Chapter 15
American Options

American options are financial derivatives that can be exercised at any time before maturity, in contrast to European options which have fixed maturities. The prices of American options are evaluated as an optimization problem, in which one has to find the optimal time to exercise in order to maximize the claim option payoff.

15.1 Perpetual American Put Options

The price of an American put option with finite expiration time $T > 0$ and strike price K can be expressed as the expected value of its discounted payoff:

$$f(t, S_t) = \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(K - S_\tau)^+ \,\big|\, S_t \big],$$

under the risk-neutral probability measure $\mathbb{P}^*$, where the supremum is taken over stopping times between t and a fixed maturity T. Similarly, the price of a finite expiration American call option with strike price K is expressed as

$$f(t, S_t) = \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \,\big|\, S_t \big].$$

Finite expiration American options can be found for example on the SPDR S&P 500 ETF Trust (SPY) exchange-traded fund. In this section we take

$T = +\infty$, in which case we refer to these options as *perpetual* options, and the corresponding put and call options are respectively priced as

$$f(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau - t)r} (K - S_\tau)^+ \,\big|\, S_t \big],$$

and

$$f(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau - t)r} (S_\tau - K)^+ \,\big|\, S_t \big].$$

Two-choice optimal stopping at a fixed price level for perpetual put options

In this section we consider the pricing of perpetual put options. Given $L \in (0, K)$ a fixed price, consider the following choices for the exercise of a *put* option with strike price K:

1. If $S_t \leqslant L$, then exercise at time t.
2. Otherwise if $S_t > L$, wait until the first hitting time

$$\tau_L := \inf\{u \geqslant t \;:\; S_u \leqslant L\} \tag{15.1}$$

   of the level $L > 0$, and exercise the option at time τ_L if $\tau_L < \infty$.

Note that by definition of (15.1) we have $\tau_L = t$ if $S_t \leqslant L$.

In case $S_t \leqslant L$, the payoff will be

$$(K - S_t)^+ = K - S_t$$

since $K > L \geqslant S_t$, however in this case one would buy the option at price $K - S_t$ only to exercise it immediately for the same amount.

In case $S_t > L$, as $r > 0$ the price of the option is given by

$$\begin{aligned} f_L(t, S_t) &= \mathbb{E}^*\big[\mathrm{e}^{-(\tau_L - t)r} (K - S_{\tau_L})^+ \,\big|\, S_t \big] \\ &= \mathbb{E}^*\big[\mathrm{e}^{-(\tau_L - t)r} (K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < \infty\}} \,\big|\, S_t \big] \\ &= \mathbb{E}^*\big[\mathrm{e}^{-(\tau_L - t)r} (K - L)^+ \mathbb{1}_{\{\tau_L < \infty\}} \,\big|\, S_t \big] \\ &= (K - L) \mathbb{E}^*\big[\mathrm{e}^{-(\tau_L - t)r} \,\big|\, S_t \big]. \end{aligned} \tag{15.2}$$

We note that the starting date t does not matter when pricing perpetual options, which have an infinite time horizon. Hence, $f_L(t, x) = f_L(x)$, $x > 0$, does not depend on $t \in \mathbb{R}_+$, and the pricing of the perpetual put option can be performed at $t = 0$. Recall that the underlying asset price is written as

$$S_t = S_0 \,\mathrm{e}^{rt + \sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \geqslant 0, \tag{15.3}$$

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where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, r is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

Lemma 15.1. *Assume that $r > 0$. We have*

$$\mathbb{E}^*\left[e^{-(\tau_L - t)r} \,\middle|\, S_t = x \right] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}, \qquad x \geqslant L. \tag{15.4}$$

Proof. We take $t = 0$ in (15.4) without loss of generality. We note that from (15.3), for all $\lambda \in \mathbb{R}$ we have

$$(S_t)^\lambda = e^{\lambda rt + \lambda\sigma\widehat{B}_t - \lambda\sigma^2 t/2}, \qquad t \geqslant 0,$$

and the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := (S_0)^\lambda e^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2} = (S_t)^\lambda e^{-(\lambda r - \lambda(1-\lambda)\sigma^2/2)t}, \qquad t \geqslant 0, \tag{15.5}$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$. Choosing $\lambda \in \mathbb{R}$ such that

$$r = r\lambda - \lambda(1-\lambda)\frac{\sigma^2}{2}, \tag{15.6}$$

we have

$$Z_t^{(\lambda)} = (S_t)^\lambda e^{-rt}, \qquad t \geqslant 0. \tag{15.7}$$

The equation (15.6) rewrites as

$$0 = \lambda^2\frac{\sigma^2}{2} + \lambda\left(r - \frac{\sigma^2}{2}\right) - r = \frac{\sigma^2}{2}\left(\lambda + \frac{2r}{\sigma^2}\right)(\lambda - 1), \tag{15.8}$$

with solutions

$$\lambda_+ = 1 \quad \text{and} \quad \lambda_- = -\frac{2r}{\sigma^2}.$$

Choosing the negative solution* $\lambda_- = -2r/\sigma^2 < 0$ and noting that $S_t \geqslant L$ for all $t \in [0, \tau_L]$, we obtain

$$0 \leqslant Z_t^{(\lambda_-)} = e^{-rt}(S_t)^{\lambda_-} \leqslant e^{-rt}L^{\lambda_-} \leqslant L^{\lambda_-}, \qquad 0 \leqslant t \leqslant \tau_L, \tag{15.9}$$

since $r > 0$. Therefore we have $\lim_{t\to\infty} Z_t^{(\lambda_-)} = 0$, and since $\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)} = Z_{\tau_L}^{(\lambda_-)}$ on $\{\tau_L < \infty\}$, using (15.7), we find

$$L^{\lambda_-}\mathbb{E}^*\left[e^{-r\tau_L}\right] = \mathbb{E}^*\left[e^{-r\tau_L}L^{\lambda_-}\mathbb{1}_{\{\tau_L<\infty\}}\right]$$

* The bound (15.9) does not hold for the positive solution $\lambda_+ = 1$.

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$$
\begin{aligned}
&= \mathbb{E}^*\left[e^{-r\tau_L}(S_{\tau_L})^{\lambda_-}\mathbb{1}_{\{\tau_L<\infty\}}\right]\\
&= \mathbb{E}^*\left[Z_{\tau_L}^{(\lambda_-)}\mathbb{1}_{\{\tau_L<\infty\}}\right]\\
&= \mathbb{E}^*\big[\mathbb{1}_{\{\tau_L<\infty\}}\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)}\big]\\
&= \mathbb{E}^*\big[\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)}\big] \qquad (15.10)\\
&= \lim_{t\to\infty}\mathbb{E}^*\big[Z_{\tau_L\wedge t}^{(\lambda_-)}\big] \qquad (15.11)\\
&= \lim_{t\to\infty}\mathbb{E}^*\big[Z_0^{(\lambda_-)}\big]\\
&= (S_0)^{\lambda_-},
\end{aligned}
$$

where by (15.9) we applied the dominated convergence theorem from (15.10) to (15.11). Hence, we find

$$
\mathbb{E}^*\big[e^{-r\tau_L}\,\big|\,S_0=x\big] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}, \qquad x \geqslant L.
$$

Note also that by (14.15) we have $\mathbb{P}(\tau_L<\infty)=1$ if $r-\sigma^2/2\leqslant 0$, and $\mathbb{P}(\tau_L=+\infty)>0$ if $r-\sigma^2/2>0$. □

Next, we apply Lemma 15.1 in order to price the perpetual American put option.

Proposition 15.2. *Assume that $r>0$. We have*

$$
\begin{aligned}
f_L(x) &= \mathbb{E}^*\big[e^{-(\tau_L-t)r}(K-S_{\tau_L})^+\,\big|\,S_t=x\big]\\
&= \begin{cases} K-x, & 0<x\leqslant L,\\[2ex] (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x\geqslant L.\end{cases}
\end{aligned}
$$

Proof. We take $t=0$ without loss of generality.

i) The result is obvious for $S_0=x\leqslant L$ since in this case we have $\tau_L=t=0$ and $S_{\tau_L}=S_0=x$, so that we only focus on the case $x>L$.

ii) Next, we consider the case $S_0=x>L$. We have

$$
\begin{aligned}
\mathbb{E}^*\big[e^{-r\tau_L}(K-S_{\tau_L})^+\,\big|\,S_0=x\big] &= \mathbb{E}^*\big[\mathbb{1}_{\{\tau_L<\infty\}}e^{-r\tau_L}(K-S_{\tau_L})^+\,\big|\,S_0=x\big]\\
&= \mathbb{E}^*\big[\mathbb{1}_{\{\tau_L<\infty\}}e^{-r\tau_L}(K-L)\,\big|\,S_0=x\big]\\
&= (K-L)\mathbb{E}^*\big[e^{-r\tau_L}\,\big|\,S_0=x\big],
\end{aligned}
$$

and we conclude by the expression of $\mathbb{E}^*\big[e^{-r\tau_L}\,\big|\,S_0=x\big]$ given in Lemma 15.1. □

We note that taking $L = K$ would yield a payoff always equal to 0 for the option holder, hence the value of L should be strictly lower than K. On the other hand, if $L = 0$ the value of τ_L will be infinite almost surely, hence the option price will be 0 when $r \geqslant 0$ from (15.2). Therefore there should be an optimal value L^*, which should be strictly comprised between 0 and K.

Figure 15.1 shows for $K = 100$ that there exists an optimal value $L^* = 85.71$ which maximizes the option price for all values of the underlying asset price.

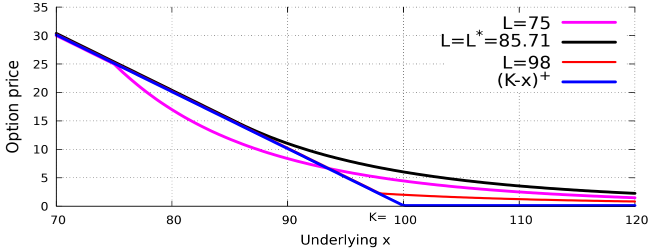

Fig. 15.1: American put by exercising at τ_L for different values of L and $K = 100$.

Smooth pasting In order to compute L^* we observe that, geometrically, the slope of $f_L(x)$ at $x = L^*$ is equal to -1, *i.e.*

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(K - L^*)\frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -1,$$

i.e.

$$\frac{2r}{\sigma^2}(K - L^*) = L^*,$$

or

$$L^* = \frac{2r}{2r+\sigma^2}K < K. \qquad (15.12)$$

We note that L^* tends to zero as σ becomes large or r becomes small, and that L^* tends to K when σ becomes small.

The same conclusion can be reached from the vanishing of the derivative of $L \mapsto f_L(x)$:

$$\frac{\partial f_L(x)}{\partial L} = -\left(\frac{x}{L}\right)^{-2r/\sigma^2} + \frac{2r}{\sigma^2}\frac{K-L}{L}\left(\frac{x}{L}\right)^{-2r/\sigma^2} = 0,$$

cf. page 351 of Shreve (2004). The next Figure 15.2 is a 2-dimensional animation that also shows the optimal value L^* of L.

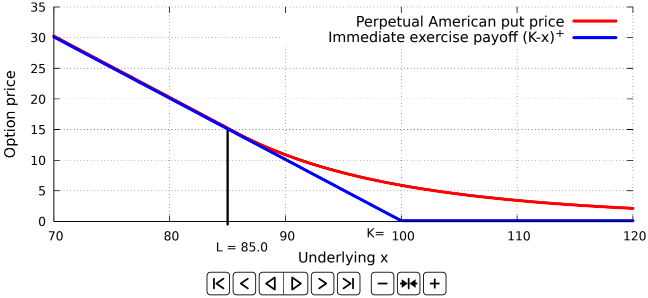

Fig. 15.2: Animated graph of American put prices depending on L with $K = 100$.*

The next Figure 15.3 gives another view of the put option prices according to different values of L, with the optimal value $L^* = 85.71$.

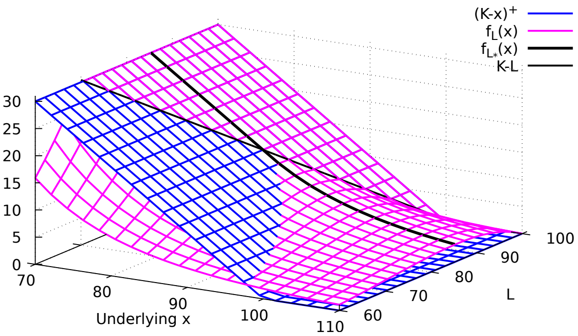

Fig. 15.3: Option price as a function of L and of the underlying asset price.

In Figure 15.4, which is based on the stock price of HSBC Holdings (0005.HK) over year 2009 as in Figures 6.8-6.15, the optimal exercise strategy for an American put option with strike price K=\$77.67 would have been to exercise whenever the underlying asset price goes above $L^* = \$62$, *i.e.* at approximately 54 days, for a payoff of \$25.67. Exercising after a longer time, *e.g.* 85 days, could yield an even higher payoff of over \$65, however, this choice is not

* The animation works in Acrobat Reader on the entire pdf file.

made because decisions are taken based on existing (past) information, and optimization is in expected value (or average) over all possible future paths.

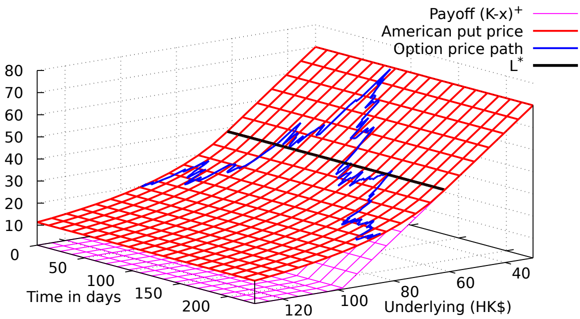

Fig. 15.4: Path of the American put option price on the HSBC stock.

See Exercise 15.6 for the pricing of perpetual American put options with dividends.

15.2 PDE Method for Perpetual Put Options

Exercise. Check by hand calculations that the function f_{L^*} defined as

$$f_{L^*}(x) := \begin{cases} K - x, & 0 < x \leqslant L^* = \dfrac{2r}{2r+\sigma^2}K, \\ \\ \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{x}{K}\right)^{-2r/\sigma^2}, & x \geqslant L^* = \dfrac{2r}{2r+\sigma^2}K, \end{cases} \tag{15.13}$$

satisfies the Partial Differential Equation (PDE)

$$\begin{aligned} -rf_{L^*}(x) + rxf'_{L^*}(x) + \frac{1}{2}\sigma^2x^2f''_{L^*}(x) &= -rK\mathbb{1}_{\{x\leqslant L^*\}} \\ &= \begin{cases} -rK < 0, & 0 < x \leqslant L^* < K, \quad \text{[Exercise now]} \\ 0, & x > L^*. \quad \text{[Wait]} \end{cases} \end{aligned} \tag{15.14}$$

in addition to the conditions

$$\begin{cases} f_{L^*}(x) = K - x, \qquad 0 < x \leqslant L^* < K, & \text{[Exercise now]} \\ \\ f_{L^*}(x) > (K-x)^+, \, x > L^*, & \text{[Wait]} \end{cases}$$

see (15.13).

The above statements can be summarized in the following proposition.

Proposition 15.3. *The function f_{L^*} satisfies the following set of differential inequalities, or variational differential equation:*

$$\begin{cases} f_{L^*}(x) \geqslant (K-x)^+, & (15.15a) \\ rxf'_{L^*}(x) + \frac{\sigma^2}{2}x^2 f''_{L^*}(x) \leqslant rf_{L^*}(x), & (15.15b) \\ \left(rf_{L^*}(x) - rxf'_{L^*}(x) - \frac{\sigma^2}{2}x^2 f''_{L^*}(x)\right)(f_{L^*}(x) - (K-x)^+) = 0. & (15.15c) \end{cases}$$

The equation (15.15c) admits an interpretation in terms of absence of arbitrage, as shown below. By (15.14) and Itô's formula applied to

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t,$$

the discounted portfolio value process

$$\widetilde{f}_{L^*}(S_t) = \mathrm{e}^{-rt} f_{L^*}(S_t), \qquad t \geqslant 0,$$

satisfies

$$\begin{aligned} & d\big(\widetilde{f}_{L^*}(S_t)\big) \\ & = \left(-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t)\right) \mathrm{e}^{-rt} dt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t) d\widehat{B}_t \\ & = -\mathbb{1}_{\{S_t \leqslant L^*\}} rK \,\mathrm{e}^{-rt} dt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t) d\widehat{B}_t \\ & = -\mathbb{1}_{\{f_{L^*}(S_t) = (K-S_t)^+\}} rK \,\mathrm{e}^{-rt} dt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t) d\widehat{B}_t, \end{aligned} \tag{15.16}$$

hence we have the relation

$$\begin{aligned} & \widetilde{f}_{L^*}(S_T) - \widetilde{f}_{L^*}(S_t) \\ & \quad = -rK \int_t^T \mathbb{1}_{\{f_{L^*}(S_u) \leqslant (K-S_u)^+\}} \mathrm{e}^{-ru} du + \int_t^T \mathrm{e}^{-ru}\sigma S_u f'_{L^*}(S_u) d\widehat{B}_u, \end{aligned}$$

for some maturity time $T > 0$, which implies

$$\begin{aligned} & \mathbb{E}^*\big[\widetilde{f}_{L^*}(S_T) - \widetilde{f}_{L^*}(S_t) \mid \mathcal{F}_t\big] = \mathbb{E}^*\big[\widetilde{f}_{L^*}(S_T) \mid \mathcal{F}_t\big] - \widetilde{f}_{L^*}(S_t) \\ & = \mathbb{E}^*\left[-rK \int_t^T \mathbb{1}_{\{f_{L^*}(S_u) \leqslant (K-S_u)^+\}} \mathrm{e}^{-ru} du + \int_t^T \mathrm{e}^{-ru}\sigma S_u f'_{L^*}(S_u) d\widehat{B}_u \;\Big|\; \mathcal{F}_t\right] \end{aligned}$$

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$$= -\mathbb{E}^*\left[rK \int_t^T \mathbb{1}_{\{f_{L^*}(S_u) \leqslant (K-S_u)^+\}} \mathrm{e}^{-ru} du \,\Big|\, \mathcal{F}_t \right],$$

hence the following decomposition of the perpetual American put price into the sum of a European put price and an early exercise premium:

$$\begin{aligned}
&\widetilde{f}_{L^*}(S_t) \\
&= \mathbb{E}^*\big[\widetilde{f}_{L^*}(S_T) \mid \mathcal{F}_t\big] + rK\mathbb{E}^*\left[\int_t^T \mathbb{1}_{\{f_{L^*}(S_u) \leqslant (K-S_u)^+\}} \mathrm{e}^{-ru} du \,\Big|\, \mathcal{F}_t\right] \\
&\geqslant \underbrace{\mathrm{e}^{-rT}\mathbb{E}^*\big[(K-S_T)^+ \mid \mathcal{F}_t\big]}_{\text{European put price}} + \underbrace{rK\mathbb{E}^*\left[\int_t^T \mathbb{1}_{\{S_u \leqslant L^*\}} \mathrm{e}^{-ru} du \,\Big|\, \mathcal{F}_t\right]}_{\text{Early exercise premium}},
\end{aligned} \tag{15.17}$$

$0 \leqslant t \leqslant T$, see also Theorem 8.4.1 in § 8.4 in Elliott and Kopp (2005) on early exercise premiums. From (15.16) we also make the following observations.

a) From Equation (15.15c), $\widetilde{f}_{L^*}(S_t)$ is a martingale when

$$f_{L^*}(S_t) > (K - S_t)^+, \quad i.e. \quad S_t > L^*, \quad [\text{Wait}]$$

and in this case the expected rate of return of the hedging portfolio value $f_{L^*}(S_t)$ equals the rate r of the riskless asset, as

$$d\big(\widetilde{f}_{L^*}(S_t)\big) = \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t) d\widehat{B}_t,$$

or

$$d\big(f_{L^*}(S_t)\big) = d\big(\mathrm{e}^{rt}\widetilde{f}_{L^*}(S_t)\big) = rf_{L^*}(S_t)dt + \sigma S_t f'_{L^*}(S_t)d\widehat{B}_t,$$

and the investor prefers to wait.

b) On the other hand, if

$$f_{L^*}(S_t) = (K - S_t)^+, \quad i.e. \quad 0 < S_t < L^*, \quad [\text{Exercise now}]$$

the return of the hedging portfolio becomes lower than r as $d\big(\widetilde{f}_{L^*}(S_t)\big) = -rK\,\mathrm{e}^{-rt}dt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t$ and

$$\begin{aligned}
d\big(f_{L^*}(S_t)\big) &= d\big(\mathrm{e}^{rt}\widetilde{f}_{L^*}(S_t)\big) \\
&= rf_{L^*}(S_t)dt - rKdt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t.
\end{aligned}$$

In this case it is not worth waiting as (15.15b)-(15.15c) show that the return of the hedging portfolio is lower than that of the riskless asset, *i.e.*:

$$-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t) = -rK < 0,$$

exercise becomes immediate since the process $\widetilde{f}_{L^*}(S_t)$ becomes a (strict) *super*martingale, and (15.15c) implies $f_{L^*}(x) = (K-x)^+$.

In view of the above derivation, it should make sense to assert that $f_{L^*}(S_t)$ is the price at time t of the perpetual American put option. The next proposition confirms that this is indeed the case, and that the optimal exercise time is $\tau^* = \tau_{L^*}$.

Proposition 15.4. *The price of the perpetual American put option is given for all $t \geqslant 0$ by*

$$\begin{aligned} f_{L^*}(S_t) &= \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(K-S_\tau)^+ \,\big|\, S_t \big] \\ &= \mathbb{E}^*\big[e^{-(\tau_{L^*}-t)r}(K-S_{\tau_{L^*}})^+ \,\big|\, S_t \big] \\ &= \begin{cases} K - S_t, & 0 < S_t \leqslant L^*, \\ (K-L^*)\left(\dfrac{S_t}{L^*}\right)^{-2r/\sigma^2} = \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{S_t}{K}\right)^{-2r/\sigma^2}, & S_t \geqslant L^*. \end{cases} \end{aligned}$$

Proof. *i*) Since the drift

$$-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t)$$

in Itô's formula (15.16) is nonpositive by the inequality (15.15b), the discounted portfolio value process

$$u \mapsto e^{-ru} f_{L^*}(S_u), \qquad u \in [t, \infty),$$

is a *super*martingale. As a consequence, for all (*a.s.* finite) stopping times $\tau \in [t, \infty)$ we have, by (14.13),

$$e^{-rt} f_{L^*}(S_t) \geqslant \mathbb{E}^*\big[e^{-r\tau} f_{L^*}(S_\tau) \,\big|\, S_t \big] \geqslant \mathbb{E}^*\big[e^{-r\tau}(K-S_\tau)^+ \,\big|\, S_t \big],$$

from (15.15a), which implies

$$e^{-rt} f_{L^*}(S_t) \geqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-r\tau}(K-S_\tau)^+ \,\big|\, S_t \big]. \qquad (15.18)$$

ii) The converse inequality is obvious by Proposition 15.2, as

$$f_{L^*}(S_t) = \mathbb{E}^*\big[e^{-(\tau_{L^*}-t)r}(K-S_{\tau_{L^*}})^+ \,\big|\, S_t \big]$$

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$$\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(K-S_\tau)^+ \,\big|\, S_t\big], \qquad (15.19)$$

since τ_{L^*} is a stopping time larger than $t \in \mathbb{R}_+$. The inequalities (15.18) and (15.19) allow us to derive the equality

$$f_{L^*}(S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(K-S_\tau)^+ \,\big|\, S_t\big].$$

□

Remark. We note that the converse inequality (15.19) can also be obtained from the variational PDE (15.15a)-(15.15c) itself, without relying on Proposition 15.2. For this, taking $\tau = \tau_{L^*}$ we note that the process

$$u \mapsto e^{-(u\wedge\tau_{L^*})r} f_{L^*}(S_{u\wedge\tau_{L^*}}), \qquad u \geqslant t,$$

is not only a *super*martingale, it is also a martingale until exercise at time τ_{L^*} by (15.14) since $S_{u\wedge\tau_{L^*}} \geqslant L^*$, hence we have

$$e^{-rt} f_{L^*}(S_t) = \mathbb{E}^*\big[e^{-(u\wedge\tau_{L^*})r} f_{L^*}(S_{u\wedge\tau_{L^*}}) \,\big|\, S_t\big], \qquad u \geqslant t,$$

hence after letting u tend to infinity we obtain

$$\begin{aligned} e^{-rt} f_{L^*}(S_t) &= \mathbb{E}^*\big[e^{-r\tau_{L^*}} f_{L^*}(S_{\tau_{L^*}}) \,\big|\, S_t\big] \\ &= \mathbb{E}^*\big[e^{-r\tau_{L^*}} f_{L^*}(L^*) \,\big|\, S_t\big] \\ &= \mathbb{E}^*\big[e^{-r\tau_{L^*}} (K - S_{\tau_{L^*}})^+ \,\big|\, S_t\big] \\ &\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-r\tau_{L^*}} (K - S_{\tau_{L^*}})^+ \,\big|\, S_t\big], \end{aligned}$$

which recovers (15.19) as

$$e^{-rt} f_{L^*}(S_t) \leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-r\tau} (K - S_\tau)^+ \,\big|\, S_t\big], \qquad t \geqslant 0.$$

15.3 Perpetual American Call Options

In this section we consider the pricing of perpetual call options.

Two-choice optimal stopping at a fixed price level for perpetual call options

Given $L > K$ a fixed price, consider the following choices for the exercise of a *call* option with strike price K:

1. If $S_t \geqslant L$, then exercise at time t.

2. Otherwise, wait until the first hitting time

$$\tau_L = \inf\{u \geqslant t \ : \ S_u = L\}$$

and exercise the option and time τ_L.

In case $S_t \geqslant L$, the immediate exercise (or intrinsic) payoff will be

$$(S_t - K)^+ = S_t - K,$$

since $K < L \leqslant S_t$.

In case $S_t < L$, as $r > 0$ the price of the option will be given by

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\big[e^{-(\tau_L - t)r}(S_{\tau_L} - K)^+ \,\big|\, S_t\big] \\ &= \mathbb{E}^*\big[e^{-(\tau_L - t)r}(S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \,\big|\, S_t\big] \\ &= \mathbb{E}^*\big[e^{-(\tau_L - t)r}(L - K)^+ \,\big|\, S_t\big] \\ &= (L-K)\mathbb{E}^*\big[e^{-(\tau_L - t)r} \,\big|\, S_t\big]. \end{aligned}$$

Lemma 15.5. *Assume that $r > 0$. We have*

$$\mathbb{E}^*\big[e^{-r\tau_L}\big] = \frac{S_0}{L}.$$

Proof. We only need to consider the case $S_0 = x < L$. Note that for all $\lambda \in \mathbb{R}$, the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := (S_t)^\lambda e^{-r\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} = (S_0)^\lambda e^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2}, \quad t \geqslant 0,$$

defined in (15.5) is a martingale under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Hence the stopped process $\big(Z_{t\wedge\tau_L}^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ is a martingale and it has constant expectation, *i.e.*, we have

$$\mathbb{E}^*\big[Z_{t\wedge\tau_L}^{(\lambda)}\big] = \mathbb{E}^*\big[Z_0^{(\lambda)}\big] = (S_0)^\lambda, \qquad t \geqslant 0. \tag{15.20}$$

Choosing λ such that

$$r = r\lambda - \lambda\frac{\sigma^2}{2} + \lambda^2\frac{\sigma^2}{2},$$

i.e.

$$0 = \lambda^2\frac{\sigma^2}{2} + \lambda\left(r - \frac{\sigma^2}{2}\right) - r = \frac{\sigma^2}{2}\left(\lambda + \frac{2r}{\sigma^2}\right)(\lambda - 1),$$

Relation (15.20) rewrites as

$$\mathbb{E}^*\big[(S_{t\wedge\tau_L})^\lambda e^{-(t\wedge\tau_L)r}\big] = (S_0)^\lambda, \qquad t \geqslant 0. \tag{15.21}$$

Choosing the positive solution* $\lambda_+ = 1$ yields the bound

$$0 \leqslant Z_t^{(\lambda_+)} = \mathrm{e}^{-rt} S_t \leqslant S_t \leqslant L, \qquad 0 \leqslant t \leqslant \tau_L, \tag{15.22}$$

since $r > 0$ and $S_t \leqslant L$ for all $t \in [0, \tau_L]$. Hence we have $\lim_{t\to\infty} Z_t^{(\lambda_+)} = 0$, and since $\lim_{t\to\infty} Z_{\tau_L \wedge t}^{(\lambda_+)} = Z_{\tau_L}^{(\lambda_+)}$ on $\{\tau_L < \infty\}$, by (15.21)-(15.22) and the dominated convergence theorem, we get

$$\begin{aligned}
L\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}\big] &= \mathbb{E}^*\big[\mathrm{e}^{-r\tau_L} S_{\tau_L} \mathbb{1}_{\{\tau_L < \infty\}}\big] \\
&= \mathbb{E}^*\big[\lim_{t\to\infty} \mathrm{e}^{-(\tau_L \wedge t) r} S_{\tau_L \wedge t}\big] \\
&= \mathbb{E}^*\big[\lim_{t\to\infty} Z_{\tau_L \wedge t}^{(\lambda_+)}\big] \\
&= \lim_{t\to\infty} \mathbb{E}^*\big[Z_{\tau_L \wedge t}^{(\lambda_+)}\big] \\
&= \lim_{t\to\infty} \mathbb{E}^*\big[Z_0^{(\lambda_+)}\big] \\
&= S_0,
\end{aligned}$$

which yields

$$\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}\big] = \frac{S_0}{L}. \tag{15.23}$$

Note also that by (14.17) we have $\mathbb{P}(\tau_L < \infty) = 1$ if $r - \sigma^2/2 \geqslant 0$, and $\mathbb{P}(\tau_L = +\infty) > 0$ if $r - \sigma^2/2 < 0$. □

Next, we apply Lemma 15.5 in order to price the perpetual American call option.

Proposition 15.6. *Assume that $r > 0$. The price of the perpetual American call option is given by $f_L(S_t)$ when $S_t < L$, where*

$$f_L(x) = \begin{cases} x - K, & x \geqslant L \geqslant K, \\ (L - K)\dfrac{x}{L}, & 0 < x \leqslant L. \end{cases} \tag{15.24}$$

Proof. $i)$ The result is obvious for $S_0 = x \geqslant L$ since in this case we have $\tau_L = t = 0$ and $S_{\tau_L} = S_0 = x$, so that we only focus on the case $x < L$.
$ii)$ Next, we consider the case $S_0 = x < L$. We have

$$\begin{aligned}
\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}(S_{\tau_L} - K)^+ \,\big|\, S_0 = x\big] &= \mathbb{E}^*\big[\mathbb{1}_{\{\tau_L < \infty\}} \mathrm{e}^{-r\tau_L}(S_{\tau_L} - K)^+ \,\big|\, S_0 = x\big] \\
&= \mathbb{E}^*\big[\mathbb{1}_{\{\tau_L < \infty\}} \mathrm{e}^{-r\tau_L}(L - K) \,\big|\, S_0 = x\big]
\end{aligned}$$

* The bound (15.22) does not hold for the negative solution $\lambda_- = -2r/\sigma^2$.

$$= (L-K)\mathbb{E}^*\left[e^{-r\tau_L} \,\middle|\, S_0 = x\right],$$

and we conclude by the expression of $\mathbb{E}^*\left[e^{-r\tau_L} \,\middle|\, S_0 = x\right]$ given in Lemma 15.1. □

One can check from Figures 15.5 and 15.6 that the situation completely differs from the perpetual put option case, as there does not exist an optimal value L^* that would maximize the option price for all values of the underlying asset price.

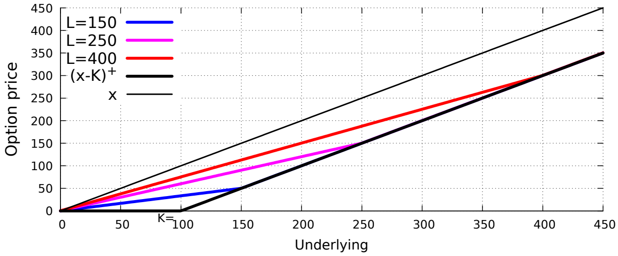

Fig. 15.5: American call prices by exercising at τ_L for different values of L and $K = 100$.

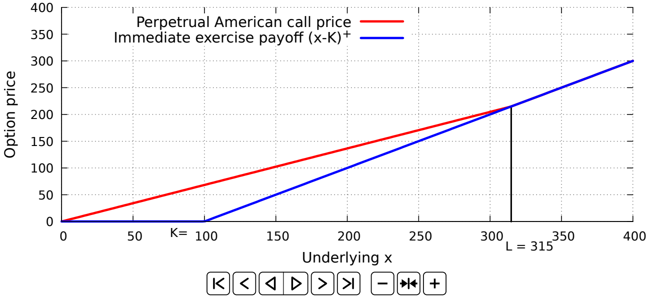

Fig. 15.6: Animated graph of American option prices depending on L with $K = 100$.*

The intuition behind this picture is that there is no upper limit above which one should exercise the option, and in order to price the American perpetual call option we have to let L go to infinity, *i.e.* the "optimal" exercise strategy is to wait indefinitely.

* The animation works in Acrobat Reader on the entire pdf file.

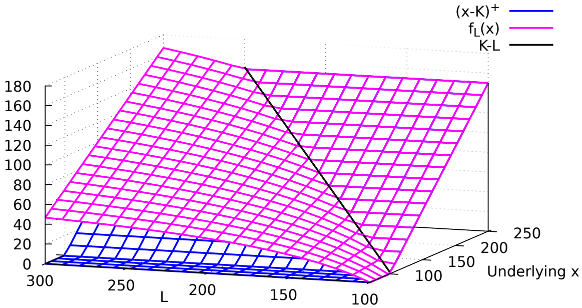

Fig. 15.7: American call prices for different values of L.

We check from (15.24) that

$$\lim_{L\to\infty} f_L(x) = x - \lim_{L\to\infty} K\frac{x}{L} = x, \qquad x > 0. \tag{15.25}$$

As a consequence we have the following proposition.

Proposition 15.7. *Assume that $r \geqslant 0$. The price of the perpetual American call option is given by*

$$\operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(S_\tau - K)^+ \bigm| S_t\big] = S_t, \qquad t \geqslant 0. \tag{15.26}$$

Proof. For all $L > K$ we have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\big[e^{-(\tau_L-t)r}(S_{\tau_L} - K)^+ \bigm| S_t\big] \\ &\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(S_\tau - K)^+ \bigm| S_t\big], \qquad t \geqslant 0, \end{aligned}$$

hence from (15.25), taking the limit as $L \to \infty$ yields

$$S_t \leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[e^{-(\tau-t)r}(S_\tau - K)^+ \bigm| S_t\big]. \tag{15.27}$$

On the other hand, since $u \mapsto e^{-(u-t)r}S_u$ is a martingale, by (14.13) we have, for all stopping times $\tau \in [t, \infty)$,

$$\mathbb{E}^*\big[e^{-(\tau-t)r}(S_\tau - K)^+ \bigm| S_t\big] \leqslant \mathbb{E}^*\big[e^{-(\tau-t)r}S_\tau \bigm| S_t\big] \leqslant S_t, \quad t \geqslant 0,$$

hence

$$\underset{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \,\big|\, S_t\big] \leqslant S_t, \qquad t \geqslant 0,$$

which shows (15.26) by (15.27). □

We may also check that since $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, the process $t \mapsto (\mathrm{e}^{-rt}S_t - K)^+$ is a *sub*martingale since the function $x \mapsto (x-K)^+$ is convex, hence for all bounded stopping times τ such that $t \leqslant \tau$ we have

$$(S_t - K)^+ \leqslant \mathbb{E}^*\big[(\mathrm{e}^{-(\tau-t)r}S_\tau - K)^+ \,\big|\, S_t\big] \leqslant \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \,\big|\, S_t\big],$$

$t \geqslant 0$, showing that it is always better to wait than to exercise at time t, and the optimal exercise time is $\tau^* = +\infty$. This argument does not apply to American put options.

See Exercise 15.7 for the pricing of perpetual American call options with dividends.

15.4 Finite Expiration American Options

In this section we consider finite expiration American put and call options with strike price K. The prices of such options can be expressed as

$$f(t, S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(K - S_\tau)^+ \,\big|\, S_t\big],$$

and

$$f(t, S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \,\big|\, S_t\big].$$

Two-choice optimal stopping at fixed times with finite expiration

We start by considering the optimal stopping problem in a simplified setting where $\tau \in \{t, T\}$ is allowed at time t to take only *two* values t (which corresponds to immediate exercise) and T (wait until maturity).

Proposition 15.8. *Assume that $r \geqslant 0$. For any stopping time $\tau \geqslant t$, the price of the European call option exercised at time τ satisfies the bound*

$$(x - K)^+ \leqslant \mathbb{E}^*[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \mid S_t = x], \qquad x, t > 0. \qquad (15.28)$$

Proof. Since the function $x \mapsto x^+ = \mathrm{Max}(x, 0)$ is convex non-decreasing and the process $(\mathrm{e}^{-rt}S_t - \mathrm{e}^{-rt}K)_{t\in\mathbb{R}_+}$ is a *sub*martingale under $\mathbb{P}^*$ since $r \geqslant 0$, Proposition 14.5-(b) shows that that $t \mapsto (\mathrm{e}^{-rt}S_t - \mathrm{e}^{-rt}K)^+$ is a *sub*martingale by the Jensen (1906) inequality (14.3). Hence, by (14.13) applied to *sub*martingales, for any stopping time τ bounded by $t > 0$ we have

$$\begin{aligned}(S_t - K)^+ &= \mathrm{e}^{rt}(\mathrm{e}^{-rt}S_t - \mathrm{e}^{-rt}K)^+ \\ &\leqslant \mathrm{e}^{rt}\mathbb{E}^*[(\mathrm{e}^{-r\tau}S_\tau - \mathrm{e}^{-r\tau}K)^+ \mid \mathcal{F}_t] \\ &= \mathbb{E}^*[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \mid \mathcal{F}_t],\end{aligned}$$

which yields (15.28). □

In particular, for the deterministic time $\tau := T \geqslant t$ we get

$$(x-K)^+ \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid S_t = x], \qquad x, t > 0.$$

as illustrated in Figure 15.8 using the Black-Scholes formula for European call options, see also Figure 6.16a. In other words, taking $x = S_t$, the payoff $(S_t - K)^+$ of immediate exercise at time t is always lower than the expected payoff $\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid S_t = x]$ given by exercise at maturity T. As a consequence, the optimal strategy for the investor is to wait until time T to exercise an American call option, rather than exercising earlier at time t. Note that the situation is completely different when $r < 0$, see Figure 6.17a.

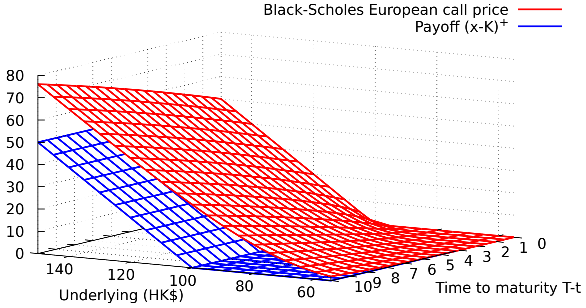

Fig. 15.8: Black-Scholes call option price with $r = 3\% > 0$ *vs.* $(x,t) \mapsto (x-K)^+$.

More generally, it can be shown that the price of the American call option equals the price of the corresponding European call option with maturity T, *i.e.*

$$f(t, S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T - K)^+ \,\big|\, S_t\big],$$

i.e. T is the optimal exercise date, see Proposition 15.10 below or §14.4 of Steele (2001) for a proof.

Put options

For put options the situation is entirely different. The Black-Scholes formula for European put options shows that the inequality

$$(K-x)^+ \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid S_t = x],$$

does not always hold, as illustrated in Figure 15.9, see also Figure 6.16b.

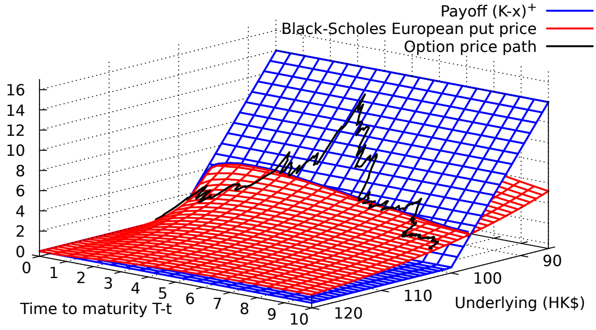

Fig. 15.9: Black-Scholes put option price with $r = 3\% > 0$ *vs.* $(x, t) \mapsto (K - x)^+$.

As a consequence, the optimal exercise decision for a put option depends on whether $(K - S_t)^+ \leqslant e^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid S_t]$ (in which case one chooses to exercise at time T) or $(K - S_t)^+ > e^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid S_t]$ (in which case one chooses to exercise at time t).

A view from above of the graph of Figure 15.9 shows the existence of an optimal frontier depending on time to maturity and on the price of the underlying asset, instead of being given by a constant level L^* as in Section 15.1, see Figure 15.10.

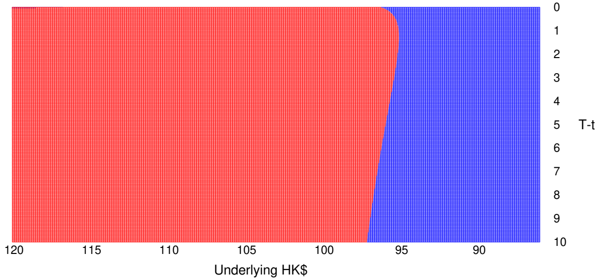

Fig. 15.10: Optimal frontier for the exercise of a put option.

At a given time t, one will choose to exercise immediately if $(S_t, T - t)$ belongs to the blue area on the right, and to wait until maturity if $(S_t, T - t)$ belongs to the red area on the left.

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When $r = 0$ we have $L^* = 0$, and the next remark shows that in this case it is always better to exercise an finite expiration American put option at maturity T, see also Exercise 15.9.

Proposition 15.9. *Assume that $r = 0$ and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative convex function. Then the price of the finite expiration American option with payoff function ϕ on the underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ coincides with the corresponding vanilla option price:*

$$f(t, S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\phi(S_\tau) \,\big|\, S_t\big] = \mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big],$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option, and $\tau^* = T$.

Proof. Since the function ϕ is convex and $(S_{t+s})_{s\in[0,T-t]}$ is a martingale under the risk-neutral measure $\mathbb{P}^*$, we know from Proposition 14.5-(a) that the process $(\phi(S_{t+s}))_{s\in[0,T-t]}$ is a *sub*martingale. Therefore, for all (bounded) stopping times τ comprised between t and T we have

$$\mathbb{E}^*[\phi(S_\tau) \mid \mathcal{F}_t] \leqslant \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

hence it is always better to wait until time T than to exercise at time $\tau \in [t, T]$, and this yields

$$\operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\phi(S_\tau) \,\big|\, S_t\big] \leqslant \mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big].$$

Since the constant T is a stopping time, it attains the above supremum. □

15.5 PDE Method with Finite Expiration

Let us describe the PDE associated to American put options. After discretization $\{0 = t_0 < t_1 < \ldots < t_N = T\}$ of the time interval $[0, T]$, the optimal exercise strategy for the American put option can be described as follow at each time step:

If $f(t, S_t) > (K - S_t)^+$, wait.

If $f(t, S_t) = (K - S_t)^+$, exercise the option at time t.

Note that we cannot have $f(t, S_t) < (K - S_t)^+$.

If $f(t, S_t) > (K - S_t)^+$ the expected return of the hedging portfolio equals the return r of the riskless asset. In this case, $f(t, S_t)$ follows the Black-Scholes PDE

$$rf(t, S_t) = \frac{\partial f}{\partial t}(t, S_t) + rS_t\frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t, S_t),$$

whereas if $f(t, S_t) = (K - S_t)^+$ it is not worth waiting as the return of the hedging portfolio is lower than that of the riskless asset:

$$rf(t, S_t) \geqslant \frac{\partial f}{\partial t}(t, S_t) + rS_t\frac{\partial f}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t, S_t).$$

As a consequence, $f(t, x)$ should solve the following variational PDE, see Jaillet et al. (1990), Theorem 8.5.9 in Elliott and Kopp (2005) and Theorem 5 in Üstünel (2009):

$$\begin{cases} f(t, x) \geqslant f(T, x) = (K - x)^+, & (15.29a) \\ \\ \dfrac{\partial f}{\partial t}(t, x) + rx\dfrac{\partial f}{\partial x}(t, x) + \dfrac{\sigma^2}{2}x^2\dfrac{\partial^2 f}{\partial x^2}(t, x) \leqslant rf(t, x), & (15.29b) \\ \\ \left(\dfrac{\partial f}{\partial t}(t, x) + rx\dfrac{\partial f}{\partial x}(t, x) + \dfrac{\sigma^2}{2}x^2\dfrac{\partial^2 f}{\partial x^2}(t, x) - rf(t, x)\right) & \\ \qquad\qquad\qquad\qquad \times \left(f(t, x) - (K - x)^+\right) = 0, & (15.29c) \end{cases}$$

$x > 0$, $0 \leqslant t \leqslant T$, subject to the terminal condition $f(T, x) = (K - x)^+$. In other words, equality holds either in (15.29a) or in (15.29b) due to the presence of the term $(f(t, x) - (K - x)^+)$ in (15.29c).

The optimal exercise strategy consists in exercising the put option as soon as the equality $f(u, S_u) = (K - S_u)^+$ holds, *i.e.* at the time

$$\tau^* = T \wedge \inf\{u \geqslant t \; : \; f(u, S_u) = (K - S_u)^+\},$$

after which the process $\widetilde{f}_{L^*}(S_t)$ ceases to be a martingale and becomes a (strict) *super*martingale.

A simple procedure to compute numerically the price of an American put option is to use a finite difference scheme while simply enforcing the condition $f(t, x) \geqslant (K - x)^+$ at every iteration by adding the condition

$$f(t_i, x_j) := \mathrm{Max}(f(t_i, x_j), (K - x_j)^+)$$

right after the computation of $f(t_i, x_j)$.

The next Figure 15.11 shows a numerical resolution of the variational PDE (15.29a)-(15.29c) using the above simplified (implicit) finite difference scheme, see also Jacka (1991) for properties of the optimal boundary function. In comparison with Figure 15.4, one can check that the PDE solution becomes time-dependent in the finite expiration case.

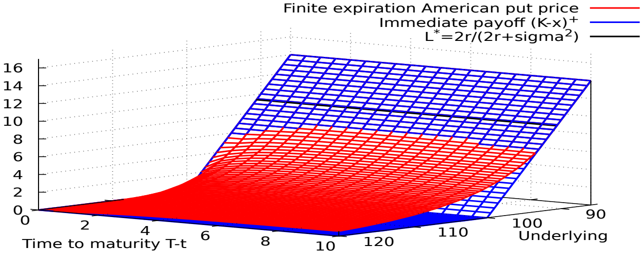

Fig. 15.11: PDE estimates of finite expiration American put option prices.

In general, one will choose to exercise the put option when

$$f(t,S_t) = (K - S_t)^+,$$

i.e. within the blue area in Figure (15.11). We check that the optimal threshold $L^* = 90.64$ of the corresponding perpetual put option is within the exercise region, which is consistent since the perpetual optimal strategy should allow one to wait longer than in the finite expiration case.

The numerical computation of the American put option price

$$f(t,S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K - S_\tau)^+ \,\middle|\, S_t \right]$$

can also be done by dynamic programming and backward optimization using the Longstaff and Schwartz (2001) (or Least Square Monte Carlo, LSM) algorithm as in Figure 15.12.

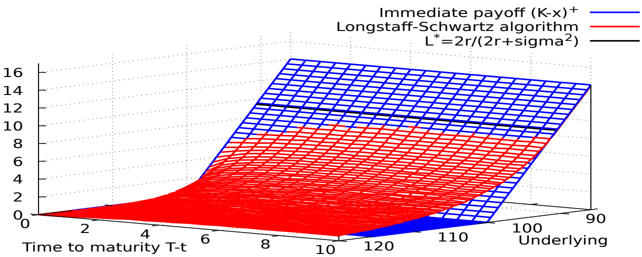

Fig. 15.12: Longstaff-Schwartz estimates of finite expiration American put option prices.

In Figure 15.12 above and Figure 15.13 below the optimal threshold of the corresponding perpetual put option is again $L^* = 90.64$ and falls within the exercise region. Also, the optimal threshold is closer to L^* for large time to maturities, which shows that the perpetual option approximates the finite expiration option in that situation. In the next Figure 15.13 we compare the numerical computation of the American put option price by the finite difference and Longstaff-Schwartz methods.

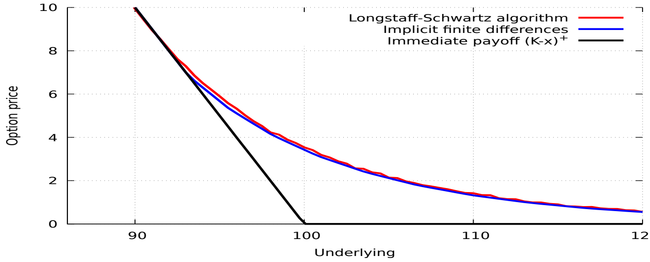

Fig. 15.13: Comparison between Longstaff-Schwartz and finite differences.

It turns out that, although both results are very close, the Longstaff-Schwartz method performs better in the critical area close to exercise at it yields the expected continuously differentiable solution, and the simple numerical PDE solution tends to underestimate the optimal threshold. Also, a small error in the values of the solution translates into a large error on the value of the optimal exercise threshold.

The fOptions package in R contains a finite expiration American put option pricer based on the Barone-Adesi and Whaley (1987) approximation, see Exercise 15.4, however, the approximation is valid only for certain parameter ranges. See also Allegretto et al. (1995) for a related approximation of the early exercise premium (15.17).

The finite expiration American call option

In the next proposition we compute the price of a finite expiration American call option with an arbitrary convex payoff function ϕ.

Proposition 15.10. *Assume that $r \geqslant 0$ and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative convex function such that $\phi(0) = 0$. The price of the finite expiration American call option with payoff function ϕ on the underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ is given by*

$$f(t, S_t) = \underset{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ Stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}\phi(S_\tau) \,\middle|\, S_t \right] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi(S_T) \,\middle|\, S_t \right],$$

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i.e. the optimal strategy is to wait until the maturity time T to exercise the option, and $\tau^* = T$.

Proof. Since the function ϕ is convex and $\phi(0) = 0$, we have

$$\phi(px) = \phi((1-p) \times 0 + px) \leqslant (1-p) \times \phi(0) + p\phi(x) = p\phi(x), \quad (15.30)$$

for all $p \in [0,1]$ and $x \geqslant 0$. Next, taking $p := \mathrm{e}^{-rs}$ in (15.30) we note that

$$\begin{aligned} \mathrm{e}^{-rs}\mathbb{E}^*\big[\phi(S_{t+s}) \,\big|\, \mathcal{F}_t\big] &\geqslant \mathrm{e}^{-rs}\phi\big(\mathbb{E}^*\big[S_{t+s} \,\big|\, \mathcal{F}_t\big]\big) \\ &\geqslant \phi\big(\mathrm{e}^{-rs}\mathbb{E}^*\big[S_{t+s} \,\big|\, \mathcal{F}_t\big]\big) \\ &= \phi(S_t), \end{aligned}$$

where we used Jensen's inequality Proposition 14.4 applied to the convex function ϕ. Hence the process $s \mapsto \mathrm{e}^{-rs}\phi(S_{t+s})$ is a *sub*martingale, and by the optional stopping theorem for *sub*martingales, see (14.9), for all (bounded) stopping times τ comprised between t and T we have

$$\mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}\phi(S_\tau) \mid \mathcal{F}_t\big] \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T) \mid \mathcal{F}_t\big].$$

In other words, it is always better to wait until time T than to exercise at time $\tau \in [t,T]$, and this yields

$$\operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}\phi(S_\tau) \,\big|\, S_t\big] \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big].$$

The converse inequality

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big] \leqslant \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}\phi(S_\tau) \,\big|\, S_t\big],$$

is obvious because T is a stopping time. □

As a consequence of Proposition 15.10 applied to the convex function $\phi(x) = (x-K)^+$, the price of the finite expiration American call option is given by

$$\begin{aligned} f(t,S_t) &= \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau - K)^+ \,\big|\, S_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T-K)^+ \,\big|\, S_t\big], \end{aligned}$$

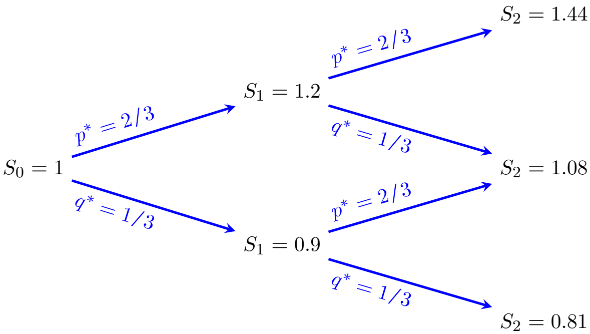
i.e. the optimal strategy is to wait until the maturity time T to exercise the option. In the following Table 15.1 we summarize the optimal exercise strategies for the pricing of American options.

Option type	Perpetual		Finite expiration	
	Pricing	Optimal time	Pricing	Optimal time
Put option	$\begin{cases} K - S_t, & 0 < S_t \leqslant L^*, \\ (K - L^*) \left(\frac{S_t}{L^*} \right)^{-2r/\sigma^2}, & S_t \geqslant L^*. \end{cases}$	$\tau^* = \tau_{L^*}$	Solve the PDE (15.29a)-(15.29c) for $f(t, x)$ or use Longstaff and Schwartz (2001)	$\tau^* = T \wedge \inf\{u \geqslant t : f(u, S_u) = (K - S_u)^+\}$
Call option	S_t	$\tau^* = +\infty$	$e^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ \mid S_t]$	$\tau^* = T$

Table 15.1: Optimal exercise strategies.

Exercises

Exercise 15.1 Consider a two-step binomial model $(S_k)_{k=0,1,2}$ with interest rate $r = 0\%$ and risk-neutral probabilities (p^*, q^*):



a) At time $t = 1$, would you exercise the American put option with strike price $K = 1.25$ if $S_1 = 1.2$? If $S_1 = 0.9$?
b) What would be your investment allocation at time $t = 0$?*

Exercise 15.2 Let $r > 0$ and $\sigma > 0$.

a) Show that for every $C > 0$, the function $f(x) := Cx^{-2r/\sigma^2}$ solves the differential equation

$$\begin{cases} rf(x) = rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x), \\ \lim_{x \to \infty} f(x) = 0. \end{cases}$$

* Download the corresponding discrete-time **IPython notebook** that can be run **here** or **here**.

b) Show that for every $K > 0$ there exists a unique level $L^* \in (0, K)$ and constant $C > 0$ such that $f(x)$ also solves the smooth fit conditions $f(L^*) = K - L^*$ and $f'(L^*) = -1$.

Exercise 15.3 Consider an American butterfly option with the following payoff function.

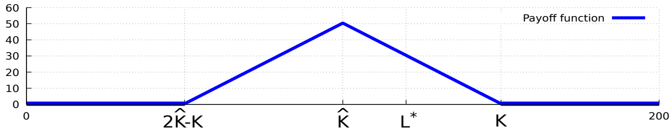


Fig. 15.14: Butterfly payoff function.

Price the *perpetual* American butterfly option with $r > 0$ in the following cases.

a) $\widehat{K} \leqslant L^* \leqslant S_0$.
b) $\widehat{K} \leqslant S_0 \leqslant L^*$.
c) $0 \leqslant S_0 \leqslant \widehat{K}$.

Exercise 15.4 (Barone-Adesi and Whaley (1987)) We approximate the finite expiration American put option price with strike price K as

$$f(x,T) \simeq \begin{cases} \mathrm{BS_p}(x,T) + \alpha (x/S^*)^{-2r/\sigma^2}, & x > S^*, \quad (15.31) \\ K - x, & x \leqslant S^*, \quad (15.32) \end{cases}$$

where $\alpha > 0$ is a parameter, $S^* > 0$ is called the *critical price*, and $\mathrm{BS}_p(x,T) = \mathrm{e}^{-rT} K \Phi(-d_-(x,T)) - x\Phi(-d_+(x,T))$ is the Black-Scholes *put* pricing function.

a) Find the value α^* of α which achieves a smooth fit (equality of derivatives in x) between (15.31) and (15.32) at $x = S^*$.
b) Derive the equation satisfied by the critical price S^*.

Exercise 15.5 Consider the process $(X_t)_{t\in\mathbb{R}_+}$ given by $X_t := tZ$, $t \in \mathbb{R}_+$, where $Z \in \{0, 1\}$ is a Bernoulli random variable with $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 0) = 1/2$. Given $\epsilon \geqslant 0$, let the random time τ_ϵ be defined as

$$\tau_\epsilon := \inf\{t > 0 \ : \ X_t > \epsilon\},$$

with $\inf \emptyset = +\infty$, and let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ denote the filtration generated by $(X_t)_{t\in\mathbb{R}_+}$.

a) Give the possible values of τ_ϵ in $[0,\infty]$ depending on the value of Z.
b) Take $\epsilon = 0$. Is $\tau_0 := \inf\{t > 0 \ : \ X_t > 0\}$ an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time?
Hint: Consider the event $\{\tau_0 > 0\}$.
c) Take $\epsilon > 0$. Is $\tau_\epsilon := \inf\{t > 0 \ : \ X_t > \epsilon\}$ an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time?
Hint: Consider the event $\{\tau_\epsilon > t\}$ for $t \geqslant 0$.

Exercise 15.6 American put options with dividends, cf. Exercise 8.5 in Shreve (2004). Consider a dividend-paying asset priced as

$$S_t = S_0 \,\mathrm{e}^{(r-\delta)t+\sigma\widehat{B}_t-\sigma^2t/2}, \qquad t \geqslant 0,$$

where $r > 0$ is the risk-free interest rate, $\delta \geqslant 0$ is a continuous dividend rate, $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, and $\sigma > 0$ is the volatility coefficient. Consider the American put option with payoff

$$(\kappa - S_\tau)^+ = \begin{cases} \kappa - S_\tau & \text{if } S_\tau \leqslant \kappa, \\ \\ 0 & \text{if } S_\tau > \kappa, \end{cases}$$

when exercised at the stopping time $\tau > 0$. Given $L \in (0,\kappa)$ a fixed level, consider the following exercise strategy for the above option:

- If $S_t \leqslant L$, then exercise at time t.
- If $S_t > L$, wait until the hitting time $\tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and exercise the option at time τ_L.

a) Give the intrinsic option value at time $t = 0$ in case $S_0 \leqslant L$.

In what follows we work with $S_0 = x > L$.
b) Show that for all $\lambda \in \mathbb{R}$ the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := \left(\frac{S_t}{S_0}\right)^\lambda \mathrm{e}^{-((r-\delta)\lambda-\lambda(1-\lambda)\sigma^2/2)t}$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$.
c) Show that $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ can be rewritten as

$$Z_t^{(\lambda)} = \left(\frac{S_t}{S_0}\right)^\lambda \mathrm{e}^{-rt}, \qquad t \geqslant 0,$$

for two values $\lambda_- \leqslant 0 \leqslant \lambda_+$ of λ that can be computed explicitly.
d) Choosing the negative solution λ_-, show that

$$0 \leqslant Z_t^{(\lambda_-)} \leqslant \left(\frac{L}{S_0}\right)^{\lambda_-}, \qquad 0 \leqslant t \leqslant \tau_L.$$

e) Let τ_L denote the hitting time

$$\tau_L = \inf\{u \in \mathbb{R}_+ \ : \ S_u \leqslant L\}.$$

By application of the Stopping Time Theorem 14.8 to the martingale $(Z_t)_{t\in\mathbb{R}_+}$, show that

$$\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}\big] = \left(\frac{S_0}{L}\right)^{\lambda_-}, \tag{15.33}$$

with

$$\lambda_- := \frac{-(r-\delta-\sigma^2/2) - \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}. \tag{15.34}$$

f) Show that for all $L \in (0, K)$ we have

$$\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \bigm| S_0 = x\big] = \begin{cases} K - x, & 0 < x \leqslant L, \\ (K-L)\left(\dfrac{x}{L}\right)^{\frac{-(r-\delta-\sigma^2/2)-\sqrt{(r-\delta-\sigma^2/2)^2+4r\sigma^2/2}}{\sigma^2}}, & x \geqslant L. \end{cases}$$

g) Show that the value L^* of L that maximizes

$$f_L(x) := \mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \bigm| S_0 = x\big]$$

for every $x > 0$ is given by

$$L^* = \frac{\lambda_-}{\lambda_- - 1}K.$$

h) Show that

$$f_{L^*}(x) = \begin{cases} K - x, & 0 < x \leqslant L^* = \dfrac{\lambda_-}{\lambda_- - 1}K, \\ \left(\dfrac{1-\lambda_-}{K}\right)^{\lambda_- - 1}\left(\dfrac{x}{-\lambda_-}\right)^{\lambda_-}, & x \geqslant L^* = \dfrac{\lambda_-}{\lambda_- - 1}K, \end{cases}$$

i) Show by hand computation that $f_{L^*}(x)$ satisfies the variational differential equation

$$\begin{cases} f_{L^*}(x) \geqslant (K-x)^+, & (15.35a) \\ (r-\delta)xf'_{L^*}(x) + \frac{1}{2}\sigma^2x^2f''_{L^*}(x) \leqslant rf_{L^*}(x), & (15.35b) \\ \left(rf_{L^*}(x) - (r-\delta)xf'_{L^*}(x) - \frac{1}{2}\sigma^2x^2f''_{L^*}(x)\right) & (15.35c) \\ \qquad\qquad \times\,(f_{L^*}(x) - (K-x)^+) = 0. \end{cases}$$

j) Using Itô's formula, check that the discounted portfolio value process

$$t \mapsto \mathrm{e}^{-rt} f_{L^*}(S_t)$$

is a *super*martingale.

k) Show that we have

$$f_{L^*}(S_0) \geqslant \operatorname*{Sup}_{\tau \text{ Stopping time}} \mathbb{E}^*\big[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \,\big|\, S_0\big].$$

l) Show that the stopped process

$$s \mapsto \mathrm{e}^{-(s\wedge\tau_{L^*})r} f_{L^*}(S_{s\wedge\tau_{L^*}}), \qquad s \geqslant 0,$$

is a martingale, and that

$$f_{L^*}(S_0) \leqslant \operatorname*{Sup}_{\tau \text{ Stopping time}} \mathbb{E}^*\big[\mathrm{e}^{-r\tau}(K-S_\tau)^+\big].$$

m) Fix $t \in \mathbb{R}_+$ and let τ_{L^*} denote the hitting time

$$\tau_{L^*} = \inf\{u \geqslant t \ : \ S_u = L^*\}.$$

Conclude that the price of the perpetual American put option with dividend is given for all $t \in \mathbb{R}_+$ by

$$\begin{aligned} f_{L^*}(S_t) &= \mathbb{E}^*\big[\mathrm{e}^{-(\tau_{L^*}-t)r}(K-S_{\tau_{L^*}})^+ \,\big|\, S_t\big] \\ &= \begin{cases} K - S_t, & 0 < S_t \leqslant \dfrac{\lambda_-}{\lambda_- - 1}K, \\ \left(\dfrac{1-\lambda_-}{K}\right)^{\lambda_- - 1}\left(\dfrac{S_t}{-\lambda_-}\right)^{\lambda_-}, & S_t \geqslant \dfrac{\lambda_-}{\lambda_- - 1}K, \end{cases} \end{aligned}$$

where $\lambda_- < 0$ is given by (15.34), and

$$\tau_{L^*} = \inf\{u \geqslant t \ : \ S_u \leqslant L\}.$$

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Exercise 15.7 American call options with dividends, see § 9.3 of Wilmott (2006). Consider a dividend-paying asset priced as $S_t = S_0 \, \mathrm{e}^{(r-\delta)t+\sigma \widehat{B}_t-\sigma^2 t/2}$, $t \geqslant 0$, where $r > 0$ is the risk-free interest rate, $\delta \geqslant 0$ is a continuous dividend rate, and $\sigma > 0$.

a) Show that for all $\lambda \in \mathbb{R}$ the process $Z_t^{(\lambda)} := (S_t)^\lambda \, \mathrm{e}^{-((r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2)t}$ is a martingale under $\mathbb{P}^*$.
b) Show that we have $Z_t^{(\lambda)} = (S_t)^\lambda \, \mathrm{e}^{-rt}$ for two values $\lambda_- \leqslant 0$, $1 \leqslant \lambda_+$ of λ satisfying a certain equation.
c) Show that $0 \leqslant Z_t^{(\lambda_+)} \leqslant L^{\lambda_+}$ for $0 \leqslant t \leqslant \tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and compute $\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}(S_{\tau_L} - K)^+ \,\big|\, S_0 = x\big]$ when $S_0 = x < L$ and $K < L$.

Exercise 15.8 Optimal stopping for exchange options (Gerber and Shiu (1996)). We consider two risky assets S_1 and S_2 modeled by

$$S_1(t) = S_1(0) \, \mathrm{e}^{\sigma_1 W_t + rt - \sigma_2^2 t/2} \qquad \text{and} \qquad S_2(t) = S_2(0) \, \mathrm{e}^{\sigma_2 W_t + rt - \sigma_2^2 t/2}, \tag{15.36}$$

$t \geqslant 0$, with $\underline{\sigma_2 > \sigma_1 \geqslant 0}$ and $r > 0$, and the perpetual optimal stopping problem

$$\operatorname*{Sup}_{\tau \text{ Stopping time}} \mathbb{E}\big[\mathrm{e}^{-r\tau}(S_1(\tau) - S_2(\tau))^+\big],$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$.

a) Find $\alpha > 1$ such that the process

$$Z_t := \mathrm{e}^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha}, \qquad t \geqslant 0, \tag{15.37}$$

is a martingale.

b) For some fixed $L \geqslant 1$, consider the hitting time

$$\tau_L = \inf\big\{t \in \mathbb{R}_+ \ : \ S_1(t) \geqslant L S_2(t)\big\},$$

and show that

$$\mathbb{E}\big[\mathrm{e}^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\big] = (L-1)\mathbb{E}\big[\mathrm{e}^{-r\tau_L} S_2(\tau_L)\big].$$

c) By an application of the Stopping Time Theorem 14.8 to the martingale (15.37), show that we have

$$\mathbb{E}\big[\mathrm{e}^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\big] = \frac{L-1}{L^\alpha} S_1(0)^\alpha S_2(0)^{1-\alpha}.$$

d) Show that the price of the perpetual exchange option is given by

$$\operatorname*{Sup}_{\tau \text{ Stopping time}} \mathbb{E}\big[\mathrm{e}^{-r\tau}(S_1(\tau) - S_2(\tau))^+\big] = \frac{L^* - 1}{(L^*)^\alpha} S_1(0)^\alpha S_2(0)^{1-\alpha},$$

where
$$L^* = \frac{\alpha}{\alpha - 1}.$$

e) As an application of Question (d), compute the perpetual American put option price
$$\operatorname*{Sup}_{\tau \text{ Stopping time}} \mathbb{E}\big[\mathrm{e}^{-r\tau}(\kappa - S_2(\tau))^+\big]$$
when $r = \sigma_2^2/2$.

Exercise 15.9 Consider an underlying asset whose price is written as
$$S_t = S_0 \,\mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$
where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $\sigma > 0$ denotes the volatility coefficient, and $r \in \mathbb{R}$ is the risk-free interest rate. For any $\lambda \in \mathbb{R}$ we consider the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined by
$$\begin{aligned} Z_t^{(\lambda)} &:= \mathrm{e}^{-rt}(S_t)^\lambda \\ &= (S_0)^\lambda \,\mathrm{e}^{\lambda\sigma B_t - \lambda^2\sigma^2 t/2 + (\lambda-1)(\lambda+2r/\sigma^2)\sigma^2 t/2}, \quad t \geqslant 0. \end{aligned} \tag{15.38}$$

a) Assume that $r \geqslant -\sigma^2/2$. Show that, under $\mathbb{P}^*$, the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ is a *super*martingale when $-2r/\sigma^2 \leqslant \lambda \leqslant 1$, and that it is a *sub*martingale when $\lambda \in (-\infty, -2r/\sigma^2] \cup [1, \infty)$.
b) Assume that $r \leqslant -\sigma^2/2$. Show that, under $\mathbb{P}^*$, the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ is a *super*martingale when $1 \leqslant \lambda \leqslant -2r/\sigma^2$, and that it is a *sub*martingale when $\lambda \in (-\infty, 1] \cup [-2r/\sigma^2, \infty)$.
c) From this question onwards, we assume that $r < 0$. Given $L > 0$, let τ_L denote the hitting time
$$\tau_L = \inf\{u \in \mathbb{R}_+ \ : \ S_u = L\}.$$
By application of the Stopping Time Theorem 14.8 to $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ to suitable values of λ, show that
$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \leqslant \begin{cases} \left(\dfrac{x}{L}\right)^{\mathrm{Max}(1,-2r/\sigma^2)}, & x \geqslant L, \\[2ex] \left(\dfrac{x}{L}\right)^{\min(1,-2r/\sigma^2)}, & 0 < x \leqslant L. \end{cases}$$
d) Deduce an upper bound on the price

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$$\mathbb{E}^*\left[e^{-r\tau_L}(K - S_{\tau_L})^+ \,\middle|\, S_0 = x\right]$$

of the American put option exercised in finite time under the stopping strategy τ_L when $L \in (0, K)$ and $x \geqslant L$.

e) Show that when $r \leqslant -\sigma^2/2$, the upper bound of Question (d) increases and tends to $+\infty$ when L decreases to 0.

f) Find an upper bound on the price

$$\mathbb{E}^*\left[e^{-r\tau_L}(S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \,\middle|\, S_0 = x\right]$$

of the American call option exercised in finite time under the stopping strategy τ_L when $L \geqslant K$ and $x \leqslant L$.

g) Show that when $-\sigma^2/2 \leqslant r < 0$, the upper bound of Question (f) increases in $L \geqslant K$, and tends to S_0 as L increases to $+\infty$.

Exercise 15.10 Perpetual American binary options.

a) Compute the price

$$C_b^{\mathrm{Am}}(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\left[e^{-(\tau - t)r} \mathbb{1}_{\{S_\tau \geqslant K\}} \,\middle|\, S_t\right]$$

of the perpetual American binary call option.

b) Compute the price

$$P_b^{\mathrm{Am}}(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\left[e^{-(\tau - t)r} \mathbb{1}_{\{S_\tau \leqslant K\}} \,\middle|\, S_t\right]$$

of the perpetual American binary put option.

Exercise 15.11 Finite expiration American binary options. An American binary (or digital) call (resp. put) option with maturity $T > 0$ on an underlying asset process $(S_t)_{t \in \mathbb{R}_+} = (e^{rt + \sigma B_t - \sigma^2 t/2})_{t \in \mathbb{R}_+}$ can be exercised at any time $t \in [0, T]$, at the choice of the option holder.

The call (resp. put) option exercised at time t yields the payoff $\mathbb{1}_{[K,\infty)}(S_t)$ (resp. $\mathbb{1}_{[0,K]}(S_t)$), and the option holder wants to find an exercise strategy that will maximize his payoff.

a) Consider the following possible situations at time t:

   i) $S_t \geqslant K$,
   ii) $S_t < K$.

   In each case (i) and (ii), tell whether you would choose to exercise the call option immediately, or to wait.

b) Consider the following possible situations at time t:

i) $S_t > K$,
ii) $S_t \leqslant K$.

In each case (i) and (ii), tell whether you would choose to exercise the put option immediately, or to wait.

c) The price $C_d^{\rm Am}(t,T,S_t)$ of an American binary call option is known to satisfy the Black-Scholes PDE

$$rC_d^{\rm Am}(t,T,x) = \frac{\partial C_d^{\rm Am}}{\partial t}(t,T,x) + rx\frac{\partial C_d^{\rm Am}}{\partial x}(t,T,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_d^{\rm Am}}{\partial x^2}(t,T,x).$$

Based on your answers to Question (a), how would you set the boundary conditions $C_d^{\rm Am}(t,T,K)$, $0 \leqslant t < T$, and $C_d^{\rm Am}(T,T,x)$, $0 \leqslant x < K$?

d) The price $P_d^{\rm Am}(t,T,S_t)$ of an American binary put option is known to satisfy the same Black-Scholes PDE

$$rP_d^{\rm Am}(t,T,x) = \frac{\partial P_d^{\rm Am}}{\partial t}(t,T,x) + rx\frac{\partial P_d^{\rm Am}}{\partial x}(t,T,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P_d^{\rm Am}}{\partial x^2}(t,T,x). \tag{15.39}$$

Based on your answers to Question (b), how would you set the boundary conditions $P_d^{\rm Am}(t,T,K)$, $0 \leqslant t < T$, and $P_d^{\rm Am}(T,T,x)$, $x > K$?

e) Show that the optimal exercise strategy for the American binary call option with strike price K is to exercise as soon as the price of the underlying asset reaches the level K, *i.e.* at time

$$\tau_K := \inf\{u \geqslant t \ : \ S_u = K\},$$

starting from any level $S_t \leqslant K$, and that the price $C_d^{\rm Am}(t,T,S_t)$ of the American binary call option is given by

$$C_d^{\rm Am}(t,x) = \mathbb{E}\big[{\rm e}^{-(\tau_K - t)r} \mathbb{1}_{\{\tau_K < T\}} \mid S_t = x \big].$$

f) Show that the price $C_d^{\rm Am}(t,T,S_t)$ of the American binary call option is equal to

$$C_d^{\rm Am}(t,T,x) = \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant x \leqslant K,$$

that this formula is consistent with the answer to Question (c), and that it recovers the answer to Question (a) of Exercise 15.10 as T tends to infinity.

g) Show that the optimal exercise strategy for the American binary put option with strike price K is to exercise as soon as the price of the underlying asset reaches the level K, *i.e.* at time

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$$\tau_K := \inf\{u \geqslant t \ : \ S_u = K\},$$

starting from any level $S_t \geqslant K$, and that the price $P_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary put option is

$$P_d^{\mathrm{Am}}(t,T,x) = \mathbb{E}\big[\mathrm{e}^{-(\tau_K - t)r}\mathbb{1}_{\{\tau_K<T\}} \mid S_t = x\big], \qquad x \geqslant K.$$

h) Show that the price $P_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary put option is equal to

$$P_d^{\mathrm{Am}}(t,T,x) = \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad x \geqslant K,$$

that this formula is consistent with the answer to Question (d), and that it recovers the answer to Question (b) of Exercise 15.10 as T tends to infinity.

i) Does the standard call-put parity relation hold for American binary options?

Exercise 15.12 American forward contracts. Consider $(S_t)_{t\in\mathbb{R}_+}$ an asset price process given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

where $r > 0$ and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

a) Compute the price

$$f(t,S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(K-S_\tau)\,\big|\,S_t\big],$$

and optimal exercise strategy of a finite expiration American-type short forward contract with strike price K on the underlying asset priced $(S_t)_{t\in\mathbb{R}_+}$, with payoff $K - S_\tau$ when exercised at time $\tau \in [0,T]$.

b) Compute the price

$$f(t,S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}(S_\tau-K)\,\big|\,S_t\big],$$

and optimal exercise strategy of a finite expiration American-type long forward contract with strike price K on the underlying asset priced $(S_t)_{t\in\mathbb{R}_+}$, with payoff $S_\tau - K$ when exercised at time $\tau \in [0,T]$.

c) How are the answers to Questions (a) and (b) modified in the case of perpetual options with $T = +\infty$?

Exercise 15.13 Consider an underlying asset price process written as

$$S_t = S_0 \,\mathrm{e}^{rt+\sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, with $\sigma, r > 0$.

a) Show that the processes $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ defined as

$$Y_t := \mathrm{e}^{-rt} S_t^{-2r/\sigma^2} \quad \text{and} \quad Z_t := \mathrm{e}^{-rt} S_t, \qquad t \geqslant 0,$$

are both martingales under $\mathbb{P}^*$.

b) Let τ_L denote the hitting time

$$\tau_L = \inf\{u \in \mathbb{R}_+ \ : \ S_u = L\}.$$

By application of the Stopping Time Theorem 14.8 to the martingales $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$, show that

$$\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L} \bigm| S_0 = x\big] = \begin{cases} \dfrac{x}{L}, & 0 < x \leqslant L, \\[2ex] \left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L. \end{cases}$$

c) Compute the price $\mathbb{E}^*[\,\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})]$ of a short forward contract under the exercise strategy τ_L.

d) Show that for every value of $S_0 = x$ there is an optimal value L_x^* of L that maximizes $L \mapsto \mathbb{E}[\,\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})]$.

e) Would you use the stopping strategy

$$\tau_{L_x^*} = \inf\{u \in \mathbb{R}_+ \ : \ S_u = L_x^*\}$$

as an optimal exercise strategy for the short forward contract with payoff $K - S_\tau$?

Exercise 15.14 Let $p \geqslant 1$ and consider a power put option with payoff

$$((\kappa - S_\tau)^+)^p = \begin{cases} (\kappa - S_\tau)^p & \text{if } S_\tau \leqslant \kappa, \\ 0 & \text{if } S_\tau > \kappa, \end{cases}$$

exercised at time τ, on an underlying asset whose price S_t is written as

$$S_t = S_0 \,\mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $r \geqslant 0$ is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

Given $L \in (0, \kappa)$ a fixed price, consider the following choices for the exercise of a *put* option with strike price κ:

i) If $S_t \leqslant L$, then exercise at time t.

ii) Otherwise, wait until the first hitting time $\tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and exercise the option at time τ_L.

a) Under the above strategy, what is the option payoff equal to if $S_t \leqslant L$?
b) Show that in case $S_t > L$, the price of the option is equal to

$$f_L(S_t) = (\kappa - L)^p \mathbb{E}^*\big[\mathrm{e}^{-(\tau_L - t)r} \bigm| S_t\big].$$

c) Compute the price $f_L(S_t)$ of the option at time t.

Hint: Recall that by (15.4) we have $\mathbb{E}^*[\mathrm{e}^{-(\tau_L-t)r} \mid S_t = x] = (x/L)^{-2r/\sigma^2}$ for $x \geqslant L$.

d) Compute the optimal value L^* that maximizes $L \mapsto f_L(x)$ for all fixed $x > 0$.

Hint: Observe that, geometrically, the slope of $x \mapsto f_L(x)$ at $x = L^*$ is equal to $-p(\kappa - L^*)^{p-1}$.

e) How would you compute the American put option price

$$f(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ Stopping time}}} \mathbb{E}^*\big[\mathrm{e}^{-(\tau-t)r}((\kappa - S_\tau)^+)^p \bigm| S_t\big] \ \ ?$$

Exercise 15.15 Same questions as in Exercise 15.14, this time for the option with payoff $\kappa - (S_\tau)^p$ exercised at time τ, with $p > 0$.