyshev partial sum*

$$\widetilde{S}_n^{(\alpha)}(x) := \sum_{k=0}^{n}{}' \tilde{f}_k^{(\alpha)}\, T_k(x), \qquad \tilde{f}_k^{(\alpha)} = \frac{2}{\pi}\int_{-1}^{1} \frac{f_\alpha(x) T_k(x)}{\sqrt{1-x^2}}\mathrm{d}x, \tag{2.22}$$

where $\sum'$ *means the first term is halved. Then for real* $\alpha > 0$ *and integer* $n \geq \alpha + 1$, *we have the identity*

$$\| f_\alpha - \widetilde{S}_n^{(\alpha)} \|_{L^\infty} = \frac{\Gamma(\alpha)}{2^{\alpha-1}\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|\frac{\Gamma(\hat{n}-\alpha/2)}{\Gamma(\hat{n}+\alpha/2)}, \tag{2.23}$$

where $\hat{n} = \lceil \frac{n+1}{2} \rceil$ *as before. Consequently, the Bernstein-type constant is*

$$\widetilde{B}_\infty^{(\alpha)} := \lim_{n\to\infty} n^\alpha \| f_\alpha - \widetilde{S}_n^{(\alpha)} \|_{L^\infty} = B_\infty^{(\alpha)} = \frac{2\Gamma(\alpha)}{\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|. \tag{2.24}$$

Proof. Like (1.9), we have the explicit formula (cf. [15, (4.58)])

$$\tilde{f}_k^{(\alpha)} = \big\{1 + (-1)^k\big\}\frac{\Gamma(\alpha+1)}{2^\alpha\pi}\frac{\Gamma(k/2-\alpha/2)}{\Gamma(k/2+\alpha/2+1)}\sin\Big(\frac{(k-\alpha)\pi}{2}\Big). \tag{2.25}$$

Thus by (2.22), (2.25) and the fact $|T_k(x)| \leq 1$,

$$\begin{aligned}\| f_\alpha - \widetilde{S}_n^{(\alpha)} \|_{L^\infty} &\leq \sum_{j=\hat{n}}^{\infty}\Big|\tilde{f}_{2j}^{(\alpha)}\Big| = \frac{\Gamma(\alpha+1)}{2^{\alpha-1}\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|\sum_{j=\hat{n}}^{\infty}\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)} \\ &= \frac{\Gamma(\alpha)}{2^{\alpha-1}\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|\sum_{j=\hat{n}}^{\infty}\Big(\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2)} - \frac{\Gamma(j-\alpha/2+1)}{\Gamma(j+\alpha/2+1)}\Big) \\ &= \frac{\Gamma(\alpha)}{2^{\alpha-1}\pi}\Big|\sin\frac{\alpha\pi}{2}\Big|\frac{\Gamma(\hat{n}-\alpha/2)}{\Gamma(\hat{n}+\alpha/2)},\end{aligned} \tag{2.26}$$

where we used the following simple equality derived from the property $\Gamma(z+1) = z\Gamma(z)$:

$$\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2+1)} = \frac{1}{\alpha}\Big(\frac{\Gamma(j-\alpha/2)}{\Gamma(j+\alpha/2)} - \frac{\Gamma(j-\alpha/2+1)}{\Gamma(j+\alpha/2+1)}\Big).$$

On the other hand, using (2.22), (2.25) and $T_k(0) = \cos(k\pi/2)$, we obtain from (2.26) that

$$\| f_\alpha - \widetilde{S}_n^{(\alpha)} \|_{L^\infty} \geq \big|(f_\alpha - \widetilde{S}_n^{(\alpha)})(0)\big| = \sum_{j=\hat{n}}^{\infty}\big|\tilde{f}_{2j}^{(\alpha)}\big| = \frac{\Gamma(\alpha)}{2^{\alpha-1}\pi}\frac{\Gamma(\hat{n}-\alpha/2)}{\Gamma(\hat{n}+\alpha/2)}\Big|\sin\frac{\alpha\pi}{2}\Big|. \tag{2.27}$$

This ends the derivation of (2.23). Using (2.13), we obtain the Bernstein-type constant (2.24). □

Remark 2.2. It is noteworthy that Varga [30] derived the constant (2.24) for $\alpha = 1$.

We infer from the bounds in (1.2) (from Bernstein [5]) and (2.4) that for $\alpha > 2$,

$$\frac{B_\infty^{(\alpha)}}{2}\Big(1 - \frac{1}{\alpha-1}\Big) \leq B_{\infty,\alpha}^{*} \leq \frac{B_\infty^{(\alpha)}}{2}. \tag{2.28}$$

As a direct consequence of (2.28), we have the following relationship between two constants.

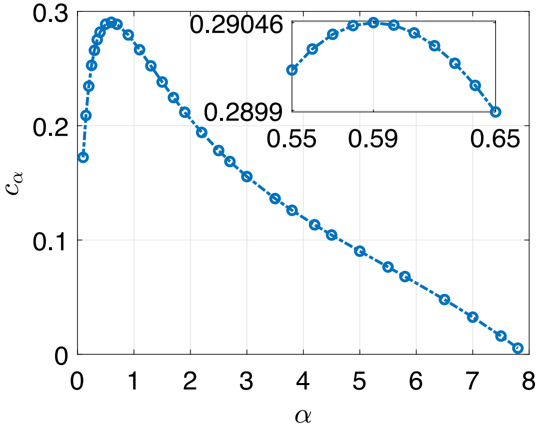

Fig. 2.1. Plot of c_α for various values of $\alpha \in (0, 8)$.

Proposition 2.1. *For $\alpha > 2$, there hold*

$$B_\infty^{(\alpha)} = (2 + c_\alpha) B_{\infty,\alpha}^* \quad \text{and} \quad 0 \le c_\alpha \le \frac{2}{\alpha - 2}, \tag{2.29}$$

so $c_\alpha \to 0$ as $\alpha \to \infty$.

We now extend the definition of $c_\alpha = B_\infty^{(\alpha)}/B_{\infty,\alpha}^* - 2$ to $\alpha \in (0, 2)$. It is noteworthy that Carpenter and Varga [8, Table 1.1] provided some very accurate values of $B_{\infty,\alpha}^*$, and here we also use the Chebfun [21] to compute more values. In Fig. 2.1, we plot the graph of c_α for $\alpha \in (0, 8)$.

From the above numerical evidences, we conjecture the following property of c_α, but the proof is open.

Conjecture 2.1. *Let $\alpha > 0$ not be an even integer. Then c_α has a global maximum point $(\alpha_0, c_{\alpha_0}) \approx (0.59, 0, 29046)$, and c_α is monotonically increasing for $\alpha < \alpha_0$ but decreasing for $\alpha > \alpha_0$.*

Data availability

No data was used for the research described in the article.

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