# Modified Laguerre pseudospectral method refined by multidomain Legendre pseudospectral approximation ${ }^{2}$ 

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Dedicated to Professor Roderick S.C. Wong on the occasion of his 60th birthday


#### Abstract

A modified Laguerre pseudospectral method is proposed for differential equations on the half-line. The numerical solutions are refined by multidomain Legendre pseudospectral approximation. Numerical results show the spectral accuracy of this approach. Some approximation results on the modified Laguerre and Legendre interpolations are established. The convergence of proposed method is proved.


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## 1. Introduction

Spectral methods have high accuracy. The usual spectral methods are only available for bounded domains. However it is also interesting to consider spectral methods for differential equations on unbounded domains, such as the KdV equation and the Klein-Gordon equation on the half-line and exterior problems. Some authors developed the Laguerre spectral method for the half-line, see [3,4,6,9,11]. In actual

[^0]computation, it is more preferable to use the Laguerre pseudospectral method, since it only needs to evaluate numerical solutions at the Laguerre interpolation nodes, say $x_{j}, 0 \leqslant j \leqslant N$. It is also much easier to deal with nonlinear terms. However, the distances between the adjacent interpolation nodes increase very fast as the mode $N$ increases, especially for large $x_{j}$. Therefore, although numerical solutions fit exact solutions well at the nodes, but as global approximate solutions, they only can simulate exact solutions roughly between the large interpolation nodes.

In this paper, we propose a combined Laguerre and multidomain Legendre pseudospectral method for the half-line. We first use the modified Laguerre pseudospectral method to obtain numerical solutions on the half-line, which fit exact solutions well at the interpolation nodes. Then we refine them locally by using multidomain Legendre pseudospectral approximation. This technique can be also regarded as recovering the accuracy by reconstructing numerical solutions.

The paper is organized as follows. In the next section, we describe the combined Laguerre and multidomain Legendre pseudospectral method. In Section 3, we discuss its implementation and present some numerical results demonstrating its efficiency. In Section 4, we establish some results on the modified Laguerre and Legendre interpolations, which play important role in the analysis of spectral methods. In Section 5, we prove the convergence of proposed method. The final section is for concluding discussions.

## 2. Combined spectral method

Let $I=(a, b), 0 \leqslant a<b \leqslant \infty$, and $\chi(x)$ be a certain weight function. For real $r \geqslant 0$, we define the weighted Sobolev space $H_{\chi}^{r}(I)$ as usual, with the semi-norm $|v|_{m, \chi, I}$ and the norm $\|v\|_{m, \chi, I}$. In particular, we denote by $(u, v)_{\chi, I}$ and $\|v\|_{\chi, I}$ the inner product and norm of $L_{\chi}^{2}(I)$. The space $H_{0, \chi}^{r}(I)$ stands for the closure of set $\mathscr{D}(I)$ consisting of all infinitely differential functions with compact support in $I$. For $\chi(x) \equiv 1$, we drop the subscript $\chi$ in notations.

For any integer $M \geqslant 0, \mathscr{P}_{M}(I)$ denotes the set of all algebraic polynomials of degree at most $M$. Moreover ${ }_{0} \mathscr{P}_{M}=\left\{v \in \mathscr{P}_{M} \mid v(a)=0\right\}$ and $\mathscr{P}_{M}^{0}=\left\{v \in{ }_{0} \mathscr{P}_{M} \mid v(b)=0\right\}$. Denote by $c$ a generic positive constant independent of any function and $M$.

Now let $\Lambda=(0, \infty)$ and $\omega(x)=\mathrm{e}^{-x}$. Denote by $\mathscr{L}_{l}(x)$ the Laguerre polynomials of degree $l, l=0,1, \ldots$ . They satisfy the recurrence relation

$$
\begin{equation*}
\mathscr{L}_{l}(x)=\partial_{x} \mathscr{L}_{l}(x)-\partial_{x} \mathscr{L}_{l+1}(x), \quad l=0,1, \ldots, \tag{1}
\end{equation*}
$$

and form the $L_{\omega}^{2}(\Lambda)$-orthogonal system, i.e., $\left(\mathscr{L}_{l}, \mathscr{L}_{m}\right)_{\omega, \Lambda}=\delta_{l, m}$.
In this paper, we introduce the modified Laguerre interpolation. Firstly let ${ }_{0} H^{1}(\Lambda)=\left\{v \in H^{1}(\Lambda) \mid v(0)=\right.$ $0\}$ and take the approximation spaces as

$$
\mathscr{S}_{M}(\Lambda)=\left\{\phi(x) \mid \phi(x)=\omega^{1 / 2}(x) \psi(x), \quad \forall \psi(x) \in \mathscr{P}_{M}(\Lambda)\right\}, \quad{ }_{0} \mathscr{S}_{M}(\Lambda)=\mathscr{S}_{M}(\Lambda) \cap_{0} H^{1}(\Lambda) .
$$

Next, let $\left\{x_{j}\right\}_{j=0}^{M}$ and $\left\{\omega_{j}\right\}_{j=0}^{M}$ be the nodes and weights of the standard Laguerre-Gauss-Radau interpolation $\mathscr{I}_{\mathscr{L}, M}$. Indeed, $x_{j}$ are the zeros of $x \partial_{x} \mathscr{L}_{M+1}(x)$, and $\omega_{j}=1 /(M+1) \mathscr{L}_{M}^{2}\left(x_{j}\right)$. Further, let $\widetilde{\omega}_{j}=\omega_{j} \omega^{-1}\left(x_{j}\right)$. Then by the property of Laguerre-Gauss-Radau interpolation (see [12]),

$$
\begin{equation*}
(\phi, \psi)_{\Lambda}=(\phi, \psi)_{M, \Lambda}:=\sum_{j=0}^{M} \phi\left(x_{j}\right) \psi\left(x_{j}\right) \widetilde{\omega}_{j}, \quad \forall \phi \cdot \psi \in \mathscr{S}_{2 M}(\Lambda) \tag{2}
\end{equation*}
$$

For any $v \in C(\bar{\Lambda})$, the modified Laguerre-Gauss-Radau interpolant $\tilde{\mathscr{I}}_{\mathscr{L}} v(x) \in \mathscr{S}_{M}(\Lambda)$, satisfying

$$
\begin{equation*}
\left(\tilde{\mathscr{I}}_{\mathscr{L}, M} v\right)\left(x_{j}\right)=v\left(x_{j}\right), \quad 0 \leqslant j \leqslant M . \tag{3}
\end{equation*}
$$

We now turn to the Legendre-Gauss-Lobatto interpolation. Let the reference interval $\hat{I}=(-1,1)$, and $L_{l}(x)$ be the Legendre polynomials of degree $l, l=0,1, \ldots$. They satisfy the recurrence relation

$$
\begin{equation*}
(2 l+1) L_{l}(x)=\partial_{x} L_{l+1}(x)-\partial_{x} L_{l-1}(x), \quad l=1,2, \ldots \tag{4}
\end{equation*}
$$

and form the $L^{2}(\hat{I})$-orthogonal system, i.e., $\left(L_{l}, L_{m}\right)_{\hat{I}}=(2 /(2 l+1)) \delta_{l, m}$.
Let $\{\hat{x}\}_{j=0}^{N}$ and $\{\hat{\omega}\}_{j=0}^{N}$ be the nodes and weights of the Legendre-Gauss-Lobatto interpolation. Indeed, $\hat{x}_{j}$ are the zeros of $\left(1-x^{2}\right) \partial_{x} L_{N}(x)$, and $\hat{\omega}_{j}=2 / N(N+1) L_{N}^{2}\left(\hat{x}_{j}\right)$. By Szegö [12]

$$
\begin{equation*}
(\phi, \psi)_{\hat{I}}=(\phi, \psi)_{N, \hat{I}}:=\sum_{j=0}^{M} \phi\left(\hat{x}_{j}\right) \psi\left(\hat{x}_{j}\right) \hat{\omega}_{j}, \quad \forall \phi \cdot \psi \in \mathscr{P}_{2 N-1}(\hat{I}) \tag{5}
\end{equation*}
$$

For any $v \in C(\overline{\hat{I}})$, the Legendre-Gauss-Lobatto interpolant $\hat{\mathscr{I}}_{L, N} v(x) \in \mathscr{P}_{N}(\hat{I})$, satisfying

$$
\begin{equation*}
\left(\hat{\mathscr{I}}_{L, N} v\right)\left(\hat{x}_{j}\right)=v\left(\hat{x}_{j}\right), \quad 0 \leqslant j \leqslant N . \tag{6}
\end{equation*}
$$

Now, we describe the combined pseudospectral method. For clearness, we focus on the simple model problem

$$
\begin{align*}
-\partial_{x}^{2} U(x)+\lambda U(x) & =f(x), \quad \lambda>0, \quad x \in \Lambda \\
\lim _{x \rightarrow \infty} U(x)=U(0) & =0 \tag{7}
\end{align*}
$$

We may derive a weighted variational formulation of (7), and then solve it by the standard Laguerre spectral method as in [6]. But in this paper, we prefer a natural (not weighted) variational formulation, and solve it by using the Laguerre functions as in [11], which is easier to match the multidomain Legendre pseudospectral approximation for refining numerical results. To do this, let $a_{\Lambda}(u, v)=\left(\partial_{x} u, \partial_{x} v\right)_{\Lambda}+$ $\lambda(u, v)_{\Lambda}$. A weak formulation of (7) is to find $U(x) \in{ }_{0} H^{1}(\Lambda)$ such that

$$
\begin{equation*}
a_{\Lambda}(U, v)=(f, v)_{\Lambda}, \quad \forall v \in{ }_{0} H^{1}(\Lambda) \tag{8}
\end{equation*}
$$

If $f(x) \in H^{-1}(\Lambda)$, then (8) has a unique solution $U(x) \in{ }_{0} H^{1}(\Lambda)$.
Next, Let $a_{M, \Lambda}(u, v)=\left(\partial_{x} u, \partial_{x} v\right)_{M, \Lambda}+\lambda(u, v)_{M, \Lambda}$. The numerical solution $u_{M}^{0}(x) \in{ }_{0} \mathscr{S}_{M}(\Lambda)$ is uniquely determined by

$$
\begin{equation*}
a_{M, \Lambda}\left(u_{M}^{0}, \phi\right)=(f, \phi)_{M, \Lambda}, \quad \phi \in{ }_{0} \mathscr{S}_{M}(\Lambda) . \tag{9}
\end{equation*}
$$

We shall use the multidomain Legendre pseudospectral method to refine $u_{M}^{0}(x)$. Let $\Lambda_{k}=\left(a_{k}, b_{k}\right) \subset$ $(0, \infty), 1 \leqslant k \leqslant K$, such that $\Lambda_{k} \cap \Lambda_{l}=\varnothing$ for $k \neq l$. The endpoints $a_{k}$ and $b_{k}$ are some Laguerre-Gauss-Radau nodes usually. But in this case, the lengths of $\left(a_{k}, b_{k}\right)$ might be very different, since the distribution of the interpolation nodes is not uniform, see [12]. In particular, for large $M$, the length of ( $a_{k}, b_{k}$ ) increases very fast as $k$ increases. Thus, for refining numerical results on $\Lambda_{k}$, we should split $\Lambda_{k}$ into $J_{k}$ subintervals $I_{j}^{k}=\left(d_{j-1}^{k}, d_{j}^{k}\right)$ where $a_{k}=d_{0}^{k}<d_{1}^{k}<\cdots<d_{J_{k}-1}^{k}<d_{J_{k}}^{k}=b_{k}$. Their lengths $h_{j}^{k}=d_{j}^{k}-d_{j-1}^{k}$.

In each $I_{j}^{k}$, we use the Legendre-Gauss-Lobatto interpolation with the nodes $x_{i}^{k, j}=\left(h_{j}^{k} / 2\right) \hat{x}_{i}+\left(d_{j-1}^{k}+\right.$ $\left.d_{j}^{k}\right) / 2$ and the weights $\omega_{i}^{k, j}=\left(2 / h_{j}^{k}\right) \hat{\omega}_{i}, \quad 0 \leqslant i \leqslant N_{j}^{k}$. For any $u, v \in C\left(\bar{\Lambda}_{k}\right)$, we define the following discrete inner product and norm on each $I_{j}^{k}$,

$$
(u, v)_{N_{j}^{k}, I_{j}^{k}}=\sum_{i=0}^{N_{j}^{k}} u^{k, j}\left(x_{i}^{k, j}\right) v^{k, j}\left(x_{i}^{k, j}\right) \omega_{i}^{k, j}, \quad\|v\|_{N_{j}^{k}, I_{j}^{k}}=(v, v)_{N_{j}^{k}, I_{j}^{k}}^{1 / 2}
$$

Further, let $\mathbf{N}_{k}=\left(N_{1}^{k}, N_{2}^{k}, \ldots, N_{J_{k}}^{k}\right)$, and define the discrete inner product and norm on $\Lambda_{k}$ by

$$
(u, v)_{\mathbf{N}_{k}, \Lambda_{k}}=\sum_{j=1}^{J_{k}}(u, v)_{N_{j}^{k}, I_{j}^{k}}, \quad\|v\|_{\mathbf{N}_{k}, \Lambda_{k}}=(v, v)_{\mathbf{N}_{k}, \Lambda_{k}}^{1 / 2} .
$$

For any fixed $\Lambda_{k}$, we take the approximation spaces as

$$
\mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right)=\left\{v \in H^{1}\left(\Lambda_{k}\right) \mid v^{k, j} \in \mathscr{P}_{N_{j}}\left(I_{j}^{k}\right), \quad 1 \leqslant j \leqslant J_{k}\right\}, \quad \mathscr{V}_{\mathbf{N}_{k}}^{0}\left(\Lambda_{k}\right)=\mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right) \cap H_{0}^{1}\left(\Lambda_{k}\right)
$$

Let $v^{k, j}:=\left.v\right|_{I_{j}^{k}}$. For any $v \in C\left(\bar{\Lambda}_{k}\right)$, the multidomain Legendre-Gauss-Lobatto interpolant $\mathscr{I}_{L, \mathbf{N}_{k}} v(x) \in$ $\mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right)$ is defined by

$$
\begin{equation*}
\left.\left(\mathscr{I}_{L, \mathbf{N}_{k}} v\right)\right|_{I_{j}^{k}}\left(x_{i}^{k, j}\right)=v^{k, j}\left(x_{i}^{k, j}\right), \quad 0 \leqslant i \leqslant N_{j}^{k} \tag{10}
\end{equation*}
$$

If $\left.\phi \cdot \psi\right|_{I_{j}^{k}} \in \mathscr{P}_{N_{j}-1}\left(I_{j}^{k}\right), \quad 1 \leqslant j \leqslant J_{k}$, then by (5),

$$
\begin{equation*}
(\phi, \psi)_{\mathbf{N}_{k}, \Lambda_{k}}=(\phi, \psi)_{\Lambda_{k}}, \quad \forall \phi \cdot \psi \in \mathscr{V}_{2 \mathbf{N}_{k}-1}\left(\Lambda_{k}\right) . \tag{11}
\end{equation*}
$$

We now derive the algorithm for refining numerical results. Let $U^{k}:=\left.U\right|_{\Lambda_{k}}, f^{k}:=\left.f\right|_{\Lambda_{k}}$ and

$$
W^{k}(x)=\frac{b_{k}-x}{b_{k}-a_{k}} U^{k}\left(a_{k}\right)+\frac{x-a_{k}}{b_{k}-a_{k}} U^{k}\left(b_{k}\right), \quad x \in \Lambda_{k} .
$$

Due to (7), $U^{k}(x) \in H^{1}\left(\Lambda_{k}\right)$ satisfies

$$
\begin{equation*}
a_{\Lambda_{k}}\left(U^{k}, v\right)=\left(f^{k}, v\right)_{\Lambda_{k}}, \quad \forall v \in H_{0}^{1}\left(\Lambda_{k}\right) \tag{12}
\end{equation*}
$$

Equivalently, it is to seek $U_{*}^{k}(x)=U^{k}(x)-W^{k}(x) \in H_{0}^{1}\left(\Lambda_{k}\right)$ such that

$$
\begin{equation*}
a_{\Lambda_{k}}\left(U_{*}^{k}, v\right)=\left(f^{k}, v\right)_{\Lambda_{k}}-a_{\Lambda_{k}}\left(W^{k}, v\right), \quad \forall v \in H_{0}^{1}\left(\Lambda_{k}\right) \tag{13}
\end{equation*}
$$

Next, let $u_{\mathbf{N}_{k}}^{k}$ be the approximation of $U^{k}$ on $\Lambda_{k}$, and

$$
w_{M}^{k}(x)=\frac{b_{k}-x}{b_{k}-a_{k}} u_{M}^{0}\left(a_{k}\right)+\frac{x-a_{k}}{b_{k}-a_{k}} u_{M}^{0}\left(b_{k}\right), \quad x \in \Lambda_{k}
$$

Let $a_{\mathbf{N}_{k}, \Lambda_{k}}(u, v)=\left(\partial_{x} u, \partial_{x} v\right)_{\mathbf{N}_{k}, \Lambda_{k}}+\lambda(u, v)_{\mathbf{N}_{k}, \Lambda_{k}}$. Then the multidomain Legendre pseudospectral approximation of (12) is to find $u_{\mathbf{N}_{k}}^{k}(x) \in \mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right)$ such that

$$
\begin{equation*}
a_{\mathbf{N}_{k}, \Lambda_{k}}\left(u_{\mathbf{N}_{k}}^{k}, \phi\right)=\left(f^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}, \quad \forall \phi \in \mathscr{V}_{\mathbf{N}_{k}}^{0}\left(\Lambda_{k}\right) . \tag{14}
\end{equation*}
$$

Equivalently, it is to seek $u_{\mathbf{N}_{k}, *}^{k}(x)=u_{\mathbf{N}_{k}}^{k}(x)-w_{M}^{k}(x) \in \mathscr{V}_{\mathbf{N}_{k}}^{0}\left(\Lambda_{k}\right)$ such that

$$
\begin{equation*}
a_{\mathbf{N}_{k}, \Lambda_{k}}\left(u_{\mathbf{N}_{k}, *}^{k}, \phi\right)=B_{\mathbf{N}_{k}, \Lambda_{k}}(f, \phi), \quad \forall \phi \in \mathscr{V}_{\mathbf{N}_{k}}^{0}\left(\Lambda_{k}\right), \tag{15}
\end{equation*}
$$

where $B_{\mathbf{N}_{k}, \Lambda_{k}}(f, \phi):=\left(f^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(w_{M}^{k}, \phi\right)$.

## 3. Implementation and numerical results

We first consider the implementation of (9). Let $\hat{\mathscr{L}}_{l}(x)=\omega^{1 / 2}(x) \mathscr{L}_{l}(x)$ and $\hat{\phi}_{l}(x)=\hat{\mathscr{L}}_{l}(x)-\hat{\mathscr{L}}_{l+1}(x)$. Since $\hat{\mathscr{L}}_{l}(0)=1$ for all $l$, we have that ${ }_{0} \mathscr{S}_{M}(\Lambda)=\operatorname{span}\left\{\hat{\phi}_{0}, \hat{\phi}_{1}, \ldots, \hat{\phi}_{M-1}\right\}$.

Now, let $u_{M}^{0}(x)=\sum_{l=0}^{M-1} \hat{u}_{l}^{0} \hat{\phi}_{l}(x)$ and $\mathbf{u}=\left(\hat{u}_{0}^{0}, \hat{u}_{1}^{0}, \ldots, \hat{u}_{M-1}^{0}\right)^{\mathrm{T}}$. Also set $\mathbf{A}=\left(a_{i l}\right)_{i, l=0,1, \ldots, M-1}, \mathbf{B}=$ $\left(b_{i l}\right)_{i, l=0,1, \ldots, M-1}$ and $\mathbf{r}=\left(r_{0}, r_{1}, \ldots, r_{M-1}\right)^{\mathrm{T}}$, with the elements $a_{i l}=\left(\partial_{x} \hat{\phi}_{l}, \partial_{x} \hat{\phi}_{i}\right)_{\Lambda}, b_{i l}=\left(\hat{\phi}_{l}, \hat{\phi}_{i}\right)_{A}$ and $r_{i}=\left(f, \hat{\phi}_{i}\right)_{M, \Lambda}$. Due to (1), we have $\partial_{x} \hat{\phi}_{l}(x)=\frac{1}{2}\left(\hat{\mathscr{L}}_{l}(x)+\hat{\mathscr{L}}_{l+1}(x)\right)$. This with the orthogonality of Laguerre polynomials leads to $a_{i l}=a_{l i}=\frac{1}{2} \delta_{i, l}+\frac{1}{4} \delta_{i-1, l}$. It is easy to show that $b_{i l}=b_{l i}=2 \delta_{i, l}-\delta_{i-1, l}$. Finally, we obtain the matrix form of scheme (9) as $(\mathbf{A}+\lambda \mathbf{B}) \mathbf{u}=\mathbf{r}$. This system is symmetric and tridiagonal.

We now turn to the implementation of (15). As in [8], we let $\mathscr{P}_{N_{j}^{k}}^{0}\left(I_{j}^{k}\right):=\mathscr{P}_{N_{j}^{k}}\left(I_{j}^{k}\right) \cap H_{0}^{1}\left(I_{j}^{k}\right)$, and $\widetilde{L}_{l}^{k, j}(x)=L_{l}(\hat{x})$ where $\hat{x}=\left(2 / h_{j}^{k}\right) x-\left(d_{j-1}^{k}+d_{j}^{k}\right) / h_{j}^{k}, \quad x \in I_{j}^{k}$. We define the local polynomials such that for $1 \leqslant k \leqslant K, \quad 1 \leqslant j \leqslant J_{k}$ and $0 \leqslant l \leqslant N_{j}-2$,

$$
\widetilde{\phi}_{l}^{k, j}(x)= \begin{cases}c_{l}^{k, j}\left(\widetilde{L}_{l}^{k, j}(x)-\widetilde{L}_{l+2}^{k, j}(x)\right), & x \in \bar{I}_{j}^{k}, \quad c_{l}^{k, j}=\frac{1}{2} \sqrt{h_{j}^{k} /(2 l+3)}  \tag{16}\\ 0, & x \in \Lambda_{k} \backslash \bar{I}_{j}^{k}\end{cases}
$$

Clearly, the set $\left\{\tilde{\phi}_{l}^{k, j}\right\}_{l=0}^{N_{j}^{k}-2}$ forms a basis of $\mathscr{P}_{N_{j}^{k}}^{0}\left(I_{j}^{k}\right)$. Next, we define the subinterval matching functions as $\widetilde{\psi}^{k, 0}(x)=\widetilde{\psi}^{k, J_{k}}(x) \equiv 0$, and for $1 \leqslant j \leqslant J_{k}-1$,

$$
\widetilde{\psi}^{k, j}(x)= \begin{cases}\left(\widetilde{L}_{0}^{k, j}(x)+\widetilde{L}_{1}^{k, j}(x)\right) / 2, & x \in \bar{I}_{j}^{k}  \tag{17}\\ \left(\widetilde{L}_{0}^{k, j+1}(x)-\widetilde{L}_{1}^{k, j+1}(x)\right) / 2, & x \in \bar{I}_{j+1}^{k} \\ 0, & x \in \Lambda_{k} \backslash \bar{I}_{j}^{k} \cup \bar{I}_{j+1}^{k}\end{cases}
$$

Obviously $\tilde{\psi}^{k, j}\left(d_{j}^{k}\right)=1$ and $\operatorname{supp} \tilde{\psi}^{k, j} \subset\left[d_{j-1}^{k}, d_{j+1}^{k}\right]$.

Now let $\eta_{j}^{k}=u_{\mathbf{N}_{k}, *}^{k}\left(d_{j}^{k}\right)$. We expand the numerical solution $u_{\mathbf{N}_{k}, *}^{k}(x)$ as

$$
\begin{equation*}
u_{\mathbf{N}_{k}, *}^{k}(x)=\sum_{j=1}^{J_{k}} v^{k, j}(x)+\sum_{j=1}^{J_{k}-1} \eta_{j}^{k} \widetilde{\psi}^{k, j}(x), \quad v^{k, j}(x)=\sum_{l=0}^{N_{j}^{k}-2} \hat{v}_{l}^{k, j} \widetilde{\phi}_{l}^{k, j}(x) \tag{18}
\end{equation*}
$$

Taking $\phi=\widetilde{\phi}_{i}^{k, j}, 0 \leqslant i \leqslant N_{j}^{k}-2$ in (15) and using (16)-(18), we obtain that

$$
\begin{equation*}
a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v^{k, j}, \widetilde{\phi}_{i}^{k, j}\right)=B_{\mathbf{N}_{k}, \Lambda_{k}}\left(f, \widetilde{\phi}_{i}^{k, j}\right)-\eta_{j-1}^{k} a_{\mathbf{N}_{k}, \Lambda_{k}}\left(\widetilde{\psi}^{k, j-1}, \widetilde{\phi}_{i}^{k, j}\right)-\eta_{j}^{k} a_{\mathbf{N}_{k}, \Lambda_{k}}\left(\widetilde{\psi}^{k, j}, \widetilde{\phi}_{i}^{k, j}\right) . \tag{19}
\end{equation*}
$$

System (19) is the basic algorithm for refining numerical results in which all unknown variables are coupled. So it is not convenient for actual computation. To remedy this deficiency, we split this system into four subsystems. Firstly, we try to find the three functions, $v_{m}^{k, j}(x)=\sum_{l=0}^{N_{j}^{k}-2} \hat{v}_{m, l}^{k, j} \widetilde{\phi}_{l}^{k, j}(x), m=1,2,3$, such that

$$
\begin{align*}
& a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v_{1}^{k, j}, \widetilde{\phi}_{i}^{k, j}\right)=B_{\mathbf{N}_{k}, \Lambda_{k}}\left(f, \widetilde{\phi}_{i}^{k, j}\right), \quad 0 \leqslant i \leqslant N_{j}^{k}-2,  \tag{20}\\
& a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v_{2}^{k, j}, \widetilde{\phi}_{i}^{k, j}\right)=-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(\widetilde{\psi}^{k, j-1}, \widetilde{\phi}_{i}^{k, j}\right), \quad 0 \leqslant i \leqslant N_{j}^{k}-2,  \tag{21}\\
& a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v_{3}^{k, j}, \widetilde{\phi}_{i}^{k, j}\right)=-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(\widetilde{\psi}^{k, j}, \widetilde{\phi}_{i}^{k, j}\right), \quad 0 \leqslant i \leqslant N_{j}^{k}-2 . \tag{22}
\end{align*}
$$

Obviously Eqs. (20)-(22) can be solved separately in each subinterval $I_{j}^{k}$. In other words, all the computations can be carried out in parallel. Moreover by (19)-(22),

$$
\begin{equation*}
v^{k, j}(x)=v_{1}^{k, j}(x)+\eta_{j-1}^{k} v_{2}^{k, j}(x)+\eta_{j}^{k} v_{3}^{k, j}(x), \quad 1 \leqslant j \leqslant J_{k} \tag{23}
\end{equation*}
$$

Inserting (23) into (18), we deduce that

$$
\begin{equation*}
u_{\mathbf{N}_{k}, *}^{k}(x)=\sum_{j=1}^{J_{k}-1}\left(v_{2}^{k, j+1}(x)+v_{3}^{k, j}(x)+\widetilde{\psi}^{k, j}(x)\right) \eta_{j}^{k}+\sum_{j=1}^{J_{k}} v_{1}^{k, j}(x) \tag{24}
\end{equation*}
$$

Thus it remains to evaluate the unknown values $\left\{\eta_{j}^{k}\right\}_{j=1}^{J_{k}-1}$. For this purpose, we take $\phi=\widetilde{\psi}^{k, i}, 1 \leqslant i \leqslant J_{k}-1$ in (15), and obtain that

$$
\begin{equation*}
\sum_{j=1}^{J_{k}-1} a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v_{2}^{k, j+1}+v_{3}^{k, j}+\widetilde{\psi}^{k, j}, \widetilde{\psi}^{k, i}\right) \eta_{j}^{k}=B_{\mathbf{N}_{k}, \Lambda_{k}}\left(f, \widetilde{\psi}^{k, l}\right)-\sum_{j=1}^{J_{k}} a_{\mathbf{N}_{k}, \Lambda_{k}}\left(v_{1}^{k, j}, \widetilde{\psi}^{k, i}\right) \tag{25}
\end{equation*}
$$

Finally we resolve (25) and obtain the refined solution $u_{\mathbf{N}_{k}, *}^{k}(x)$ by (24).
We can also derive the simple matrix forms for subsystems (20)-(22).
We now present some numerical results. We take the exact solution $U(x)=\mathrm{e}^{-\left(x-\xi_{0}\right)^{2} / h_{0}}$ with $h_{0}>0$. As shown in Fig. 1, the interested information of $U(x)$ is mainly contained in the subinterval $\Delta\left(\xi_{0}, h_{0}\right)=$ $\left[\xi_{0}-3 \sqrt{h_{0} / 2}, \xi_{0}+3 \sqrt{h_{0} / 2}\right]$.


Fig. 1. $U(x)$ vs. $u_{32}^{0}(x)$.

We take $\lambda=1, \xi_{0}=95$ and $h_{0}=20$, and use (9) with $M=32$ to solve (7) numerically. In Fig. 1, we plot the exact solution $U(x)$ by solid line, the numerical solution $u_{M}^{0}(x)$ by dash dotted line, and the numerical solution at the interpolation nodes by ' $o$ '. Clearly $u_{M}^{0}(x)$ fits $U(x)$ very well at the nodes. But the errors are very big for large $x$ which are not nodes. So the pure modified Laguerre pseudospectral method can not identify the interested structure of exact solution in detail.

We next refine the numerical solution $u_{M}^{0}(x)$. Since the numerical gradients between $x_{25}$ and $x_{32}$ are very big, we take $K=1, J_{k}=1, a_{1}=x_{25}$ and $b_{1}=x_{32}$. We use (15) with $N_{1}^{1}=32$ to solve (13) in the subinterval $\left[a_{1}, b_{1}\right]$, with the approximate boundary values $u_{M}^{0}\left(a_{1}\right)$ and $u_{M}^{0}\left(b_{1}\right)$. In Fig. 2, we plot the exact solution $U(x)$ by solid line, the numerical solution $u_{M}^{0}(x)$ by dash dotted line, and the refined numerical solution $u_{32}^{1}(x)$ by dotted line. Clearly, the refined numerical result fits the exact solution very well, even for large $x$ which are not the interpolation nodes. In fact we cannot distinguish $U(x)$ and $u_{32}^{1}(x)$ in Fig. 2.

The proposed method is also suitable for solutions with several peaks. For instance, we consider the exact solution as $U(x)=\sum_{j=0}^{3} \mathrm{e}^{-\left(x-\xi_{j}\right)^{2} / h_{j}}, h_{j}>0$. We take $\xi_{0}=20, \xi_{1}=100, \xi_{2}=140, \xi_{3}=180$ and $h_{j}=10, j=0,1,2,3$, and use (9) with $M=64$ to solve (7) numerically. In Fig. 3, we plot the exact solution $U(x)$ by solid line, the numerical solution $u_{M}^{0}(x)$ by dash dotted line, and the numerical solution at interpolation nodes by ' $o$ '. Fig. 3 indicates again that for large $x$, the numerical solution only fits the exact solution well at the nodes.

We now refine the numerical solution $u_{64}^{0}(x)$. Since the numerical gradients and residuals are very big between $x_{44}$ and $x_{64}$, we take $K=1, J_{k}=2, a_{1}=x_{44}, b_{1}=x_{64}, d_{0}^{1}=a_{1}, d_{1}^{1}=\left(b_{1}-a_{1}\right) / 2$ and $d_{2}^{1}=b_{1}$. We use (15) with $N_{1}^{1}=N_{2}^{1}=45$ to resolve (13) in the subinterval [ $a_{1}, b_{1}$ ], with the approximate


Fig. 2. $U(x), u_{32}^{0}(x)$ vs. $u_{32}^{1}(x)$.


Fig. 3. $U(x)$ vs. $u_{64}^{0}(x)$.


Fig. 4. $U(x)$ vs. $u_{45,45}^{1}(x)$.
boundary values $u_{64}^{0}\left(a_{1}\right)$ and $u_{64}^{0}\left(b_{1}\right)$. In Fig. 4, we plot the exact solution $U(x)$ by solid line, and the refined numerical solution $u_{45,45}^{1}(x)$ by dash dotted line. Obviously, the refined numerical solution fits the exact solution very well.

## 4. Some approximation results

### 4.1. Modified Laguerre interpolation

We first consider the orthogonal projection $P_{M}: L_{\omega}^{2}(\Lambda) \rightarrow \mathscr{P}_{M}(\Lambda)$, defined by

$$
\left(P_{M} v-v, \phi\right)_{\omega, \Lambda}=0, \quad \forall \phi \in \mathscr{P}_{M}(\Lambda)
$$

To describe approximation result, we introduce the weighted Sobolev space $A^{r}(\Lambda)$. Let $\omega_{r}(x)=x^{r} \mathrm{e}^{-x}$. For integer $r \geqslant 0$, the semi-norm and norm of $A^{r}(\Lambda)$ are given by $|v|_{A^{r}, \Lambda}=\left\|\partial_{x}^{r} v\right\|_{\omega_{r}, \Lambda}$ and $\|v\|_{A^{r}, \Lambda}=$ $\left(\sum_{k=0}^{r}|v|_{A^{k}, \Lambda}^{2}\right)^{1 / 2}$, respectively. For real $r>0$, we define the space $A^{r}(\Lambda)$ and its norm by space interpolation. We can prove the following result in the same manner as in [13].

Theorem 1. If $v \in A^{r}(\Lambda)$ and integer $r \geqslant \mu \geqslant 0$, then $\left\|P_{M} v-v\right\|_{A^{\mu}, \Lambda} \leqslant c M^{(\mu-r) / 2}|v|_{A^{r}, \Lambda}$.
Theorem 2. If $v \in H_{\omega}^{1}(\Lambda), \partial_{x} v \in A^{r-1}(\Lambda)$ and integer $r \geqslant 1$, then

$$
\left|P_{M} v-v\right|_{1, \omega, \Lambda} \leqslant c M^{1-r / 2}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda} .
$$

Proof. We have

$$
\left|P_{M} v-v\right|_{1, \omega, \Lambda} \leqslant\left\|P_{M} \partial_{x} v-\partial_{x} v\right\|_{\omega, \Lambda}+\left\|P_{M} \partial_{x} v-\partial_{x} P_{M} v\right\|_{\omega, \Lambda} .
$$

By Theorem 1, $\left\|P_{M} \partial_{x} v-\partial_{x} v\right\|_{\omega, \Lambda} \leqslant c M^{1 / 2-r / 2}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda}$. Thus it suffices to estimate $\| P_{M} \partial_{x} v-$ $\partial_{x} P_{M} v \|_{\omega, \Lambda}$. Let $\partial_{x} v(x)=\sum_{l=0}^{\infty} \hat{\hat{v}}_{l} \mathscr{L}_{l}(x)$. By (1),

$$
\partial_{x} v(x)=\sum_{l=0}^{\infty} \hat{\hat{v}}_{l}\left(\partial_{x} \mathscr{L}_{l}(x)-\partial_{x} \mathscr{L}_{l+1}(x)\right)=\sum_{l=1}^{\infty}\left(\hat{\hat{v}}_{l}-\hat{\hat{v}}_{l-1}\right) \partial_{x} \mathscr{L}_{l}(x) .
$$

On the other hand, $\partial_{x} v(x)=\sum_{l=1}^{\infty} \hat{v}_{l} \partial_{x} \mathscr{L}_{l}(x)$. Therefore $\hat{v}_{l}=\hat{\hat{v}}_{l}-\hat{\hat{v}}_{l-1}$, and so $\hat{\hat{v}}_{l}=-\sum_{p=l+1}^{\infty} \hat{v}_{p}$. Moreover, (1) implies that $\partial_{x} \mathscr{L}_{l}(x)=-\sum_{p=0}^{l-1} \mathscr{L}_{p}(x)$. The above statements lead to that

$$
P_{M} \partial_{x} v(x)=-\sum_{l=0}^{M} \mathscr{L}_{l}(x)\left(\sum_{p=l+1}^{\infty} \hat{v}_{p}\right), \quad \partial_{x} P_{M} v(x)=-\sum_{l=0}^{M-1} \mathscr{L}_{l}(x)\left(\sum_{p=l+1}^{M} \hat{v}_{p}\right) .
$$

Thus

$$
P_{M} \partial_{x} v(x)-\partial_{x} P_{M} v(x)=-\left(\sum_{l=0}^{M} \mathscr{L}_{l}(x)\right)\left(\sum_{p=M+1}^{\infty} \hat{v}_{p}\right)=\hat{\hat{v}}_{M} \sum_{l=0}^{M} \mathscr{L}_{l}(x) .
$$

Accordingly, we use Theorem 1 to obtain that

$$
\left\|P_{M} \partial_{x} v-\partial_{x} P_{M} v\right\|_{\omega, \Lambda}^{2}=\hat{\hat{v}}_{M}^{2} \sum_{l=0}^{M}\left\|\mathscr{L}_{l}\right\|_{\omega, \Lambda}^{2} \leqslant c M \hat{\hat{v}}_{M}^{2} \leqslant c M\left\|P_{M-1} v-v\right\|_{\omega, \Lambda}^{2} \leqslant c M^{2-r}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda}^{2}
$$

This completes the proof.
We next consider the orthogonal projection ${ }_{0} P_{M}^{1}:{ }_{0} H_{\omega}^{1}(\Lambda) \rightarrow{ }_{0} \mathscr{P}_{M}(\Lambda)$, defined by

$$
\left(\partial_{x}\left({ }_{0} P_{M}^{1} v-v\right), \partial_{x} \phi\right)_{\omega, \Lambda}=0, \quad \forall \phi \in_{0} \mathscr{P}_{M}(\Lambda) .
$$

Lemma 1. If $v \in{ }_{0} H^{1}(\Lambda), \partial_{x} v \in A^{r-1}(\Lambda)$ and integer $r \geqslant 1$, then

$$
\left\|\left\|_{0} P_{M}^{1} v-v\right\|_{1, \omega, \Lambda} \leqslant c M^{1 / 2-r / 2}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda} .\right.
$$

Proof. Let $\phi(x)=\int_{0}^{x} P_{M-1} \partial_{y} v(y) \mathrm{d} y$. By projection theorem and Theorem 1,

$$
\left.\right|_{0} P_{M}^{1} v-\left.v\right|_{1, \omega, \Lambda} \leqslant|\phi-v|_{1, \omega, \Lambda}=\left\|P_{M-1} \partial_{x} v-\partial_{x} v\right\|_{\omega, \Lambda} \leqslant c M^{1 / 2-r / 2}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda} .
$$

According to Lemma 2.2 of [6], $\|v\|_{\omega, \Lambda} \leqslant 2|v|_{1, \omega, \Lambda}$ for any $v \in{ }_{0} H_{\omega}^{1}(\Lambda)$. A combination of the above two estimates leads to the desired result.

We now derive an important result on the modified Laguerre approximation.

Theorem 3. For any $v \in{ }_{0} H^{1}(\Lambda)$, let ${ }_{0} \bar{P}_{M}^{1} v(x)=\mathrm{e}^{-x / 2}{ }_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v(x)\right)$. Then the projector ${ }_{0} \bar{P}_{M}^{1}$ : ${ }_{0} H^{1}(\Lambda) \rightarrow{ }_{0} \mathscr{S}_{M}(\Lambda)$ satisfies

$$
\begin{equation*}
\left(\partial_{x}\left({ }_{0} \bar{P}_{M}^{1} v-v\right), \partial_{x} \phi\right)_{\Lambda}+\frac{1}{4}\left({ }_{0} \bar{P}_{M}^{1} v-v, \phi\right)_{\Lambda}=0, \quad \forall \phi \in{ }_{0} \mathscr{S}_{M}(\Lambda) . \tag{26}
\end{equation*}
$$

Moreover, for integer $r \geqslant 1$,

$$
\begin{equation*}
\left\|_{0} \bar{P}_{M}^{1} v-v\right\|_{1, \Lambda} \leqslant c M^{1 / 2-r / 2}\left|\partial_{x}\left(\mathrm{e}^{x / 2} v\right)\right|_{A^{r-1}, \Lambda} \tag{27}
\end{equation*}
$$

Proof. Result (26) comes from Lemma 3.2 of [11]. Next, by Lemmas 1 and 2.2 of [6],

$$
\begin{align*}
\left.\right|_{0} \bar{P}_{M}^{1} v-\left.v\right|_{1, \Lambda} & \leqslant \frac{1}{2}\left\|_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v\right)-\mathrm{e}^{x / 2} v\right\|_{\omega, \Lambda}+\left.\right|_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v\right)-\left.\mathrm{e}^{x / 2} v\right|_{1, \omega, \Lambda} \\
& \leqslant\left. 2\right|_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v\right)-\left.\mathrm{e}^{x / 2} v\right|_{1, \omega, \Lambda} \leqslant c M^{1 / 2-r / 2}\left|\partial_{x}\left(\mathrm{e}^{x / 2} v\right)\right|_{A^{r-1}, \Lambda} \tag{28}
\end{align*}
$$

Furthermore, by the definition of ${ }_{0} \bar{P}_{M}^{1}$,

$$
\begin{equation*}
\left\|_{0} \bar{P}_{M}^{1} v-v\right\|_{\Lambda}=\left\|_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v\right)-\mathrm{e}^{x / 2} v\right\|_{\omega, \Lambda} \leqslant\left. 2\right|_{0} P_{M}^{1}\left(\mathrm{e}^{x / 2} v\right)-\left.\mathrm{e}^{x / 2} v\right|_{1, \omega, \Lambda} \tag{29}
\end{equation*}
$$

A combination of (28) and (29) leads to the desired result.
In the end of this subsection, we deal with the interpolations $\mathscr{I}_{\mathscr{L}, M}$ and $\tilde{\mathscr{I}}_{\mathscr{L}, M}$, respectively. We first study the stability of the interpolation $\mathscr{I}_{\mathscr{L}, M}$.

Lemma 2. For any $v \in H_{\omega}^{1}(\Lambda) \cap A^{1}(\Lambda)$,

$$
\left\|\mathscr{I}_{\mathscr{L}, M} v\right\|_{\omega, \Lambda} \leqslant c\left(\|v\|_{\omega, \Lambda}+\|v\|_{\omega, \Lambda}^{1 / 2}|v|_{1, \omega, \Lambda}^{1 / 2}+(\ln M)^{1 / 2}\|v\|_{A^{1}, \Lambda}\right)
$$

Proof. We have $\left\|\mathscr{I} \mathscr{L}_{, M} v\right\|_{\omega, \Lambda}^{2}=A_{M}+B_{M}$ where $A_{M}=\sum_{0 \leqslant x_{j} \leqslant 1} v^{2}\left(x_{j}\right) \omega_{j}$ and $B_{M}=\sum_{x_{j}>1} v^{2}\left(x_{j}\right) \omega_{j}$. As shown in the proof of Lemma 2.1 of [14], for any $v \in H_{\omega}^{1}(\Lambda)$,

$$
\left\|\mathrm{e}^{-x / 2} v\right\|_{L^{\infty}(\Lambda)} \leqslant\|v\|_{\omega, \Lambda}+\sqrt{2}\|v\|_{\omega, \Lambda}^{1 / 2}|v|_{1, \omega, \Lambda}^{1 / 2} .
$$

Since $\sum_{j=0}^{M} \omega_{j}=1$, we deduce that

$$
A_{M} \leqslant\|v\|_{L^{\infty}(0,1)}^{2} \sum_{0 \leqslant x_{j}<1} \omega_{j} \leqslant c\left\|\mathrm{e}^{-x / 2} v\right\|_{L^{\infty}(\Lambda)}^{2} \leqslant c\left(\|v\|_{\omega, \Lambda}^{2}+2\|v\|_{\omega, \Lambda}|v|_{1, \omega, \Lambda}\right)
$$

Next, let $\mathscr{L}_{l}^{(1)}(x)$ be the generalized Laguerre polynomial of degree $l$, i.e., $\mathscr{L}_{l}^{(1)}(x)=1 / l!\mathrm{e}^{x} / x \mathrm{~d}^{l} /$ $\mathrm{d} x^{l}\left(\mathrm{e}^{-x} x^{l+1}\right)$. Denote by $\sigma_{j}, 1 \leqslant j \leqslant M$ the zeros of $\mathscr{L}_{M}^{(1)}(x)$, arranged in increasing order. Since $\partial_{x} \mathscr{L}_{M+1}(x)=-\mathscr{L}_{M}^{(1)}(x)\left(\right.$ see [12]), we have $x_{j}=\sigma_{j}, \quad 1 \leqslant j \leqslant M$. Moreover by the properties of the

Laguerre-Gauss and Laguerre-Gauss-Radau quadratures (see [12]),

$$
\begin{aligned}
\omega_{j} & =\frac{1}{\left.\partial_{x}\left(x \partial_{x} \mathscr{L}_{M+1}(x)\right)\right|_{x=x_{j}}} \int_{\Lambda} \frac{x \partial_{x} \mathscr{L}_{M+1}(x)}{x-x_{j}} \omega(x) \mathrm{d} x \\
& =\frac{1}{\left.\partial_{x}\left(x \mathscr{L}_{M}^{(1)}(x)\right)\right|_{x=x_{j}}} \int_{\Lambda} \frac{x \mathscr{L}_{M}^{(1)}(x)}{x-x_{j}} \omega(x) \mathrm{d} x \\
& =\frac{1}{\sigma_{j} \partial_{x} \mathscr{L}_{M}^{(1)}\left(\sigma_{j}\right)} \int_{\Lambda} \frac{\mathscr{L}_{M}^{(1)}(x)}{x-\sigma_{j}} x \omega(x) \mathrm{d} x=\sigma_{j}^{-1} \rho_{j}
\end{aligned}
$$

By (2.4) and (2.7) of [10], we know that $\rho_{j} \leqslant c \sigma_{j}^{3 / 2} \mathrm{e}^{-\sigma_{j}}\left(4 M-\sigma_{j}\right)^{-1 / 2}$ and $\sigma_{j+1}-\sigma_{j} \cong \sigma_{j}^{1 / 2}\left(4 M-\sigma_{j}\right)^{-1 / 2}$.
Moreover, we have from Lemma 2.6 of [14] that $\left\|x^{1 / 2} \mathrm{e}^{-x / 2} v\right\|_{L^{\infty}(0, \infty)}^{2} \leqslant 2\|v\|_{A^{1}, \Lambda}^{2}$. Thus

$$
\begin{aligned}
B_{M} & =\sum_{x_{j}>1} v^{2}\left(x_{j}\right) \sigma_{j}^{-1} \rho_{j} \leqslant c \sum_{x_{j}>1} v^{2}\left(x_{j}\right) \sigma_{j}^{1 / 2} \mathrm{e}^{-\sigma_{j}}\left(4 M-\sigma_{j}\right)^{-1 / 2} \\
& \leqslant c\|v\|_{A^{1}, \Lambda}^{2} \sum_{x_{j}>1} \sigma_{j}^{-1 / 2}\left(4 M-\sigma_{j}\right)^{-1 / 2} \cong c\|v\|_{A^{1}, \Lambda}^{2} \sum_{x_{j}>1} \sigma_{j}^{-1}\left(\sigma_{j+1}-\sigma_{j}\right) \\
& \leqslant c\|v\|_{A^{1}, \Lambda}^{2} \int_{1}^{M} \sigma^{-1} \mathrm{~d} \sigma \leqslant c \ln M\|v\|_{A^{1}, \Lambda}^{2} .
\end{aligned}
$$

The proof is completed.
Theorem 4. If $v \in A^{r}(\Lambda), \partial_{x} v \in A^{r-1}(\Lambda)$ and integer $r \geqslant 1$, then for $0 \leqslant \mu \leqslant 1$,

$$
\|\mathscr{I} \mathscr{L}, M v-v\|_{\mu, \omega, \Lambda} \leqslant c(\ln M)^{1 / 2} M^{\mu+1 / 2-r / 2}\left(|v|_{A^{r}, \Lambda}+\left|\partial_{x} v\right|_{A^{r-1}, \Lambda}\right) .
$$

Proof. By Theorems 1 and 2 and Lemma 2,

$$
\begin{aligned}
\left\|\mathscr{I}_{\mathscr{L}, M} v-P_{M} v\right\|_{\omega, \Lambda}= & \| \mathscr{\mathscr { L }} \mathscr{L}^{\prime} M \\
\leqslant & c\left(\| P_{M} v-v\right)\left\|_{\omega, \Lambda} v-v\right\|_{\omega, \Lambda}+\left\|P_{M} v-v\right\|_{\omega, \Lambda}^{1 / 2}\left|P_{M} v-v\right|_{1, \omega, \Lambda}^{1 / 2} \\
& \left.+(\ln M)^{1 / 2}\left\|P_{M} v-v\right\|_{A^{1}, \Lambda}\right) \\
\leqslant & c\left(M^{-r / 2}|v|_{A^{r}, \Lambda}+M^{1 / 2-r / 2}|v|_{A^{r}, \Lambda}^{1 / 2}\left|\partial_{x} v\right|_{A^{r-1}, \Lambda}^{1 / 2}\right. \\
& \left.+(\ln M)^{1 / 2} M^{1 / 2-r / 2}|v|_{A^{r}, \Lambda}\right) .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathscr{I} \mathscr{L}, M v-v\|_{\omega, \Lambda} & \leqslant\left\|\mathscr{I} \mathscr{L}, M v-P_{M} v\right\|_{\omega, \Lambda}+\left\|P_{M}-v\right\|_{\omega, \Lambda} \\
& \leqslant c(\ln M)^{1 / 2} M^{1 / 2-r / 2}\left(|v|_{A^{r}, \Lambda}+\left|\partial_{x} v\right|_{A^{r-1}, \Lambda}\right) .
\end{aligned}
$$

Next, for any $\phi \in \mathscr{P}_{M}(\Lambda),\left\|\partial_{x} \phi\right\|_{\omega, \Lambda} \leqslant c M\|\phi\|_{\omega, \Lambda}$ (see [2]). This with Theorem 2 leads to that

$$
\begin{aligned}
\left|\mathscr{I}_{\mathscr{L}, M} v-v\right|_{1, \omega, \Lambda} & \leqslant \mid \mathscr{I} \mathscr{L}, M \\
& \leqslant c M(\| \mathscr{\mathscr { I }} \mathscr{L}, M \\
& \leqslant c(\ln M)^{1 / 2} M^{3 / 2-r / 2}\left(|v|_{A^{r}, \Lambda}+\left|\left.\right|_{\omega, \Lambda}+\left|P_{M} v\right|_{A^{r-1}, \Lambda}\right)\right.
\end{aligned}
$$

Finally we use space interpolation to complete the proof.
We now derive an important result on the modified Laguerre interpolation. We introduce the space $B^{r}(\Lambda)$. For integer $r \geqslant 0$, its norm is given by

$$
\|v\|_{B^{r}, \Lambda}=\left(\sum_{k=0}^{r}\left\|x^{(r-1) / 2}(x+1)^{1 / 2} \partial_{x}^{r} v\right\|_{\Lambda}^{2}\right)^{1 / 2}
$$

For any real $r>0$, we define the space $B^{r}(\Lambda)$ by space interpolation.
Theorem 5. For any $v \in B^{r}, r \geqslant 1$ and $0 \leqslant \mu \leqslant 1$,

$$
\begin{equation*}
\left\|\tilde{\mathscr{I}}_{\mathscr{L}, M} v-v\right\|_{\mu, \Lambda} \leqslant c(\ln M)^{1 / 2} M^{\mu+1 / 2-r / 2}\|v\|_{B^{r}, \Lambda} \tag{30}
\end{equation*}
$$

Proof. Let $u(x)=\mathrm{e}^{x / 2} v(x)$. Then $\tilde{\mathscr{I}}_{\mathscr{L}, M} v(x)=\mathrm{e}^{-x / 2} \mathscr{I}_{\mathscr{L}, M} u(x)$. By Theorem 4,

$$
\begin{aligned}
\left\|\tilde{\mathscr{I}}_{\mathscr{L}, M} v-v\right\|_{\Lambda} & =\|\mathscr{\mathscr { }} \mathscr{\mathscr { Q } , M} \bar{u}-u\|_{\omega, \Lambda} \leqslant c(\ln M)^{1 / 2} M^{1 / 2-r / 2}\left(|u|_{A^{r}, \Lambda}+\left|\partial_{x} u\right|_{A^{r-1}, \Lambda}\right) \\
& \leqslant c(\ln M)^{1 / 2} M^{1 / 2-r / 2}\|v\|_{B^{r}, \Lambda}
\end{aligned}
$$

We can estimate $\left|\widetilde{\mathscr{I}}_{\mathscr{L}, M} v-v\right|_{1, \Lambda}$ similarly. Finally we use space interpolation to complete the proof.
In the error estimations, we need an imbedding inequality. In fact, for any $v \in H^{1}(\Lambda)$, we have that $v(x) \rightarrow 0$, a e as $x \rightarrow \infty$. Therefore

$$
\begin{equation*}
\sup _{x \in \bar{A}}|v(x)| \leqslant \sqrt{2}\|v\|_{\Lambda}^{1 / 2}|v|_{1, \Lambda}^{1 / 2} . \tag{31}
\end{equation*}
$$

### 4.2. Multidomain Legendre interpolation

We first derive some results on the Legendre approximation, which are more precise than those in the existing literatures. Let $\mathscr{P}_{N}^{0}(\hat{I})=\left\{v \in \mathscr{P}_{N}(\hat{I}) \mid v( \pm 1)=0\right\}$. We introduce the orthogonal projections $\hat{P}_{N}, \hat{P}_{N}^{1}$ and $\hat{P}_{N}^{1,0}$, defined by

$$
\begin{aligned}
& \left(\hat{P}_{N} v-v, \phi\right)_{\hat{I}}=0, \quad \forall v \in L^{2}(\hat{I}), \quad \phi \in \mathscr{P}_{N}(\hat{I}) \\
& \left(\partial_{x} \hat{P}_{N}^{1} v-\partial_{x} v, \partial_{x} \phi\right)_{\hat{I}}+\left(\hat{P}_{N}^{1} v-v, \phi\right)_{\hat{I}}=0, \quad \forall v \in H^{1}(\hat{I}), \quad \phi \in \mathscr{P}_{N}(\hat{I}), \\
& \left(\partial_{x} \hat{P}_{N}^{1,0} v-\partial_{x} v, \partial_{x} \phi\right)_{\hat{I}}=0, \quad v \in H_{0}^{1}(\hat{I}), \quad \phi \in \mathscr{P}_{N}^{0}(\hat{I}) .
\end{aligned}
$$

Lemma 3. If $v,\left(1-x^{2}\right)^{r / 2} \partial_{x}^{r} v \in L^{2}(\hat{I})$ and integer $r \geqslant 0$, then

$$
\begin{equation*}
\left\|\hat{P}_{N} v-v\right\|_{\hat{I}} \leqslant c N^{-r}\left\|\left(1-x^{2}\right)^{r / 2} \partial_{x}^{r} v\right\|_{\hat{I}} . \tag{32}
\end{equation*}
$$

Moreover, if $v \in H^{\mu}(\hat{I}),\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v \in L^{2}(\hat{I})$ and integer $r \geqslant 1$, then for $0 \leqslant \mu \leqslant 1$,

$$
\begin{equation*}
\left\|\hat{P}_{N}^{1} v-v\right\|_{\mu, \hat{I}} \leqslant c N^{\mu-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} \tag{33}
\end{equation*}
$$

Proof. The first result comes from [1,5]. Next, due to [5, Lemma 2.3], for any $v \in H^{1}(\hat{I})$ with $v(0)=0$, we have $\|v\|_{\hat{I}} \leqslant c|v|_{1, \hat{I}}$. Let $\phi(x)=\int_{-1}^{x} \hat{P}_{N-1} \partial_{y} v(y) \mathrm{d} y+\xi$, where $\xi$ is chosen in such a way that $v(0)=\phi(0)$. Then by projection theorem and (32),

$$
\left\|\hat{P}_{N}^{1} v-v\right\|_{1, \hat{I}} \leqslant\|\phi-v\|_{1, \hat{I}} \leqslant c|\phi-v|_{1, \hat{I}} \leqslant c\left\|\hat{P}_{N-1} \partial_{x} v-\partial_{x} v\right\|_{\hat{I}} \leqslant c N^{1-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} .
$$

This leads to (33) with $\mu=1$. Moreover, we can derive (33) with $\mu=0$ by the previous result and a duality argument as usual. Finally, result (33) with $0<\mu<1$ follows from space interpolation.

Lemma 4. If $v \in H_{0}^{1}(\hat{I}),\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v \in L^{2}(\hat{I})$ and integer $r \geqslant 1$, then for any $0 \leqslant \mu \leqslant 1$,

$$
\left\|\hat{P}_{N}^{1,0} v-v\right\|_{\mu, \hat{I}} \leqslant c N^{\mu-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}}
$$

Proof. We first prove the desired result with $\mu=1$. Let

$$
v_{N}^{0}(x)=\int_{-1}^{x}\left(\hat{P}_{N-1} \partial_{y} v(y)-\frac{1}{2} \int_{\hat{I}} \hat{P}_{N-1} \partial_{z} v(z) \mathrm{d} z\right) \mathrm{d} y .
$$

Clearly $v_{N}^{0} \in \mathscr{P}_{N}^{0}(\hat{I})$. By the Poincaré inequality, projection theorem and (32),

$$
\begin{align*}
\left\|\hat{P}_{N}^{1,0} v-v\right\|_{1, \hat{I}} & \leqslant c\left|\hat{P}_{N}^{1,0} v-v\right|_{1, \hat{I}} \leqslant c\left|v_{N}^{0}-v v_{N}^{0}\right|_{1, \hat{I}} \\
& \leqslant c\left(\left\|\hat{P}_{N-1} \partial_{x} v-\partial_{x} v\right\|_{\hat{I}}+\left|\int_{\hat{I}}\left(\hat{P}_{N-1} \partial_{x} v(x)-\partial_{x} v(x)\right) \mathrm{d} x\right|\right) \\
& \leqslant c\left\|\hat{P}_{N-1} \partial_{x} v-\partial_{x} v\right\|_{\hat{I}} \leqslant c N^{1-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} . \tag{34}
\end{align*}
$$

Next, using (34) and a duality argument as in the proof of Lemma 3.16 of [7], we reach that $\| \hat{P}_{N}^{1,0} v-$ $v\left\|_{\hat{I}} \leqslant c N^{-r}\right\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v \|_{\hat{I}}$. Finally, the desired result follows from the above statements and space interpolation.

Lemma 5. If $v \in H^{1}(\hat{I}),\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v \in L^{2}(\hat{I})$ and integer $r \geqslant 1$, then

$$
\left\|\hat{\mathscr{I}}_{L, N} v-v\right\|_{\hat{I}} \leqslant c N^{-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} .
$$

Proof. Let

$$
v_{N}^{*}(x)=\hat{P}_{N}^{1,0}\left(v(x)-v^{*}(x)\right)+v^{*}(x), \quad v^{*}(x)=\frac{1-x}{2} v(-1)+\frac{1+x}{2} v(1)
$$

Clearly, $v_{N}^{*}(x) \in \mathscr{P}_{N}(\hat{I})$ and $v_{N}^{*}( \pm 1)=v( \pm 1)$. Moreover, $\left|v^{*}\right|_{1, \hat{I}}=\frac{1}{2}|v(-1)-v(1)| \leqslant c|v|_{1, \hat{I}}$. Thus, we have from Lemma 4 that for $0 \leqslant \mu \leqslant 1 \leqslant r$,

$$
\begin{align*}
\left\|v_{N}^{*}-v\right\|_{\mu, \hat{I}} & =\left\|\hat{P}_{N}^{1,0}\left(v-v^{*}\right)-\left(v-v^{*}\right)\right\|_{\mu, \hat{I}} \\
& \leqslant c N^{\mu-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r}\left(v-v^{*}\right)\right\|_{\hat{I}} \leqslant c N^{\mu-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} \tag{35}
\end{align*}
$$

Since $\hat{\mathscr{I}}_{L, N} v_{N}^{*}=v_{N}^{*}$, we use Lemma 4.9 of [7] and (35) to obtain that

$$
\begin{aligned}
\left\|\hat{\mathscr{I}}_{L, N} v-v_{N}^{*}\right\|_{\hat{I}} & =\left\|\hat{\mathscr{I}}_{L, N}\left(v_{N}^{*}-v\right)\right\|_{\hat{I}} \leqslant c\left(\left\|v_{N}^{*}-v\right\|_{\hat{I}}+N^{-1}\left\|\left(1-x^{2}\right)^{1 / 2} \partial_{x}\left(v_{N}^{*}-v\right)\right\|_{\hat{I}}\right) \\
& \leqslant c N^{-r}\left\|\left(1-x^{2}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{\hat{I}} .
\end{aligned}
$$

The above with (35) leads to the desired result.
Finally, we present the main results of this subsection. Define the affine mapping

$$
\begin{equation*}
v(x)=\hat{v}(\hat{x}), \quad x=\frac{h_{j}^{k}}{2} \hat{x}+\frac{d_{j-1}^{k}+d_{j}^{k}}{2}, \quad \hat{x} \in \hat{I}, \quad x \in I_{j}^{k} . \tag{36}
\end{equation*}
$$

Let $\chi_{j}^{k}(x)=\frac{1}{4}\left(x-d_{j-1}^{k}\right)\left(d_{j}^{k}-x\right)\left(h_{j}^{k}\right)^{-2} \leqslant 1$. By (36), we have that $1-\hat{x}^{2}=\chi_{j}^{k}(x)$ and $\partial_{\hat{x}} \hat{v}(\hat{x})=\frac{1}{2} h_{j}^{k} \partial_{x} v(x)$. Thus we have that

$$
\begin{equation*}
\left\|\left(1-\hat{x}^{2}\right)^{(r-1) / 2} \partial_{\hat{x}}^{r} \hat{v}\right\|_{\hat{I}} \leqslant c\left(h_{j}^{k}\right)^{r-1 / 2}\left\|\left(\chi_{j}^{k}\right)^{(r-1) / 2} \partial_{x}^{r} v\right\|_{I_{j}^{k}} . \tag{37}
\end{equation*}
$$

For description of the multidomain Legendre approximation, we introduce several piecewise Sobolev spaces. Let $v^{k, j}(x)=\left.v(x)\right|_{I_{j}^{k}}$, and define $\hat{H}^{1}\left(\Lambda_{k}\right)=\left\{v \mid v^{k, j}(x) \in H^{1}\left(I_{j}^{k}\right), \quad 1 \leqslant j \leqslant J_{k}\right\}$, equipped with the following norm and semi-norm

$$
\|v\|_{\hat{H}^{1}\left(A_{k}\right)}=\left(\sum_{j=0}^{J_{k}}\left\|v^{k, j}\right\|_{H^{1}\left(I_{j}^{k}\right)}^{2}\right)^{1 / 2}, \quad|v|_{\hat{H}^{1}\left(\Lambda_{k}\right)}=\left(\sum_{j=0}^{J_{k}}\left\|\partial_{x} v^{k, j}\right\|_{I_{j}^{k}}^{2}\right)^{1 / 2}
$$

We also define the spaces $\bar{H}^{r}\left(\Lambda_{k}\right)(r \geqslant 0)$ and $\widetilde{H}^{r}\left(\Lambda_{k}\right)(r \geqslant 1)$. For integer $r$, their semi-norms are given by

$$
|v|_{\bar{H}^{r}\left(\Lambda_{k}\right)}=\left(\sum_{j=1}^{J_{k}}\left\|\left(\chi_{j}^{k}\right)^{r / 2} \partial_{x}^{r} v^{k, j}\right\|_{I_{j}^{k}}^{2}\right)^{1 / 2}, \quad|v|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}=\left(\sum_{j=1}^{J_{k}}\left\|\left(\chi_{j}^{k}\right)^{(r-1) / 2} \partial_{x}^{r} v^{k, j}\right\|_{I_{j}^{k}}^{2}\right)^{1 / 2}
$$

For any real $r \geqslant 0$, we define these spaces by space interpolation.
Next, we introduce the operators $\widetilde{P}_{\mathbf{N}_{k}}: L^{2}\left(\Lambda_{k}\right) \rightarrow \mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right)$ and $\widetilde{P}_{\mathbf{N}_{k}}^{1}: \widetilde{H}^{1}\left(\Lambda_{k}\right) \rightarrow \mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right)$, defined by

$$
\left.\left(\widetilde{P}_{\mathbf{N}_{k}} v\right)\right|_{I_{j}^{k}}(x)=\left(\widetilde{P}_{\mathbf{N}_{k}} v\right)^{k, j}=\hat{P}_{N_{j}^{k}} \hat{v}^{k, j}(\hat{x}),\left.\quad\left(\widetilde{P}_{\mathbf{N}_{k}}^{1} v\right)\right|_{I_{j}^{k}}(x)=\left(\hat{P}_{N_{j}^{k}}^{1} \hat{v}^{k, j}\right)(\hat{x}), \quad 1 \leqslant j \leqslant J_{k}
$$

Let $\hbar_{k}=\max _{j \in ป_{k}}\left\{h_{j}^{k} \cdot\left(N_{j}^{k}\right)^{-1}\right\}$. We have the following important result.
Theorem 6. For any $v \in \bar{H}^{r}\left(\Lambda_{k}\right)$ and integer $r \geqslant 0$,

$$
\begin{equation*}
\left\|\widetilde{P}_{\mathbf{N}_{k}} v-v\right\|_{\Lambda_{k}} \leqslant c \hbar_{k}^{r}|v|_{\bar{H}^{r}\left(\Lambda_{k}\right)} . \tag{38}
\end{equation*}
$$

For any $v \in \tilde{H}^{r}\left(\Lambda_{k}\right)$ and integer $r \geqslant 1$,

$$
\begin{equation*}
\left\|\widetilde{P}_{\mathbf{N}_{k}}^{1} v-v\right\|_{\Lambda_{k}}+\hbar_{k}\left|\widetilde{P}_{\mathbf{N}_{k}}^{1} v-v\right|_{\hat{H}^{1}(\Lambda)} \leqslant c \hbar_{k}^{r}|v|_{\bar{H}^{r}\left(\Lambda_{k}\right)} . \tag{39}
\end{equation*}
$$

Proof. By using (32), (36) and (37), we verify that

$$
\begin{align*}
\left\|\widetilde{P}_{\mathbf{N}_{k}} v-v\right\|_{\Lambda_{k}}^{2} & =\sum_{j=1}^{J_{k}}\left\|\left(\widetilde{P}_{\mathbf{N}_{k}} v\right)^{k, j}-v^{k, j}\right\|_{I_{j}^{k}}^{2} \leqslant c \sum_{j=1}^{J_{k}} h_{j}^{k}\left\|\hat{P}_{N_{j}^{k}} \hat{v}^{k, j}-\hat{v}^{k, j}\right\|_{\hat{I}}^{2} \\
& \leqslant c \sum_{j=1}^{J} h_{j}^{k}\left(N_{j}^{k}\right)^{-2 r}\left\|\left(1-\hat{x}^{2}\right)^{r / 2} \partial_{\hat{x}}^{r} \hat{v}^{k, j}\right\|_{\hat{I}}^{2} \\
& \leqslant c \sum_{j=1}^{J} h_{j}^{k}\left(N_{j}^{k}\right)^{-2 r}\left(h_{j}^{k}\right)^{2 r-1}\left\|\left(\chi_{j}^{k}\right)^{r / 2} \partial_{x}^{r} v^{k, j}\right\|_{I_{j}^{k}}^{2} \leqslant c \hbar_{k}^{2 r}|v|_{\bar{H}^{r}\left(\Lambda_{k}\right)}^{2} \tag{40}
\end{align*}
$$

We can use (34) to prove the second result similarly.
The main result on the multidomain Legendre interpolation $\mathscr{I}_{L, \mathbf{N}_{k}}$ is stated below.
Theorem 7. For any $v \in \widetilde{H}^{r}\left(\Lambda_{k}\right)$ and integer $r \geqslant 1$, we have $\left\|\mathscr{I}_{L, \mathbf{N}_{k}} v-v\right\|_{\Lambda_{k}} \leqslant c \hbar_{k}^{r}|v|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}$.
Proof. By Definitions (6) and (10), we find that $\left(\widehat{\mathscr{I}_{L, \mathbf{N}_{k}} v}\right)^{k, j}=\left.\widehat{\mathscr{I}_{L, \mathbf{N}_{k}} v}\right|_{I_{j}^{k}}=\hat{\mathscr{I}}_{L, N_{j}^{k}} \hat{v}^{k, j}$. In view of this fact, we can use Lemma 5 and a similar argument as in the derivation of (40) to obtain the desired result.

In numerical analysis, we also need the following results.
Lemma 6. For any $\phi \in \mathscr{V} \mathbf{N}_{k}\left(\Lambda_{k}\right)$,

$$
\begin{equation*}
\|\phi\|_{\Lambda_{k}} \leqslant\|\phi\|_{\mathbf{N}_{k}, \Lambda_{k}} \leqslant \sqrt{3}\|\phi\|_{\Lambda_{k}} \tag{41}
\end{equation*}
$$

Moreover, for any $v \in \widetilde{H}^{r}\left(\Lambda_{k}\right)$ and integer $r \geqslant 1$,

$$
\begin{equation*}
\left|(v, \phi)_{\mathbf{N}_{k}, \Lambda_{k}}-(v, \phi)_{\Lambda_{k}}\right| \leqslant c \hbar_{k}^{r}|v|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{\Lambda_{k}}, \quad \forall \phi \in \mathscr{V}_{\mathbf{N}_{k}}\left(\Lambda_{k}\right) . \tag{42}
\end{equation*}
$$

Proof. The proof of the first result is simple. Next, by (11), (38), Theorem 7 and (41),

$$
\begin{aligned}
\left|(v, \phi)_{\Lambda_{k}}-(v, \phi)_{\mathbf{N}_{k}, \Lambda_{k}}\right| & \leqslant\left|(v, \phi)_{\Lambda_{k}}-\left(\widetilde{P}_{\mathbf{N}_{k}-1} v, \phi\right)_{\Lambda_{k}}\right|+\left|\left(\widetilde{P}_{\mathbf{N}_{k}-1} v, \phi\right)_{\Lambda_{k}}-\left(\mathscr{I}_{L, \mathbf{N}_{k}} v, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}\right| \\
& \leqslant\left(\left\|\widetilde{P}_{\mathbf{N}_{k}-1} v-v\right\|_{\Lambda_{k}}+\left\|\widetilde{P}_{\mathbf{N}_{k}-1} v-\mathscr{I}_{L, \mathbf{N}_{k}} v\right\|_{\mathbf{N}_{k}, \Lambda_{k}}\right)\|\phi\|_{\Lambda_{k}} \\
& \leqslant\left(2\left\|\widetilde{P}_{\mathbf{N}_{k}-1} v-v\right\|_{\Lambda_{k}}+\left\|\mathscr{I}_{L, \mathbf{N}_{k}} v-v\right\|_{\Lambda_{k}}\right)\|\phi\|_{\Lambda_{k}} \\
& \leqslant c \hbar_{k}^{r} \mid v \widetilde{H}_{\tilde{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{\Lambda_{k}} . \quad \square
\end{aligned}
$$

## 5. Error estimation

In this section, we prove the convergence of the mixed pseudospectral method proposed in Section 2. We first consider (9).

Theorem 8. Let $U(x)$ and $u_{M}^{0}(x)$ be the solutions of (8) and (9), respectively. If $U \in{ }_{0} H^{1}(\Lambda) \cap B^{s}(\Lambda)$, $f \in B^{\sigma}(\Lambda)$ and $\sigma \geqslant 1$, then

$$
\left\|u_{M}^{0}-U\right\|_{1, \Lambda}+\sup _{x \in \bar{\Lambda}}\left|\left(u_{M}^{0}-U\right)(x)\right| \leqslant c\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, \Lambda}\right)
$$

Proof. Let $\widetilde{\mathscr{I}}_{\mathscr{L}, M}$ and ${ }_{0} \bar{P}_{M}^{1}$ be the same as in (3) and (26). Set $U_{M}^{*}={ }_{0} \bar{P}_{M}^{1} U$. By (8) and (26),

$$
a_{\Lambda}\left(U_{M}^{*}, \phi\right)=a_{\Lambda}(U, \phi)+\left(\lambda-\frac{1}{4}\right)\left(U_{M}^{*}-U, \phi\right)_{\Lambda}, \quad \forall \phi \in \mathscr{S}_{M}^{0}(\Lambda)
$$

Thus, by (2), (8) and (9),

$$
\begin{equation*}
a_{\Lambda}\left(u_{M}^{0}-U_{M}^{*}, \phi\right)=\left(\frac{1}{4}-\lambda\right)\left(U_{M}^{*}-U, \phi\right)_{\Lambda}+\left(\tilde{\mathscr{I}}_{\mathscr{L}, M} f-f, \phi\right)_{\Lambda}, \quad \phi \in \mathscr{S}_{M}^{0}(\Lambda) \tag{43}
\end{equation*}
$$

Taking $\phi=u_{M}^{0}-U_{M}^{*}$ in (43) and using Theorems 3 and 5, we verify that

$$
\begin{aligned}
\left\|u_{M}^{0}-U_{M}^{*}\right\|_{1, \Lambda} & \leqslant c\left(\left\|U_{M}^{*}-U\right\|_{1, \Lambda}+\left\|\tilde{\mathscr{I}}_{\mathscr{L}, M} f-f\right\|_{\Lambda}\right) \\
& \leqslant c\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, \Lambda}\right)
\end{aligned}
$$

This with (31) leads to the desired result.
We next deal with the convergence of (12).
Theorem 9. Let $U^{k}(x)$ and $u_{\mathbf{N}_{k}}^{k}(x)$ be the solutions of (12) and (14), respectively. If $U \in{ }_{0} H^{1}(\Lambda) \cap$ $B^{s}(\Lambda), U^{k} \in \tilde{H}^{r}\left(\Lambda_{k}\right), f \in B^{\sigma}(\Lambda), f^{k} \in \widetilde{H}^{\sigma^{\prime}}\left(\Lambda_{k}\right)$ with integers $r \geqslant 1, \sigma^{\prime} \geqslant 0$ and real numbers $s, \sigma \geqslant 1$, then we have

$$
\begin{aligned}
\left\|u_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}}+\left|u_{\mathbf{N}_{k}}^{k}-U^{k}\right|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \leqslant & c_{k}^{*}\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, \Lambda}\right. \\
& \left.+\hbar_{k}^{r-1 / 2}\left|U^{k}\right|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}+\hbar_{k}^{\sigma^{\prime}}\left|f^{k}\right|_{\tilde{H}^{\sigma^{\prime}}\left(\Lambda_{k}\right)}\right)
\end{aligned}
$$

where $c_{k}^{*}$ is positive constant only depending on $\lambda$ and the length of $\Lambda_{k}$.
Proof. Let $U_{\mathbf{N}_{k}}^{k}=\widetilde{P}_{\mathbf{N}_{k}}^{1} U^{k}, U_{\mathbf{N}_{k}, *}^{k}(x)=U_{\mathbf{N}_{k}}^{k}(x)-W_{\mathbf{N}_{k}}^{k}(x)$, and

$$
W_{\mathbf{N}_{k}}^{k}(x)=\frac{b_{k}-x}{b_{k}-a_{k}} U_{\mathbf{N}_{k}}^{k}\left(a_{k}\right)+\frac{x-a_{k}}{b_{k}-a_{k}} U_{\mathbf{N}_{k}}^{k}\left(b_{k}\right), \quad x \in \Lambda_{k}
$$

Then by (13) and (15),

$$
\begin{aligned}
\min (\lambda, 1)\left\|u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right\|_{\hat{H}^{1}\left(\Lambda_{k}\right)}^{2} \leqslant & a_{\mathbf{N}_{k}, \Lambda_{k}}\left(u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right) \\
= & B_{\mathbf{N}_{k}, \Lambda_{k}}\left(u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right) \\
= & \left(f^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right)_{\mathbf{N}_{k}, \Lambda_{k}}-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(w_{M}^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right) \\
& -a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right)+a_{\Lambda_{k}}\left(U_{*}^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right) \\
& +a_{\Lambda_{k}}\left(W^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right)-\left(f^{k}, u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right)_{\Lambda_{k}} .
\end{aligned}
$$

Thus, we have that

$$
\begin{align*}
& \min (\lambda, 1)\left\|u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \\
& \leqslant \sup _{\substack{\phi \in \mathcal{V}_{\mathbf{N}_{k}}^{0} \\
\phi \neq 0}} \frac{\left|a_{\Lambda_{k}}\left(U_{*}^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, \phi\right)\right|}{\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)}}+\sup _{\substack{\phi \in V_{\mathbf{N}_{k}} \\
\phi \neq 0}} \frac{\left|a_{\Lambda_{k}}\left(W^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(w_{M}^{k}, \phi\right)\right|}{\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)}} \\
& \quad+\sup _{\substack{\phi \in \mathscr{V}_{\mathbf{N}_{k}} \\
\phi \neq 0}} \frac{\left|\left(f^{k}, \phi\right)_{\Lambda_{k}}-\left(f^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}\right|}{\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)}} . \tag{44}
\end{align*}
$$

We now estimate the terms at the right-hand side of (44). Firstly, by the definitions of $U_{*}^{k}$ and $U_{\mathbf{N}_{k}, *}^{k}$,

$$
a_{\Lambda_{k}}\left(U_{*}^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, \phi\right)=a_{\Lambda_{k}}\left(U^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)+A_{1}(\phi),
$$

where $A_{1}(\phi)=a_{\mathbf{N}_{k}, \Lambda_{k}}\left(W_{\mathbf{N}_{k}}^{k}, \phi\right)-a_{\Lambda_{k}}\left(W^{k}, \phi\right)$. Moreover, thanks to (11) and $U_{\mathbf{N}_{k}}^{k}=\widetilde{P}_{\mathbf{N}_{k}}^{1} U^{k}$,

$$
\begin{aligned}
a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}}^{k}, \phi\right) & =\left(\partial_{x} U_{\mathbf{N}_{k}}^{k}, \partial_{x} \phi\right)_{\Lambda_{k}}+\lambda\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}} \\
& =\left(\partial_{x} U^{k}, \partial_{x} \phi\right)_{\Lambda_{k}}+\left(U^{k}, \phi\right)_{\Lambda_{k}}+\lambda\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}-\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\Lambda_{k}} \\
& =a_{\Lambda_{k}}\left(U^{k}, \phi\right)+A_{2}(\phi)+A_{3}(\phi),
\end{aligned}
$$

where $A_{2}(\phi)=\lambda\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}-\lambda\left(U^{k}, \phi\right)_{\Lambda_{k}}$ and $A_{3}(\phi)=\left(U^{k}, \phi\right)_{\Lambda_{k}}-\left(U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\Lambda_{k}}$. The previous statements imply that

$$
\begin{equation*}
\left|a_{\Lambda_{k}}\left(U_{*}^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, \phi\right)\right| \leqslant\left|A_{1}(\phi)\right|+\left|A_{2}(\phi)\right|+\left|A_{3}(\phi)\right| . \tag{45}
\end{equation*}
$$

So it suffices to estimate $\left|A_{j}(\phi)\right|, j=1,2,3$. By the Sobolev inequality, for any $v \in H^{1}(a, b)$,

$$
\begin{equation*}
\max _{x \in[a, b]}|v(x)| \leqslant\left(\frac{1}{b-a}+2\right)^{1 / 2}\|v\|_{L^{2}(a, b)}^{1 / 2}|v|_{H^{1}(a, b)}^{1 / 2} \tag{46}
\end{equation*}
$$

In view of (11), (39) and (46), a direct calculation gives that

$$
\begin{aligned}
\left|A_{1}(\phi)\right| & \leqslant c_{k}^{*}\left(\left|\left(U_{\mathbf{N}_{k}}^{k}-U^{k}\right)\left(a_{k}\right)\right|+\left|\left(U_{\mathbf{N}_{k}}^{k}-U^{k}\right)\left(b_{k}\right)\right|\right)\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \\
& \leqslant c_{k}^{*}\left\|U_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}}^{1 / 2}\left|U_{\mathbf{N}_{k}}^{k}-U^{k}\right|_{\hat{H}^{1}\left(\Lambda_{k}\right)}^{1 / 2}\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \leqslant c_{k}^{*} \hbar_{k}^{r-1 / 2}\left|U^{k}\right|_{\widetilde{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} .
\end{aligned}
$$

By (11) and Theorem 6,

$$
\begin{aligned}
\left|A_{2}(\phi)\right| & \leqslant \lambda\left|\left(U^{k}-\widetilde{P}_{\mathbf{N}_{k}-1} U^{k}, \phi\right)_{\Lambda_{k}}\right|+\lambda\left|\left(\widetilde{P}_{\mathbf{N}_{k}-1} U^{k}-U_{\mathbf{N}_{k}}^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}\right| \\
& \leqslant c\left(\left\|\widetilde{P}_{\mathbf{N}_{k}-1} U^{k}-U^{k}\right\|_{\Lambda_{k}}+\left\|\widetilde{P}_{\mathbf{N}_{k}-1} U^{k}-U_{\mathbf{N}_{k}}^{k}\right\|_{\Lambda_{k}}\right)\|\phi\|_{\Lambda_{k}} \leqslant c \hbar_{k}^{r}\left|U^{k}\right|_{\bar{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{\Lambda_{k}}
\end{aligned}
$$

Using Theorem 6 again yields that $\left|A_{3}(\phi)\right| \leqslant c \hbar_{k}^{r}\left|U^{k}\right|_{\bar{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{A_{k}}$. Substituting the estimates for $\left|A_{j}(\phi)\right|$ into (45), we assert that

$$
\begin{equation*}
\left|a_{\Lambda_{k}}\left(U_{*}^{k}, \phi\right)-a_{\mathbf{N}_{k}, \Lambda_{k}}\left(U_{\mathbf{N}_{k}, *}^{k}, \phi\right)\right| \leqslant c_{k}^{*} \hbar_{k}^{r-1 / 2}\left|U^{k}\right|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \tag{47}
\end{equation*}
$$

Next, by Theorem 8,

$$
\begin{align*}
\left|a_{\mathbf{N}_{k}, \Lambda_{k}}\left(w_{M}^{k}, \phi\right)-a_{\Lambda_{k}}\left(W^{k}, \phi\right)\right| & \leqslant c_{k}^{*}\left(\left|\left(u_{M}^{0}-U^{k}\right)\left(a_{k}\right)\right|+\left|\left(u_{M}^{0}-U^{k}\right)\left(b_{k}\right)\right|\right)\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \\
& \leqslant c_{k}^{*}\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, A}\right)\|\phi\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \tag{48}
\end{align*}
$$

Moreover, (42) implies that

$$
\begin{equation*}
\left|\left(f^{k}, \phi\right)_{\Lambda_{k}}-\left(f^{k}, \phi\right)_{\mathbf{N}_{k}, \Lambda_{k}}\right| \leqslant c \hbar_{k}^{\sigma^{\prime}}\left|f^{k}\right|_{\tilde{H}^{\sigma^{\prime}}\left(\Lambda_{k}\right)}\|\phi\|_{\Lambda_{k}} \tag{49}
\end{equation*}
$$

Inserting (47)-(49) into (44), we obtain that

$$
\begin{gather*}
\left\|u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right\|_{\hat{H}^{1}\left(\Lambda_{k}\right)} \leqslant \\
c_{k}^{*}\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, \Lambda}\right.  \tag{50}\\
+ \\
\left.+\hbar_{k}^{r-1 / 2}\left|U^{k}\right|_{\tilde{H}^{r}\left(\Lambda_{k}\right)}+\hbar_{k}^{\sigma^{\prime}}\left|f^{k}\right|_{\tilde{H}^{\sigma^{\prime}}\left(\Lambda_{k}\right)}\right)
\end{gather*}
$$

Furthermore,

$$
\begin{align*}
\left\|u_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}} \leqslant & \left\|U_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}}+\left\|u_{\mathbf{N}_{k}}^{k}-U_{\mathbf{N}_{k}}^{k}\right\|_{\Lambda_{k}} \\
\leqslant & \left\|U_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}}+\left\|u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right\|_{\Lambda_{k}}+\left\|w_{M}^{k}-W_{\mathbf{N}_{k}}^{k}\right\|_{\Lambda_{k}} \\
\leqslant & \left\|U_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}}+\left\|u_{\mathbf{N}_{k}, *}^{k}-U_{\mathbf{N}_{k}, *}^{k}\right\|_{\Lambda_{k}}+c_{k}^{*}\left(\left|\left(u_{M}^{0}-U^{k}\right)\left(a_{k}\right)\right|\right. \\
& \left.+\left|\left(U_{\mathbf{N}_{k}}^{k}-U^{k}\right)\left(a_{k}\right)\right|+\left|\left(u_{M}^{0}-U^{k}\right)\left(b_{k}\right)\right|+\left|\left(U_{\mathbf{N}_{k}}^{k}-U^{k}\right)\left(b_{k}\right)\right|\right) . \tag{51}
\end{align*}
$$

Due to $U_{\mathbf{N}_{k}}^{k}=\tilde{P}_{\mathbf{N}_{k}}^{1} U^{k}$, we can use Theorem 6 and (50) to estimate the first two terms at the right-hand side of the above inequality, and estimate the last four terms as in the derivations of (48) and the upper-bound of $\left|A_{1}(\phi)\right|$. Accordingly

$$
\begin{aligned}
\left\|u_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{\Lambda_{k}} \leqslant & c_{k}^{*}\left(M^{1 / 2-s / 2}\|U\|_{B^{s}, \Lambda}+(\ln M)^{1 / 2} M^{1 / 2-\sigma / 2}\|f\|_{B^{\sigma}, \Lambda}\right. \\
& \left.+\hbar_{k}^{r-1 / 2}\left|U^{k}\right|_{\widetilde{H}^{r}\left(\Lambda_{k}\right)}+\hbar_{k}^{\sigma^{\prime}}\left|f^{k}\right|_{\widetilde{H}^{\sigma^{\prime}}\left(\Lambda_{k}\right)}\right)
\end{aligned}
$$

By replacing the norm $\|\cdot\|_{\Lambda_{k}}$ in (51) by $|\cdot|_{\hat{H}^{1}\left(\Lambda_{k}\right)}$, we can prove the second result similarly.
Using the above theorem and (46), we have the same upper-bound for $\left\|u_{\mathbf{N}_{k}}^{k}-U^{k}\right\|_{L^{\infty}\left(\Lambda_{k}\right)}$.

## 6. Concluding discussions

In this paper, the modified Laguerre interpolation was first proposed for differential equations on the half-line. It keeps the natural weight as in continuous version, and so simplifies computation and numerical analysis. But like the standard Laguerre interpolation, the distance between the large interpolation nodes increases fast as the mode $N$ increases. So the numerical solutions can not describe the character of exact solutions well, if the exact solution varies rapidly between the large nodes. To remedy this deficiency, we used the multidomain Legendre pseudospectral method to refine numerical solution. In other words, we reconstruct the numerical solutions by the multidomain Legendre pseudospectral approximation to recover the accuracy on certain subintervals where the exact solutions vary rapidly. These two techniques matched each other very well. Numerical results demonstrated the efficiency of this method, even for oscillated solutions. This method can be also regarded as a cascade multigrid pseudospectral method on the half line, which is also available for many other problems, such as nonlinear problems and exterior problems.

We improved some results on the standard Laguerre interpolation, and first built up the results on the modified Laguerre interpolation. These results are applied successfully to analyzing the proposed method. In fact, they play important role in numerical analysis of pseudospectral methods for various problems on unbounded domains.

We established some results on the multidomain Legendre pseudospectral approximation, which served as one of basic tools in the error estimates. In particular, in the expressions of norms appearing in the error estimates, there exist piecewise Jacobi-type weights which tend to zero as $x$ goes to the endpoints of subintervals. Thus the conditions on the smoothness of unknown functions and numerical solutions at the endpoints of subintervals are weekened. The related results seem very appropriate for numerical analysis of domain decomposition spectral method.

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