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LETTER TO THE EDITOR

The proper analytical solution of the Korteweg–de Vries–Burgers equation

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Abstract. On using variable transformations and proofs of theorems, the asymptotic behaviour and the proper analytical solution of the Korteweg-de Vries-Burgers equation have been found in this letter.

It is common knowledge that many physical problems (such as non-linear shallow-water waves and wave motion in plasmas) can be described by the KdV equation [1]. Solitons and solitary waves, one class of special solutions of the KdV equation, have been known for some time. In order to study the problems of the flow of liquids containing gas bubbles [2], the fluid flow in elastic tubes, and so on [3-5], the control equation can be reduced to the so-called KdV-Burgers equation as follows [6]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}+\delta\frac{\partial^3 u}{\partial x^3}=0. \tag{1}$$

This equation is equal to the KdV equation if a viscous dissipation term (νu_{xx}) is added. At present, studies of the KdV equation ($\nu=0$) and Burgers equation ($\delta=0$) have been undertaken, but studies of the KdV-Burgers equation are just beginning. The exact solution for the general case of equation (1) ($\nu\neq 0$, $\delta\neq 0$) has still not been calculated.

In studying the theory of ordinary differential equations and applications, it is clear that the asymptotic solution behaviour is highly important. The asymptotic behaviour and the proper analytical solution of equation (1) are presented in this letter. It would be useful to thoroughly research the properties of the solution to the KdV-Burgers equation. At present, numerical studies of equation (1) are gradually increasing in frequency [5], and so the proper analytical solution would provide a reliable basis for estimating the advantages, and disadvantages of the numerical method.

Jeffrey and Kakutani [7] have thoroughly analysed the qualitative characteristics of the solution of equation (1). They introduce the following new variables [6, 7]:

$$\zeta=x-\lambda t \qquad t'=t. \tag{2}$$

Equation (1) can be written as

$$u_{t'}+(u-\lambda)u_\zeta-\nu u_{\zeta\zeta}+\delta u_{\zeta\zeta\zeta}=0. \tag{3}$$

We shall only consider the so-called travelling-wave solution, i.e. $u=f(\zeta)$. Integrating formula (3) for ζ, we can obtain a non-linear differential equation as follows:

$$\delta u_{\zeta\zeta}-\nu u_\zeta+\tfrac{1}{2}u^2-\lambda u+A=0 \tag{4}$$

where A is an integral constant, less than $\frac{1}{2}\lambda^2$. Equation (4) is equal to a non-linear equation set of first order as follows:

$$
\begin{aligned}
&\mathrm{d}u/\mathrm{d}\zeta = v \\
&\mathrm{d}v/\mathrm{d}\zeta = -\tfrac{1}{2\delta}(u-u_1)(u-u_2) + \nu v/\delta
\end{aligned}
\tag{5}
$$

where $u_{1,2} = \lambda \pm \sqrt{\lambda^2 - 2A}$ and $u_1 > u_2$.

According to the qualitative theory of ordinary differential equations [8], the equation set (5) has two singular points, $(u_1, 0)$ and $(u_2, 0)$. The singular point $(u_2, 0)$ is invariably a saddle point. The singular point $(u_1, 0)$ has three different cases which depend on the values of ν, 1 δ and λ.

(a) If $\nu^2 \geqslant 4\delta\sqrt{\lambda^2 - 2A}$, $(u_1, 0)$ is a nodal point.

(b) If $0 < \nu^2 < 4\delta\sqrt{\lambda^2 - 2A}$, $(u_1, 0)$ is a focal point.

(c) If $\nu = 0$, $(u_1, 0)$ is a central point.

The three classes of solutions are roughly shown in figure 1.

A proper solution is one which is real and continuous if the argument is greater than a certain value.

We use some variable transformations to reduce equation (4) to a simple form.

With $\delta \neq 0$ and $\nu \neq 0$, equation (4) can be written as

$$u_{\zeta\zeta} + au_\zeta + bu^2 + cu + E = 0 \tag{6}$$

where $a = -\nu/\delta$, $b = 1/2\delta$, $c = \lambda/\delta$ and $E = A/\delta$.

To observe the general character, we can consider $E = 0$. For if $E \neq 0$, we can make a simple translation transformation $u = \tilde{u} + D$ ($D = \lambda \pm \sqrt{\lambda^2 - 2A}$). $\tilde{u}$ satisfies the following equation:

$$\tilde{u}_{\zeta\zeta} + a\tilde{u}_\zeta + b\tilde{u}^2 + (c + 2bD)\tilde{u} = 0.$$

From now on, we shall confine ourselves to consideration of $E = 0$ (i.e. $A = 0$) alone.

We can further assume that $\lambda \geqslant 0$. For if $\lambda < 0$, we can make $\tilde{\lambda} = -\lambda$ and discuss it in the same manner.

We make the following variable transformation:

$$\tau = \mathrm{e}^{-a\zeta} = \mathrm{e}^{\nu\zeta/\delta}. \tag{7}$$

Equation (6) ($E = 0$) can be written as

$$a^2\tau^2 \mathrm{d}^2u/\mathrm{d}\tau^2 + bu^2 + cu = 0. \tag{8}$$

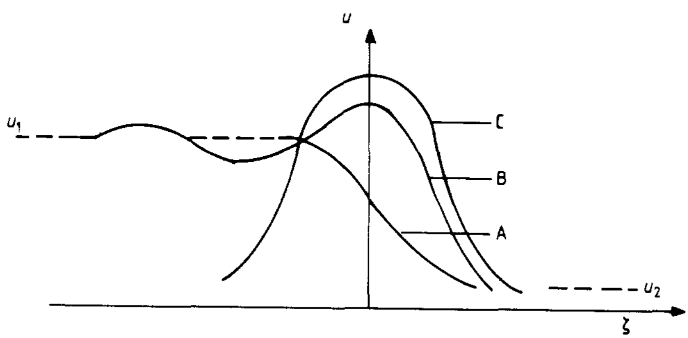

Figure 1. The typical solution of the KdV-Burgers equation. A is a dissipation-dominant solution for a monotonic shock wave, B is a chromatic dispersion-dominant solution for an oscillatory shock wave and C is a solitary-wave solution of the KdV equation.

Now we make the following variable transformation:

$$u = \tau^{1/2(1-k)}\eta(\xi) \qquad \xi = \tau^k \tag{9}$$

where $k = (1-4c/a^2)^{1/2} = (1+4\lambda\delta/\nu^2)^{1/2}$ is a constant, and obviously $k \geqslant 1$.

Equation (8) can be written as

$$\frac{d^2\eta}{d\xi^2} = -\frac{b}{a^2k^2}\xi^{(1-5k)/2k}\eta^2 \tag{10}$$

i.e.

$$\frac{d^2\eta}{d\xi^2} = -\frac{\delta}{2\nu^2k^2}\xi^{(1-5k)/2k}\eta^2. \tag{11}$$

Next we introduce the following new variables:

$$\eta = -2\nu^2k^2y/\delta \qquad x = \xi \tag{12}$$

and we let $\sigma = (1-5k)/2k$. It is obvious that if $\lambda = 0$, $\sigma = -2$, and if $\lambda > 0$, $-\frac{5}{2} < \sigma < -2$. Equation (11) can be reduced to

$$d^2y/dx^2 = x^\sigma y^2. \tag{13}$$

We can use equation (13) to derive some characteristics of the KdV–Burgers equation.

Definition. Let $f(x)$ be an arbitrary function of x. If x_0 is a zero point and no other zero point except x_0 exists in the open interval $(x_0-\varepsilon, x_0+\varepsilon)$ for some $\varepsilon > 0$, then x_0 is called an isolated zero point of the function $f(x)$.

Theorem 1. The KdV–Burgers equation (4) has finite isolated zero points only.

Proof. Since $y'' = x^\sigma y^2$, y'' does not change its sign for $x \in (0, +\infty)$. Equation (13) has finite zero points only, except that it identically vanishes for some intervals. Equation (13) has finite zero points only. This theorem can be proved by inverse transformation.

From theorem 1, we see that the solution of the KdV–Burgers equation is consistently positive, negative or zero for large arguments. It depends upon the conditions of the infinite point.

In order to obtain the proper solution of the KdV–Burgers equation, we introduce the following three lemmas, which are not proved here.

Lemma 1. The integral rule for asymptotic formulae. Let $\varphi(t) \sim f(t)$, where $f(t) \neq 0$ and does not change in sign. If

$$\int_{t_0}^{+\infty} |f(t)| dt = +\infty$$

then

$$\int_{t_0}^{t} \varphi(t)\, dt \sim \int_{t_0}^{t} f(t)\, dt$$

and if

$$\int_{t_0}^{+\infty} |f(t)|\, dt < +\infty$$

then

$$\int_t^{+\infty} \varphi(t)\,\mathrm{d}t \sim \int_t^{+\infty} f(t)\,\mathrm{d}t.$$

Lemma 2. The character of the proper solution. If $f(t)>0$, and if f' is a continuous and non-negative function as $t \geq t_0$, then $f' \leq f^{1+\varepsilon}$ for $t \geq t_0$ for any $\varepsilon > 0$, except perhaps in a set of intervals of finite total length which depends upon ε.

Lemma 3. Hardy's theorem. Any solution of the equation

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{P(f,t)}{Q(f,t)}$$

continuous for $t \geq t_0$, is ultimately monotonic, together with all its derivatives and satisfies one of the relations

$$f \sim at^b\,\mathrm{e}^{E(t)} \qquad f \sim at^b(\ln t)^{1/c}$$

where $E(t)$ is a polynomial in t and c is an integer.

We can use the above three lemmas to obtain the asymptotic expressions of the proper solution of the KdV–Burgers equation.

Theorem 2. If $\lambda > 0$, every negative proper solution of the KdV–Burgers equation has the following asymptotic form:

$$u=-\frac{2k^2\nu^2 a_\infty}{\delta}\exp\{-[(k-1)\nu\zeta/2\delta]\}$$
$$-\frac{8k^4\nu^2 a_\infty^2}{(k-1)(3k-1)\delta}\exp\{-[(k-1)\nu\zeta/\delta]\}[1+\mathrm{O}(1)] \qquad (14)$$

as $\zeta \to +\infty$, where $k=(1+4\lambda\delta/\nu^2)^{1/2}$, and $a_\infty > 0$ is constant.

Proof. First of all, we consider equation (13) ($-\frac{5}{2}<\sigma<-2$). If u is a negative proper solution, then y is a positive proper solution. Since $y''=x^\sigma y^2>0$ for $x>0$, y' must be strictly monotonically increasing for $x \in (0,+\infty)$ and y must be a monotone function for large x.

Thus y' has three possible cases as $x \to +\infty$:

(1) $y' \to 0$

(2) $y' \to y_0' > 0$

(3) $y' \to +\infty$.

Let us show that case (2) is impossible.

If $y' \to y_0' > 0$, then $y \sim y_0' x$, and from equation (13)

$$y''=x^\sigma y^2 \sim y_0'^2 x^{\sigma+2} > \tfrac{1}{2} y_0'^2 x^{\sigma+2}$$

whose integration yields

$$y' > \frac{y_0'^2}{2(\sigma+3)} x^{\sigma+3} - c_0 \to +\infty$$

for large x, where c_0 is constant, which is a contradiction.

We now show that case (3) is impossible.

If $y' \to +\infty$, then $y' > M$ for large x for some $M > 0$, and hence $y > Mx$. Reverting to equation (13), $y'' = x^\sigma y^2 > M^2 x^{\sigma+2}$, $y' > [M^2/2(\sigma+3)]x^{\sigma+3}$, $y > [M^2/4(\sigma+3)(\sigma+4)]x^{\sigma+4}$. Continuing in this fashion, we obtain $y > C_0 x^{b_0}$ for large x for some $c_0 > 0$. Hence from equation (13), $y'' = x^\sigma y^2 > \sqrt{c_0} y^{3/2}$ for large x. Since y' is positive, we have

$$y'y'' > \sqrt{c_0} y^{3/2} y'$$

whose integration yields

$$y'^2 > \tfrac{4}{5}\sqrt{c_0} y^{5/2} \qquad \text{or} \qquad y' > 2c_0^{1/4} y^{5/4}/\sqrt{5}.$$

Using lemma 2, we know this is not possible.

Consequently we are left with case (1), where $y' \to 0$.

Since y' is strictly monotone increasing for $x \in (0, +\infty)$, $y' < 0$ for $x \in (0, +\infty)$, or y is strictly monotone decreasing for $x \in (0, +\infty)$. Since $y > 0$ for large x, y has a finite limit $a_\infty \geqslant 0$ as $x \to +\infty$.

Let us show that $a_\infty \neq 0$.

If $a_\infty = 0$, then let $y(x_0) = \delta > 0$ be small. Since y is strictly monotone decreasing,

$$\delta = y(x_0) = \int_{x_0}^{+\infty} \left(\int_t^{+\infty} t_1^\sigma y^2 \, \mathrm{d}t_1 \right) \mathrm{d}t < \delta^2 \int_{x_0}^{+\infty} \left(\int_t^{+\infty} t_1^\sigma \, \mathrm{d}t_1 \right) \mathrm{d}t$$

or

$$\delta^2 > (\sigma+1)(\sigma+2)/x_0^{\sigma+2}$$

and we have a contradiction for δ sufficiently small.

Then let $y(+\infty) = a_\infty > 0$, $y(x) = a_\infty + \mathrm{O}(1)$ as $x \to +\infty$:

$$y'(x) = -\int_x^{+\infty} y'' \, \mathrm{d}t = -\int_x^{+\infty} t^\sigma y^2 \, \mathrm{d}t = \frac{a_\infty^2}{\sigma+1} x^{\sigma+1}[1 + \mathrm{O}(1)]$$

and thus

$$y(x) = a_\infty - \int_x^{+\infty} y' \, \mathrm{d}t = a_\infty + \frac{a_\infty^2}{(\sigma+1)(\sigma+2)} x^{\sigma+2}[1 + \mathrm{O}(1)].$$

This theorem can then be proved by inverse transformation.

Theorem 3. If $\lambda = 0$, every negative proper solution of the KdV-Burgers equation has the following asymptotic form:

$$u \sim -\frac{2k\nu}{\zeta} \exp -[(k-1)\nu\zeta/2\delta] \tag{15}$$

as $\zeta \to +\infty$, where $k = (1 + 4\lambda\delta/\nu^2)^{1/2}$.

Proof. First of all, we consider equation (13) ($\sigma = -2$). If u is a negative proper solution, then y is a positive proper solution.

Let $x = \mathrm{e}^s$, obtaining from equation (13)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s} - y^2 = 0. \tag{16}$$

For if $dy/ds=0$ at s_0, then $d^2y/ds^2=y^2>0$, and y can only have a minimum at s_0. Hence y is a monotone function for large x.

Thus y has three possible cases as $s\to+\infty$:

(1) $y\to 0$

(2) $y\to y_0>0$

(3) $y\to+\infty$.

Let us show that case (2) is impossible.

If $y\to y_0>0$, then $d^2y/ds^2-dy/ds\sim y_0^2$. Integrating, we obtain $dy/ds-y\sim y_0^2 s$. Since $y\to y_0>0$, this implies $dy/ds\sim y_0^2 s$, from which $y\sim\frac{1}{2}y_0^2s^2$, which contradicts $y\to y_0$.

Let us show that case (3) is impossible.

If $y\to+\infty$, then let $p=dy/ds$. Equation (16) becomes

$$p\,dp/dy-p-y^2=0. \tag{17}$$

Since $y\to+\infty$, $p=dy/ds>0$. Using lemma 3, we see that p has two possible cases for large y:

(i) $p\sim a_1y^{b_\infty}e^{E_\infty(y)}$

(ii) $p\sim a_2y^{b_\infty}(\ln y)^{m_\infty}$

where a_1 and a_2 are two positive numbers and $E_\infty(y)$ is a polynomial in y.

Let us show that case (i) is impossible.

If $E_\infty(y)\to-\infty$, then $p\to 0$ and $dp/dy\to 0$. By referring to equation (17), we see that this is a contradiction. If $E_\infty(y)\to+\infty$, then $p>y^2$ for large y. Using lemma 2, we know that this is not possible. Hence $E_\infty(y)\equiv$ constant.

If $b_\infty>1$, then $p>y^{1+(b_\infty-1)/2}$ for large y. Using lemma 2, we know that this is not possible. If $b_\infty\leqslant 1$, then from equation (17) we obtain $p\,dp/dy\sim y^2$. Integrating, we obtain $\frac{1}{2}p^2\sim\frac{1}{3}y^3$, where $2b_\infty=3$, or $b_\infty=\frac{3}{2}>1$, which is a contradiction.

Let us show that case (ii) is impossible.

If $l_\infty>1$, then $p>y^{1+(l_\infty-1)/2}$ for large y. Using lemma 2, we know this is not possible. If $l_\infty\leqslant 1$, then from equation (17) we obtain $p\,dp/dy\sim y^2$. Integrating, we obtain $\frac{1}{2}p^2\sim\frac{1}{3}y^3$, whence $2l_\infty=3$, or $l_\infty=\frac{3}{2}>1$, which is a contradiction.

Consequently we are left with case (1), where $y\to 0$.

Letting $v=1/y$ and $w=dv/ds$, we obtain from equation (16)

$$w\,dw/dv-2w^2/v-w+1=0. \tag{18}$$

Since $y\to 0$, $v\to+\infty$ and $dy/ds<0$, we obtain $w=dv/ds=-(1/y^2)(dy/ds)>0$. Using lemma 3, we see that w has two possible cases for large v:

(iii) $\quad w\sim a_1v^{b_\infty}e^{E_\infty(v)}$

(iv) $\quad w\sim a_2v^{l_\infty}(\ln v)^{m_\infty}$

where a_1 and a_2 are two positive numbers and $E_\infty(v)$ is a polynomial in v.

We now show that if case (iii) is satisfied, then $E_\infty(v)\equiv$ constant and $b_\infty=0$.

Similar to above, $E_\infty(v)\equiv$ constant and $b_\infty\leqslant 1$.

If $b_\infty=1$, then $dw/dv\sim a_1>0$. From equation (18), we obtain $a_1-2a_1-1=0$, or $a_1=-1$, which is a contradiction.

If $0<b_\infty<1$, then from equation (18) we obtain $dw/dv\sim 1$. Integrating, we obtain $w\sim v$, from which $b_\infty=1$, which is a contradiction.

If $b_\infty<0$, then from equation (18) we obtain $w\,dw/dv\sim-1$. Integrating, we obtain $\frac{1}{2}w^2\sim -v$, which is a contradiction.

Let us show that if case (iv) is satisfied, then $l_\infty=0$ and $m_\infty=0$.

Similar to above, either $l_\infty=1$ and $m_\infty\neq 0$, or $l_\infty=0$.

If $l_\infty = 1$, $m_\infty < 0$ or $m_\infty > 0$, then from equation (18) we obtain $\mathrm{d}w/\mathrm{d}v \sim 1$ or $\mathrm{d}w/\mathrm{d}v \sim 2w/v$. Integrating, we obtain $w \sim v$ or $w \sim v^2$, whence $m_\infty = 0$, which is a contradiction. Hence $l_\infty = 0$.

If $m_\infty < 0$, then from equation (18) we obtain $w\,\mathrm{d}w/\mathrm{d}v \sim -1$. Integrating, we obtain $\frac{1}{2}w^2 \sim -v$, which is a contradiction.

If $m_\infty > 0$, then from equation (18) we obtain $\mathrm{d}w/\mathrm{d}v \sim 1$. Integrating, we obtain $w \sim v$, whence $m_\infty = 0$, which is a contradiction.

Summing up, we obtain $w \sim a_\infty > 0$. From equation (18) we obtain $a_\infty = 1$ or $w \sim 1$, whence $\mathrm{d}v/\mathrm{d}s \sim 1$ as $s \to +\infty$. integrating, we obtain $v \sim s$ as $s \to +\infty$, whence $y \sim 1/\ln x$ as $x \to +\infty$.

This theorem can be proved by inverse transformation.

Developing theorem 2, we obtain the approximate proper analytical solution of the KdV-Burgers equation.

Theorem 4. If $\lambda > 0$, the negative proper solution of the KdV-Burgers equation can be written as

$$u = -\frac{2k^2\nu^2 a_\infty}{\delta}\exp\{-[(k-1)\nu\zeta/2\delta]\} - \frac{2k^4\nu^2}{\delta}\sum_{i=1}^{+\infty}\frac{(2a_\infty)^{i+1}\exp\{-[(i+1)(k-1)\nu\zeta/2\delta]\}}{\Pi_{j=1}^{i}[j(k-1)+2k][j(k-1)]} \tag{19}$$

where $k = (1 + 4\lambda\delta/\nu^2)^{1/2}$ and $a_\infty > 0$ is constant.

Proof. Since $\exp\{-[(i+1)(k-1)\zeta/2\delta]\}$ exists, the infinite series converges. Let

$$u_m = -\frac{2k^2\nu^2 a_\infty}{\delta}\exp\{-[(k-1)\nu\zeta/2\delta]\} - \frac{2k^2\nu^2}{\delta}\sum_{i=1}^{m}\frac{(2a_\infty)^{i+1}\exp\{-[(i+1)(k-1)\nu\zeta/2\delta]\}}{\Pi_{j=1}^{i}[j(k-1)+2k][j(k-1)]}$$

whence

$$y_m = a_\infty + \sum_{i=1}^{m}\frac{2^{i-1}a_\infty^{i+1}x^{i(\sigma+2)}}{\Pi_{j=1}^{i}[j(\sigma+2)-1][j(\sigma+2)]}.$$

We can obtain

$$y_{m+1}[1+\mathrm{O}(1)] = a_\infty + \int_x^{+\infty}\left(\int_t^{+\infty} t_1^\sigma y_m^2[1+\mathrm{O}(1)]^2\,\mathrm{d}t_1\right)\mathrm{d}t$$

for an arbitrary integer m.

Since $u_m \to u$ as $m \to +\infty$, $y_m \to y_{+\infty}$ as $m \to +\infty$, whence

$$y_{+\infty} = a_\infty + \int_x^{+\infty}\left(\int_t^{+\infty} t_1^\sigma y_{+\infty}^2\,\mathrm{d}t_1\right)\mathrm{d}t$$

and $y_{+\infty}$ is the positive proper solution of equation (13).

This theorem can be proved by inverse transformation.

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