# POINTWISE ERROR ESTIMATES AND LOCAL SUPERCONVERGENCE OF JACOBI EXPANSIONS 

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#### Abstract

As one myth of polynomial interpolation and quadrature, Trefethen [Math. Today (Southend-on-Sea) 47 (2011), pp. 184-188] revealed that the Chebyshev interpolation of $|x-a|$ (with $|a|<1$ ) at the Clenshaw-Curtis points exhibited a much smaller error than the best polynomial approximation (in the maximum norm) in about $95 \%$ range of $[-1,1]$ except for a small neighbourhood near the singular point $x=a$. In this paper, we rigorously show that the Jacobi expansion for a more general class of $\Phi$-functions also enjoys such a local convergence behaviour. Our assertion draws on the pointwise error estimate using the reproducing kernel of Jacobi polynomials and the Hilb-type formula on the asymptotic of the Bessel transforms. We also study the local superconvergence and show the gain in order and the subregions it occurs. As a by-product of this new argument, the undesired $\log n$-factor in the pointwise error estimate for the Legendre expansion recently stated in Babus̆ka and Hakula [Comput. Methods Appl. Mech Engrg. 345 (2019), pp. 748-773] can be removed. Finally, all these estimates are extended to the functions with boundary singularities. We provide ample numerical evidences to demonstrate the optimality and sharpness of the estimates.


## 1. Introduction

Approximation by polynomials plays a fundamental role in algorithm development and numerical analysis of many computational methods. It is known that for a given continuous function $f(x)$ defined on $[-1,1]$, the best polynomial approximation of $f(x)$ in the maximum norm is a unique polynomial $p_{n}^{*} \in \mathcal{P}_{n}$ (denotes the set of polynomials of degree at most $n$ ) that minimizes

$$
\left\|f-p_{n}^{*}\right\|_{\infty}=\min _{p \in \mathcal{P}_{n}}\|f-p\|_{\infty} .
$$

The best polynomial approximation $p_{n}^{*}(x)$ is optimal, but its computation is nontrivial for a general nonlinear function $f(x)$ [30]. In fact, Trefethen [29] pointed out that for $f(x)=\left|x-\frac{1}{4}\right|$, the pointwise error $\left|f(x)-p_{n}(x)\right|$ by the polynomial

[^0]interpolation $p_{n}$ at the Clenshaw-Curtis points $\left\{x_{j}=\cos \left(\frac{j \pi}{n}\right)\right\}_{j=0}^{n}$ is much smaller than that of the best polynomial: $\left|f(x)-p_{n}^{*}(x)\right|$ for most values of $x$, except for a small subinterval centred around the singular point $x=\frac{1}{4}$ (see Figure 1.1 (left) for an illustration of $n=100$ ).


Figure 1.1. Pointwise error curves of the best polynomial approximation $f(x)-p_{n}^{*}(x)$, Chebyshev interpolation $f(x)-p_{n}(x)$ (left), and Chebyshev truncation $f(x)-S_{n}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}[f](x)$ (right), where $n=100$

Needless to say, the pointwise error is a very useful indication of the approximability and approximation quality of a numerical tool in solving partial differential equations [4, 11, 16. In the past several decades, the error estimates of spectral approximation in Sobolev norms have been intensively studied and well-documented in e.g., [2, 3, 18, 24, 30, 31. However, whenever possible, one would wish to estimate the pointwise error of the approximation [5, 17, 19, 31, 33, 34, 37, 39, though it is usually more challenging.

Compared with the aforementioned Chebyshev interpolation $p_{n}(x)$, the pointwise error of Chebyshev spectral projection $S_{n}^{(-1 / 2,-1 / 2)}[f]$ of $f(x)=\left|x-\frac{1}{4}\right|$ is also much smaller than the best polynomial approximation except for the subinterval near the singular point $x=\frac{1}{4}$, and it is more localized than $f(x)-p_{n}(x)$ near the singularity (see Figure 1.1). This implies some underlying local superconvergence at the points slightly away from the singularity.

Interestingly, such a superconvergence phenomenon also occurs in the Legendre and more general Jacobi expansions (except for an additional small neighbourhood of the endpoints $x= \pm 1$, see Figure (1.2). In fact, we are not the first to unfold this convergence behaviour. In a recent work, Babuška and Hakula [5 provided deep insights into this phenomenon for the Legendre expansion of the class of $\Phi$-functions defined by

$$
f(x)=(x-a)_{+}^{\lambda}=\left\{\begin{array}{ll}
0, & -1 \leq x<a,  \tag{1.1}\\
(x-a)^{\lambda}, & a<x \leq 1,
\end{array} \quad \lambda>-1, \quad a \in(-1,1)\right.
$$



Figure 1.2. Pointwise error curves of the best polynomial approximation $f(x)-p_{n}^{*}(x)$ and the Jacobi truncation $f(x)-S_{n}^{(\alpha, \beta)}[f](x)$ with $\alpha=\beta=0$ (left) and $\alpha=\beta=\frac{1}{2}$ (right), where $n=100$
which appears frequently in various applications [5]. More precisely, define the truncated Legendre series and the pointwise error as

$$
\begin{equation*}
S_{n}^{(0,0)}[f](x)=\sum_{k=0}^{n} a_{k} P_{k}(x), \quad e_{f}(n, x)=\left|f(x)-S_{n}^{(0,0)}[f](x)\right|, \tag{1.2}
\end{equation*}
$$

where $P_{k}(x)$ is the Legendre polynomial of degree $k$ as in [27] and $a_{k}$ are the Legendre expansion coefficients. Taking into account the convergence rates on the piecewise analytic functions in Saff and Totik [23], Babus̆ka and Hakula [5] derived the following estimates.

Theorem 1.1 (See [5]). Let $f(x)$ be a $\Phi$-function defined by (1.1) with $\lambda=0$, i.e., a step function.
(i) For $x \in(-1, a) \cup(a, 1)$, we have $e_{f}(n, x) \leq C(x) n^{-1}$, where $C(x)$ is independent of $n$ and has the behaviours near $x=a, \pm 1$ as follows

$$
\begin{equation*}
C(-1+\xi) \leq D(-1) \xi^{-\frac{1}{4}}, \quad C(1-\xi) \leq D(1) \xi^{-\frac{1}{4}}, \quad C(a \pm \xi) \leq D(a) \xi^{-1} \tag{1.3}
\end{equation*}
$$

for $0<\xi \leq \delta$, where $D( \pm 1), D(a)>0$ and $\delta>0$ are independent of $n$.
(ii) At $x= \pm 1$, a, we have

$$
\begin{equation*}
e_{f}(n, \pm 1) \leq C n^{-\frac{1}{2}}, \quad e_{f}(n, a) \leq C n^{-1} . \tag{1.4}
\end{equation*}
$$

In (1.4) and what follows, we denote by $C$ a generic positive constant independent of $n$ which may have a different value in a different context.

Following Wahlbin [32] and Bary [6, Babuška and Hakula [5] further obtained the following estimates for $\lambda \neq 0$.

Theorem 1.2 (See [5]). Let $f(x)$ be a $\Phi$-function defined by (1.1) with $\lambda>-1$ but $\lambda \neq 0$.
(i) For $x \in(-1, a) \cup(a, 1)$, we have

$$
\begin{equation*}
e_{f}(n, x) \leq C(x) n^{-\lambda-1} \log n \tag{1.5}
\end{equation*}
$$

where $C(x)>0$ is independent of $n$, and has the same behaviour as $C(x)$ in (1.3).
(ii) At $x= \pm 1$, a, we have

$$
\begin{align*}
& e_{f}(n, \pm 1) \leq C n^{-\lambda-\frac{1}{2}} \log n, \\
& e_{f}(n, a) \leq \begin{cases}C n^{-\lambda-1} \log n, & \lambda \text { even }, \\
C n^{-\lambda} \log n, & \lambda>0 \text { and non-even. }\end{cases} \tag{1.6}
\end{align*}
$$

Some remarks are in order.

- From ample delicate numerical experiments, Babus̆ka and Hakula [5] conjectured that the multiplicative factor $\log n$ in Theorem 1.2 seems to be a defect of the analysis technique employed in the proof, and Theorem 1.2 should hold without the $\log n$ factor. This was stated as a hypothesis and claimed "in spite of many attempts, the hypothesis underlines the need for new theory" in 5].
- It is worthy of mentioning that Kruglov extended the numerical study in [5] for the Legendre expansions to the more general Jacobi polynomial cases in the master thesis [15], but the log-term remained as a conjecture in the results therein.
- It is seen from Theorem 1.2 that if $\lambda$ is not an even integer, we have the superconvergence

$$
e_{f}(n, x) \leq C(x) n^{-1} \log n\left\|f-S_{n}^{(0,0)}[f]\right\|_{\infty}
$$

with a gain of convergence rate $\mathcal{O}\left(n^{-1} \log n\right)$ on any closed subinterval that excludes $x=a, \pm 1$.
The main purposes of this paper are twofold. Firstly, using a new technique, we shall show that the log-factor can be removed. Secondly, we shall conduct the optimal pointwise convergence and superconvergence analysis for the Jacobi expansions of the following generalised $\Phi$-functions

$$
f(x)=z(x) \cdot\left\{\begin{array}{ll}
0, & -1 \leq x<a,  \tag{1.8}\\
(x-a)^{\lambda}, & a<x \leq 1,
\end{array} \quad a \in(-1,1), \quad \lambda>-1,\right.
$$

and

$$
\begin{equation*}
f(x)=|x-a|^{\lambda} z(x) \quad(\lambda>-1 \text { is not an even integer }), \tag{1.9}
\end{equation*}
$$

where we set in (1.8) $f(a)=0$ for $\lambda>0$ and $f(a)=\frac{z(a)}{2}$ for $\lambda=0$, and the given function $z(x)$ involved is assumed to be smooth with $z(a) \neq 0$. Denote the Jacobi expansion of $f(x)$ in (1.8) or (1.9) and the pointwise error respectively by

$$
\begin{equation*}
S_{n}^{(\alpha, \beta)}[f](x)=\sum_{k=0}^{n} a_{k}^{(\alpha, \beta)} P_{k}^{(\alpha, \beta)}(x), \quad e_{f}^{(\alpha, \beta)}(n, x)=\left|f(x)-S_{n}^{(\alpha, \beta)}[f](x)\right|, \tag{1.10}
\end{equation*}
$$

where $P_{k}^{(\alpha, \beta)}(x)$ is the Jacobi polynomial of degree $k$ and

$$
\begin{align*}
& a_{k}^{(\alpha, \beta)}=\frac{1}{\sigma_{k}^{(\alpha, \beta)}} \int_{-1}^{1} f(x) P_{k}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) \mathrm{d} x \\
& \omega^{(\alpha, \beta)}(x)=(1-x)^{\alpha}(1+x)^{\beta}  \tag{1.11}\\
& \sigma_{k}^{(\alpha, \beta)}=\frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{k!(2 k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)} .
\end{align*}
$$

Using the reproducing kernel of Jacobi polynomials, together with the Hilbtype formula and van der Corput-type Lemma on the asymptotic of the Bessel transforms, we are able to derive the following main results.

Theorem 1.3. Let $f(x)$ be a generalised $\Phi$-function defined in (1.8) or (1.9). Then for $\alpha, \beta>-1$ and $\lambda>-1$, we have the following pointwise error estimates.
(i) For $x \in(-1, a) \cup(a, 1)$, we have

$$
\begin{equation*}
e_{f}^{(\alpha, \beta)}(n, x) \leq C(x) n^{-\lambda-1}, \tag{1.12}
\end{equation*}
$$

where $C(x)$ is independent of $n$ and has the behaviours near $x=a, \pm 1$ as follows

$$
\begin{align*}
& C(-1+\xi) \leq D(-1) \xi^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} \\
& C(1-\xi) \leq D(1) \xi^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}  \tag{1.13}\\
& C(a \pm \xi) \leq D(a) \xi^{-1}, \quad 0<\xi \leq \delta
\end{align*}
$$

Here $D( \pm 1), D(a)>0$ and $\delta>0$ are independent of $n$.
(ii) At $x= \pm 1$, we have

$$
\begin{equation*}
e_{f}^{(\alpha, \beta)}(n, 1) \leq C n^{-\lambda+\alpha-\frac{1}{2}}, \quad e_{f}^{(\alpha, \beta)}(n,-1) \leq C n^{-\lambda+\beta-\frac{1}{2}} \tag{1.14}
\end{equation*}
$$

(iii) At $x=a$ and for $\lambda>0$, we have

$$
\begin{align*}
& e_{f}^{(\alpha, \beta)}(n, a) \leq\left\{\begin{array}{ll}
C n^{-\lambda-1}, & \lambda \text { even, } \\
C n^{-\lambda}, & \text { otherwise, }
\end{array} \text { for } f(x)\right. \text { defined by (1.8); }  \tag{1.15}\\
& e_{f}^{(\alpha, \beta)}(n, a) \leq C n^{-\lambda} \text { for } f(x) \text { defined by (1.9) and non-even } \lambda .
\end{align*}
$$

We emphasize that all the above estimates are optimal in the sense that the convergence order cannot be improved, which will be illustrated numerically in Section 3 As a special case, the multiplicative factor $\log n$ in Theorem 1.2 for the Legendre expansion is removed. The asymptotic behaviour of the pointwise error around the endpoints is described clearly. Indeed, we infer from (1.13) and (1.14) that $e_{f}^{(\alpha, \beta)}(n, x)$ achieves the best convergence rate around $x= \pm 1$ when $\alpha, \beta \leq-\frac{1}{2}$. As an example, we consider $f(x)=|x-a|$. It is known that the pointwise error of the best polynomial approximation equally oscillates $N \geq n+2$ times and converges linearly as $n \rightarrow \infty$, i.e., there exist at least $N \geq n+2$ distinct points $x_{1}, x_{2}, \cdots, x_{N}$ on $[-1,1]$ such that

$$
f\left(x_{i}\right)-p_{n}^{*}\left(x_{i}\right)=\varepsilon(-1)^{i}\left\|f-p_{n}^{*}\right\|_{\infty}, \quad\left\|f-p_{n}^{*}\right\|_{\infty} \sim \sigma \sqrt{1-a^{2}} n^{-1}
$$

where $\varepsilon= \pm 1$ and $\sigma \approx 1 / 2 \sqrt{\pi}$ is the Bernstein constant (see [7, 30]). As a comparison, $e_{f}^{(\alpha, \beta)}(n, x)$ shares the same order of convergence $\mathcal{O}\left(n^{-1}\right)$ at $x=a$, but somehow worse in magnitude than that of the best polynomial approximation. Nevertheless, superconvergence appears when $x \in(-1, a) \cup(a, 1)$, where it follows from Theorem 1.3 that $e_{f}^{(\alpha, \beta)}(n, x)=\mathcal{O}\left(n^{-2}\right)$ (also see Figure 1.2).

Incidentally, from the viewpoint of the maximum norm (i.e., the worst-case behaviour of $\left.e_{f}^{(\alpha, \beta)}(n, x)\right)$, the Jacobi truncation $S_{n}^{(\alpha, \beta)}[f](x)\left(\alpha, \beta \leq \frac{1}{2}\right)$ performs as excellent as the best polynomial approximation in the sense of asymptotic rate


Figure 1.3. Comparison of Jacobi expansion and the optimal polynomial approximation to $f(x)=(x-1 / 4)_{+}^{1 / 2}$ by the convergence rates of $\left\|\hat{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}$ (red) versus $\left\|f-p_{n}^{*}\right\|_{\infty}$ (blue)
when $n \rightarrow \infty$ for functions defined in (1.8) and (1.9) where $\lambda>0$, that is,

$$
\left\|f-S_{n}^{(\alpha, \beta)}[f]\right\|_{\infty}=\left\{\begin{array}{lr}
\mathcal{O}\left(n^{\max \left\{\alpha-\frac{1}{2}, \beta-\frac{1}{2}\right\}-\lambda}\right), & \text { if } \max \{\alpha, \beta\}>\frac{1}{2}  \tag{1.16}\\
\mathcal{O}\left(n^{-\lambda}\right), & \text { if } \max \{\alpha, \beta\} \leq \frac{1}{2}
\end{array}\right.
$$

Taking the local behaviour of $e_{f}^{(\alpha, \beta)}(n, x)$ around the boundaries $x= \pm 1$ and singularity $x=a$ into consideration, we consider a new weighted pointwise error function

$$
\begin{equation*}
\hat{e}_{f}^{(\alpha, \beta)}(n, x)=(1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}(x-a) e_{f}^{(\alpha, \beta)}(n, x) . \tag{1.17}
\end{equation*}
$$

Then we deduce from Theorem 1.3 the uniform convergence order

$$
\begin{equation*}
\left\|\hat{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}=\mathcal{O}\left(n^{-\lambda-1}\right) \tag{1.18}
\end{equation*}
$$

which also testifies to the optimality of the estimates on $C(x)$ in (1.13). As a result, a global superconvergence is attained by $\hat{e}_{f}^{(\alpha, \beta)}(x)$, which gains one order higher in convergence rate than the best polynomial approximation (see Figure 1.3).

The rest of this paper is largely devoted to the proof of the results stated in Theorem [1.3 In Section 2 we present the pointwise error formula and some asymptotic results on the Jacobi polynomials. Applying the Hilb-type formula and van der Corput-type Lemma for Bessel transforms, in Section 3, we prove the optimal pointwise error estimates on Jacobi truncation of functions defined in (1.8) and (1.9). Errors in maximum norm and weighted maximum norm (superconvergence analysis) are considered in Section 4 Finally we extend the analysis to functions with boundary singularities and conclude the paper with some remarks in Section 5

## 2. The reproducing kernel and pointwise error formula

Let $\mathrm{d} \omega(x)$ be a given distribution in the Stieltjes sense. Assume that $\left\{p_{k}\right\}_{k=0}^{\infty}$ with $\operatorname{deg}\left(p_{k}\right)=k$ is the set of orthonormal polynomials associated with $\mathrm{d} \omega(x)$

$$
\int_{-1}^{1} p_{j}(x) p_{k}(x) \mathrm{d} \omega(x)=\delta_{j k}, \quad j, k=0,1,2, \cdots
$$

where $\delta_{j k}$ is the Kronecker Delta symbol. In view of the Christoffel-Darboux formula, the reproducing kernel $K_{n}(x, y)$ is defined by (see [27. Theorem 3.22] and Lubinsky [20])

$$
\begin{equation*}
K_{n}(x, y)=\sum_{k=0}^{n} p_{k}(x) p_{k}(y)=\frac{\kappa_{n}}{\kappa_{n+1}} \frac{p_{n+1}(x) p_{n}(y)-p_{n}(x) p_{n+1}(y)}{x-y}, \tag{2.1}
\end{equation*}
$$

where $\kappa_{n}$ is the leading coefficient of $p_{n}(x)$ and $\lim _{n \rightarrow \infty} \frac{\kappa_{n}}{\kappa_{n+1}}=\frac{1}{2}$. It is easy to verify from the orthogonality that

$$
\begin{equation*}
\int_{-1}^{1} K_{n}(x, y) q(y) \mathrm{d} \omega(y)=q(x), \quad \forall q \in \mathcal{P}_{n} . \tag{2.2}
\end{equation*}
$$

We intend to estimate the pointwise error of the Jacobi orthogonal projection $e_{f}^{(\alpha, \beta)}(n, x)$ defined in (1.10). According to Szegö [27, (4.5.2)] and Hesthaven, Gottlieb and Gottlieb [14, Theorem 4.4], the reproducing kernel of the Jacobi polynomials can be represented as follows

$$
\begin{align*}
K_{n}(x, y) & =\sum_{k=0}^{n} \frac{1}{\sigma_{k}^{(\alpha, \beta)}} P_{k}^{(\alpha, \beta)}(x) P_{k}^{(\alpha, \beta)}(y)  \tag{2.3}\\
& =\rho_{n}^{(\alpha, \beta)} \frac{P_{n+1}^{(\alpha, \beta)}(x) P_{n}^{(\alpha, \beta)}(y)-P_{n}^{(\alpha, \beta)}(x) P_{n+1}^{(\alpha, \beta)}(y)}{x-y}
\end{align*}
$$

where

$$
\begin{equation*}
\rho_{n}^{(\alpha, \beta)}:=\frac{2^{-\alpha-\beta}}{2 n+\alpha+\beta+2} \cdot \frac{\Gamma(n+2) \Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)} \tag{2.4}
\end{equation*}
$$

From (2.2) with $q(x) \equiv 1$ and (2.3), we can derive the following pointwise error formula, which plays a fundamental role in the error analysis.

Theorem 2.1. Let $f(x)$ be a suitably smooth function on $[-1,1]$. For every $x \in$ $[-1,1]$, we denote the Jacobi expansion coefficients of the following quotient in y by

$$
\begin{align*}
& a_{n}^{(\alpha, \beta)}(x ; g)=\frac{1}{\sigma_{k}^{(\alpha, \beta)}} \int_{-1}^{1} g(x, y) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y,  \tag{2.5}\\
& g(x, y):=\frac{f(x)-f(y)}{x-y}
\end{align*}
$$

Then the pointwise error of the Jacobi expansion has the compact representation

$$
\begin{equation*}
f(x)-S_{n}^{(\alpha, \beta)}[f](x)=A_{n}^{(\alpha, \beta)} a_{n}^{(\alpha, \beta)}(x ; g) P_{n+1}^{(\alpha, \beta)}(x)-B_{n}^{(\alpha, \beta)} a_{n+1}^{(\alpha, \beta)}(x ; g) P_{n}^{(\alpha, \beta)}(x), \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
A_{n}^{(\alpha, \beta)} & =\frac{2(n+1)(n+\alpha+\beta+1)}{(2 n+\alpha+\beta+2)(2 n+\alpha+\beta+1)},  \tag{2.7}\\
B_{n}^{(\alpha, \beta)} & =\frac{2(n+\alpha+1)(n+\beta+1)}{(2 n+\alpha+\beta+2)(2 n+\alpha+\beta+3)} .
\end{align*}
$$

Proof. From (1.11) and (2.1)-(2.2), we find readily that

$$
\begin{aligned}
S_{n}^{(\alpha, \beta)}[f](x) & =\sum_{k=0}^{n} \frac{P_{k}^{(\alpha, \beta)}(x)}{\sigma_{k}^{(\alpha, \beta)}} \int_{-1}^{1} f(y) P_{k}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\int_{-1}^{1} K_{n}(x, y) f(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y
\end{aligned}
$$

and

$$
f(x)=\int_{-1}^{1} K_{n}(x, y) f(x) \omega^{(\alpha, \beta)}(y) \mathrm{d} y
$$

Thus, we obtain from (1.11) and (2.3) that

$$
\begin{aligned}
& f(x)-S_{n}^{(\alpha, \beta)}[f](x) \\
& =\int_{-1}^{1} K_{n}(x, y)[f(x)-f(y)] \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\rho_{n}^{(\alpha, \beta)} \int_{-1}^{1} \frac{f(x)-f(y)}{x-y}\left[P_{n+1}^{(\alpha, \beta)}(x) P_{n}^{(\alpha, \beta)}(y)-P_{n}^{(\alpha, \beta)}(x) P_{n+1}^{(\alpha, \beta)}(y)\right] \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\rho_{n}^{(\alpha, \beta)}\left[\sigma_{n}^{(\alpha, \beta)} a_{n}^{(\alpha, \beta)}(x ; g) P_{n+1}^{(\alpha, \beta)}(x)-\sigma_{n+1}^{(\alpha, \beta)} a_{n+1}^{(\alpha, \beta)}(x ; g) P_{n}^{(\alpha, \beta)}(x)\right] .
\end{aligned}
$$

Then, the identity (2.6) follows from directly working out the constants $A_{n}^{(\alpha, \beta)}=$ $\rho_{n}^{(\alpha, \beta)} \sigma_{n}^{(\alpha, \beta)}$ and $B_{n}^{(\alpha, \beta)}=\rho_{n}^{(\alpha, \beta)} \sigma_{n+1}^{(\alpha, \beta)}$ by using (1.11) and (2.4).

Now, we take $f(x)$ in the above to be the generalised $\Phi$-function (1.8), and obtain from (2.5) that

$$
a_{n}^{(\alpha, \beta)}(x ; g)=\frac{1}{\sigma_{n}^{(\alpha, \beta)}} \cdot \begin{cases}\int_{a}^{1} g_{1}(x, y) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y, & x<a  \tag{2.8}\\ \int_{a}^{1} g_{2}(y) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y, & x=a \\ \int_{-1}^{1} g_{3}(x, y) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y, & x>a\end{cases}
$$

where $g=g_{i}, i=1,2,3$, are given by

$$
\begin{align*}
& g_{1}(x, y)=\frac{z(y)(y-a)^{\lambda}}{y-x} ; \\
& g_{2}(y)=g_{1}(a, y)=z(y)(y-a)^{\lambda-1},  \tag{2.9}\\
& \lambda>0 ; \\
& g_{3}(x, y)= \begin{cases}\frac{z(x)(x-a)^{\lambda}}{x-y}, & y \leq a, \\
\frac{z(x)(x-a)^{\lambda}-z(y)(y-a)^{\lambda}}{x-y}, & y>a .\end{cases}
\end{align*}
$$

It is important to point out that in (2.6), the convergence rate of $e_{f}^{(\alpha, \beta)}(n, x)$ depends on both $a_{n}^{(\alpha, \beta)}(x ; g)$ and $P_{n}^{(\alpha, \beta)}(x)$. Moreover, the two terms in the right hand side of (2.6) do not cancel each other for almost all $x$, except for the function (1.8) with $\lambda$ being even. Accordingly, we can estimate $a_{n}^{(\alpha, \beta)}(x ; g)$ and $P_{n}^{(\alpha, \beta)}(x)$ separately. The roadmap for the pointwise error analysis is as follows.
(i) We shall bound $\left|P_{n}^{(\alpha, \beta)}(x)\right|$ pointwisely using Theorem 2.3 and Corollary 2.1
(ii) We shall estimate $\left|a_{n}^{(\alpha, \beta)}(x ; g)\right|$ by using the Hilb-type formula in Theorem 2.2 and the van der Corput-type Lemma to be presented in Section 3. together with the regularity analysis of the underlying function $g(x, y)$ in $y$ given in (2.9).
We first recall that Darboux [12] and Szegö [27, Theorem 8.21.12] introduced the following Hilb-type formula on the asymptotics of $P_{n}^{(\alpha, \beta)}(x)$ in terms of a highly oscillatory Bessel function.

Theorem 2.2 (See [12,27]). For $\alpha, \beta>-1$ and $n \gg 1$, we have

$$
\begin{align*}
\theta^{-\frac{1}{2}} & \sin ^{\alpha+\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos ^{\beta+\frac{1}{2}}\left(\frac{\theta}{2}\right) P_{n}^{(\alpha, \beta)}(\cos \theta) \\
& =\frac{\Gamma(n+\alpha+1)}{\sqrt{2} n!\tilde{N}^{\alpha}} J_{\alpha}(\tilde{N} \theta)+ \begin{cases}\theta^{\frac{1}{2}} \mathcal{O}\left(\tilde{N}^{-\frac{3}{2}}\right), & c n^{-1} \leq \theta \leq \pi-\epsilon \\
\theta^{\alpha+2} \mathcal{O}\left(\tilde{N}^{\alpha}\right), & 0<\theta \leq c n^{-1}\end{cases} \tag{2.10}
\end{align*}
$$

where $\tilde{N}=n+(\alpha+\beta+1) / 2$, c and $\epsilon$ are fixed positive numbers, and $J_{\alpha}(z)$ is the first kind of Bessel function of order $\alpha$. The constants in the $\mathcal{O}$-terms depend on $\alpha, \beta, c$, and $\epsilon$, but do not depend on $n$.

Using [27, Theorem 7.32.2] and $P_{n}^{(\alpha, \beta)}(-x)=(-1)^{n} P_{n}^{(\beta, \alpha)}(x)$, Muckenhoupt [21] derived the following pointwise bound.

Theorem 2.3 (See [21, (2.6)-(2.7)]). Let $\alpha, \beta>-1$ and $d$ be a fixed integer. Then for $n \geq \max \{0,-d\}$, we have

$$
\begin{equation*}
\left|P_{n+d}^{(\alpha, \beta)}(x)\right| \leq C E_{n}^{(\alpha, \beta)}(x) \tag{2.11}
\end{equation*}
$$

where $C$ is a positive constant independent of $n$ and $x$, and

$$
E_{n}^{(\alpha, \beta)}(x)= \begin{cases}(n+1)^{\alpha}, & 1-(n+1)^{-2} \leq x \leq 1  \tag{2.12}\\ (n+1)^{-\frac{1}{2}}(1-x)^{-\frac{\alpha}{2}-\frac{1}{4}}, & 0 \leq x \leq 1-(n+1)^{-2} \\ (n+1)^{-\frac{1}{2}}(1+x)^{-\frac{\beta}{2}-\frac{1}{4}}, & -1+(n+1)^{-2} \leq x \leq 0 \\ (n+1)^{\beta}, & -1 \leq x \leq-1+(n+1)^{-2}\end{cases}
$$

As a direct consequence of Theorem 2.3, we have the following useful pointwise upper bound.

Corollary 2.1. For $\alpha, \beta>-1$ and $x \in[-1,1]$, we have

$$
\begin{equation*}
\left|P_{n}^{(\alpha, \beta)}(x)\right| \leq C_{0}(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} \tag{2.13}
\end{equation*}
$$

where $C_{0}=2^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4}, 0\right\}} C$ and $C$ is the same as in (2.11) with $d=0$.
Proof. It is evident that if $-1<\alpha \leq-1 / 2$, then $\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}=0$, and by (2.12),
$\begin{cases}(n+1)^{\alpha} \leq(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}, & 1-(n+1)^{-2} \leq x \leq 1, \\ (n+1)^{-\frac{1}{2}}(1-x)^{-\frac{\alpha}{2}-\frac{1}{4}} \leq(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}, & 0 \leq x \leq 1-(n+1)^{-2} .\end{cases}$
If $\alpha>-1 / 2$, then we have $\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}=\frac{\alpha}{2}+\frac{1}{4}>0$, so apparently (2.12) is valid. Therefore, for $x \in[0,1]$, and $\alpha, \beta>-1$, we have

$$
\begin{aligned}
E_{n}^{(\alpha, \beta)}(x) & \leq(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}} \\
& \leq 2^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}
\end{aligned}
$$

Following the same lines as above, we can show that for $x \in[-1,0]$ and $\alpha, \beta>-1$,

$$
\begin{aligned}
E_{n}^{(\alpha, \beta)}(x) & \leq(n+1)^{-\frac{1}{2}}(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} \\
& \leq 2^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(n+1)^{-\frac{1}{2}}(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} .
\end{aligned}
$$

This completes the proof.
Remark 2.1. In some special cases, the constant $C$ in (2.13) is explicitly known. Indeed, we find from [22, (18.14.3)] that for $-\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$, we have

$$
\left(\frac{1-x}{2}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\left(\frac{1+x}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\left|P_{n}^{(\alpha, \beta)}(x)\right| \leq \frac{\Gamma(\max (\alpha, \beta)+n+1)}{\pi^{\frac{1}{2}} n!\left(n+\frac{\alpha+\beta+1}{2}\right)^{\max (\alpha, \beta)+\frac{1}{2}}}, x \in[-1,1] .
$$

Moreover, Förster [13] stated the bound for the Gegenbauer polynomials with $\alpha \geq 1$, that is,

$$
\left(1-x^{2}\right)^{\frac{\alpha}{2}}\left|C_{n}^{(\alpha)}(x)\right| \leq \frac{(2 \alpha-1) \Gamma\left(\frac{n}{2}+\alpha\right)}{\Gamma(\alpha) \Gamma\left(\frac{n}{2}+1\right)}, \quad x \in[-1,1],
$$

where

$$
C_{n}^{(\alpha)}(x)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right) \Gamma(n+2 \alpha)}{\Gamma(2 \alpha) \Gamma\left(n+\alpha+\frac{1}{2}\right)} P_{n}^{\left(\alpha-\frac{1}{2}, \alpha-\frac{1}{2}\right)}(x)
$$

Remark 2.2. The above bound of $P_{n}^{(\alpha, \beta)}(x)$ can precisely characterise its behaviour near the endpoints, which allows us to describe $C(1-\xi)$ and $C(-1+\xi)$ for $\xi \in(0, \delta)$ in (1.13).

## 3. Pointwise error estimate for the Jacobi expansion of the GENERALISED $\Phi$-FUNCTION

This section is devoted to the asymptotic analysis of the Jacobi expansion coefficient $a_{n}^{(\alpha, \beta)}(x ; g)$ of $g(x, y)$ in (2.8)-(2.9). With this and the preparations in Section [2] we shall be able to prove the main results stated in Theorem 1.3 ,
3.1. Useful lemmas. A critical tool for the analysis is the following asymptotic formulas involving a highly oscillatory Bessel functions related to the Jacobi polynomials in (2.10), which extend the classical van der Corput Lemma on the Fourier transform [25, pp. 332-334] to the Bessel transform. They are therefore dubbed as the generalised van der Corput-type Lemmas.

Let $\Omega=(a, b) \subset \mathbb{R}$ be a finite open interval. Denote by $\operatorname{AC}(\bar{\Omega})$ the space of absolutely continuous functions on $\bar{\Omega}$. We further introduce the space

$$
W_{\mathrm{AC}}(\Omega)=\left\{\psi: \psi \in \operatorname{AC}(\bar{\Omega}), \quad \psi^{\prime} \in L^{1}(\Omega)\right\}
$$

equipped with the norm

$$
\|\psi\|_{W_{\mathrm{AC}}(\Omega)}=\|\psi\|_{L^{\infty}(\Omega)}+\left\|\psi^{\prime}\right\|_{L^{1}(\Omega)} .
$$

Indeed, according to Stein and Shakarchi [26, pp. 130] and Tao [28, pp. 143-145], we have the integral representation for any $\psi \in W_{\mathrm{AC}}(\Omega)$ :

$$
\psi(x)=\psi(a)+\int_{a}^{x} \psi^{\prime}(t) \mathrm{d} t
$$

and any continuous function of bounded variation on $\Omega$ which maps each set of measure zero into a set of measure zero is also absolutely continuous.

It is also noteworthy that the AC-type space with different regularity on the highest derivative, e.g., BV (functions of bounded variation) has been used in Trefethen [30] in the context of Chebyshev polynomial approximation of singular functions.

Lemma 3.1. Given $\alpha+\nu>-1$ and $\beta>-1$, the following asymptotic estimates hold for $\omega \gg 1$.
(i) For $\psi(x) \in W_{\mathrm{AC}}(0, b)$, we have

$$
\begin{equation*}
\int_{0}^{b} x^{\alpha}(b-x)^{\beta} J_{\nu}(\omega x) \psi(x) \mathrm{d} x=\|\psi\|_{W_{\mathrm{AC}}(0, b)} \cdot \mathcal{O}\left(\omega^{-\min \left\{\alpha+1, \beta+\frac{3}{2}, \frac{3}{2}\right\}}\right) . \tag{3.1}
\end{equation*}
$$

(ii) For $b>c>0$ and $\psi(x) \in W_{\mathrm{AC}}(c, b)$, we have

$$
\begin{equation*}
\int_{c}^{b}(b-x)^{\beta} J_{\nu}(\omega x) \psi(x) \mathrm{d} x=\|\psi\|_{W_{\mathrm{AC}}(c, b)} \cdot \mathcal{O}\left(\omega^{-\min \left\{\beta+\frac{3}{2}, \frac{3}{2}\right\}}\right) \tag{3.2}
\end{equation*}
$$

Here, the constant in the Big $\mathcal{O}$ is independent of $\omega$ and $\psi$.
Proof. The estimate (3.1) is a special case of (35, Lemma 2.5]. Now we show the improved estimate (3.2) on the closed subinterval $[c, b] \subset[0, b]$. From the asymptotic property of the Bessel function [1] pp. 362]:

$$
J_{\nu}(z)=\sqrt{\frac{2}{\pi z}} \cos (z-\nu \pi / 2-\pi / 4)+\mathcal{O}\left(z^{-\frac{3}{2}}\right), \quad z \rightarrow \infty
$$

we have by using $\cos (\omega x-\nu \pi / 2-\pi / 4)=\cos (-\omega x+\nu \pi / 2+\pi / 4)$ that

$$
\begin{align*}
& \int_{c}^{b}(b-x)^{\beta} \psi(x) J_{\nu}(\omega x) \mathrm{d} x \\
& =\sqrt{\frac{2}{\pi \omega}} \int_{c}^{b}(b-x)^{\beta} \cos (\omega x-\nu \pi / 2-\pi / 4) x^{-\frac{1}{2}} \psi(x) \mathrm{d} x+\mathcal{O}\left(\omega^{-\frac{3}{2}}\right)  \tag{3.3}\\
& =\sqrt{\frac{2}{\pi \omega}} \Re\left\{e^{\mathrm{i}(\nu \pi / 2+\pi / 4)} \int_{c}^{b}(b-x)^{\beta} e^{-\mathrm{i} \omega x} x^{-\frac{1}{2}} \psi(x) \mathrm{d} x\right\}+\mathcal{O}\left(\omega^{-\frac{3}{2}}\right) \\
& =\sqrt{\frac{2}{\pi \omega}} \Re\left\{e^{\mathrm{i}(\nu \pi / 2+\pi / 4-b \omega)} \int_{0}^{b-c} u^{\beta} e^{\mathrm{i} \omega u}(b-u)^{-\frac{1}{2}} \psi(b-u) \mathrm{d} u\right\}+\mathcal{O}\left(\omega^{-\frac{3}{2}}\right),
\end{align*}
$$

where $\Re\{z\}$ denotes the real part of $z$. Setting $F(x)=\int_{0}^{x} u^{\beta} e^{\mathrm{i} \omega u} \mathrm{~d} u$ and applying the integration by parts, we obtain

$$
\begin{align*}
& \left|\int_{0}^{b-c} u^{\beta} e^{\mathrm{i} \omega u}(b-u)^{-\frac{1}{2}} \psi(b-u) \mathrm{d} u\right| \\
& \quad=\left|\int_{0}^{b-c}(b-u)^{-\frac{1}{2}} \psi(b-u) \mathrm{d} F(u)\right| \\
& \quad=\left|\left[(b-u)^{-\frac{1}{2}} \psi(b-u) F(u)\right]_{0}^{b-c}-\int_{0}^{b-c} F(u)\left((b-u)^{-\frac{1}{2}} \psi(b-u)\right)^{\prime} \mathrm{d} u\right|  \tag{3.4}\\
& \quad \leq\left(\frac{|\psi(c)|}{\sqrt{c}}+\int_{c}^{b}\left|\left(x^{-\frac{1}{2}} \psi(x)\right)^{\prime}\right| \mathrm{d} x\right) \max _{u \in[0, b-c]}|F(u)| \\
& \quad \leq C\|\psi\|_{W_{\mathrm{AC}}(c, b)}^{\max _{x \in[0, b-c]}|F(x)|,}
\end{align*}
$$

where $C$ is a constant independent of $\omega$. Finally, we use an asymptotic behaviour of $F(x)$ in terms of the hypergeometric function ${ }_{1} \mathrm{~F}_{1}(\cdot)$ in [1, (13.5.1)] to claim that

$$
\left.|F(x)|=\left.\frac{x^{\beta+1}}{\beta+1}\right|_{1} \mathrm{~F}_{1}(\beta+1 ; \beta+2 ; \mathrm{i} \omega x) \right\rvert\,=\mathcal{O}\left(\omega^{-1}+\omega^{-\beta-1}\right),
$$

which, together with (3.3) and (3.4), leads to (3.2).
With Lemma 3.1 at our disposal, we now associate the Bessel function in (3.1) and (3.2) with the Jacobi polynomial through the Hilb-type formula (2.10) and derive the asymptotic estimates in Lemma 3.2 and Lemma 3.3. These allow us to deal with the integrals involved the Jacobi polynomials in $a_{n}^{(\alpha, \beta)}(x ; g)$.

Lemma 3.2. Let $\alpha, \beta, \gamma, \delta>-1$. If $\psi(x) \in W_{\mathrm{AC}}(a, 1)$ with $a \in(-1,1)$, then for $n \gg 1$, we have

$$
\begin{equation*}
\int_{a}^{1}(x-a)^{\gamma}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x=\|\psi\|_{W_{\mathrm{AC}}(a, 1)} \cdot \mathcal{O}\left(n^{-\min \left\{2 \delta-\alpha+2, \gamma+\frac{3}{2}, \frac{3}{2}\right\}}\right) . \tag{3.5}
\end{equation*}
$$

If $\psi(x) \in W_{\mathrm{AC}}(a, b)$ with $-1<a<b<1$, then for $n \gg 1$, we have

$$
\begin{equation*}
\int_{a}^{b}(x-a)^{\gamma} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x=\|\psi\|_{W_{\mathrm{AC}}(a, b)} \cdot \mathcal{O}\left(n^{-\min \left\{\gamma+\frac{3}{2}, \frac{3}{2}\right\}}\right) . \tag{3.6}
\end{equation*}
$$

Proof. We make a change of variable $x=\cos \theta$ and denote $\theta_{0}=\arccos a$. Then it follows from the Hilb-type formula (2.10) that

$$
\begin{align*}
& \int_{a}^{1}(x-a)^{\gamma}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x \\
& =\int_{0}^{\theta_{0}} 2^{\delta+1} \sin ^{2 \delta+1}\left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)\left(\cos \theta-\cos \theta_{0}\right)^{\gamma} P_{n}^{(\alpha, \beta)}(\cos \theta) \psi(\cos \theta) \mathrm{d} \theta  \tag{3.7}\\
& =\frac{\Gamma(n+\alpha+1)}{n!\tilde{N}^{\alpha}} \int_{0}^{\theta_{0}} \theta^{2 \delta+1-\alpha}\left(\theta_{0}-\theta\right)^{\gamma} J_{\alpha}(\tilde{N} \theta) \Psi(\theta) \mathrm{d} \theta+\mathcal{O}\left(n^{-3 / 2}\right),
\end{align*}
$$

where $\Psi(\theta)=h(\theta) \psi(\cos \theta)$ and

$$
h(\theta)=2^{\alpha-\delta}\left(\frac{\sin (\theta / 2)}{\theta / 2}\right)^{2 \delta-\alpha+1 / 2} \cos ^{1 / 2-\beta}\left(\frac{\theta}{2}\right)\left(\frac{\cos \theta-\cos \theta_{0}}{\theta_{0}-\theta}\right)^{\gamma} .
$$

One verifies readily that $\Psi(\theta)$ is absolutely continuous on $\left[0, \theta_{0}\right]$ and $\Psi^{\prime}(\theta) \in$ $L^{1}\left(0, \theta_{0}\right)$. Then using (3.1) in Lemma 3.1 and the asymptotic property of the Gamma function (see [1])

$$
\lim _{n \rightarrow \infty} \frac{\Gamma(n+\alpha+1)}{n!\tilde{N}^{\alpha}}=1
$$

we obtain from (3.7) that

$$
\begin{equation*}
\int_{a}^{1}(x-a)^{\gamma}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x=\|\Psi\|_{W_{\mathrm{AC}}\left(0, \theta_{0}\right)} \cdot \mathcal{O}\left(n^{-\min \left\{2 \delta-\alpha+2, \gamma+\frac{3}{2}, \frac{3}{2}\right\}}\right) \tag{3.8}
\end{equation*}
$$

Since for any $\theta \in\left[0, \theta_{0}\right]$, we have

$$
|\Psi(\theta)| \leq C_{0}|\psi(\cos \theta)|, \quad\left|\Psi^{\prime}(\theta)\right| \leq C_{1}\left(|\psi(\cos \theta)|+\left|\psi^{\prime}(\cos \theta)\right| \sin \theta\right),
$$

where $C_{0}, C_{1}$ are some constants independent of $\theta$. Then it follows from direct calculation that

$$
\begin{align*}
\|\Psi\|_{W_{\mathrm{AC}}\left(0, \theta_{0}\right)} & \leq C\left(\|\psi(\cos \theta)\|_{L^{\infty}\left(0, \theta_{0}\right)}+\int_{0}^{\theta_{0}}|\psi(\cos \theta)| \mathrm{d} \theta+\int_{0}^{\theta_{0}}\left|\psi^{\prime}(\cos \theta)\right| \sin \theta \mathrm{d} \theta\right)  \tag{3.9}\\
& =C\left(\|\psi\|_{L^{\infty}(a, 1)}+\int_{a}^{1}|\psi(x)| \frac{\mathrm{d} x}{\sqrt{1-x^{2}}}+\int_{a}^{1}\left|\psi^{\prime}(x)\right| \mathrm{d} x\right) \\
& \leq C\left((\pi+1)\|\psi\|_{L^{\infty}(a, 1)}+\int_{a}^{1}\left|\psi^{\prime}(x)\right| \mathrm{d} x\right) \\
& \leq C(\pi+1)\|\psi\|_{W_{\mathrm{AC}}(a, 1)} .
\end{align*}
$$

Thus we claim (3.5) from (3.8) and (3.9).
Next for fixed $b<1$, we set $\theta_{1}=\arccos b$. Using (3.2) in Lemma 3.1] on $\left[\theta_{1}, \theta_{0}\right]$, we derive (3.6) directly from the second identity in (3.7) with $\delta=0$ and $b=\cos \theta_{1}$, since
$\int_{a}^{b}(x-a)^{\gamma} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x=\frac{\Gamma(n+\alpha+1)}{n!\tilde{N}^{\alpha}} \int_{\theta_{1}}^{\theta_{0}}\left(\theta_{0}-\theta\right)^{\gamma} J_{\alpha}(\tilde{N} \theta) \bar{\Psi}(\theta) \mathrm{d} \theta+\mathcal{O}\left(n^{-3 / 2}\right)$,
where

$$
\bar{\Psi}(\theta)=\sqrt{2 \theta} \sin ^{\frac{1}{2}-\alpha}\left(\frac{\theta}{2}\right) \cos ^{\frac{1}{2}-\beta}\left(\frac{\theta}{2}\right)\left(\frac{\cos \theta-\cos \theta_{0}}{\theta_{0}-\theta}\right)^{\gamma} \psi(\cos \theta) \in W_{\mathrm{AC}}\left(\theta_{1}, \theta_{0}\right) .
$$

Thus, following the same lines as the above for (3.9), we can obtain (3.6).
If $\psi(x)$ has more regularity, we denote $W_{\mathrm{AC}}^{m}(\Omega)$ for some positive integer $m$ as

$$
W_{\mathrm{AC}}^{m}(\Omega)=\left\{\psi: \psi^{(k)} \in W_{\mathrm{AC}}(\Omega), k=0, \cdots, m\right\}
$$

equipped with the norm

$$
\|\psi\|_{W_{\mathrm{AC}}^{m}(\Omega)}=\sum_{k=0}^{m}\left\|\psi^{(k)}\right\|_{W_{\mathrm{AC}}(\Omega)} .
$$

In particular, for $m=0$, we have $W_{\mathrm{AC}}(\Omega)=W_{\mathrm{AC}}^{0}(\Omega)$. Then we can further derive the following estimate using Lemma 3.2 ,

Lemma 3.3. Let $\alpha, \beta, \gamma>-1, a \in(-1,1)$ and $n \gg 1$. Denote by $m=\lfloor\gamma\rfloor$ the greatest integer that is less than $\gamma$. If $\psi \in W_{\mathrm{AC}}^{m+1}\left(a, \frac{1+a}{2}\right) \cap W_{\mathrm{AC}}^{m+2}\left(\frac{1+a}{2}, 1\right)$, then we have

$$
\begin{align*}
& \int_{a}^{1}(x-a)^{\gamma} \psi(x) P_{n}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) \mathrm{d} x  \tag{3.10a}\\
& \quad=\left(\|\psi\|_{W_{\mathrm{AC}}^{m+1}\left(a, \frac{1+a}{2}\right)}+\|\psi\|_{W_{\mathrm{AC}}^{m+2}\left(\frac{1+a}{2}, 1\right)}\right) \cdot \mathcal{O}\left(n^{-\gamma-\frac{3}{2}}\right) .
\end{align*}
$$

If $\psi \in W_{\mathrm{AC}}^{m+1}\left(\frac{-1+a}{2}, a\right) \cap W_{\mathrm{AC}}^{m+2}\left(-1, \frac{-1+a}{2}\right)$, then

$$
\begin{align*}
& \int_{-1}^{a}(a-x)^{\gamma} \psi(x) P_{n}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) \mathrm{d} x  \tag{3.10b}\\
& \quad=\left(\|\psi\|_{W_{\mathrm{AC}}^{m+1}\left(\frac{-1+a}{2}, a\right)}+\|\psi\|_{W_{\mathrm{AC}}^{m+2}\left(-1, \frac{-1+a}{2}\right)}\right) \cdot \mathcal{O}\left(n^{-\gamma-\frac{3}{2}}\right) .
\end{align*}
$$

Proof. Recall the Rodrigues' formula of Jacobi polynomials (see [27, pp. 94]):

$$
\begin{equation*}
\omega^{(\alpha, \beta)}(y) P_{n}^{(\alpha, \beta)}(y)=\frac{(-1)^{k}}{2^{k}(n)_{k}} \frac{\mathrm{~d}^{k}}{\mathrm{~d} y^{k}}\left(\omega^{(\alpha+k, \beta+k)}(y) P_{n-k}^{(\alpha+k, \beta+k)}(y)\right), \quad k \in \mathbb{N}, \tag{3.11}
\end{equation*}
$$

where $(n)_{k}=n(n-1) \cdots(n-k+1)$ denotes the falling factorial. Using (3.11) and integration by parts, we find from Lemma 3.2 and the fact $P_{n}^{(\alpha, \beta)}\left(\frac{1+a}{2}\right)=\mathcal{O}\left(n^{-\frac{1}{2}}\right)$ (see Theorem [2.3) that

$$
\begin{align*}
& \int_{a}^{1}(x-a)^{\gamma} \psi(x) P_{n}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) \mathrm{d} x  \tag{3.12}\\
& = \\
& =\frac{1}{2^{m+1}(n)_{m+1}} \int_{a}^{1}\left[(x-a)^{\gamma} \psi(x)\right]^{(m+1)} P_{n-m-1}^{(\alpha+m+1, \beta+m+1)}(x) \omega^{(\alpha+m+1, \beta+m+1)}(x) \mathrm{d} x \\
& = \\
& \frac{1}{2^{m+1}(n)_{m+1}} \int_{a}^{\frac{1+a}{2}}\left[(x-a)^{\gamma} \psi(x)\right]^{(m+1)} P_{n-m-1}^{(\alpha+m+1, \beta+m+1)}(x) \omega^{(\alpha+m+1, \beta+m+1)}(x) \mathrm{d} x \\
& \quad-\left.\frac{1}{2^{m+2}(n)_{m+2}}\left[(x-a)^{\gamma} \psi(x)\right]^{(m+1)} P_{n-m-2}^{(\alpha+m+2, \beta+m+2)}(x) \omega^{(\alpha+m+2, \beta+m+2)}(x)\right|_{\frac{1+a}{2}} ^{1} \\
& \quad+\frac{1}{2^{m+2}(n)_{m+2}} \int_{\frac{1+a}{2}}^{1}\left[(x-a)^{\gamma} \psi(x)\right]^{(m+2)} P_{n-m-2}^{(\alpha+m+2, \beta+m+2)}(x) \omega^{(\alpha+m+2, \beta+m+2)}(x) \mathrm{d} x \\
& =\|\psi\|_{W_{\mathrm{AC}}^{m+1}\left(a, \frac{1+a}{2}\right)} \cdot \mathcal{O}\left(n^{-\gamma-\frac{3}{2}}\right)+\left.\left[(x-a)^{\gamma} \psi(x)\right]^{(m+1)}\right|_{x=\frac{1+a}{2}} \cdot \mathcal{O}\left(n^{-m-\frac{5}{2}}\right) \\
& \quad+\|\psi\|_{W_{\mathrm{AC}}^{m+2}\left(\frac{1+a}{2}, 1\right)} \cdot \mathcal{O}\left(n^{-m-\frac{7}{2}}\right) .
\end{align*}
$$

This leads to (3.10a) by the fact that $\gamma+\frac{3}{2} \leq m+\frac{5}{2}$.
We can obtain (3.10b) directly using $P_{n}^{(\alpha, \beta)}(-x)=(-1)^{n} P_{n}^{(\beta, \alpha)}(x)$.
3.2. Analysis of $a_{n}^{(\alpha, \beta)}(x ; g)$. We now turn to the optimal asymptotic estimates of $a_{n}^{(\alpha, \beta)}(x ; g)$ in (2.8)-(2.9), which is of paramount importance in Theorem 1.3

Theorem 3.1. For $\alpha, \beta>-1$ and $a_{n}^{(\alpha, \beta)}(x ; g)$ defined in (2.8), we have

$$
a_{n}^{(\alpha, \beta)}(x ; g)= \begin{cases}|x-a|^{-1} \mathcal{O}\left(n^{-\lambda-\frac{1}{2}}\right), & x \in[-1, a) \cup(a, 1], \quad \lambda>-1,  \tag{3.13}\\ \mathcal{O}\left(n^{-\lambda+\frac{1}{2}}\right), & x=a, \lambda>0,\end{cases}
$$

and

$$
\begin{equation*}
a_{n}^{(\alpha, \beta)}(x ; g)=\mathcal{O}\left(n^{-\lambda+\frac{1}{2}}\right) \quad \text { for } \lambda>0 \text { and } \forall x \in[-1,1] \text {, } \tag{3.14}
\end{equation*}
$$

where the constants in $\mathcal{O}$-terms are independent of $x$.
Proof. For clarity, we carry out the proof in three cases: (i) $x<a$; (ii) $x=a$ and (iii) $x>a$. Below we just provide the detailed proof for the cases (i) and (ii), but sketch that of the third case in Appendix A to avoid unnecessary repetition.
(i) $x<a$ : For each fixed $x<a$, one verifies readily from (3.10) and $\sigma_{n}^{(\alpha, \beta)}=$ $\mathcal{O}\left(n^{-1}\right)$ that

$$
\begin{aligned}
a_{n}^{(\alpha, \beta)}(x ; g) & =\frac{1}{\sigma_{n}^{(\alpha, \beta)}} \int_{a}^{1}(y-a)^{\lambda} \frac{z(y)}{y-x} P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\left\|\frac{z(y)}{y-x}\right\|_{W_{\mathrm{AC}}^{m+2}(a, 1)} \cdot \mathcal{O}\left(n^{-\lambda-\frac{1}{2}}\right),
\end{aligned}
$$

where $m=\lfloor\lambda\rfloor$ is defined as that in Lemma 3.3. Although the asymptotic order $n^{-\lambda-\frac{1}{2}}$ in above is optimal, but the constant before it is not well controlled when $x$ is closed to $a$. To obtain (3.13) and (3.14), for simplicity, we redefine $m$ as $m=\lambda-1$ if $\lambda$ is an integer, otherwise $m=\lfloor\lambda\rfloor$.

Applying the Rodrigues' formula (3.11), we obtain

$$
\begin{align*}
a_{n}^{(\alpha, \beta)}(x ; g)= & \frac{1}{2^{m+1}(n)_{m+1} \sigma_{n}^{(\alpha, \beta)}} \int_{a}^{1}(y-a)^{\lambda-m-1} \frac{\phi_{m+1}(x, y)}{y-x}  \tag{3.15}\\
& \quad \times P_{n-m-1}^{(\alpha+m+1, \beta+m+1)}(y) \omega^{(\alpha+m+1, \beta+m+1)}(y) \mathrm{d} y,
\end{align*}
$$

where Leibniz's rule is used that

$$
\begin{equation*}
\partial_{y}^{m+1} g_{1}(x, y)=(y-a)^{\lambda-m-1} \frac{\phi_{m+1}(x, y)}{y-x} \tag{3.16}
\end{equation*}
$$

$$
\phi_{m+1}(x, y)=\sum_{k=0}^{m+1} \sum_{j=0}^{k} \frac{(-1)^{k-j}(m+1)!(\lambda)_{j}}{j!(m+1-k)!} z^{(m+1-k)}(y)(y-a)^{m+1-k}\left(\frac{y-a}{y-x}\right)^{k-j}
$$

It is not difficult to verify that there exist two constants $C_{1}$ and $C_{2}$ independent of $x$ such that

$$
\begin{equation*}
\max _{y \in\left[a, \frac{1+a}{2}\right]}\left|\phi_{m+1}(x, y)\right| \leq C_{1}, \quad \max _{y \in\left[a, \frac{1+a}{2}\right]}\left|\partial_{y} \phi_{m+1}(x, y)\right| \leq \frac{C_{2}}{y-x} \tag{3.17}
\end{equation*}
$$

As a result, we can derive the estimate (3.13) from Lemma 3.3 since

$$
\left\|\frac{\phi_{m+1}}{y-x}\right\|_{W_{\mathrm{AC}}\left(a, \frac{1+a}{2}\right)} \leq C|x-a|^{-1}
$$

for some constant $C$ independent of $x$, and $\frac{\phi_{m+1}(x, y)}{y-x}$ is smooth for $x<a$ and $\frac{a+1}{2} \leq$ $y \leq 1$ (i.e. $\left\|\frac{\phi_{m+1}}{y-x}\right\|_{W_{\mathrm{AC}}^{1}\left(\frac{1+a}{2}, 1\right)}$ is uniformly bounded by a constant independent of $x)$.

In order to obtain the uniformly estimate (3.14) for $a_{n}^{(\alpha, \beta)}(x ; g)$ for any $x \in$ [ $-1, a$ ), we conduct integration by parts till $m$ instead of $m+1$ in (3.15) and find

$$
\partial_{y}^{m} g_{1}(x, y)=(y-a)^{\lambda-m-1}\left(\frac{y-a}{y-x} \phi_{m}(x, y)\right)
$$

Note that these two functions in $x$

$$
\frac{y-a}{y-x} \phi_{m}(x, y) \quad \text { and } \quad \int_{a}^{1}\left|\partial_{y}\left(\frac{y-a}{y-x} \phi_{m}(x, y)\right)\right| \mathrm{d} y
$$

are uniformly bounded on $(x, y) \in[-1, a) \times(a, 1)$, and $\left\|\frac{\phi_{m+1}}{y-x}\right\|_{W_{\mathrm{AC}}^{1}\left(\frac{1+a}{2}, 1\right)}$ is uniformly bounded independent of $x$ too. Thus we obtain the uniform bound (3.14).
(ii) $x=a$ : It follows from (3.10) directly that

$$
a_{n}^{(\alpha, \beta)}(a ; g)=\frac{1}{\sigma_{n}^{(\alpha, \beta)}} \int_{a}^{1}(y-a)^{\lambda-1} z(y) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y=\mathcal{O}\left(n^{-\lambda+\frac{1}{2}}\right)
$$

(iii) $x>a$ : See Appendix A for a sketch.
3.3. Proof of Theorem 1.3 for (1.8). From (2.6) and (3.13), it follows that for $x \in[-1, a) \cup(a, 1]$,

$$
\begin{align*}
e_{f}^{(\alpha, \beta)}(n, x)= & \frac{2(n+1)(n+\alpha+\beta+1)}{(2 n+\alpha+\beta+2)(2 n+\alpha+\beta+1)} a_{n}^{(\alpha, \beta)}(x ; g) P_{n+1}^{(\alpha, \beta)}(x) \\
& -\frac{2(n+\alpha+1)(n+\beta+1)}{(2 n+\alpha+\beta+2)(2 n+\alpha+\beta+3)} a_{n+1}^{g}(x, \alpha, \beta) P_{n}^{(\alpha, \beta)}(x)  \tag{3.18}\\
= & |x-a|^{-1} \mathcal{O}\left(n^{-\lambda-\frac{1}{2}}\right)\left(\left|P_{n}^{(\alpha, \beta)}(x)\right|+\left|P_{n+1}^{(\alpha, \beta)}(x)\right|\right),
\end{align*}
$$

while for $x=a$,

$$
\begin{equation*}
e_{f}^{(\alpha, \beta)}(n, a)=\left(\left|P_{n}^{(\alpha, \beta)}(a)\right|+\left|P_{n+1}^{(\alpha, \beta)}(a)\right|\right) \mathcal{O}\left(n^{-\lambda+\frac{1}{2}}\right), \tag{3.19}
\end{equation*}
$$

which, together with Theorem 2.3 on $P_{k}^{(\alpha, \beta)}(x)=\mathcal{O}\left(n^{-\frac{1}{2}}\right)(k=n, n+1)$ for fixed $x \in(-1,1)$, yields

$$
e_{f}^{(\alpha, \beta)}(n, x) \leq C(x) n^{-\lambda-1}(x \neq a), \quad e_{f}^{(\alpha, \beta)}(n, a) \leq C n^{-\lambda} .
$$

Here, $C$ and $C(x)$ are independent of $n$. Furthermore, from Corollary 2.1 on $P_{n}^{(\alpha, \beta)}(1 \pm \xi)$, we deduce that $C(x)$ behaves like (1.13) near $x=a, \pm 1$. For $x= \pm 1$, using the properties [27, (7.32.2)]:

$$
P_{n}^{(\alpha, \beta)}(1)=\mathcal{O}\left(n^{\alpha}\right), \quad P_{n}^{(\alpha, \beta)}(-1)=\mathcal{O}\left(n^{\beta}\right),
$$

and (3.18), we obtain (1.14), i.e.,

$$
e_{f}^{(\alpha, \beta)}(n, 1) \leq C n^{\alpha-\frac{1}{2}-\lambda}, \quad e_{f}^{(\alpha, \beta)}(n,-1) \leq C n^{\beta-\frac{1}{2}-\lambda},
$$

where $C$ is a positive constant independent of $n$.
At this point, we have completed the proof of Theorem 1.3 for (1.8) except for the case where $x=a$ and $\lambda>0$ is an even integer.

Indeed, if $\lambda$ is an even integer, we apply the Rodrigues formula (3.11) $\lambda$ times, and use Lemma 3.3, leading to

$$
\begin{align*}
& a_{n}^{(\alpha, \beta)}(a ; g)=-\left.\frac{g_{2}^{(\lambda-1)}(y) P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(y) \omega^{(\alpha+\lambda, \beta+\lambda)}(y)}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}}\right|_{a} ^{1} \\
& \quad+\frac{1}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}} \int_{a}^{1} g_{2}^{(\lambda)}(y) P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(y) \omega^{(\alpha+\lambda, \beta+\lambda)}(y) \mathrm{d} y \\
& =\frac{(\lambda-1)!(1-a)^{\alpha+\lambda}(1+a)^{\beta+\lambda} P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) z(a)}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}}  \tag{3.20}\\
& \quad+\frac{1}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}} \int_{a}^{1} g_{2}^{(\lambda)}(y) P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(y) \omega^{(\alpha+\lambda, \beta+\lambda)}(y) \mathrm{d} y \\
& =\frac{(\lambda-1)!(1-a)^{\alpha+\lambda}(1+a)^{\beta+\lambda} P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) z(a)}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}}+\mathcal{O}\left(n^{-\lambda-\frac{1}{2}}\right),
\end{align*}
$$

where we used $g_{2}$ is smooth for $y \in[a, 1]$ and

$$
\begin{aligned}
g_{2}^{(\lambda-1)}(a) & =\left[(y-a)^{\lambda-1} z(y)\right]^{(\lambda-1)} \\
& =\left.\sum_{k=0}^{\lambda-1}\binom{\lambda-1}{k}(\lambda-1)_{k}(y-a)^{\lambda-1-k} z^{(\lambda-1-k)}(y)\right|_{y=a} \\
& =(\lambda-1)!z(a) .
\end{aligned}
$$

Then from (3.20) and (3.18) we have that for $x=a$,

$$
\begin{align*}
& e_{f}^{(\alpha, \beta)}(n, a) \\
&= A_{n}^{(\alpha, \beta)} \frac{(\lambda-1)!(1-a)^{\alpha+\lambda}(1+a)^{\beta+\lambda} z(a)}{2^{\lambda}} \frac{P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n+1}^{(\alpha, \beta)}(a)}{(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}} \\
&-B_{n}^{(\alpha, \beta)} \frac{(\lambda-1)!(1-a)^{\alpha+\lambda}(1+a)^{\beta+\lambda} z(a)}{2^{\lambda}} \frac{P_{n+1-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n}^{(\alpha, \beta)}(a)}{(n+1)_{\lambda} \sigma_{n+1}^{(\alpha, \beta)}}  \tag{3.21}\\
&+\mathcal{O}\left(n^{-\lambda-1}\right) \\
&= A_{n}^{(\alpha, \beta)} \frac{(\lambda-1)!(1-a)^{\alpha+\lambda}(1+a)^{\beta+\lambda} z(a)}{2^{\lambda}(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}}\left[P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n+1}^{(\alpha, \beta)}(a)\right. \\
&\left.-P_{n+1-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n}^{(\alpha, \beta)}(a)\right]+\mathcal{O}\left(n^{-\lambda-1}\right),
\end{align*}
$$

where we used the factor $P_{n+1-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n}^{(\alpha, \beta)}(a)=\mathcal{O}\left(n^{-1}\right)$ and

$$
B_{n}^{(\alpha, \beta)}=A_{n}^{(\alpha, \beta)}\left(1+\mathcal{O}\left(n^{-2}\right)\right), \quad \frac{1}{(n+1)_{\lambda} \sigma_{n+1}^{(\alpha, \beta)}}=\frac{1}{(n)_{\lambda} \sigma_{n}^{(\alpha, \beta)}}\left(1+\mathcal{O}\left(n^{-1}\right)\right)
$$

Moreover, we shall show that for even integer $\lambda>0$,

$$
\begin{equation*}
P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n+1}^{(\alpha, \beta)}(a)-P_{n+1-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n}^{(\alpha, \beta)}(a)=\mathcal{O}\left(n^{-2}\right) \tag{3.22}
\end{equation*}
$$

which implies a cancellation happens and gains one order higher in convergence rate. To this end, we use the asymptotic property in [27, Theorem 8.21.8]: For any $\alpha, \beta \in \mathbb{R}$ and $\theta \in(0, \pi)$, it holds that

$$
\begin{equation*}
P_{n}^{(\alpha, \beta)}(\cos \theta)=(n \pi)^{-\frac{1}{2}} \sin ^{-\alpha-\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos ^{-\beta-\frac{1}{2}}\left(\frac{\theta}{2}\right) \cos (\tilde{N} \theta+\gamma)+\mathcal{O}\left(n^{-\frac{3}{2}}\right) \tag{3.23}
\end{equation*}
$$

where $\tilde{N}=n+(\alpha+\beta+1) / 2, \gamma=-\frac{2 \alpha+1}{4} \pi$. The bound for the error term holds uniformly in the interval $[\epsilon, \pi-\epsilon]$ for fixed positive number $\epsilon$. Thus by (3.23), we have

$$
\begin{align*}
& \frac{P_{n-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n+1}^{(\alpha, \beta)}(a)-P_{n+1-\lambda}^{(\alpha+\lambda, \beta+\lambda)}(a) P_{n}^{(\alpha, \beta)}(a)}{(2 \pi)^{-1} \sin ^{-2 \alpha-\lambda-1}\left(\frac{\theta_{0}}{2}\right) \cos ^{-2 \beta-\lambda-1}\left(\frac{\theta_{0}}{2}\right)} \\
& \quad=\frac{\cos \left(\theta_{0}+\lambda \pi / 2\right)}{\sqrt{(n-\lambda)(n+1)}}-\frac{\cos \left(\theta_{0}+\lambda \pi / 2\right)}{\sqrt{(n+1-\lambda) n}}  \tag{3.24}\\
& \quad+\frac{\cos \left(\theta_{1}\right)}{\sqrt{(n-\lambda)(n+1)}}-\frac{\cos \left(\theta_{1}\right)}{\sqrt{(n+1-\lambda) n}}+\mathcal{O}\left(n^{-2}\right)
\end{align*}
$$

with

$$
\theta_{1}=(2 n+2+\alpha+\beta) \theta_{0}-\frac{(2 \alpha+\lambda+1) \pi}{2}
$$

Note that if $\lambda$ is an even integer, the first four $\mathcal{O}\left(n^{-1}\right)$ terms in (3.24) can be cancelled. This yields (3.22). Then we derive from (3.21) and (3.24) that for even integer $\lambda$,

$$
\begin{equation*}
e_{f}^{(\alpha, \beta)}(n, a)=\mathcal{O}\left(n^{-\lambda-1}\right) \tag{3.25}
\end{equation*}
$$

3.4. Extension to (1.9). Let

$$
f^{*}(x)=z(x) \cdot\left\{\begin{array}{ll}
(a-x)^{\lambda}, & -1 \leq x<a,  \tag{3.26}\\
0, & a<x \leq 1
\end{array} \quad a \in(-1,1)\right.
$$

where $\lambda>-1, z \in C^{\infty}[-1,1]$ and $f^{*}(a)=0$ for $\lambda>0$ and $f^{*}(a)=\frac{z(a)}{2}$ for $\lambda=0$. Using an analogous argument as for (1.8), we can show that Theorem 1.3 is also valid for $f^{*}(x)$.

Corollary 3.1. Given $f^{*}(x)$ in (3.26) and $\alpha, \beta>-1$, we have the following pointwise error estimates.
(i) For $x \in(-1, a) \cup(a, 1)$, we have $e_{f^{*}}^{(\alpha, \beta)}(n, x) \leq C(x) n^{-\lambda-1}$, where $C(x)$ is independent of $n$ and has the behaviours near $x=a, \pm 1$ as follows

$$
\begin{align*}
& C(-1+\xi) \leq D(-1) \xi^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}, \\
& C(1-\xi) \leq D(1) \xi^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}},  \tag{3.27}\\
& C(a \pm \xi) \leq D(a) \xi^{-1},
\end{align*}
$$

for $0<\xi \leq \delta$, where $D( \pm 1), D(a)>0$ and $\delta>0$ are independent of $n$.
(ii) At $x= \pm 1$, we have

$$
\begin{equation*}
e_{f^{*}}^{(\alpha, \beta)}(n, 1) \leq C n^{-\lambda+\alpha-\frac{1}{2}} ; \quad e_{f^{*}}^{(\alpha, \beta)}(n,-1) \leq C n^{-\lambda+\beta-\frac{1}{2}} \tag{3.28}
\end{equation*}
$$

(iii) At $x=a$ and $\lambda>0$, we have

$$
e_{f^{*}}^{(\alpha, \beta)}(n, a) \leq \begin{cases}C n^{-\lambda-1}, & \lambda \text { is even }  \tag{3.29}\\ C n^{-\lambda}, & \text { otherwise } .\end{cases}
$$

Notice that $z(x)|x-a|^{\lambda}=f(x)+f^{*}(x)$, which implies Theorem 1.3 also holds for functions defined by (1.9).

To check the error bounds (1.12)-(1.15) numerically, we illustrate the convergence orders of the pointwise errors $e_{f}^{(\alpha, \beta)}(n, x)$ for functions $f(x)=\left(x-\frac{1}{4}\right)_{+}^{\lambda}$ and $f(x)=\left|x-\frac{1}{4}\right|^{\lambda}$ with different values of $\lambda$ and $\alpha, \beta$. From Figures 3.1 and 3.2, we observe that these convergence orders are attainable and in accordance with the estimates stated in Theorem 1.3, even for the divergent cases (see the last column of the first row in Figure 3.2 and Figure 4.1).

Note that the function $C(x)$ near $x= \pm 1$ and $x=a$ is described in (1.13). We further demonstrate that the estimate agrees well with the pointwise errors through a test on $f(x)=(x-a)_{+}^{\lambda}$. Pointwise errors around $x= \pm 1$ and $x=a$ are plotted in Figure 3.3, which implies the optimality on the estimates (1.13) in the sense that the orders on $\xi$ cannot be improved.


Figure 3.1. Pointwise errors $e_{f}^{(\alpha, \beta)}(n, x)$ for $f(x)=\left(x-\frac{1}{4}\right)_{+}^{\lambda}$ with $\lambda=-\frac{1}{3}$ at $x=-\frac{1}{2}$ (first row) and $x=\frac{1}{2}$ (second row), respectively, for $n=1: 5000$. In the dashed dotted lines, the constants in $\mathcal{O}$ may have different values for different cases, and likewise for the figures hereinafter.


Figure 3.2. Pointwise errors $e_{f}^{(\alpha, \beta)}(n, x)$ for $f(x)=\left(x-\frac{1}{4}\right)_{+}^{\lambda}$ with $\lambda=\frac{1}{3}$ at endpoint $x=1$ (first row) and singular point $x=\frac{1}{4}$ (second row), respectively, for $n=100: 100: 5000$


Figure 3.3. Pointwise error plots of $e_{f}^{(\alpha, \beta)}(n, x)$ around $x=-1$ (left), $x=a$ (middle) and $x=1$ (right), where $f(x)=\left(x-\frac{1}{10}\right)_{+}^{1 / 2}$, $\alpha=\frac{1}{2}, \beta=\frac{2}{5}$ and $n=2000$

## 4. Convergence rates in the maximum norm

From the proof of Theorem 3.1 and Theorem 1.3 for function (1.8) or (1.9) with $\lambda>0$, we have the following asymptotic convergence rates.

Corollary 4.1. Suppose that $\lambda>0$ and $f(x)$ is a function defined in (1.8) (or (1.9) where $\lambda$ is not an even number), then for $\alpha, \beta>-1$, (1.16) holds, that is,

$$
\left\|f-S_{n}^{(\alpha, \beta)}[f]\right\|_{\infty}= \begin{cases}\mathcal{O}\left(n^{\max \left\{\alpha-\frac{1}{2}, \beta-\frac{1}{2}\right\}-\lambda}\right), & \text { if } \max \{\alpha, \beta\}>\frac{1}{2} \\ \mathcal{O}\left(n^{-\lambda}\right), & \text { if } \max \{\alpha, \beta\} \leq \frac{1}{2}\end{cases}
$$

Proof. We assume here that $f(x)$ is defined in (1.8), while for functions in (1.9) a similar proof can be done by the fact $z(x)|x-a|^{\lambda}=f(x)+f^{*}(x)$.

If $x$ belongs to a closed subset of $[-1, a) \cup(a, 1]$, that is, $x \in\left[-1, a-\delta_{0}\right] \cup\left[a+\delta_{0}, 1\right]$ for any fixed $\delta_{0}>0$ such that $a-\delta_{0}>-1$ and $a+\delta_{0}<1$, then $a_{n}^{(\alpha, \beta)}(x ; g)$ will be uniformly bounded by $a_{n}^{(\alpha, \beta)}(x ; g)=\mathcal{O}\left(n^{-\lambda-1 / 2}\right)$. This further leads to

$$
\begin{align*}
e_{f}^{(\alpha, \beta)}(n, x) & =\mathcal{O}\left(n^{-\lambda-\frac{1}{2}}\right) \max \left\{\left\|P_{n}^{(\alpha, \beta)}\right\|_{\infty},\left\|P_{n+1}^{(\alpha, \beta)}\right\|_{\infty}\right\}  \tag{4.1}\\
& =\mathcal{O}\left(n^{\max \left\{\alpha-\frac{1}{2}, \beta-\frac{1}{2},-1\right\}-\lambda}\right) .
\end{align*}
$$

While if $x \in\left[a-\delta_{0}, a+\delta_{0}\right]$, from the uniform estimate (3.14) that $a_{n}^{(\alpha, \beta)}(x ; g)=$ $\mathcal{O}\left(n^{-\lambda+1 / 2}\right)$ and $P_{n}^{(\alpha, \beta)}(x)=\mathcal{O}\left(n^{-1 / 2}\right)$ (see from Theorem [2.3), we obtain

$$
\begin{equation*}
e_{f}^{(\alpha, \beta)}(n, x)=\mathcal{O}\left(n^{-\lambda+\frac{1}{2}}\right)\left(\left|P_{n}^{(\alpha, \beta)}(x)\right|+\left|P_{n+1}^{(\alpha, \beta)}(x)\right|\right)=\mathcal{O}\left(n^{-\lambda}\right) \tag{4.2}
\end{equation*}
$$

which together with (4.1) leads to (1.16).
In Figure 4.1, we demonstrate the convergence rates of the maximum error of $S_{n}^{(\alpha, \beta)}[f]$ and $p_{n}^{*}$ for $f(x)=\left|x-\frac{1}{4}\right|^{\lambda}$. Obviously, all these numerical results are consistent with the theoretical estimate (1.16).

Accordingly, it is interesting to examine the weighted pointwise error defined in (1.17), which is one order higher than the optimal polynomial approximation as stated below.

Corollary 4.2. Suppose that $\lambda>0$ and $f(x)$ is a function defined in (1.8) (or (1.9) where $\lambda$ is not an even number), then for $\alpha, \beta>-1$, (1.18) holds, i.e., $\left\|\hat{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}=\mathcal{O}\left(n^{-\lambda-1}\right)$.


Figure 4.1. Maximum error $\left\|f-S_{n}^{(\alpha, \beta)}[f]\right\|_{\infty}$ (red circle) and $\left\|f-p_{n}^{*}\right\|_{\infty}$ (blue square) for $f(x)=\left|x-\frac{1}{4}\right|^{\lambda}$ with $\lambda=1 / 2$ (first row) and $\lambda=1$ (second row), respectively

Proof. From (2.13), we have

$$
\begin{equation*}
(1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} P_{n}^{(\alpha, \beta)}(x)=\mathcal{O}\left(n^{-\frac{1}{2}}\right), \tag{4.3}
\end{equation*}
$$

which, together with (3.13), leads to
$\hat{e}_{f}^{(\alpha, \beta)}(n, x)=(1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}(x-a) e_{f}^{(\alpha, \beta)}(n, x)=\mathcal{O}\left(n^{-\lambda-1}\right)$,
when $x \neq a$, where the constant in $\mathcal{O}$-term is independent of $x$. While when $x=a$, it is obvious that $\hat{e}_{f}^{(\alpha, \beta)}(n, a)=0$. This completes the proof.

Corollary 4.2 indicates that the weighted pointwise error of $S_{n}^{(\alpha, \beta)}[f]$ in the uniform norm is one order higher in convergence rate than the optimal polynomial approximation. However, if we consider the weighted pointwise error by removing the factor $(x-a)$ :

$$
\tilde{e}_{f}^{(\alpha, \beta)}(n, x)=(1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}} e_{f}^{(\alpha, \beta)}(n, x),
$$

then we can obtain similarly the following asymptotic estimate.
Corollary 4.3. Suppose that $\lambda>0$ and $f(x)$ is defined by (1.8) (or (1.9) where $\lambda$ is not an even number), then for $\alpha, \beta>-1$, we have the following estimate

$$
\begin{equation*}
\left\|\tilde{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}=\mathcal{O}\left(n^{-\lambda}\right) \tag{4.4}
\end{equation*}
$$

Proof. This is obtained by the uniform estimate $a_{n}^{(\alpha, \beta)}(x ; g)=\mathcal{O}\left(n^{-\lambda+1 / 2}\right)$ and (4.3).

Numerical results in Figure 4.2 illustrate the optimal estimates on these two kinds of weighted pointwise errors.


Figure 4.2. Plots of $\left\|\hat{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}$ (black pentagram), $\left\|\tilde{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}$ (red circle) and the best approximation $\left\|f-p^{*}\right\|_{\infty}$ (blue square) for $f(x)=(x-1 / 4)_{+}^{\lambda}$ with $\lambda=1 / 2$ (first row) and $\lambda=2$ (second row)

## 5. Extensions to functions with endpoint singularities

In this section, we intend to study the pointwise error estimates and local superconvergence of Jacobi expansions to functions with endpoint singularities, which are stated in the following theorems.

In fact, the estimates in Lemma 3.2 can be generalised to the case with $a=-1$.
Lemma 5.1. Let $\alpha, \beta, \gamma, \delta>-1$ and $\psi(x) \in W_{\mathrm{AC}}(-1,1)$, then for $n \gg 1$ we have

$$
\begin{align*}
& \int_{-1}^{1}(1+x)^{\gamma}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x) \psi(x) \mathrm{d} x  \tag{5.1}\\
& \quad=\|\psi\|_{W_{\mathrm{AC}}(-1,1)} \cdot \mathcal{O}\left(n^{-\min \{2 \delta-\alpha+2,2 \gamma-\beta+2,3 / 2\}}\right)
\end{align*}
$$

Proof. In order to obtain the estimate, we split the integral in (5.1) into two parts as follows

$$
\int_{-1}^{0}(1+x)^{\gamma} P_{n}^{(\alpha, \beta)}(x)(1-x)^{\delta} \psi(x) \mathrm{d} x+\int_{0}^{1}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x)(1+x)^{\gamma} \psi(x) \mathrm{d} x .
$$

Using Lemma 3.2, we can derive the estimate

$$
\begin{aligned}
& \int_{0}^{1}(1-x)^{\delta} P_{n}^{(\alpha, \beta)}(x)(1+x)^{\gamma} \psi(x) \mathrm{d} x \\
& \quad=\left\|(1+x)^{\gamma} \psi(x)\right\|_{W_{\mathrm{AC}}(0,1)} \cdot \mathcal{O}\left(n^{-\min \{2 \delta-\alpha+2,3 / 2\}}\right) \\
& \quad=\|\psi\|_{W_{\mathrm{AC}}(0,1)} \cdot \mathcal{O}\left(n^{-\min \{2 \delta-\alpha+2,3 / 2\}}\right)
\end{aligned}
$$

Similarly, we can show that

$$
\int_{-1}^{0}(1+x)^{\gamma} P_{n}^{(\alpha, \beta)}(x)(1-x)^{\delta} \psi(x) \mathrm{d} x=\|\psi\|_{W_{\mathrm{AC}}(-1,0)} \cdot \mathcal{O}\left(n^{-\min \{2 \gamma-\beta+2,3 / 2\}}\right)
$$

Then the estimate (5.1) follows immediately from the above.

We have the following results on the pointwise estimates for the endpoint singularities.

Theorem 5.1. Define $f_{1}(x)=(1-x)^{\lambda} z(x)(\lambda+\alpha>-1)$ and $f_{2}(x)=(1+$ $x)^{\lambda} z(x)(\lambda+\beta>-1)$ where the given function $z(x)$ is smooth with $z( \pm 1) \neq 0$. Then for $\lambda>-1$ not an integer, we have the following pointwise error estimates.
(i) For $x \in(-1,1)$, we have

$$
\begin{equation*}
e_{f_{1}}^{(\alpha, \beta)}(n, x) \leq C_{1}(x) n^{-2 \lambda-\alpha-\frac{3}{2}}, \quad e_{f_{2}}^{(\alpha, \beta)}(n, x) \leq C_{2}(x) n^{-2 \lambda-\beta-\frac{3}{2}}, \tag{5.2}
\end{equation*}
$$

where $C_{i}(x)$ is independent of $n$ and has the behaviours near $x= \pm 1$ as follows for some $\delta>0$ and $0<\xi \leq \delta$

$$
\begin{align*}
& C_{1}(1-\xi) \leq D(1) \xi^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}-1}, \quad C_{1}(-1+\xi) \leq D(-1) \xi^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}, \\
& C_{2}(1-\xi) \leq D(1) \xi^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}, \quad C_{2}(-1+\xi) \leq D(-1) \xi^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}-1}, \tag{5.3}
\end{align*}
$$ with $D( \pm 1)$ independent of $\xi$ and $n$.

(ii) At $x= \pm 1$, we have

$$
\begin{equation*}
e_{f_{1}}^{(\alpha, \beta)}(n, 1) \leq C n^{-2 \lambda}, \quad e_{f_{2}}^{(\alpha, \beta)}(n,-1) \leq C n^{-2 \lambda} \tag{5.4}
\end{equation*}
$$

(iii) For the weighted pointwise error

$$
\begin{align*}
& (1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}(1-x) e_{f_{1}}^{(\alpha, \beta)}(n, x)=\mathcal{O}\left(n^{-2 \lambda-\alpha-\frac{3}{2}}\right), \\
& (1-x)^{\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}}(1+x)^{\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}}(1+x) e_{f_{2}}^{(\alpha, \beta)}(n, x)=\mathcal{O}\left(n^{-2 \lambda-\beta-\frac{3}{2}}\right) \tag{5.5}
\end{align*}
$$

where the $\mathcal{O}$-terms involved are independent of $x$.
Proof. For simplicity, we only consider the estimates for $f_{1}$. By $P_{n}^{(\alpha, \beta)}(-x)=$ $(-1)^{n} P_{n}^{(\beta, \alpha)}(x)$ it leads to the desired results for $f_{2}$.

Notice that

$$
\begin{align*}
a_{n}^{(\alpha, \beta)}(x ; g)= & \frac{1}{\sigma_{n}^{(\alpha, \beta)}}\left[\int_{-1}^{1} \frac{z(x)-z(y)}{x-y}(1-y)^{\lambda} P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y\right.  \tag{5.6}\\
& \left.+\int_{-1}^{1} \frac{(1-x)^{\lambda}-(1-y)^{\lambda}}{x-y} z(x) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y\right]
\end{align*}
$$

Obviously, $\frac{z(x)-z(y)}{x-y}$ is smooth in $[-1,1]$ and satisfies for some $\xi$ between $x$ and $y$ that

$$
\begin{equation*}
\partial_{y}^{k}\left(\frac{z(x)-z(y)}{x-y}\right)=k!\frac{z(x)-\sum_{j=0}^{k} \frac{z^{(j)}(y)}{j!}(x-y)^{j}}{(x-y)^{k+1}}=\frac{1}{k+1} z^{(k+1)}(\xi) \tag{5.7}
\end{equation*}
$$

which is uniformly bounded independent of $x$ and $y$ for any fixed positive integer $k$. Then it follows from the proof of [35, Theorem 3.1] with $\mu=0$ that

$$
\begin{equation*}
\frac{1}{\sigma_{n}^{(\alpha, \beta)}} \int_{-1}^{1} \frac{z(x)-z(y)}{x-y}(1-y)^{\lambda} P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y=\mathcal{O}\left(n^{-2 \lambda-\alpha-1}\right), \tag{5.8}
\end{equation*}
$$

where the constant in $\mathcal{O}$-term is independent of $x$ and $n$.
Now we turn to the second term in (5.6). Firstly, take $m$ to be a positive integer such that $\alpha+2 \lambda+2-m \leq \frac{3}{2} \leq \beta+m+2$ and $m>\lambda$, then it follows from 35]
that

$$
\begin{align*}
& \frac{z(x)}{\sigma_{n}^{(\alpha, \beta)}} \int_{-1}^{1} \frac{(1-x)^{\lambda}-(1-y)^{\lambda}}{x-y} P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\frac{z(x)}{2^{m}(n)_{m} \sigma_{n}^{(\alpha, \beta)}} \int_{-1}^{1}(1-y)^{\lambda-m} P_{n-m}^{(\alpha+m, \beta+m)}(y) \psi_{3}(x, y) \omega^{(\alpha+m, \beta+m)}(y) \mathrm{d} y, \tag{5.9}
\end{align*}
$$

where

$$
\psi_{3}(x, y)=(1-y)^{m-\lambda} \partial_{y}^{m}\left(\frac{(1-x)^{\lambda}-(1-y)^{\lambda}}{x-y}\right)
$$

Analogously, $\psi_{3}(x, y)$ is continuous and for any integer $k>\lambda$, there exists $\xi_{1}$ between $x$ and $y$ such that

$$
\begin{align*}
& (-1)^{\lfloor\lambda\rfloor+1} \partial_{y}^{k}\left(\frac{(1-x)^{\lambda}-(1-y)^{\lambda}}{x-y}\right)  \tag{5.10}\\
& \quad=\frac{(-1)^{\lfloor\lambda\rfloor+1}}{k+1}(\lambda)_{k+1}\left(1-\xi_{1}\right)^{\lambda-k-1}(-1)^{k+1}>0 .
\end{align*}
$$

In addition, by an argument similar to the proof of the monotonicity of $\psi_{2}$ in Appendix but with $1-y$ in place of $y-a$ and $m$ in place of $m+1$, we can readily prove that

$$
(-1)^{\lfloor\lambda\rfloor+1} \partial_{y} \psi_{3}(x, y)>0 .
$$

This together with (5.10) indicates that $\psi_{3}$ is positive and increasing w.r.t. $y \in$ $[-1,1]$ if $\lfloor\lambda\rfloor$ is odd, otherwise $\psi_{3}$ will be negative and decreasing. As a result, it leads to

$$
\begin{aligned}
& \max _{y \in[-1,1]}\left|\psi_{3}(x, y)\right|=\left|\psi_{3}(x, 1)\right| \leq C(1-x)^{-1} \\
& \int_{-1}^{1}\left|\partial_{y} \psi_{3}(x, y)\right| \mathrm{d} y=\left|\psi_{3}(x, 1)-\psi_{3}(x,-1)\right| \leq C(1-x)^{-1}
\end{aligned}
$$

so $\psi_{3}(x, \cdot) \in W_{\mathrm{AC}}(-1,1)$. Then by Lemma 5.1 it establishes from (5.6) and (5.9) that

$$
\begin{equation*}
a_{n}^{(\alpha, \beta)}(x ; g)=(1-x)^{-1} \cdot \mathcal{O}\left(n^{-2 \lambda-\alpha-1}\right), \tag{5.11}
\end{equation*}
$$

which together with (2.6) and Corollary 2.1 leads to the desired results (5.2) and (5.3).

When $x=1$, it is difficult to establish (5.4) from (2.6), but it can be derived from the estimate (see [35, Theorem 3.1])

$$
a_{n}^{(\alpha, \beta)}\left(x ; f_{1}\right)=\mathcal{O}\left(n^{-2 \lambda-\alpha-1}\right)
$$

and Theorem 2.3 that is,

$$
e_{f_{1}}^{(\alpha, \beta)}(n, 1)=\left|\sum_{j=n+1}^{\infty} a_{n}^{(\alpha, \beta)}\left(x ; f_{1}\right) P_{n}^{(\alpha, \beta)}(1)\right| \leq C \sum_{j=n+1}^{\infty} j^{-2 \lambda-\alpha-1} j^{\alpha} \leq C n^{-2 \lambda}
$$

Finally, we obtain the weighted pointwise error estimate (5.5) by analogous arguments as the proof of Theorem 1.3 and Corollary 2.1

Based on (5.11) and Theorem 2.1 we may give some improved and optimal bounds for $\left\|f_{i}-S_{n}^{(\alpha, \beta)}\left[f_{i}\right]\right\|_{\infty}$ than those in [36].

Theorem 5.2. Let $f_{1}(x)=(1-x)^{\lambda} z(x)(\lambda+\alpha>-1)$ and $f_{2}(x)=(1+x)^{\lambda} z(x)(\lambda+$ $\beta>-1)$ with $\lambda>0$ not an integer and $z(x)$ defined as above. Then for $\min \{\alpha, \beta\} \geq$ $-\frac{1}{2}$, it holds

$$
\begin{aligned}
& \left\|f_{1}-S_{n}^{(\alpha, \beta)}\left[f_{1}\right]\right\|_{\infty}=\mathcal{O}\left(n^{-2 \lambda+\max \{0, \beta-\alpha-1\}}\right) ; \\
& \left\|f_{2}-S_{n}^{(\alpha, \beta)}\left[f_{2}\right]\right\|_{\infty}=\mathcal{O}\left(n^{-2 \lambda+\max \{0, \alpha-\beta-1\}}\right) .
\end{aligned}
$$

Proof. Note by $\left\|P_{n}^{(\alpha, \beta)}\right\|_{[0,1]}:=\max _{0 \leq x \leq 1}\left|P_{n}^{(\alpha, \beta)}\right|=\mathcal{O}\left(n^{\alpha}\right)$ (see Theorem 2.3) that for $f_{1}$ and $x \in[0,1]$ it yields

$$
\begin{aligned}
e_{f_{1}}^{(\alpha, \beta)}(n, x) & =\left|\sum_{j=n+1}^{\infty} a_{j}^{(\alpha, \beta)}\left(x ; f_{1}\right) P_{j}^{(\alpha, \beta)}(x)\right| \\
& \leq C \sum_{j=n+1}^{\infty} j^{-2 \lambda-\alpha-1}\left\|P_{j}^{(\alpha, \beta)}\right\|_{[0,1]} \\
& \leq C_{1} n^{-2 \lambda}
\end{aligned}
$$

for some constants $C$ and $C_{1}$ independent of $x \in[0,1]$ and $n$. While for $x \in[-1,0]$, by (2.6), (5.11) and $\left\|P_{n}^{(\alpha, \beta)}\right\|_{[-1,0]}:=\max _{-1 \leq x \leq 0}\left|P_{n}^{(\alpha, \beta)}\right|=\mathcal{O}\left(n^{\beta}\right)$ (see Theorem (2.3), it implies

$$
\begin{aligned}
e_{f_{1}}^{(\alpha, \beta)}(n, x) & =\left|A_{n}^{(\alpha, \beta)} a_{n}^{(\alpha, \beta)}(x ; g) P_{n+1}^{(\alpha, \beta)}(x)-B_{n}^{(\alpha, \beta)} a_{n+1}^{(\alpha, \beta)}(x ; g) P_{n}^{(\alpha, \beta)}(x)\right| \\
& \leq C n^{-2 \lambda-\alpha-1}\left\|P_{n}^{(\alpha, \beta)}\right\|_{[-1,0]} \\
& \leq C_{2} n^{-2 \lambda-\alpha+\beta-1}
\end{aligned}
$$

for some constants $C_{2}$ independent of $x \in[0,1]$ and $n$. These together lead to the desired results for $f_{1}$. Analogously, the bound for $f_{2}$ is also satisfied.

In Figure 5.1, we show the maximum error of $S_{n}^{(\alpha, \beta)}[f]$ and $p_{n}^{*}[f]$ as a function of $n$ for two functions $f_{1}(x)=(1-x)^{1 / 2}$ and $f_{2}(x)=(1+x)^{2 / 3}$. In order to consider the boundary behaviours around $x= \pm 1$, we also show a weighted maximum error defined by (5.5). Clearly, numerical results are in good agreement with our theoretical estimates in Theorem 5.1 and Theorem 5.2, which implies as well the optimality of our estimates in the sense that the orders derived can no more be improved.

Remark 5.1.
(i) The local behaviours of Legendre series have been extensively studied in Wahlbin [32, Theorem 3.3], and were illustrated by numerical examples 6.2.b, 6.3.ab and 6.4.a-b in 32 for some specific $x_{0}$

$$
f(x)=\left\{\begin{array}{ll}
0, & -1 \leq x \leq 0,  \tag{5.12}\\
1, & 0<x \leq 1
\end{array} \text { with } \quad e_{f}\left(n, \frac{\sqrt{2}}{2}\right) \leq C \frac{\ln n}{n}, e_{f}(n, 1) \leq C \frac{\ln n}{\sqrt{n}}\right.
$$

or for $x_{0}$ a "unit" distance away from 0

$$
\begin{align*}
& f(x)=|x|^{\frac{1}{2}} \quad \text { with } \quad e_{f}\left(n, x_{0}\right) \leq C \sigma_{n}\left(x_{0}\right) n^{-\frac{3}{2}}(\ln n)^{\frac{3}{2}}  \tag{5.13a}\\
& f(x)=\sqrt{1-x} \quad \text { with } \quad e_{f}\left(n, x_{0}\right) \leq C \sigma_{n}\left(x_{0}\right) n^{-\frac{5}{2}}(\ln n)^{\frac{5}{2}} \tag{5.13b}
\end{align*}
$$



Figure 5.1. Plots of $\left\|e_{f}^{(\alpha, \beta)}\right\|_{\infty}$ (red circle), $\left\|f-p_{n}^{*}\right\|_{\infty}$ (blue square) and the weighted maximum error $\left\|\hat{e}_{f}^{(\alpha, \beta)}\right\|_{\infty}$ (black pentagram) for $f_{1}(x)=(1-x)^{1 / 2}$ (first row) and $f_{2}(x)=(1+x)^{2 / 3}$ (second row)

$$
\begin{gather*}
f(x)=|x|^{-\frac{1}{2}} \quad \text { with } \quad e_{f}\left(n, x_{0}\right) \leq C \sigma_{n}\left(x_{0}\right) n^{-\frac{1}{2}}(\ln n)^{\frac{1}{2}}  \tag{5.14a}\\
f(x)=\frac{1}{\sqrt{1-x^{2}}} \quad \text { with } \quad e_{f}\left(n, x_{0}\right) \leq C \sigma_{n}\left(x_{0}\right) n^{-\frac{1}{2}}(\ln n)^{\frac{1}{2}} \tag{5.14b}
\end{gather*}
$$

where $C$ is a constant independent of $n$, and $\sigma_{n}\left(x_{0}\right)=\min \left\{\left(1-x_{0}^{2}\right)^{-\frac{1}{4}}, n^{\frac{1}{2}}\right\}$ (see [32] for more details).
(ii) For $f(x)=(x-a)_{+}^{\lambda}$, in more recently work by Babuška and Hakula [5], the above log-term in (5.12) is omitted without the assumption $\delta \geq C_{4} \frac{\ln n}{n}$ in Theorem 3.3 [32] if $\lambda=0$, and the $\log$-term $(\ln n)^{\frac{3}{2}}$ in (5.13a) and $(\ln n)^{\frac{1}{2}}$ in (5.14a) is replaced by $\ln n$ respectively if $\lambda \neq 0$.
(iii) Following Wahlbin [32, Theorem 3.3], from Theorem [2.3 and Corollary 2.1] we may define $\widetilde{C}(n ; x)=|x-a|^{-1} \sigma_{n}^{(\alpha, \beta)}(x)$ related to $n$ and $x$ instead of $C(x)$ in (1.12) of Theorem 1.3, while for boundary singularities, we define $\widetilde{C}_{1}(n ; x)=$ $(1-x)^{-1} \sigma_{n}^{(\alpha, \beta)}(x)$ and $\widetilde{C}_{2}(n ; x)=(1+x)^{-1} \sigma_{n}^{(\alpha, \beta)}(x)$ instead of $C_{1}(x)$ and $C_{2}(x)$ in (5.3) in Theorem 5.1 respectively, where

$$
\begin{aligned}
& \sigma_{n}^{(\alpha, \beta)}(x)=\left\{\begin{array}{lll}
\min \left\{(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}},(n+1)^{\frac{1}{2}}\right\}, & \alpha<-\frac{1}{2}, & x \in[0,1] ; \\
\min \left\{(1-x)^{-\max \left\{\frac{\alpha}{2}+\frac{1}{4}, 0\right\}},(n+1)^{\alpha+\frac{1}{2}}\right\}, & \alpha \geq-\frac{1}{2}, & \\
\sigma_{n}^{(\alpha, \beta)}(x)=\left\{\begin{array}{lll}
\min \left\{(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}},(n+1)^{\frac{1}{2}}\right\}, & \beta<-\frac{1}{2}, & x \in[-1,0] \\
\min \left\{(1+x)^{-\max \left\{\frac{\beta}{2}+\frac{1}{4}, 0\right\}},(n+1)^{\beta+\frac{1}{2}}\right\}, & \beta \geq-\frac{1}{2},
\end{array}\right.
\end{array} . \begin{array}{l}
\end{array}\right.
\end{aligned}
$$

satisfied that $\sigma_{n}^{(0,0)}(x) \sim \sigma_{n}(x)$, i.e., with the same order on $x$ and $n$ for Legendre series. Then from Theorems 1.3 and 5.1, all the logarithimic factors in (5.12), (5.13a), (5.13b), (5.14a) and (5.14b) can be removed, improvement of 32] in Legendre series. As mentioned in Wahlbin [32], these bounds without the logarithimic factors are sharp.

Remark 5.2. From Theorem 5.2 we see that if $\beta-\alpha-1 \leq 0$ (resp. $\alpha-\beta-1 \leq 0$ ) for $f_{1}(x)\left(\right.$ resp. $\left.f_{2}(x)\right),\left\|f_{i}-S_{n}^{(\alpha, \beta)}\left[f_{i}\right]\right\|_{\infty}=\mathcal{O}\left(n^{-2 \lambda}\right)$ has the same order as $\left\|f_{i}-p_{n}^{*}\right\|_{\infty}$, ( $i=1,2$ ).

## Appendix A. Proof of case (iii) in Theorem 3.1

In Section 3 we presented detailed proofs of (3.13) for $x<a$ and $x=a$, respectively. In the following, we sketch the proof for Case (iii).

Case (iii) $x>a$ : A routine computation from (2.8) gives rise to

$$
\begin{align*}
a_{n}^{(\alpha, \beta)}(x ; g)= & \frac{1}{\sigma_{n}^{(\alpha, \beta)}}\left[\int_{-1}^{a} \frac{z(x)(a-y)^{\lambda}}{x-y} P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y\right. \\
& +\int_{a}^{1} \omega^{(\alpha, \beta)}(y) \frac{z(x)-z(y)}{x-y}(y-a)^{\lambda} P_{n}^{(\alpha, \beta)}(y) \mathrm{d} y  \tag{A.1}\\
& \left.+\int_{-1}^{1} \frac{(x-a)^{\lambda}-|y-a|^{\lambda}}{x-y} z(x) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y\right] .
\end{align*}
$$

Similar to the proof in case (i), it is not difficult to verify that the first term in the right hand side of (A.1) satisfies (3.13) and (3.14). Since $z(x)$ is smooth on $[-1,1]$, and for any integer $k, \partial_{y}^{k}\left(\frac{z(x)-z(y)}{x-y}\right)$ is uniformly bounded by a constant independent of $x$. As a result, the second term in (A.1) satisfies (3.13)-(3.14) as well by Lemma 3.3. While for the third term, recalling the definition of $m$ ( $m=\lambda-1$ if $\lambda$ is an integer, otherwise $m=\lfloor\lambda\rfloor$ ), and applying the Rodrigues' formula (3.11), we have

$$
\begin{align*}
& \int_{-1}^{1} \frac{(x-a)^{\lambda}-|y-a|^{\lambda}}{x-y} z(x) P_{n}^{(\alpha, \beta)}(y) \omega^{(\alpha, \beta)}(y) \mathrm{d} y \\
& =\frac{z(x)}{2^{m+1}(n)_{m+1} \sigma_{n}^{(\alpha, \beta)}}  \tag{A.2}\\
& \quad \times\left\{\int_{-1}^{a}(a-y)^{\lambda-m-1} P_{n-m-1}^{(\alpha+m+1, \beta+m+1)}(y) \psi_{1}(x, y) \omega^{(\alpha+m+1, \beta+m+1)}(y) \mathrm{d} y\right. \\
& \left.\quad+\int_{a}^{1}(y-a)^{\lambda-m-1} P_{n-m-1}^{(\alpha+m+1, \beta+m+1)}(y) \psi_{2}(x, y) \omega^{(\alpha+m+1, \beta+m+1)}(y) \mathrm{d} y\right\},
\end{align*}
$$

where

$$
\begin{array}{ll}
\psi_{1}(x, y)=(a-y)^{m+1-\lambda} \partial_{y}^{m+1}\left(\frac{(x-a)^{\lambda}-(a-y)^{\lambda}}{x-y}\right), & y \in[-1, a], \\
\psi_{2}(x, y)=(y-a)^{m+1-\lambda} \partial_{y}^{m+1}\left(\frac{(x-a)^{\lambda}-(y-a)^{\lambda}}{x-y}\right), & y \in[a, 1] .
\end{array}
$$

It is evident that $\psi_{1}(x, y)$ and $\psi_{2}(x, y)$ are smooth on $(a, 1) \times\left[-1, \frac{-1+a}{2}\right]$ and $(a, 1) \times$ $\left[\frac{1+a}{2}, 1\right]$, respectively. Similar to (3.16), $\psi_{1}(x, y)$ can be written as

$$
\begin{aligned}
\psi_{1}(x, y)=\frac{(m+1)!}{x-y}[ & \frac{(x-a)^{\lambda}-(a-y)^{\lambda}}{(x-y)^{\lambda}}\left(\frac{a-y}{x-y}\right)^{m+1-\lambda} \\
& \left.+\sum_{k=1}^{m+1} \frac{(-1)^{m+1}(\lambda)_{k}}{k!}\left(\frac{a-y}{x-y}\right)^{m+1-k}\right]
\end{aligned}
$$

and satisfies

$$
\left|\psi_{1}(x, y)\right| \leq \frac{C_{1}}{x-y}, \quad\left|\partial_{y} \psi_{1}(x, y)\right| \leq \frac{C_{2}}{(x-y)^{2}}, \quad \forall y \in[(-1+a) / 2, a]
$$

for some constants $C_{1}$ and $C_{2}$ independent of $x$. Thus, we can immediately derive that $\psi_{1}(x, \cdot) \in W_{\mathrm{AC}}\left(\frac{-1+a}{2}, a\right)$ and $\left\|\psi_{1}(x, \cdot)\right\|_{W_{\mathrm{AC}}^{1}\left(-1, \frac{-1+a}{2}\right)}$ is uniformly bounded for all $x$, which, together with Lemma 3.3 leads to the desired result.

Now we turn to $\psi_{2}(x, y)$, after a calculation by the Leibniz's formula, it leads to

$$
\begin{equation*}
\psi_{2}(x, y)=(m+1)!(y-a)^{m+1-\lambda} \frac{h(x, y)}{(x-y)^{m+2}} \tag{A.3}
\end{equation*}
$$

where

$$
h(x, y)=(x-a)^{\lambda}-\sum_{k=0}^{m+1} \frac{(\lambda)_{k}}{k!}(y-a)^{\lambda-k}(x-y)^{k} .
$$

Similar to (5.7), we have for some $\xi$ between $x$ and $y$ that

$$
\begin{align*}
\frac{h(x, y)}{(x-y)^{m+2}} & =\frac{(x-a)^{\lambda}-(y-a)^{\lambda}-\sum_{k=1}^{m+1} \frac{\left((y-a)^{\lambda}\right)^{(k)}}{k!}(x-y)^{k}}{(x-y)^{m+2}}  \tag{A.4}\\
& =\frac{(\lambda)_{m+2}(\xi-a)^{\lambda-m-2}}{(m+2)!}<0 .
\end{align*}
$$

Moreover, we find that $\psi_{2}(x, y)$ is monotonically increasing w.r.t. $y$ (see the proof at the end of this section and numerical illustration in Figure A.1). Therefore, we get from (A.3) and (A.4) by letting $y \rightarrow a$ that

$$
\max _{y \in\left[a, \frac{1+a}{2}\right]}\left|\psi_{2}(x, y)\right|=\left|\psi_{2}(x, a)\right|=(\lambda)_{m+1}(x-a)^{-1}
$$

and

$$
\int_{a}^{\frac{1+a}{2}}\left|\partial_{y} \psi_{2}(x, y)\right| \mathrm{d} y=\left|\psi_{2}\left(x, \frac{1+a}{2}\right)-\psi_{2}(x, a)\right| \leq 2(\lambda)_{m+1}(x-a)^{-1}
$$

So the second integral in the right hand side of (A.2) also satisfies (3.13) as $\psi_{2}(x, \cdot) \in$ $W_{\mathrm{AC}}\left(a, \frac{1+a}{2}\right)$ and $\left\|\psi_{2}(x, \cdot)\right\|_{W_{\mathrm{AC}}^{1}\left(\frac{1+a}{2}, 1\right)}$ is uniformly bounded for all $x$. To sum up all the results above, we complete the proof of (3.13).

In order to obtain the uniformly estimate (3.14), we conduct integration by parts till $m$ instead of $m+1$. The proof is analogous to that in Case (i) and omitted here.

We end this section after providing a rigorous proof of the monotonicity of $\psi_{2}(x, y)$ with respect to $y$. For any fixed $x \in(a, 1)$, we have

$$
\partial_{y} \psi_{2}(x, y)=(m+1)!\frac{u(y)}{(x-y)^{m+3}}
$$

where

$$
u(y)=(x-y) \partial_{y}\left((y-a)^{m+1-\lambda} h(x, y)\right)+(m+2)(y-a)^{m+1-\lambda} h(x, y) .
$$

Interestingly, we can show that $u^{(k)}(x)=0$ for any $k \in\{0,1, \cdots, m+1\}$, that is,

$$
\begin{aligned}
& u^{(k)}(x)=\left.(m+2-k) \partial_{y}^{k}\left((y-a)^{m+1-\lambda} h(x, y)\right)\right|_{y=x} \\
&=(m+2-k) \partial_{y}^{k}\left[(x-a)^{\lambda}(y-a)^{m+1-\lambda}\right. \\
&\left.-\sum_{j=0}^{m+1} \frac{(\lambda)_{j}}{j!}(y-a)^{m+1-j}(x-y)^{j}\right]_{y=x} \\
&=(m+2-k)\left[(x-a)^{\lambda}(m+1-\lambda)_{k}(y-a)^{m+1-\lambda-k}\right. \\
&\left.-\sum_{j=0}^{k} \frac{(\lambda)_{j}}{j!}\binom{k}{j}\left((y-a)^{m+1-j}\right)^{(k-j)}\left((x-y)^{j}\right)^{(j)}\right]_{y=x} \\
&=(m+2-k)(x-a)^{m+1-k}\left[(m+1-\lambda)_{k}\right. \\
&=\left.\quad-\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}(\lambda)_{j}(m+1-j)_{k-j}\right] \\
&
\end{aligned}
$$

and further

$$
\begin{aligned}
& u^{(m+2)}(y)=(x-y) \partial_{y}^{m+3}\left((y-a)^{m+1-\lambda} h(x, y)\right) \\
& =(x-y) \partial_{y}^{m+3}\left((x-a)^{\lambda}(y-a)^{m+1-\lambda}-\sum_{j=0}^{m+1} \frac{(\lambda)_{j}}{j!}(y-a)^{m+1-j}(x-y)^{j}\right) \\
& =(m+1-\lambda)_{m+3}(x-y)(x-a)^{\lambda}(y-a)^{-\lambda-2}
\end{aligned}
$$

Obviously, it follows from $0 \leq m+1-\lambda<1$ that

$$
\operatorname{sgn}\left((-1)^{m+2} u^{(m+2)}(y)\right)=\operatorname{sgn}(x-y)
$$

As a consequence, it deduces from Taylor's theorem that

$$
\begin{align*}
\partial_{y} \psi_{2}(x, y) & =\frac{(m+1)!}{(x-y)^{m+3}}\left(\sum_{k=0}^{m+1} \frac{u^{(k)}(x)}{k!}(y-x)^{k}+\frac{u^{(m+2)}(\xi)}{(m+2)!}(y-x)^{m+2}\right)  \tag{A.5}\\
& =\frac{(-1)^{m+2} u^{(m+2)}(\xi)}{(m+2)(x-y)} \geq 0
\end{align*}
$$

where $\xi=x+\eta(y-x)$ and $0<\eta<1$. While if $y=x$, we have

$$
\begin{aligned}
\partial_{y} \psi_{2}(x, x) & =\lim _{y \rightarrow x} \partial_{y} \psi_{2}(x, y)=\lim _{y \rightarrow x}\left\{(m+1)!\frac{u(y)}{(x-y)^{m+3}}\right\} \\
& =\frac{(-1)^{m}(m+1-\lambda)_{m+3}}{(m+3)(m+2)}(x-a)^{-2} \geq 0,
\end{aligned}
$$

which together with (A.5) completes the proof.


Figure A.1. The monotonicity of $\psi_{2}(x, y)$ with respect to the argument $y$ on $[a, 1]$, where $a=0$ and $x=1 / 2$.

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## References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, D.C., 1964.
[2] I. Babuška and B. Guo, Optimal estimates for lower and upper bounds of approximation errors in the p-version of the finite element method in two dimensions, Numer. Math. 85 (2000), no. 2, 219-255, DOI 10.1007/PL00005387. MR 1754720
[3] I. Babuška and B. Guo, Direct and inverse approximation theorems for the p-version of the finite element method in the framework of weighted Besov spaces. I. Approximability of functions in the weighted Besov spaces, SIAM J. Numer. Anal. 39 (2001/02), no. 5, 15121538, DOI 10.1137/S0036142901356551. MR 1885705
[4] I. Babuška and B. Guo, Direct and inverse approximation theorems for the p-version of the finite element method in the framework of weighted Besov spaces. II. Optimal rate of convergence of the p-version finite element solutions, Math. Models Methods Appl. Sci. 12 (2002), no. 5, 689-719, DOI 10.1142/S0218202502001854. MR1909423
[5] I. Babuška and H. Hakula, Pointwise error estimate of the Legendre expansion: the known and unknown features, Comput. Methods Appl. Mech. Engrg. 345 (2019), 748-773, DOI 10.1016/j.cma.2018.11.017. MR 3892019
[6] N. K. Bary, A Treatise on Trigonometric Series. Vols. I, II, A Pergamon Press Book, The Macmillan Company, New York, 1964. Authorized translation by Margaret F. Mullins. MR 0171116
[7] S. Bernstein, On the best approximation of $|x-c|^{p}$, Doll. Akad. Nauk SSSR 18 (1938), 374-384.
[8] W. Cao, C.-W. Shu, Y. Yang, and Z. Zhang, Superconvergence of discontinuous Galerkin methods for two-dimensional hyperbolic equations, SIAM J. Numer. Anal. 53 (2015), no. 4, 1651-1671, DOI 10.1137/140996203. MR3365565
[9] W. Cao, C.-W. Shu, Y. Yang, and Z. Zhang, Superconvergence of discontinuous Galerkin method for scalar nonlinear hyperbolic equations, SIAM J. Numer. Anal. 56 (2018), no. 2, 732-765, DOI 10.1137/17M1128605. MR3780748
[10] W. Cao, Z. Zhang, and Q. Zou, Superconvergence of discontinuous Galerkin methods for linear hyperbolic equations, SIAM J. Numer. Anal. 52 (2014), no. 5, 2555-2573, DOI 10.1137/130946873. MR3270187
[11] P. Castillo, B. Cockburn, D. Schötzau, and C. Schwab, Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems, Math. Comp. 71 (2002), no. 238, 455-478, DOI 10.1090/S0025-5718-01-01317-5. MR 1885610
[12] G. Darboux, Mémoire sur l'approximation des fonctions de très-grands nombres et sur une classe étendue de développements en série, J. Math. Purer Appl. 4 (1978), 5-56, 377-416.
[13] K.-J. Förster, Inequalities for ultraspherical polynomials and application to quadrature, Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991), J. Comput. Appl. Math. 49 (1993), no. 1-3, 59-70, DOI 10.1016/0377-0427(93)90135-X. MR 1256011
[14] J. S. Hesthaven, S. Gottlieb, and D. Gottlieb, Spectral Methods for Time-Dependent Problems, Cambridge Monographs on Applied and Computational Mathematics, vol. 21, Cambridge University Press, Cambridge, 2007, DOI 10.1017/CBO9780511618352. MR2333926
[15] E. Kruglov, Pointwise convergence of Jacobi polynomials, Master's Thesis, Aalto University, 2018.
[16] R. Lin and Z. Zhang, Natural superconvergence points in three-dimensional finite elements, SIAM J. Numer. Anal. 46 (2008), no. 3, 1281-1297, DOI 10.1137/070681168. MR2390994
[17] W. Liu and L.-L. Wang, Asymptotics of the generalized Gegenbauer functions of fractional degree, J. Approx. Theory 253 (2020), 105378, 25, DOI 10.1016/j.jat.2020.105378. MR4064848
[18] W. Liu, L.-L. Wang, and H. Li, Optimal error estimates for Chebyshev approximations of functions with limited regularity in fractional Sobolev-type spaces, Math. Comp. 88 (2019), no. 320, 2857-2895, DOI 10.1090/mcom/3456. MR3985478
[19] W. Liu, L.-L. Wang, and B. Wu, Optimal error estimates for Legendre expansions of singular functions with fractional derivatives of bounded variation, Adv. Comput. Math. 47 (2021), no. 6, Paper No. 79, 32, DOI 10.1007/s10444-021-09905-3. MR4329941
[20] D. S. Lubinsky, A new approach to universality limits involving orthogonal polynomials, Ann. of Math. (2) $\mathbf{1 7 0}$ (2009), no. 2, 915-939, DOI 10.4007/annals.2009.170.915. MR2552113
[21] B. Muckenhoupt, Transplantation theorems and multiplier theorems for Jacobi series, Mem. Amer. Math. Soc. 64 (1986), no. 356, iv+86, DOI 10.1090/memo/0356. MR858466
[22] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), NIST Handbook of Mathematical Functions, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX). MR 2723248
[23] E. B. Saff and V. Totik, Polynomial approximation of piecewise analytic functions, J. London Math. Soc. (2) 39 (1989), no. 3, 487-498, DOI 10.1112/jlms/s2-39.3.487. MR 1002461
[24] J. Shen, T. Tang, and L.-L. Wang, Spectral Methods, Springer Series in Computational Mathematics, vol. 41, Springer, Heidelberg, 2011. Algorithms, analysis and applications, DOI 10.1007/978-3-540-71041-7. MR2867779
[25] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR 1232192
[26] E. M. Stein and R. Shakarchi, Real Analysis, Princeton Lectures in Analysis, vol. 3, Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces. MR2129625
[27] G. Szegö, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Vol. 23, American Mathematical Society, New York, 1939. MR0000077
[28] T. Tao, An Introduction to Measure Theory, Graduate Studies in Mathematics, vol. 126, American Mathematical Society, Providence, RI, 2011, DOI 10.1090/gsm/126. MR 2827917
[29] L. N. Trefethen, Six myths of polynomial interpolation and quadrature, Math. Today (Southend-on-Sea) 47 (2011), no. 4, 184-188. MR 2952622
[30] L. N. Trefethen, Approximation Theory and Approximation Practice, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013. MR3012510
[31] P. D. Tuan and D. Elliott, Coefficients in series expansions for certain classes of functions, Math. Comp. 26 (1972), 213-232, DOI 10.2307/2004731. MR301440
[32] L. B. Wahlbin, A comparison of the local behavior of spline $L_{2}$-projections, Fourier series and Legendre series, Singularities and Constructive Methods for Their Treatment (Oberwolfach, 1983), Lecture Notes in Math., vol. 1121, Springer, Berlin, 1985, pp. 319-346, DOI 10.1007/BFb0076279. MR806401
[33] H. Wang, How much faster does the best polynomial approximation converge than Legendre projection?, Numer. Math. 147 (2021), no. 2, 481-503, DOI 10.1007/s00211-021-01173-z. MR4215344
[34] H. Wang, On the optimal estimates and comparison of Gegenbauer expansion coefficients, SIAM J. Numer. Anal. 54 (2016), no. 3, 1557-1581, DOI 10.1137/15M102232X. MR3504991
[35] S. Xiang, Convergence rates on spectral orthogonal projection approximation for functions of algebraic and logarithmatic regularities, SIAM J. Numer. Anal. 59 (2021), no. 3, 1374-1398, DOI 10.1137/20M134407X. MR4259911
[36] S. Xiang and G. Liu, Optimal decay rates on the asymptotics of orthogonal polynomial expansions for functions of limited regularities, Numer. Math. 145 (2020), no. 1, 117-148, DOI 10.1007/s00211-020-01113-3. MR4091598
[37] Z. Zhang, Superconvergence points of polynomial spectral interpolation, SIAM J. Numer. Anal. 50 (2012), no. 6, 2966-2985, DOI 10.1137/120861291. MR3022250
[38] Z. Zhang and A. Naga, A new finite element gradient recovery method: superconvergence property, SIAM J. Sci. Comput. 26 (2005), no. 4, 1192-1213, DOI 10.1137/S1064827503402837. MR2143481
[39] X. Zhao and Z. Zhang, Superconvergence points of fractional spectral interpolation, SIAM J. Sci. Comput. 38 (2016), no. 1, A598-A613, DOI 10.1137/15M1011172. MR3463700

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