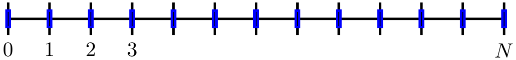
Chapter 2
Discrete-Time Market Model

The single-step model considered in Chapter 1 is extended to a discrete-time model with $N+1$ time instants $t = 0, 1, \ldots, N$. A basic limitation of the one-step model is that it does not allow for trading until the end of the first time period is reached, while the multistep model allows for multiple portfolio re-allocations over time. The Cox-Ross-Rubinstein (CRR) model, or binomial model, is considered as an example whose importance also lies with its computer implementability.

2.1 Discrete-Time Compounding

In this chapter, we work in a finite horizon discrete-time model indexed by $\{0, 1, \ldots, N\}$.



Investment plan

We invest an amount m each year in an investment plan that carries a constant interest rate r. At the end of the N-*th* year, the value of the amount m invested at the beginning of year $k = 1, 2, \ldots, N$ has turned into $(1+r)^{N-k+1}m$, and the value of the plan at the end of the N-*th* year becomes

$$\begin{aligned} A_N := m\sum_{k=1}^{N}(1+r)^{N-k+1} \qquad (2.1) \\ = m\sum_{k=1}^{N}(1+r)^k \\ = m(1+r)\frac{(1+r)^N-1}{r}, \end{aligned}$$

hence the duration N of the plan satisfies

$$N+1 = \frac{1}{\log(1+r)}\log\left(1+r+\frac{rA_N}{m}\right).$$

Loan repayment

At time $t=0$ one borrows an amount $A_1 := A$ over a period of N years at the constant interest rate r per year.

Proposition 2.1. *Constant repayments. Assuming that the loan is completely repaid at the beginning of year $N+1$, the amount m refunded every year is given by*

$$m = \frac{r(1+r)^NA}{(1+r)^N-1} = \frac{r}{1-(1+r)^{-N}}A. \qquad (2.2)$$

Proof. Denoting by A_k the amount owed by the borrower at the beginning of year n^o $k=1,2,\ldots,N$ with $A_1=A$, the amount m refunded at the end of the first year can be decomposed as

$$m = rA_1 + (m-rA_1),$$

into rA_1 paid in interest and $m-rA_1$ in principal repayment, *i.e.* there remains

$$\begin{aligned} A_2 &= A_1-(m-rA_1) \\ &= (1+r)A_1-m, \end{aligned}$$

to be refunded. Similarly, the amount m refunded at the end of the second year can be decomposed as

$$m = rA_2 + (m-rA_2),$$

into rA_2 paid in interest and $m - rA_2$ in principal repayment, *i.e.* there remains

$$\begin{aligned} A_3 &= A_2 - (m - rA_2) \\ &= (1+r)A_2 - m \\ &= (1+r)((1+r)A_1 - m) - m \\ &= (1+r)^2 A_1 - m - (1+r)m \end{aligned}$$

to be refunded. After repeating the argument we find that at the beginning of year k there remains

$$\begin{aligned} A_k &= (1+r)^{k-1}A_1 - m - (1+r)m - \cdots - (1+r)^{k-2}m \\ &= (1+r)^{k-1}A_1 - m\sum_{i=0}^{k-2}(1+r)^i \\ &= (1+r)^{k-1}A_1 + m\frac{1-(1+r)^{k-1}}{r} \end{aligned}$$

to be refunded, *i.e.*

$$A_k = \frac{m - (1+r)^{k-1}(m - rA)}{r}, \qquad k = 1, 2, \ldots, N. \tag{2.3}$$

In other words, the repayment at the end of year k can be decomposed as

$$m = rA_k + (m - rA_k),$$

with

$$rA_k = m + (1+r)^{k-1}(rA_1 - m)$$

in interest repayment, and

$$m - rA_k = (1+r)^{k-1}(m - rA_1)$$

in principal repayment. At the beginning of year $N+1$, the loan should be completely repaid, hence $A_{N+1} = 0$, which reads

$$(1+r)^N A + m\frac{1-(1+r)^N}{r} = 0,$$

and yields (2.2). □

We also have

$$\frac{A}{m} = \frac{1-(1+r)^{-N}}{r}. \tag{2.4}$$

and

$$N = \frac{1}{\log(1+r)} \log \frac{m}{m-rA} = -\frac{\log(1-rA/m)}{\log(1+r)}.$$

Remark: One needs $m > rA$ in order for N to be finite.

The next proposition is a direct consequence of (2.2) and (2.3).

Proposition 2.2. *The k-th interest repayment can be written as*

$$rA_k = m\left(1 - \frac{1}{(1+r)^{N-k+1}}\right) = mr \sum_{l=1}^{N-k+1} (1+r)^{-l},$$

and the k-th principal repayment is

$$m - rA_k = \frac{m}{(1+r)^{N-k+1}}, \qquad k = 1, 2, \ldots, N.$$

Note that the sum of *discounted* payments at the rate r is

$$\sum_{l=1}^{N} \frac{m}{(1+r)^l} = m\frac{1-(1+r)^{-N}}{r} = A.$$

In particular, the first interest repayment satisfies

$$rA = rA_1 = mr\sum_{l=1}^{N} \frac{1}{(1+r)^l} = m\left(1-(1+r)^{-N}\right),$$

and the first principal repayment is

$$m - rA = \frac{m}{(1+r)^N}.$$

2.2 Arbitrage and Self-Financing Portfolios

Stochastic processes

A *stochastic process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(X_t)_{t\in\mathfrak{T}}$ of random variables $X_t : \Omega \longrightarrow \mathbb{R}$ indexed by a set $\mathfrak{T}$. Examples include:

- the one-step (or two-instant) model: $\mathfrak{T} = \{0, 1\}$,
- the discrete-time model with finite horizon: $\mathfrak{T} = \{0, 1, \ldots, N\}$,
- the discrete-time model with infinite horizon: $\mathfrak{T} = \mathbb{N}$,
- the continuous-time model: $\mathfrak{T} = \mathbb{R}_+$.

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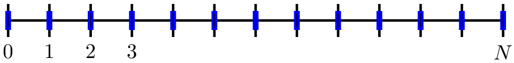
For real-world examples of stochastic processes, one can mention:

- the time evolution of a risky asset, *e.g.* X_t represents the price of the asset at time $t \in \mathfrak{T}$.
- the time evolution of a physical parameter - for example, X_t represents a temperature observed at time $t \in \mathfrak{T}$.

In this chapter, we focus on the finite horizon discrete-time model with $\mathfrak{T} = \{0, 1, \ldots, N\}$.



Asset price modeling

The prices at time $t = 0$ of $d + 1$ assets numbered $0, 1, \ldots, d$ are denoted by the *random* vector

$$\overline{S}_0 = \left(S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)}\right)$$

in $\mathbb{R}^{d+1}$. Similarly, the values at time $t = 1, 2, \ldots, N$ of assets n^o $0, 1, \ldots, d$ are denoted by the *random* vector

$$\overline{S}_t = \left(S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}\right)$$

on Ω, which forms a stochastic process $\left(\overline{S}_t\right)_{t=0,1,\ldots,N}$.

In what follows we assume that asset n^o 0 is a riskless asset (of savings account type) yielding an interest rate r, *i.e.* we have

$$S_t^{(0)} = (1+r)^t S_0^{(0)}, \qquad t = 0, 1, \ldots, N.$$

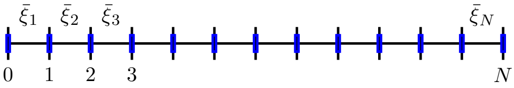
Portfolio strategies

Definition 2.3. *A portfolio strategy is a stochastic process* $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N} \subset \mathbb{R}^{d+1}$ *where* $\xi_t^{(k)}$ *denotes the (possibly fractional) quantity of asset n^o k held in the portfolio over the time interval* $(t-1, t]$, $t = 1, 2, \ldots, N$.

Note that the portfolio allocation

$$\bar{\xi}_t = \left(\xi_t^{(0)}, \xi_t^{(1)}, \ldots, \xi_t^{(d)}\right)$$

is decided at time $t - 1$ and remains constant over the interval $(t-1, t]$ while the stock price changes from $S_{t-1}^{(k)}$ to $S_t^{(k)}$ over this time interval.



In other words, the quantity

$$\xi_t^{(k)} S_{t-1}^{(k)}$$

represents the amount invested in asset n^o k at the beginning of the time interval $(t-1,t]$, and

$$\xi_t^{(k)} S_t^{(k)}$$

represents the value of this investment at the end of the time interval $(t-1,t]$, $t=1,2,\ldots,N$.

Self-financing portfolio strategies

The opening price of the portfolio at the beginning of the time interval $(t-1,t]$ is

$$\bar{\xi}_t \cdot \overline{S}_{t-1} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t-1}^{(k)},$$

when the market "opens" at time $t-1$. When the market "closes"at the end of the time interval $(t-1,t]$, it takes the closing value

$$\bar{\xi}_t \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \tag{2.5}$$

$t=1,2,\ldots,N$. After the new portfolio allocation $\bar{\xi}_{t+1}$ is designed we get the new portfolio opening price

$$\bar{\xi}_{t+1} \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \tag{2.6}$$

at the beginning of the next trading session $(t,t+1]$, $t=0,1,\ldots,N-1$.

Note that here, the stock price $\overline{S}_t$ is assumed to remain constant "overnight", *i.e.* from the end of $(t-1,t]$ to the beginning of $(t,t+1]$, $t=1,2,\ldots,N-1$.

In case (2.5) coincides with (2.6) for $t=0,1,\ldots,N-1$ we say that the portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ is *self-financing*. A non self-financing portfolio could be either bleeding money, or burning cash, for no good reason.

Definition 2.4. *A portfolio strategy* $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ *is said to be* self-financing *if*

$$\bar{\xi}_t \cdot \overline{S}_t = \bar{\xi}_{t+1} \cdot \overline{S}_t, \qquad t=1,2,\ldots,N-1, \tag{2.7}$$

i.e.

$$\underbrace{\sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}}_{\text{Closing value}} = \underbrace{\sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}}_{\text{Opening price}}, \qquad t = 1, 2, \ldots, N-1.$$

The meaning of the self-financing condition (2.7) is simply that one cannot take any money in or out of the portfolio during the "overnight" transition period at time t. In other words, at the beginning of the new trading session $(t, t+1]$ one should re-invest the totality of the portfolio value obtained at the end of the interval $(t-1, t]$.

The next figure is an illustration of the self-financing condition.

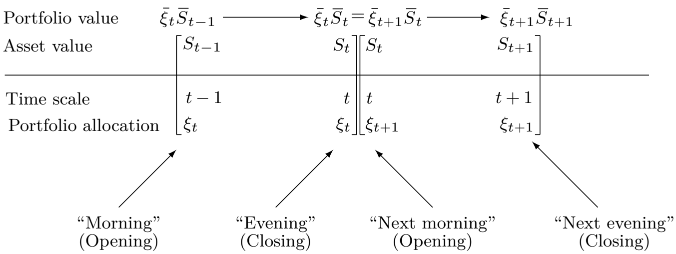


Fig. 2.1: Illustration of the self-financing condition (2.7).

By (2.5) and (2.6) the self-financing condition (2.7) can be rewritten as

$$\sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \qquad t = 0, 1, \ldots, N-1,$$

or

$$\sum_{k=0}^{d} \left(\xi_{t+1}^{(k)} - \xi_t^{(k)} \right) S_t^{(k)} = 0, \qquad t = 0, 1, \ldots, N-1.$$

Note that any portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ which is constant over time, *i.e.* $\bar{\xi}_t = \bar{\xi}_{t+1}$, $t = 1, 2, \ldots, N-1$, is self-financing by construction.

Here, portfolio re-allocation happens "overnight", during which time the global portfolio value remains the same due to the self-financing condition. The portfolio allocation ξ_t remains the same throughout the day, however, the portfolio value changes from morning to evening due to a change in the stock price. Also, $\bar{\xi}_0$ is not defined and its value is actually not needed in this framework.

In case $d = 1$ we are only trading $d+1 = 2$ assets with prices $\overline{S}_t = (S_t^{(0)}, S_t^{(1)})$ and the portfolio allocation reads $\bar{\xi}_t = (\xi_t^{(0)}, \xi_t^{(1)})$. In this case, the self-financing condition means that:

- In the event of an increase in the stock position $\xi_t^{(1)}$, the corresponding cost of purchase $(\xi_{t+1}^{(1)} - \xi_t^{(1)})S_t^{(1)} > 0$ has to be *deducted from* the savings account value $\xi_t^{(0)} S_t^{(0)}$, which becomes updated as

$$\xi_{t+1}^{(0)} S_t^{(0)} = \xi_t^{(0)} S_t^{(0)} - (\xi_{t+1}^{(1)} - \xi_t^{(1)}) S_t^{(1)},$$

  recovering (2.7).
- In the event of a decrease in the stock position $\xi_t^{(1)}$, the corresponding sale profit $(\xi_t^{(1)} - \xi_{t+1}^{(1)})S_t^{(1)} > 0$ has to be *added to* the savings account value $\xi_t^{(0)} S_t^{(0)}$, which becomes updated as

$$\xi_{t+1}^{(0)} S_t^{(0)} = \xi_t^{(0)} S_t^{(0)} + (\xi_t^{(1)} - \xi_{t+1}^{(1)}) S_t^{(1)},$$

  recovering (2.7).

Clearly, the chosen unit of time may not be the day and it can be replaced by weeks, hours, minutes, or fractions of seconds in high-frequency trading.

Portfolio value

Definition 2.5. *The portfolio opening prices at times $t = 0, 1, \ldots, N-1$ are defined as*

$$V_t := \bar{\xi}_{t+1} \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \quad t = 0, 1, \ldots, N-1.$$

Under the self-financing condition (2.7), the portfolio closing values V_t at times $t = 1, 2, \ldots, N$ rewrite as

$$V_t = \bar{\xi}_t \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t = 1, 2, \ldots, N, \tag{2.8}$$

as summarized in Table 2.1.

	V_0	V_1	V_2	$\cdots\cdots$	V_{N-1}	V_N
Opening price	$\bar{\xi}_1 \cdot \overline{S}_0$	$\bar{\xi}_2 \cdot \overline{S}_1$	$\bar{\xi}_3 \cdot \overline{S}_2$	$\cdots\cdots$	$\bar{\xi}_N \cdot \overline{S}_{N-1}$	N.A.
Closing value	N.A.	$\bar{\xi}_1 \cdot \overline{S}_1$	$\bar{\xi}_2 \cdot \overline{S}_2$	$\cdots\cdots$	$\bar{\xi}_{N-1} \cdot \overline{S}_{N-1}$	$\bar{\xi}_N \cdot \overline{S}_N$

Table 2.1: Self-financing portfolio value process.

Discounting

Summing the prices of assets considered at different times requires discounting with respect to a common date in order to compensate for possible monetary inflation. Assuming a yearly risk-free interest rate r, one dollar of year N can be added to one dollar of year $N+1$ either as $(1+r)\$1+\$1$ if pricing occurs as of year $N+1$, or as $\$1+(1+r)^{-1}\$1$ if pricing occurs as of year N.

My portfolio S_t grew by $b = 5\%$ this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My portfolio S_t grew by $b = 5\%$ this year.

The risk-free or inflation rate is $r = 10\%$.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

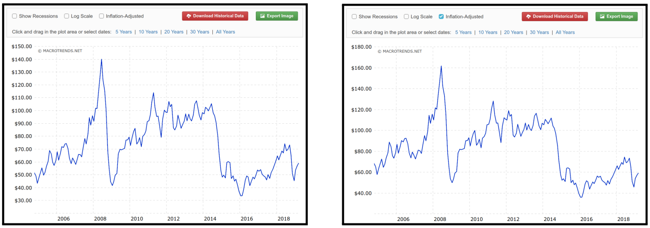
Fig. 2.2: Why apply discounting?

Definition 2.6. *Let*

$$\overline{X}_t := \left(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\right)$$

denote the vector of discounted asset prices, defined as:

$$\widetilde{S}_t^{(i)} = \frac{1}{(1+r)^t} S_t^{(i)}, \qquad i = 0, 1, \ldots, d, \quad t = 0, 1, \ldots, N.$$



(a) *Without* inflation adjustment.

(b) *With* inflation adjustment.

Fig. 2.3: Are oil prices higher in 2019 compared to 2005?

We can also write

$$\overline{X}_t := \frac{1}{(1+r)^t} \overline{S}_t, \qquad t = 0, 1, \ldots, N.$$

The *discounted* value at time 0 of the portfolio is defined by

$$\widetilde{V}_t = \frac{1}{(1+r)^t} V_t, \qquad t = 0, 1, \ldots, N.$$

For $t = 1, 2, \ldots, N$ we have

$$\begin{aligned}\widetilde{V}_t &= \frac{1}{(1+r)^t} \bar{\xi}_t \cdot \overline{S}_t \\ &= \frac{1}{(1+r)^t} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\ &= \sum_{k=0}^{d} \xi_t^{(k)} \widetilde{S}_t^{(k)} \\ &= \bar{\xi}_t \cdot \overline{X}_t,\end{aligned}$$

while for $t = 0$ we get

$$\widetilde{V}_0 = \bar{\xi}_1 \cdot \overline{X}_0 = \bar{\xi}_1 \cdot \overline{S}_0.$$

The effect of discounting from time t to time 0 is to divide prices by $(1+r)^t$, making all prices comparable at time 0.

Arbitrage

The definition of arbitrage in discrete time follows the lines of its analog in the one-step model.

Definition 2.7. *A portfolio strategy* $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ *constitutes an arbitrage opportunity if all* three *following conditions are satisfied:*

i) $V_0 \leqslant 0$ *at time* $t = 0$*,* [Start from a zero-cost portfolio, or with a debt.]

ii) $V_N \geqslant 0$ *at time* $t = N$*,* [Finish with a nonnegative amount.]

iii) $\mathbb{P}(V_N > 0) > 0$ *at time* $t = N$*.* [Profit is made with nonzero probability.]

2.3 Contingent Claims

Recall that from Definition 1.9, a contingent claim is given by the nonnegative random payoff C of an option contract at maturity time $t = N$. For example, in the case of the European call option of Definition 2, the payoff C is given by $C = \left(S_N^{(i)} - K\right)^+$ where K is called the strike (or exercise) price of the option, while in the case of the European put option of Definition 1 we have $C = \left(K - S_N^{(i)}\right)^+$.

The list given below is somewhat restrictive and there exists many more option types, with new ones appearing constantly on the markets.

Physical delivery *vs.* cash settlement

The cash settlement realized through the payoff $C = \big(S_N^{(i)} - K\big)^+$ can be replaced by the *physical delivery* of the underlying asset in exchange for the strike price K. Physical delivery occurs only when $S_N^{(i)} > K$, in which case the underlying asset can be sold at the price $S_N^{(i)}$ by the option holder, for a payoff $S_N^{(i)} - K$. When $S_N^{(i)} > K$, no delivery occurs and the payoff is 0, which is consistent with the expression $C = \big(S_N^{(i)} - K\big)^+$. A similar procedure can be applied to other option contracts.

Vanilla options - examples

Vanilla options are options whose claim payoff depends only on the terminal value S_T of the underlying risky asset price at maturity time T.

i) *European options.*

The payoff of the European call option on the underlying asset n^o i with maturity N and strike price K is

$$C = \big(S_N^{(i)} - K\big)^+ = \begin{cases} S_N^{(i)} - K & \text{if } S_N^{(i)} \geqslant K, \\ \\ 0 & \text{if } S_N^{(i)} < K. \end{cases}$$

The *moneyness* at time $t = 0, 1, \ldots, N$ of the European call option with strike price K on the asset n^o i is the ratio

$$\mathrm{M}_t^{(i)} := \frac{S_t^{(i)} - K}{S_t^{(i)}}, \qquad t = 0, 1, \ldots, N.$$

The option is said to be "*out of the money*" (OTM) when $\mathrm{M}_t^{(i)} < 0$, "*in the money*" (ITM) when $\mathrm{M}_t^{(i)} > 0$, and "*at the money*" (ATM) when $\mathrm{M}_t^{(i)} = 0$.

The payoff of the European put option on the underlying asset n^o i with exercise date N and strike price K is

$$C = \big(K - S_N^{(i)}\big)^+ = \begin{cases} K - S_N^{(i)} & \text{if } S_N^{(i)} \leqslant K, \\ \\ 0 & \text{if } S_N^{(i)} > K. \end{cases}$$

The *moneyness* at time $t = 0, 1, \ldots, N$ of the European put option with strike price K on the asset n^o i is the ratio

$$\mathrm{M}_t^{(i)} := \frac{K - S_t^{(i)}}{S_t^{(i)}}, \qquad t = 0, 1, \ldots, N.$$

ii) *Binary options.*

Binary (or digital) options, also called cash-or-nothing options, are options whose payoffs are of the form

$$C = \mathbb{1}_{[K,\infty)}\big(S_N^{(i)}\big) = \begin{cases} \$1 & \text{if } S_N^{(i)} \geqslant K, \\ \\ 0 & \text{if } S_N^{(i)} < K, \end{cases}$$

for binary call options, and

$$C = \mathbb{1}_{(-\infty,K]}\big(S_N^{(i)}\big) = \begin{cases} \$1 & \text{if } S_N^{(i)} \leqslant K, \\ \\ 0 & \text{if } S_N^{(i)} > K, \end{cases}$$

for binary put options.

iii) *Collar and spread options.*

Collar and spread options provide other examples of vanilla options, whose payoffs can be constructed using call and put option payoffs, see, *e.g.*, Exercises 3.12 and 3.13.

Exotic options - examples

i) *Asian options.*

The payoff of an Asian call option (also called option on average) on the underlying asset n^o i with exercise date N and strike price K is

$$C = \left(\frac{1}{N+1}\sum_{t=0}^{N} S_t^{(i)} - K\right)^+.$$

The payoff of an Asian put option on the underlying asset n^o i with exercise date N and strike price K is

$$C = \left(K - \frac{1}{N+1}\sum_{t=0}^{N} S_t^{(i)}\right)^+.$$

We refer to Section 13.1 for the pricing of Asian options in continuous time. It can be shown, see Exercise 3.14, that Asian call option prices can be upper bounded by European call option prices.

Other examples of such options include weather derivatives (based on averaged temperatures) and volatility derivatives (based on averaged volatilities).

ii) *Barrier options.*

The payoff of a down-an-out (or knock-out) barrier call option on the underlying asset n^o i with exercise date N, strike price K and barrier level B is

$$C = \left(S_N^{(i)} - K\right)^+ \mathbb{1}_{\left\{\min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B\right\}} = \begin{cases} \left(S_N^{(i)} - K\right)^+ & \text{if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B, \\ \\ 0 & \text{if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} \leqslant B. \end{cases}$$

This option is also called a Callable Bull Contract with no residual value, or turbo warrant with no rebate, in which B denotes the call price $B \geqslant K$.

The payoff of an up-and-out barrier put option on the underlying asset n^o i with exercise date N, strike price K and barrier level B is

$$C = \left(K - S_N^{(i)}\right)^+ \mathbb{1}_{\left\{\operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} < B\right\}} = \begin{cases} \left(K - S_N^{(i)}\right)^+ & \text{if } \operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} < B, \\ \\ 0 & \text{if } \operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} \geqslant B. \end{cases}$$

This option is also called a Callable Bear Contract with no residual value, in which the call price B usually satisfies $B \leqslant K$. See Eriksson and Persson (2006) and Wong and Chan (2008) for the pricing of type R Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants, which involve a rebate or residual value computed as the payoff of a down-and-in lookback option. We refer the reader to Chapters 11, 12, and 13 for the pricing and hedging of related options in continuous time.

iii) *Lookback options.*

The payoff of a floating strike lookback call option on the underlying asset n^o i with exercise date N is

$$C = S_N^{(i)} - \min_{t=0,1,\ldots,N} S_t^{(i)}.$$

The payoff of a floating strike lookback put option on the underlying asset n^o i with exercise date N is

$$C = \left(\operatorname*{Max}_{t=0,1,\ldots,N} S_t^{(i)} \right) - S_N^{(i)}.$$

We refer to Section 10.4 for the pricing of lookback options in continuous time.

Options in insurance and investment

Such options are involved in the statements of Exercises 2.1 and 2.2.

Vanilla *vs.* exotic options

Vanilla options such as European or binary options, have a payoff $\phi(S_N^{(i)})$ that depends only on the terminal value $S_N^{(i)}$ of the underlying asset at maturity, as opposed to exotic or path-dependent options such as Asian, barrier, or lookback options, whose payoff may depend on the whole path of the underlying asset price until expiration time.

Exotic *vs* Vanilla Options

Vanilla options are called that way because:

(A) They were first used for the trading of vanilla by the Maya beginning around the 14th century.

(B) "Plain vanilla" is the most standard and common of all ice cream flavors.

(C) To meet FDA standards, pure vanilla extract must contain 13.35 ounces of vanilla beans per gallon.

(D) Sir Charles C. Vanilla, FLS, was the early discoverer of the properties of Brownian motion in asset pricing.

Fig. 2.4: Take the Quiz.

2.4 Martingales and Conditional Expectations

Before proceeding to the definition of risk-neutral probability measures in discrete time we need to introduce more mathematical tools such as conditional expectations, filtrations, and martingales.

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Conditional expectations

Clearly, the expected value of any risky asset or random variable is dependent on the amount of available information. For example, the expected return on a real estate investment typically depends on the location of this investment.

In the probabilistic framework the available information is formalized as a collection $\mathcal{G}$ of events, which may be smaller than the collection $\mathcal{F}$ of all available events, *i.e.* $\mathcal{G} \subset \mathcal{F}$.*

The notation $\mathbb{E}[F \mid \mathcal{G}]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathcal{G}$, and it is read "the conditional expectation of F given $\mathcal{G}$". In a certain sense, $\mathbb{E}[F \mid \mathcal{G}]$ represents the best possible estimate of F in the mean-square sense, given the information contained in $\mathcal{G}$.

The conditional expectation satisfies the following five properties, cf. Section A.7 for details and proofs.

i) $\mathbb{E}[FG \mid \mathcal{G}] = G\mathbb{E}[F \mid \mathcal{G}]$ if G depends only on the information contained in $\mathcal{G}$.

ii) $\mathbb{E}[G \mid \mathcal{G}] = G$ when G depends only on the information contained in $\mathcal{G}$.

iii) $\mathbb{E}[\mathbb{E}[F \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[F \mid \mathcal{G}]$ if $\mathcal{G} \subset \mathcal{H}$, called the *tower property*, cf. also Relation (A.33).

iv) $\mathbb{E}[F \mid \mathcal{G}] = \mathbb{E}[F]$ when F "does not depend" on the information contained in $\mathcal{G}$ or, more precisely stated, when the random variable F is *independent* of the σ-algebra $\mathcal{G}$.

v) If G depends only on $\mathcal{G}$ and F is independent of $\mathcal{G}$, then

$$\mathbb{E}[h(F, G) \mid \mathcal{G}] = \mathbb{E}[h(F, x)]_{x=G}.$$

When $\mathcal{H} = \{\emptyset, \Omega\}$ is the trivial σ-algebra we have

$$\mathbb{E}[F \mid \mathcal{H}] = \mathbb{E}[F], \qquad F \in L^1(\Omega).$$

See (A.33) and (A.39) for illustrations of the tower property by conditioning with respect to discrete and continuous random variables.

Filtrations

The total amount of "information" available on the market at times $t = 0, 1, \ldots, N$ is denoted by $\mathcal{F}_t$. We assume that

$$\mathcal{F}_t \subset \mathcal{F}_{t+1}, \qquad t = 0, 1, \ldots, N-1,$$

* The collection $\mathcal{G}$ is also called a σ-algebra, cf. Section A.1.

which means that the amount of information available on the market increases over time.

Usually, $\mathcal{F}_t$ corresponds to the knowledge of the values $S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)}$, $i = 1, 2, \ldots, d$, of the risky assets up to time t. In mathematical notation we say that $\mathcal{F}_t$ is generated by $S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)}$, $i = 1, 2, \ldots, d$, and we usually write

$$\mathcal{F}_t = \sigma\big(S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)},\ i = 1, 2, \ldots, d\big), \qquad t = 1, 2, \ldots, N,$$

with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Example: Consider the simple random walk

$$Z_t := X_1 + X_2 + \cdots + X_t, \qquad t \geqslant 0,$$

where $(X_t)_{t\geqslant 1}$ is a sequence of independent, identically distributed $\{-1, 1\}$ valued random variables. The filtration (or information flow) $(\mathcal{F}_t)_{t\geqslant 0}$ generated by $(Z_t)_{t\geqslant 0}$ is given by $\mathcal{F}_0 = \big\{\emptyset, \Omega\big\}$, $\mathcal{F}_1 = \big\{\emptyset, \{X_1 = 1\}, \{X_1 = -1\}, \Omega\big\}$, and

$$\mathcal{F}_2 = \sigma\big(\{\emptyset, \{X_1 = 1, X_2 = 1\}, \{X_1 = 1, X_2 = -1\}, \{X_1 = -1, X_2 = 1\}, \\ \{X_1 = -1, X_2 = -1\}, \Omega\}\big).$$

The notation $\mathcal{F}_t$ is useful to represent a quantity of information available at time t. Note that different agents or traders may work with different filtrations. For example, an insider may have access to a filtration $(\mathcal{G}_t)_{t=0,1,\ldots,N}$ which is larger than the ordinary filtration $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ available to an ordinary agent, in the sense that

$$\mathcal{F}_t \subset \mathcal{G}_t, \qquad t = 0, 1, \ldots, N.$$

The notation $\mathbb{E}[F \mid \mathcal{F}_t]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathcal{F}_t$. Again, $\mathbb{E}[F \mid \mathcal{F}_t]$ denotes the best possible estimate of F in mean-square sense, given the information known up to time t.

We will assume that no information is available at time $t = 0$, which translates as

$$\mathbb{E}[F \mid \mathcal{F}_0] = \mathbb{E}[F]$$

for any integrable random variable F. As above, the conditional expectation with respect to $\mathcal{F}_t$ satisfies the following five properties:

i) $\mathbb{E}[FG \mid \mathcal{F}_t] = F\mathbb{E}[G \mid \mathcal{F}_t]$ if F depends only on the information contained in $\mathcal{F}_t$.

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ii) $\mathbb{E}[F \mid \mathcal{F}_t] = F$ when F depends only on the information known at time t and contained in $\mathcal{F}_t$.

iii) $\mathbb{E}[\mathbb{E}[F \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] = \mathbb{E}[F \mid \mathcal{F}_t]$ if $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ (by the *tower property*, cf. also Relation (7.1) below).

iv) $\mathbb{E}[F \mid \mathcal{F}_t] = \mathbb{E}[F]$ when F does not depend on the information contained in $\mathcal{F}_t$.

v) If F depends only on $\mathcal{F}_t$ and G is independent of $\mathcal{F}_t$, then

$$\mathbb{E}[h(F,G) \mid \mathcal{F}_t] = \mathbb{E}[h(x,G)]_{x=F}.$$

Note that by the tower property (iii) the process $t \mapsto \mathbb{E}[F \mid \mathcal{F}_t]$ is a martingale, cf. *e.g.* Relation (7.1) for details.

Martingales

A martingale is a stochastic process whose value at time $t+1$ can be estimated using conditional expectation given its value at time t. Recall that a stochastic process $(M_t)_{t=0,1,\ldots,N}$ is said to be $(\mathcal{F}_t)_{t=0,1,\ldots,N}$-adapted if the value of M_t depends only on the information available at time t in $\mathcal{F}_t$, $t = 0,1,\ldots,N$.

Definition 2.8. *A stochastic process $(M_t)_{t=0,1,\ldots,N}$ is called a discrete-time* martingale *with respect to the filtration $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ if $(M_t)_{t=0,1,\ldots,N}$ is $(\mathcal{F}_t)_{t=0,1,\ldots,N}$-adapted and satisfies the property*

$$\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] = M_t, \qquad t = 0,1,\ldots,N-1.$$

Note that the above definition implies that $M_t \in \mathcal{F}_t$, $t = 0,1,\ldots,N$. In other words, a random process $(M_t)_{t=0,1,\ldots,N}$ is a martingale if the best possible prediction of M_{t+1} in the mean-square sense given $\mathcal{F}_t$ is simply M_t.

In discrete-time finance, the martingale property can be used to characterize risk-neutral probability measures, and for the computation of conditional expectations.

Exercise. Using the *tower property* (A.33) of conditional expectation, show that Definition 2.8 can be equivalently stated by saying that

$$\mathbb{E}[M_n \mid \mathcal{F}_k] = M_k, \qquad 0 \leqslant k < n.$$

A particular property of martingales is that their expectation is constant over time.

Proposition 2.9. *Let $(Z_n)_{n\in\mathbb{N}}$ be a martingale. We have*

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \qquad n \geqslant 0.$$

Proof. From the tower property (A.33) of conditional expectation, we have:

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]] = \mathbb{E}[Z_n], \qquad n \geqslant 0,$$

hence by induction on $n \geqslant 0$ we have

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_{n-1}] = \cdots = \mathbb{E}[Z_1] = \mathbb{E}[Z_0], \qquad n \geqslant 0.$$

□

Weather forecasting can be seen as an example of application of martingales. If M_t denotes the random temperature observed at time t, this process is a martingale when the best possible forecast of tomorrow's temperature M_{t+1} given the information known up to time t is simply today's temperature M_t, $t = 0, 1, \ldots, N-1$.

Definition 2.10. *A stochastic process $(\xi_k)_{k\geqslant 1}$ is said to be* predictable *if ξ_k depends only on the information in $\mathcal{F}_{k-1}$, $k \geqslant 1$.*

When $\mathcal{F}_0$ simply takes the form $\mathcal{F}_0 = \{\emptyset, \Omega\}$ we find that ξ_1 is a constant when $(\xi_t)_{t=1,2,\ldots,N}$ is a predictable process. Recall that on the other hand, the process $\big(S_t^{(i)}\big)_{t=0,1,\ldots,N}$ is *adapted* as $S_t^{(i)}$ depends only on the information in $\mathcal{F}_t$, $t = 0, 1, \ldots, N$, $i = 1, 2, \ldots, d$.

The discrete-time stochastic integral (2.9) will be interpreted as the sum of discounted profits and losses $\xi_k\big(\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}\big)$, $k = 1, 2, \ldots, t$, in a portfolio holding a quantity ξ_k of a risky asset whose price variation is $\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}$ at time $k = 1, 2, \ldots, t$.

An important property of martingales is that the discrete-time stochastic integral (2.9) of a predictable process is itself a martingale, see also Proposition 7.1 for the continuous-time analog of the following proposition, which will be used in the proof of Theorem 3.5 below.*

In what follows, the martingale (2.9) will be interpreted as a discounted portfolio value, in which $\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}$ represents the increment in the discounted asset price and ξ_k is the amount invested in that asset, $k = 1, 2 \ldots, N$.

Theorem 2.11. Martingale transform. *Given $(X_k)_{k=0,1,\ldots,N}$ a martingale and $(\xi_k)_{k=1,2,\ldots,N}$ a (bounded) predictable process, the discrete-time process $(M_t)_{t=0,1,\ldots,N}$ defined by*

$$M_t = \sum_{k=1}^{t} \underbrace{\xi_k(X_k - X_{k-1})}_{\text{Profit/loss}}, \qquad t = 0, 1, \ldots, N, \tag{2.9}$$

is a martingale.

* See **here** for a related discussion of martingale strategies in a particular case.

Proof. Given $n > t \geqslant 0$, we have

$$\begin{aligned}
\mathbb{E}[M_n \mid \mathcal{F}_t] &= \mathbb{E}\left[\sum_{k=1}^{n} \xi_k (X_k - X_{k-1}) \,\Big|\, \mathcal{F}_t\right] \\
&= \sum_{k=1}^{n} \mathbb{E}\big[\xi_k (X_k - X_{k-1}) \,\big|\, \mathcal{F}_t\big] \\
&= \sum_{k=1}^{t} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] \\
&= \sum_{k=1}^{t} \xi_k (X_k - X_{k-1}) + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] \\
&= M_t + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right].
\end{aligned}$$

In order to conclude to $\mathbb{E}\left[M_n \mid \mathcal{F}_t\right] = M_t$ it suffices to show that

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] = 0, \qquad t + 1 \leqslant k \leqslant n.$$

First, we note that when $0 \leqslant t \leqslant k-1$ we have $\mathcal{F}_t \subset \mathcal{F}_{k-1}$, hence by the tower property (A.33) of conditional expectations, we get

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_t\right].$$

Next, since the process $(\xi_k)_{k\geqslant 1}$ is predictable, ξ_k depends only on the information in $\mathcal{F}_{k-1}$, and using Property (ii) of conditional expectations we may pull out ξ_k out of the expectation since it behaves as a constant parameter given $\mathcal{F}_{k-1}$, $k = 1, 2, \ldots, n$. This yields

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1}\right] = \xi_k \mathbb{E}\left[X_k - X_{k-1} \mid \mathcal{F}_{k-1}\right] = 0 \tag{2.10}$$

since

$$\begin{aligned}
\mathbb{E}\left[X_k - X_{k-1} \mid \mathcal{F}_{k-1}\right] &= \mathbb{E}\left[X_k \mid \mathcal{F}_{k-1}\right] - \mathbb{E}\left[X_{k-1} \mid \mathcal{F}_{k-1}\right] \\
&= \mathbb{E}\left[X_k \mid \mathcal{F}_{k-1}\right] - X_{k-1} \\
&= 0, \qquad k = 1, 2, \ldots, N,
\end{aligned}$$

because $(X_k)_{k=0,1,\ldots,N}$ is a martingale. By (2.10), it follows that

$$\begin{aligned}
\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\xi_k \mathbb{E}\left[X_k - X_{k-1} \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_t\right] \\
&= 0,
\end{aligned}$$

for $k = t + 1, \ldots, n$. □

2.5 Market Completeness and Risk-Neutral Measures

As in the two time step model, the concept of risk-neutral probability measure (or martingale measure) will be used to price financial claims under the absence of arbitrage hypothesis.*

Definition 2.12. *A probability measure* $\mathbb{P}^*$ *on* Ω *is called a risk-neutral probability measure if under* $\mathbb{P}^*$*, the expected return of each risky asset equals the return* r *of the riskless asset, that is*

$$\mathbb{E}^*\big[S_{t+1}^{(i)} \,\big|\, \mathcal{F}_t\big] = (1+r)S_t^{(i)}, \qquad t = 0, 1, \ldots, N-1, \tag{2.11}$$

$i = 0, 1, \ldots, d$. *Here,* $\mathbb{E}^*$ *denotes the expectation under* $\mathbb{P}^*$.

Since $S_t^{(i)} \in \mathcal{F}_t$, denoting by

$$R_{t+1}^{(i)} := \frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}}$$

the return of asset n^o i over the time interval $(t, t+1]$, $t = 0, 1, \ldots, N-1$, Relation (2.11) can be rewritten as

$$\begin{aligned}
\mathbb{E}^*\big[R_{t+1}^{(i)} \mid \mathcal{F}_t\big] &= \mathbb{E}^*\left[\frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\frac{S_{t+1}^{(i)}}{S_t^{(i)}} \,\middle|\, \mathcal{F}_t\right] - 1 \\
&= r, \qquad t = 0, 1, \ldots, N-1,
\end{aligned}$$

which means that the average of the return $\big(S_{t+1}^{(i)} - S_t^{(i)}\big)/S_t^{(i)}$ of asset n^o i under the risk-neutral probability measure $\mathbb{P}^*$ is equal to the risk-free interest rate r.

In other words, taking risks under $\mathbb{P}^*$ by buying the risky asset n^o i has a neutral effect, as the expected return is that of the riskless asset. The measure $\mathbb{P}^\sharp$ would yield a *positive* risk premium if we had

$$\mathbb{E}^\sharp\big[S_{t+1}^{(i)} \,\big|\, \mathcal{F}_t\big] = (1+\tilde{r})S_t^{(i)}, \qquad t = 0, 1, \ldots, N-1,$$

with $\tilde{r} > r$, and a *negative* risk premium if $\tilde{r} < r$.

In the next proposition we reformulate the definition of risk-neutral probability measure using the notion of martingale.

* See also the Efficient Market Hypothesis.

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Proposition 2.13. *A probability measure $\mathbb{P}^*$ on Ω is a risk-neutral measure if and only if the discounted price process*

$$\widetilde{S}_t^{(i)} := \frac{S_t^{(i)}}{(1+r)^t}, \qquad t = 0, 1, \ldots, N,$$

is a martingale under $\mathbb{P}^*$, *i.e.*

$$\mathbb{E}^*\big[\widetilde{S}_{t+1}^{(i)} \,\big|\, \mathcal{F}_t\big] = \widetilde{S}_t^{(i)}, \qquad t = 0, 1, \ldots, N-1, \tag{2.12}$$

$i = 0, 1, \ldots, d.$

Proof. It suffices to check that by the relation $S_t^{(i)} = (1+r)^t \widetilde{S}_t^{(i)}$, Condition (2.11) can be rewritten as

$$(1+r)^{t+1}\mathbb{E}^*\big[\widetilde{S}_{t+1}^{(i)} \,\big|\, \mathcal{F}_t\big] = (1+r)(1+r)^t \widetilde{S}_t^{(i)},$$

$i = 1, 2, \ldots, d$, which is clearly equivalent to (2.12) after division by $(1+r)^t$, $t = 0, 1, \ldots, N-1$. □

Note that, as a consequence of Propositions 2.9 and 2.13, the discounted price process $\widetilde{S}_t^{(i)} := S_t^{(i)}/(1+r)^t$, $t = 0, 1, \ldots, n$, has constant expectation under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.*

$$\mathbb{E}^*\big[\widetilde{S}_t^{(i)}\big] = \widetilde{S}_0^{(i)}, \qquad t = 1, 2, \ldots, N,$$

for $i = 0, 1, \ldots, d$.

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 2.14. *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is said to be* equivalent *to another probability measure $\mathbb{P}$ when*

$$\mathbb{P}^*(A) = 0 \quad \textit{if and only if} \quad \mathbb{P}(A) = 0, \quad \textit{for all} \quad A \in \mathcal{F}. \tag{2.13}$$

Next, we restate in discrete time the first fundamental theorem of asset pricing, which can be used to check for the existence of arbitrage opportunities.

Theorem 2.15. *A market is* without *arbitrage opportunity if and only if it admits at least one* equivalent *risk-neutral probability measure.*

Proof. See Harrison and Kreps (1979) and Theorem 5.17 of Föllmer and Schied (2004). □

Next, we turn to the notion of *market completeness*, starting with the definition of attainability for a contingent claim.

Definition 2.16. *A contingent claim with payoff C is said to be attainable (at time N) if there exists a (predictable) self-financing portfolio strategy*

$(\bar{\xi}_t)_{t=1,2,\ldots,N}$ *such that*

$$C = \bar{\xi}_N \cdot \overline{S}_N = \sum_{k=0}^{d} \xi_N^{(k)} S_N^{(k)}, \qquad \mathbb{P} - a.s. \tag{2.14}$$

In case $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ is a portfolio that attains the claim payoff C at time N, *i.e.* if (2.14) is satisfied, we also say that $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ *hedges* the claim payoff C. In case (2.14) is replaced by the condition

$$\bar{\xi}_N \cdot \overline{S}_N \geqslant C,$$

we talk of super-hedging.

When a self-financing portfolio $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ hedges a claim payoff C, the arbitrage-free price $\pi_t(C)$ of the claim at time t is given by the value

$$\pi_t(C) = \bar{\xi}_t \cdot \overline{S}_t$$

of the portfolio at time $t = 0, 1, \ldots, N$. Recall that arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value (mark to market). Note that at time $t = N$ we have

$$\pi_N(C) = \bar{\xi}_N \cdot \overline{S}_N = C,$$

i.e. since exercise of the claim occurs at time N, the price $\pi_N(C)$ of the claim equals the value C of the payoff.

Definition 2.17. *A market model is said to be* complete *if every contingent claim is attainable.*

The next result can be viewed as the second fundamental theorem of asset pricing in discrete time.

Theorem 2.18. *A market model without arbitrage opportunities is complete if and only if it admits only one* equivalent *risk-neutral probability measure.*

Proof. See Harrison and Kreps (1979) and Theorem 5.38 of Föllmer and Schied (2004). □

2.6 The Cox-Ross-Rubinstein (CRR) Market Model

We consider the discrete-time Cox-Ross-Rubinstein (Cox et al. (1979)) model, also called the *binomial model*, with $N + 1$ time instants $t = 0, 1, \ldots, N$ and $d = 1$ risky asset, see Sharpe (1978). In this setting, the price $S_t^{(0)}$ of the riskless asset evolves as

$$S_t^{(0)} = S_0^{(0)}(1 + r)^t, \qquad t = 0, 1, \ldots, N.$$

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Let the *return* of the risky asset $S^{(1)}$ be defined as

$$R_t := \frac{S_t^{(1)} - S_{t-1}^{(1)}}{S_{t-1}^{(1)}}, \qquad t = 1, 2, \ldots, N.$$

In the CRR (or binomial) model, the return R_t is random and allowed to take only two values a and b at each time step, *i.e.*

$$R_t \in \{a, b\}, \qquad t = 1, 2, \ldots, N,$$

with $-1 < a < b$ and

$$\mathbb{P}(R_t = a) > 0, \quad \mathbb{P}(R_t = b) > 0, \qquad t = 1, 2, \ldots, N.$$

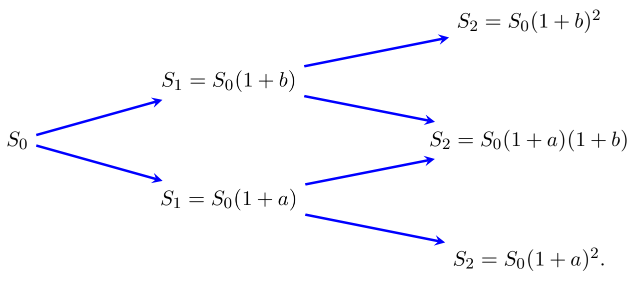
That means, the evolution of $S_{t-1}^{(1)}$ to $S_t^{(1)}$ is random and given by

$$S_t^{(1)} = \left\{ \begin{array}{ll} (1+b)S_{t-1}^{(1)} & \text{if } R_t = b \\ \\ (1+a)S_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = (1+R_t)S_{t-1}^{(1)}, \qquad t = 1, \ldots, N,$$

and

$$S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t} (1 + R_k), \qquad t = 0, 1, \ldots, N.$$

Note that the price process $\big(S_t^{(1)}\big)_{t=0,1,\ldots,N}$ evolves on a binary recombining (or binomial) tree of the following type:*



The discounted asset price is

* Download the corresponding **IPython notebook1** and **IPython notebook2** that can be run **here** or **here**.

$$\widetilde{S}_t^{(1)} = \frac{S_t^{(1)}}{(1+r)^t}, \qquad t = 0, 1, \ldots, N,$$

with

$$\widetilde{S}_t^{(1)} = \left\{ \begin{array}{ll} \dfrac{1+b}{1+r} \widetilde{S}_{t-1}^{(1)} & \text{if } R_t = b \\ \\ \dfrac{1+a}{1+r} \widetilde{S}_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = \frac{1+R_t}{1+r} \widetilde{S}_{t-1}^{(1)}, \qquad t = 1, 2, \ldots, N,$$

and

$$\widetilde{S}_t^{(1)} = \frac{S_0^{(1)}}{(1+r)^t} \prod_{k=1}^t (1+R_k) = \widetilde{S}_0^{(1)} \prod_{k=1}^t \frac{1+R_k}{1+r}.$$

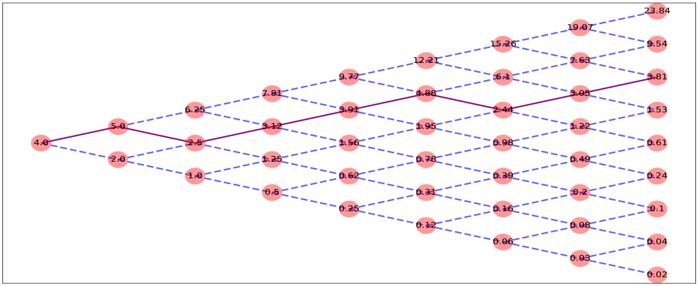


Fig. 2.5: Discrete-time asset price tree in the CRR model.

In this model, the discounted value at times $t = 1, 2, \ldots, N$ of the portfolio is given by

$$\bar{\xi}_t \bullet \overline{X}_t = \xi_t^{(0)} \widetilde{S}_0^{(0)} + \xi_t^{(1)} \widetilde{S}_t^{(1)}.$$

The information $\mathcal{F}_t$ known in the market up to time t is given by the knowledge of $S_1^{(1)}, S_2^{(1)}, \ldots, S_t^{(1)}$, which is equivalent to the knowledge of $\widetilde{S}_1^{(1)}, \widetilde{S}_2^{(1)}, \ldots, \widetilde{S}_t^{(1)}$ or $R_1, R_2, \ldots, R_t$, *i.e.* we write

$$\mathcal{F}_t = \sigma\big(S_1^{(1)}, S_2^{(1)}, \ldots, S_t^{(1)}\big) = \sigma\big(\widetilde{S}_1^{(1)}, \widetilde{S}_2^{(1)}, \ldots, \widetilde{S}_t^{(1)}\big) = \sigma(R_1, R_2, \ldots, R_t),$$

$t = 0, 1, \ldots, N$, where, as a convention, S_0 is a constant and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ contains no information.

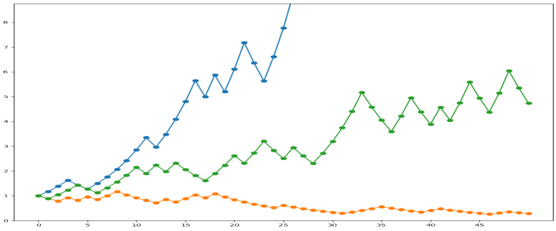


Fig. 2.6: Discrete-time asset price graphs in the CRR model.

Theorem 2.19. *The CRR model is without arbitrage opportunities if and only if $a < r < b$. In this case the market is complete and the* equivalent *risk-neutral probability measure $\mathbb{P}^*$ is given by*

$$\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a}, \qquad (2.15)$$

$t = 0, 1, \ldots, N-1$. *In particular, $(R_1, R_2, \ldots, R_N)$ forms a sequence of independent and identically distributed (i.i.d.) random variables under $\mathbb{P}^*$, with*

$$p^* := \mathbb{P}^*(R_t = b) = \frac{r-a}{b-a} \quad \text{and} \quad q^* := \mathbb{P}^*(R_t = a) = \frac{b-r}{b-a}, \qquad (2.16)$$

$t = 1, 2, \ldots, N$.

Proof. In order to check for arbitrage opportunities we may use Theorem 2.15 and look for a risk-neutral probability measure $\mathbb{P}^*$. According to the definition of a risk-neutral measure this probability $\mathbb{P}^*$ should satisfy Condition (2.11), *i.e.*

$$\mathbb{E}^*\big[S_{t+1}^{(1)} \,\big|\, \mathcal{F}_t\big] = (1+r)S_t^{(1)}, \qquad t = 0, 1, \ldots, N-1.$$

Rewriting $\mathbb{E}^*\big[S_{t+1}^{(1)} \,\big|\, \mathcal{F}_t\big]$ as

$$\begin{aligned}\mathbb{E}^*\big[S_{t+1}^{(1)} \,\big|\, \mathcal{F}_t\big] &= \mathbb{E}^*\big[S_{t+1}^{(1)} \,\big|\, S_t^{(1)}\big] \\ &= (1+a)S_t^{(1)}\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + (1+b)S_t^{(1)}\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t),\end{aligned}$$

it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases} (1+b)S_t^{(1)}\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + (1+a)S_t^{(1)}\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = (1+r)S_t^{(1)} \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

i.e.

$$\begin{cases} b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = r \\ \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

with solution

$$\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$. Since the values of $\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t)$ and $\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t)$ computed in (2.15) are non random, they are independent* of the information contained in $\mathcal{F}_t$ up to time t. As a consequence, under $\mathbb{P}^*$, the random variable R_{t+1} is independent of $R_1, R_2, \ldots, R_t$, hence the sequence of random variables $(R_t)_{t=0,1,\ldots,N}$ is made of *mutually independent* random variables under $\mathbb{P}^*$, and by (2.15) we have

$$\mathbb{P}^*(R_{t+1} = b) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a) = \frac{b-r}{b-a}.$$

Clearly, $\mathbb{P}^*$ can be equivalent to $\mathbb{P}$ only if $r - a > 0$ and $b - r > 0$. In this case the solution $\mathbb{P}^*$ of the problem is unique by construction, hence the market is complete by Theorem 2.18. □

As a consequence of Proposition 2.13, letting $p^* := (r-a)/(b-a)$, when $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{a, b\}^N$ we have

$$\mathbb{P}^*(R_1 = \epsilon_1, R_2 = \epsilon_2, \ldots, R_N = \epsilon_n) = (p^*)^l (1-p^*)^{N-l},$$

where l, resp. $N - l$, denotes the number of times the term "b", resp. "a", appears in the sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \in \{a, b\}^N$.

Exercises

Exercise 2.1 Today I went to Furong Peak Mall. After exiting the Poon Way MTR station, I was met by a friendly investment consultant from NTRC Input, who recommended that I subscribe to the following investment plan. The plan requires to invest \$2,550 per year over the first 10 years, with no contribution required from year 11 until year 20. The total projected surrender value is \$30,835 at maturity $N = 20$. The plan also includes a death benefit which is not considered here.

* The relation $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ is equivalent to the independence relation $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ of the events A and B.

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	Total Premiums	Guaranteed ($S)	Surrender Value Projected at 3.25%	
Year	Paid To-date ($S)		Non-Guaranteed ($S)	Total ($S)
1	2,550	0	0	0
2	5,100	2,460	140	2,600
3	7,650	4,240	240	4,480
4	10,200	6,040	366	6,406
5	12,750	8,500	518	9,018
10	25,499	19,440	1,735	21,175
15	25,499	22,240	3,787	26,027
20	25,499	24,000	6,835	30,835

Table 2.2: NTRC Input investment plan.

a) Compute the constant interest rate over 20 years corresponding to this investment plan.
b) Compute the projected value of the plan at the end of year 20, if the annual interest rate is $r = 3.25\%$ over 20 years.
c) Compute the projected value of the plan at the end of year 20, if the annual interest rate $r = 3.25\%$ is paid only over the first 10 years. Does this recover the total projected value $\$30,835$?

Exercise 2.2 Today I went to East Mall. After exiting the Bukit Kecil MTR station, I was approached by a friendly investment consultant from Avenda Insurance, who recommended me to subscribe to the following investment plan. The plan requires me to invest $3,581 per year over the first 10 years, with no contribution required from year 11 until year 20. The total projected surrender value is $50,862 at maturity $N = 20$. The plan also includes a death benefit which is not considered here.

	Total Premiums	Guaranteed ($S)	Surrender Value Projected at 3.25%	
Year	Paid To-date ($S)		Non-Guaranteed ($S)	Total ($S)
1	3,581	0	0	0
2	7,161	1,562	132	1,694
3	10,741	3,427	271	3,698
4	14,321	5,406	417	5,823
5	17,901	6,992	535	7,527
10	35,801	19,111	1,482	20,593
15	35,801	29,046	3,444	32,490
20	35,801	43,500	7,362	50,862

Table 2.3: Avenda Insurance investment plan.

a) Using the following graph, compute the constant interest rate over 20 years corresponding to this investment.

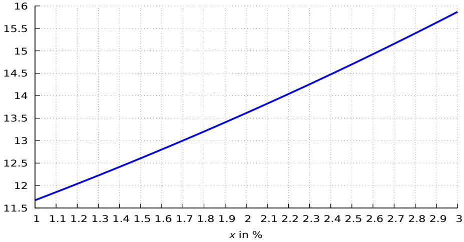

Fig. 2.7: Graph of the function $x \mapsto ((1+x)^{21} - (1+x)^{11})/x$.

b) Compute the projected value of the plan at the end of year 20, if the annual interest rate is $r = 3.25\%$ over 20 years.
c) Compute the projected value of the plan at the end of year 20, if the annual interest rate $r = 3.25\%$ is paid only over the first 10 years. Does this recover the total projected value $\$50,862$?

Exercise 2.3 A lump sum of \$100,000 is to be released through N identical yearly installment payments m at the beginning of every year, over $N = 10$ years.

a) Find the value of m.
b) Assume that interests are due at the rate $r = 2\%$ on the amount remaining at the beginning of every year. How does this affect the value of the constant yearly payment m?
c) Assume that the amount remaining at the beginning of every year is invested at the interest rate $r = 2\%$, and that m is as in Question (a). In this case, how much is left at the end of year N after the N payments have been completed?

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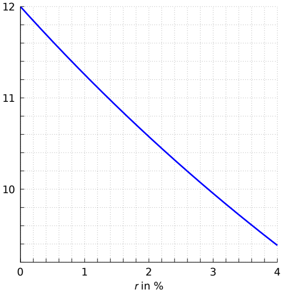
Exercise 2.4

Today I received an SMS from Jack, and I opted for the 3K loan over 12 months.

a) Compute the *monthly* interest rate earned by Jack using the below graph of the function $r \mapsto (1-(1+r)^{-12})/r$.
b) Compute the *yearly* interest rate earned by Jack.
c) Should I:

    i) Block him,
    ii) Report him,
    iii) Sue him,
    iv) All of the above.



Hi Dear Customer Im Jack Here

We Power Direct understands the frustrations of bills piling up, having difficulties to make payments on time and in need of Emergency Funds. We may offer the best/right solutions for your needs.

FREE APPLY !!!
Personal/Business Loan
NO NEED UP FRONT PAYMENT !!!
Repayment On 1Year-5Year
First 88 Dear Customer Free 1 Mth

(1 Year Loan)
3K 275 x12Month
4K 370 x12Month
5K 460 x12Month

(3 Year Loan)
10K 305 x36Month
30K 920 x36Month
50K 1530 x36Month

(5 Year Loan)
20K 380 x60Month
50K 930 x60Month
100K 1850 60Month

WhatsApp:
WhatsApp Number>JACK
+65 12345678
WhatsApp Link Click Here >Jack
https://1234.abcd

THIS IS A SYSTEM GENERATED MESSAGE. PLEASE REPLY HI TO ACTIVATE THE LINK ABOVE..

Exercise 2.5 We consider the following two scenarios:

i) In Scenario (i) we borrow the amount \$A at the rate r_{loan} to purchase a house. By renting out the house we receive investment income compounded every month at the rate r_{rent}, and we refund the loan by paying \$m at the end of every month.
ii) In Scenario (ii) at the end of every month we only invest an amount \$m on an account paying the rate r_{inv}, for the same duration as in Scenario (i).

a) How much remains on our account in Scenario (i) after the loan has been completely repaid?
   Hint: Refunding \$A over N identical payments of \$m at the rate $r_{\text{loan}} > 0$ imposes the relation $A/m = (1-(1+r_{\text{loan}})^{-N})/r_{\text{loan}}$.
b) How much remains on our account in Scenario (ii) at the end of the investment duration?

Hint: Reaching a target of \$B by investing N identical payments of \$m at the rate $r_{\rm inv} > 0$ imposes the relation $B/m = ((1+r_{\rm inv})^N - 1)/r_{\rm inv}$.

c) Taking $N = 12$ months and assuming $r_{\rm rent} = r_{\rm inv} = 2\%$ and $r_{\rm loan} = 5\%$, which of Scenario (i) and Scenario (ii) is more profitable?

Hint: Use the graph of Figure 2.8.

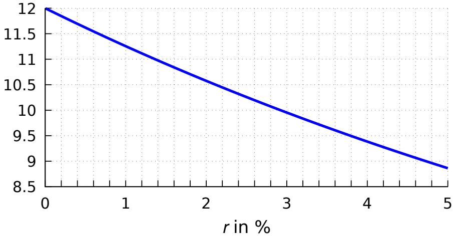


Fig. 2.8: Graph of the function $r \mapsto (1 - (1+r)^{-12})/r$.

Exercise 2.6 Consider a two-step trinomial (or ternary) market model $(S_t)_{t=0,1,2}$ with $r = 0$ and three possible return rates $R_t \in \{-1, 0, 1\}$. Show that the probability measure $\mathbb{P}^*$ given by

$$\mathbb{P}^*(R_t = -1) := \frac{1}{4}, \quad \mathbb{P}^*(R_t = 0) := \frac{1}{2}, \quad \mathbb{P}^*(R_t = 1) := \frac{1}{4}$$

is risk-neutral.

Exercise 2.7 We consider a riskless asset valued $S_k^{(0)} = S_0^{(0)}$, $k = 0, 1, \ldots, N$, where the risk-free interest rate is $r = 0$, and a risky asset $S^{(1)}$ whose returns $R_k := \dfrac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}$, $k = 1, 2, \ldots, N$, form a sequence of independent identically distributed random variables taking three values $\{-b < 0 < b\}$ at each time step, with

$$p^* := \mathbb{P}^*(R_k = b) > 0, \quad \theta^* := \mathbb{P}^*(R_k = 0) > 0, \quad q^* := \mathbb{P}^*(R_k = -b) > 0,$$

$k = 1, 2, \ldots, N$. The information known to the market up to time k is denoted by $\mathcal{F}_k$.

a) Determine all possible risk-neutral probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in terms of the parameter $\theta^* \in (0, 1)$.
b) Assume that the conditional variance

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$$\mathrm{Var}^* \left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \,\middle|\, \mathcal{F}_k \right] = \sigma^2 > 0, \qquad k = 0, 1, \ldots, N-1, \tag{2.17}$$

of the asset return is constant and equal to σ^2. Show that this condition defines a unique risk-neutral probability measure $\mathbb{P}_\sigma^*$ under a certain condition on b and σ, and determine $\mathbb{P}_\sigma^*$ explicitly.

Exercise 2.8 The Ross (2015) Recovery Theorem allows for estimates of the real-world transition probabilities of an underlying asset from option prices in a Markovian state model. We consider a one-step asset price model with N possible states $\{s_1, \ldots, s_N\}$ at time $t = 0$ and at time $t = 1$, and let $X_t \in \{s_1, \ldots, s_N\}$ denote the state of the market at times $t = 0, 1$. We also consider N binary options $\mathcal{B}_1, \ldots, \mathcal{B}_N$ maturing at $t = 1$, with respective payoffs

$$\mathbb{1}_{\{X_1 = s_k\}}, \quad k = 1, \ldots, N,$$

i.e. $\mathcal{B}_k$ pays \$1 if and only if $X_1 = s_k$, $k = 1, \ldots, N$. For $k, l \in \{1, \ldots, N\}$ we denote by $b_{k,l}$ the **known** price of the binary option $\mathcal{B}_l$ given that $X_0 = s_k$.

a) In this question, we aim at recovering the *risk-neutral* probabilities

$$P^* = (p_{k,l}^*)_{k,l=1,\ldots,N} = (\mathbb{P}^*(X_1 = s_l \mid X_0 = s_k))_{k,l=1,\ldots,N}$$

using risk-neutral pricing.

1) From Proposition 1.16, express $b_{k,l}$ using:
   i) the price d_k of the bond paying \$1 at $t = 1$ when the market starts from state s_k at $t = 0$, and
   ii) the *risk-neutral* probability $p_{k,l}^*$, $k, l = 1, \ldots, N$.
2) Write the price d_k of a bond paying \$1 at $t = 1$ in terms of the binary option prices $b_{k,l}$ of $\mathcal{B}_l$ when the market starts from state s_k at $t = 0$, $k, l = 1, \ldots, N$.
3) For $k = 1, \ldots, N$, express the *risk-neutral* probabilities $p_{k,l}^*$ and the bond prices d_k in terms of the **known** binary option prices $b_{k,l}$, $k, l = 1, \ldots, N$.

b) In this question, we aim at recovering the real-world probabilities

$$P = (p_{k,l})_{k,l=1,\ldots,N} = (\mathbb{P}(X_1 = s_l \mid X_0 = s_k))_{k,l=1,\ldots,N}$$

using *marginal utility pricing*. For this, we price the binary option $\mathcal{B}_l$ by the relation

$$u_l b_{k,l} = \delta u_k p_{k,l}, \qquad k, l = 1, \ldots, N,$$

where $\delta > 0$ is an **unknown** time discount factor and $u_k > 0$ represents an **unknown** marginal utility in state s_k, $k = 1, \ldots, N$.

1) How many equations do we have? How many unknowns do we have?

2) Show that the matrix equation $Bu^\top = \delta u^\top$ holds, where $u := [u_1, \ldots, u_N]$ and $B := (b_{k,l})_{k,l=1,\ldots,N}$.
3) Prove that the equation of Question (b2) admits a unique solution δ, $u_1, \ldots, u_N$ made of strictly positive numbers.

   Hint. Apply the Perron-Frobenius theorem for positive matrices.
4) Show that the transition probabilities $p_{k,l}$ can be recovered from the **known** binary option prices $b_{k,l}$, $k, l = 1, \ldots, N$.
5) Assume that X_1 does not depend on the initial state $X_0 = k$, $k = 1, \ldots, N$. Find the relation between δ and the bond prices d_k, and the relation between the real-world and risk-neutral probabilities $p_{k,l}$, $p^*_{k,l}$, $k, l = 1, \ldots, N$.

Figure 2.9 gives an example of estimation of transition probabilities on AMZN option chain data downloaded on March 31, 2022.

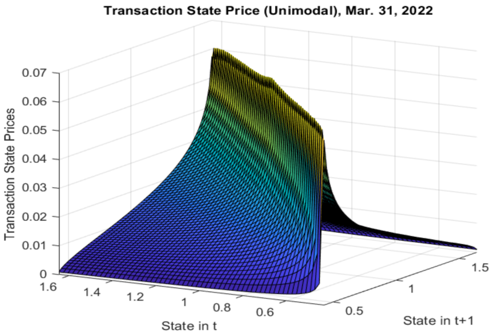

Fig. 2.9: Transition probabilities in the recovery theorem.

Exercise 2.9 We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$, with a riskless asset whose price A_t evolves as $A_t = A_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of S_{t-1} to S_t is given by

$$S_t = \begin{cases} (1+b)S_{t-1} \\ \\ (1+a)S_{t-1} \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset S is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N,$$

and $\mathcal{F}_t$ is generated by $R_1, R_2, \ldots, R_t$, $t = 1, 2, \ldots, N$.

a) What are the possible values of R_t?
b) Show that, under the probability measure $\mathbb{P}^*$ defined by

$$p^* = \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a}, \qquad q^* = \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$, the expected return $\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t]$ of S is equal to the return r of the riskless asset.
c) Show that under $\mathbb{P}^*$ the process $(S_t)_{t=0,1,\ldots,N}$ satisfies

$$\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] = (1+r)^k S_t, \qquad t = 0, 1, \ldots, N-k, \quad k = 0, 1, \ldots, N.$$

Exercise 2.10 We consider the discrete-time Cox-Ross-Rubinstein model on $N+1$ time instants $t = 0, 1, \ldots, N$, with a riskless asset whose price A_t evolves as $A_t = A_0(1+r)^t$ with $r \geqslant 0$, and a risky asset whose price S_t is given by

$$S_t = S_0 \prod_{k=1}^{t} (1 + R_k), \qquad t = 0, 1, \ldots, N,$$

where the *asset returns* R_k are independent random variables taking two possible values a and b with $-1 < a < r < b$, and $\mathbb{P}^*$ is the probability measure defined by

$$p^* = \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a}, \qquad q^* = \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$, where $(\mathcal{F}_t)_{t=0,1,\ldots,N}$ is the filtration generated by $(R_t)_{t=1,2,\ldots,N}$.

a) Compute the conditional expected return $\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t]$ under $\mathbb{P}^*$, $t = 0, 1, \ldots, N-1$.
b) Show that the discounted asset price process

$$\big(\widetilde{S}_t\big)_{t=0,1,\ldots,N} := \left(\frac{S_t}{A_t}\right)_{t=0,1,\ldots,N}$$

is a (nonnegative) $(\mathcal{F}_t)$-martingale under $\mathbb{P}^*$.
Hint: Use the independence of asset returns $(R_t)_{t=1,2,\ldots,N}$ under $\mathbb{P}^*$.
c) Compute the moment $\mathbb{E}^*[(S_N)^\beta]$ for all $\beta > 0$.
Hint: Use the independence of asset returns $(R_t)_{t=1,2,\ldots,N}$ under $\mathbb{P}^*$.
d) For any $\alpha > 0$, find an upper bound for the probability

$$\mathbb{P}^*\big(S_t \geqslant \alpha A_t \ \text{ for some } t \in \{0, 1, \ldots, N\}\big).$$

Hint: Use the fact that when $(M_t)_{t=0,1,\ldots,N}$ is a nonnegative martingale, we have

$$\mathbb{P}^*\Big(\operatorname*{Max}_{t=0,1,\ldots,N} M_t \geqslant x\Big) \leqslant \frac{\mathbb{E}^*[(M_N)^\beta]}{x^\beta}, \qquad x > 0, \quad \beta \geqslant 1. \qquad (2.18)$$

e) For any $x > 0$, find an upper bound for the probability

$$\mathbb{P}^*\Big(\operatorname*{Max}_{t=0,1,\ldots,N} S_t \geqslant x\Big).$$

Hint: Note that (2.18) remains valid for any nonnegative *sub*martingale.