Chapter 7
Martingale Approach to Pricing and Hedging

In the *martingale approach* to the pricing and hedging of financial derivatives, option prices are expressed as the expected values of discounted option payoffs. This approach relies on the construction of risk-neutral probability measures by the Girsanov theorem, and the associated hedging portfolios are obtained via stochastic integral representations.

7.1 Martingale Property of the Itô Integral

Recall (Definition 4.2) that an integrable process $(X_t)_{t\in\mathbb{R}_+}$ is said to be a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \qquad 0 \leqslant s \leqslant t.$$

In what follows,

$$L^2(\Omega) := \{F : \Omega \to \mathbb{R} \ : \ \mathbb{E}[|F|^2] < \infty\}$$

denotes the space of square-integrable random variables.

Examples of martingales (*i*)

1. Given $F \in L^2(\Omega)$ a square-integrable random variable and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ a filtration, the process $(X_t)_{t\in\mathbb{R}_+}$ defined by

$$X_t := \mathbb{E}[F \mid \mathcal{F}_t], \qquad t \geqslant 0,$$

is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-martingale under $\mathbb{P}$. Indeed, since $\mathcal{F}_s \subset \mathcal{F}_t$, $0 \leqslant s \leqslant t$, it follows from the tower property (A.33) of conditional expectations that

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[F \mid \mathcal{F}_s] = X_s, \quad 0 \leqslant s \leqslant t. \quad (7.1)$$

2. Any integrable stochastic process $(X_t)_{t\in\mathbb{R}_+}$ whose increments $(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}})$ are mutually independent and centered under $\mathbb{P}$ (*i.e.* $\mathbb{E}[X_t] = 0$, $t \in \mathbb{R}_+$) is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(X_t)_{t\in\mathbb{R}_+}$, as we have

$$\begin{aligned}\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s] \\ &= \mathbb{E}[X_t - X_s \mid \mathcal{F}_s] + \mathbb{E}[X_s \mid \mathcal{F}_s] \\ &= \mathbb{E}[X_t - X_s] + X_s \\ &= X_s, \qquad 0 \leqslant s \leqslant t. \end{aligned} \quad (7.2)$$

In particular, the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale because it has centered and independent increments. This fact is also consequence of Proposition 7.1 below as B_t can be written as

$$B_t = \int_0^t dB_s, \qquad t \geqslant 0.$$

3. The driftless geometric Brownian motion

$$X_t := X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \quad (7.3)$$

is a martingale. Indeed, using the Gaussian moment generating function identity (A.41), we have

$$\begin{aligned}\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}\big[X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \mid \mathcal{F}_s\big] \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \mathbb{E}\big[\mathrm{e}^{\sigma B_t} \mid \mathcal{F}_s\big] \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \mathbb{E}\big[\mathrm{e}^{(B_t - B_s)\sigma + \sigma B_s} \mid \mathcal{F}_s\big] \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \mathbb{E}\big[\mathrm{e}^{(B_t - B_s)\sigma} \mid \mathcal{F}_s\big] \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \mathbb{E}\big[\mathrm{e}^{(B_t - B_s)\sigma}\big] \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \exp\left(\mathbb{E}[(B_t - B_s)\sigma] + \frac{1}{2}\,\mathrm{Var}[(B_t - B_s)\sigma]\right) \\ &= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathrm{e}^{(t-s)\sigma^2/2} \\ &= X_0 \,\mathrm{e}^{\sigma B_s - \sigma^2 s/2} \\ &= X_s, \qquad 0 \leqslant s \leqslant t. \end{aligned}$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

The following result shows that the Itô integral yields a martingale with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. It is the continuous-time analog of the discrete-time Theorem 2.11.

Proposition 7.1. *The stochastic integral process* $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ *of a square-integrable adapted process* $u \in L^2_{\mathrm{ad}}(\Omega\times\mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \;\Big|\; \mathcal{F}_s\right] = \int_0^s u_\tau dB_\tau, \qquad 0 \leqslant s \leqslant t. \tag{7.4}$$

In particular, $\int_0^t u_s dB_s$ is $\mathcal{F}_t$-measurable, $t \geqslant 0$, and since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, Relation (7.4) applied with $t = 0$ recovers the fact that the Itô integral is a centered random variable:

$$\mathbb{E}\left[\int_0^t u_s dB_s\right] = \mathbb{E}\left[\int_0^t u_s dB_s \;\Big|\; \mathcal{F}_0\right] = \int_0^0 u_s dB_s = 0, \quad t \geqslant 0.$$

Proof. The statement is first proved in case $(u_t)_{t\in\mathbb{R}_+}$ is a simple predictable process, and then extended to the general case, cf. *e.g.* Proposition 2.5.7 in Privault (2009). For example, for u a predictable step process of the form

$$u_s := F\mathbb{1}_{[a,b]}(s) = \begin{cases} F \text{ if } s \in [a,b], \\ \\ 0 \ \text{ if } s \notin [a,b], \end{cases}$$

with F an $\mathcal{F}_a$-measurable random variable and $t \in [a,b]$, by Definition 4.17 we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \;\Big|\; \mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbb{1}_{[a,b]}(s) dB_s \;\Big|\; \mathcal{F}_t\right] \\
&= \mathbb{E}[(B_b - B_a)F \mid \mathcal{F}_t] \\
&= F\mathbb{E}[B_b - B_a \mid \mathcal{F}_t] \\
&= F(B_t - B_a) \\
&= \int_a^t u_s dB_s \\
&= \int_0^t u_s dB_s, \qquad a \leqslant t \leqslant b.
\end{aligned}$$

On the other hand, when $t \in [0,a]$ we have

$$\int_0^t u_s dB_s = 0,$$

and we check that

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \;\Big|\; \mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbb{1}_{[a,b]}(s) dB_s \;\Big|\; \mathcal{F}_t\right] \\
&= \mathbb{E}[F(B_b - B_a) \mid \mathcal{F}_t]
\end{aligned}$$

$$\begin{aligned}&= \mathbb{E}[\mathbb{E}[F(B_b - B_a) \mid \mathcal{F}_a] \mid \mathcal{F}_t]\\&= \mathbb{E}[F\mathbb{E}[B_b - B_a \mid \mathcal{F}_a] \mid \mathcal{F}_t]\\&= 0, \qquad 0 \leqslant t \leqslant a,\end{aligned}$$

where we used the tower property (A.33) of conditional expectations and the fact that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale:

$$\mathbb{E}[B_b - B_a \mid \mathcal{F}_a] = \mathbb{E}[B_b \mid \mathcal{F}_a] - B_a = B_a - B_a = 0.$$

The extension from simple processes to square-integrable processes in $L^2_{\rm ad}(\Omega \times \mathbb{R}_+)$ can be proved as in Proposition 4.21. Indeed, given $\big(u^{(n)}\big)_{n\in\mathbb{N}}$ be a sequence of simple predictable processes converging to u in $L^2(\Omega \times [0,T])$ cf. Lemma 1.1 of Ikeda and Watanabe (1989), pages 22 and 46, by Fatou's Lemma A.12, Jensen's inequality and the Itô isometry (4.16), we have:

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^t u_s dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right]\\&= \mathbb{E}\left[\lim_{n\to\infty}\left(\int_0^t u_s^{(n)} dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right]\\&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^{(n)} dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right]\\&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right]\\&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s\right)^2 \,\Big|\, \mathcal{F}_t\right]\right]\\&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^{(n)} - u_s) dB_s\right)^2\right]\\&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^{(n)} - u_s|^2 ds\right]\\&= \liminf_{n\to\infty} \|u^{(n)} - u\|^2_{L^2(\Omega\times[0,T])}\\&= 0,\end{aligned}$$

where we used the Itô isometry (4.16). We conclude that

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right] = \int_0^t u_s dB_s, \qquad t \geqslant 0,$$

for $u \in L^2_{\rm ad}(\Omega \times \mathbb{R}_+)$ a square-integrable adapted process, which leads to (7.4) after applying this identity to the process $(\mathbb{1}_{[0,t]} u_s)_{s\in\mathbb{R}_+}$, *i.e.*,

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \,\Big|\, \mathcal{F}_s\right] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau \,\Big|\, \mathcal{F}_s\right]$$

$$= \int_0^s \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau$$
$$= \int_0^s u_\tau dB_\tau, \qquad 0 \leqslant s \leqslant t.$$

□

Examples of martingales (*ii*)

1. The martingale property of the driftless geometric Brownian motion (7.3) can also be recovered from Proposition 7.1, since by Proposition 5.15, $(X_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dX_t = \sigma X_t dB_t,$$

which shows that X_t can be written using the Brownian stochastic integral

$$X_t = X_0 + \sigma \int_0^t X_u dB_u, \qquad t \geqslant 0.$$

2. Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \geqslant 0, \tag{7.5}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. By the *Discounting Lemma 5.13*, the discounted asset price process $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$, $t \geqslant 0$, satisfies the stochastic differential equation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t,$$

and the discounted asset price

$$\widetilde{S}_t = \mathrm{e}^{-rt} S_t = S_0 \,\mathrm{e}^{(\mu-r)t+\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}$ when $\mu = r$. The case $\mu \neq r$ will be treated in Section 7.3 using risk-neutral probability measures, see Definition 5.4, and the Girsanov Theorem 7.3, see (7.16) below.

3. The discounted value

$$\widetilde{V}_t = \mathrm{e}^{-rt} V_t, \qquad t \geqslant 0,$$

of a self-financing portfolio is given by

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \geqslant 0,$$

cf. Lemma 5.14 is a martingale when $\mu = r$ by Proposition 7.1 because

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u dB_u, \qquad t \geqslant 0,,$$

since we have

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t) = \sigma \widetilde{S}_t dB_t$$

by the *Discounting Lemma 5.13*. Since the Black-Scholes theory is in fact valid for any value of the parameter μ we will look forward to including the case $\mu \neq r$ in the sequel.

7.2 Risk-Neutral Probability Measures

Recall that by definition, a risk-neutral measure is a probability measure $\mathbb{P}^*$ under which the discounted asset price process

$$\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} := (\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$$

is a *martingale*, see Definition 5.4 and Proposition 5.5.

Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation (7.5). Note that when $\mu = r$, the discounted asset price process $\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} = (S_0 \mathrm{e}^{\sigma B_t - \sigma^2 t/2})_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^* = \mathbb{P}$, which is a risk-neutral probability measure.

In this section, we address the construction of a risk-neutral probability measure $\mathbb{P}^*$ in the general case $\mu \neq r$ using the Girsanov Theorem 7.3 below. For this, we note that by the *Discounting Lemma 5.13*, the relation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t$$

where $\mu - r$ is the risk premium, can be rewritten as

$$d\widetilde{S}_t = \sigma \widetilde{S}_t d\widehat{B}_t, \tag{7.6}$$

where $\big(\widehat{B}_t\big)_{t\in\mathbb{R}_+}$ is a drifted Brownian motion given by

$$\widehat{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad t \geqslant 0,$$

where the drift coefficient $\nu := (\mu - r)/\sigma$ is the "Market Price of Risk" (MPoR). The MPoR represents the difference between the return μ expected when investing in the risky asset S_t, and the risk-free interest rate r, measured in units of volatility σ.

From (7.6) and Propositions 5.5 and 7.1 we note that the risk-neutral probability measure can be constructed as a probability measure $\mathbb{P}^*$ under which $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

Let us come back to the informal approximation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm\sqrt{\Delta t},$$

with

$$\mathbb{P}\big(\Delta B_t = +\sqrt{\Delta t}\big) = \mathbb{P}\big(\Delta B_t = -\sqrt{\Delta t}\big) = \frac{1}{2},$$

and

$$\mathbb{E}[\Delta B_t] = \frac{1}{2}\sqrt{\Delta t} - \frac{1}{2}\sqrt{\Delta t} = 0.$$

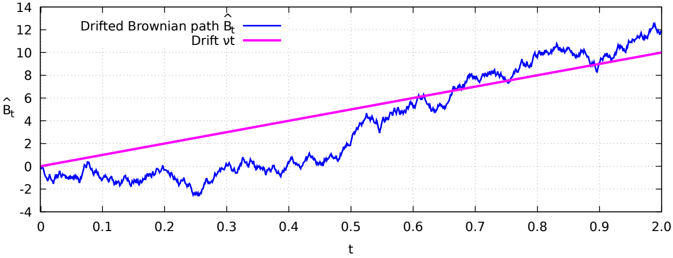


Fig. 7.1: Drifted Brownian path $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ with $\nu > 0$.

Clearly, given $\nu \in \mathbb{R}$, the drifted process

$$\widehat{B}_t := \nu t + B_t, \qquad t \geqslant 0,$$

is no longer a standard Brownian motion because it is not centered:

$$\mathbb{E}\big[\widehat{B}_t\big] = \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

cf. Figure 7.1. This identity can be formulated in terms of infinitesimal increments as

$$\mathbb{E}[\nu\Delta t + \Delta B_t] = \frac{1}{2}(\nu\Delta t + \sqrt{\Delta t}) + \frac{1}{2}(\nu\Delta t - \sqrt{\Delta t}) = \nu\Delta t \neq 0.$$

In order to make $\nu t + B_t$ a centered process (*i.e.* a standard Brownian motion, since $\nu t + B_t$ conserves all the other properties (i)-(iii) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to $1/2$.

That is, the problem is now to find two numbers $p^*, q^* \in [0, 1]$ such that

$$\begin{cases} p^*(\nu\Delta t + \sqrt{\Delta t}) + q^*(\nu\Delta t - \sqrt{\Delta t}) = 0 \\ \\ p^* + q^* = 1. \end{cases}$$

The solution to this problem is given by

$$p^* := \frac{1}{2}(1 - \nu\sqrt{\Delta t}) \quad \text{and} \quad q^* := \frac{1}{2}(1 + \nu\sqrt{\Delta t}). \tag{7.7}$$

Definition 7.2. *We say that a probability measure $\mathbb{Q}$ is absolutely continuous with respect to another probability measure $\mathbb{P}$ if there exists a nonnegative random variable $F : \Omega \longrightarrow \mathbb{R}_+$ such that $\mathbb{E}[F] = 1$, and*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = F, \qquad i.e. \qquad d\mathbb{Q} = F d\mathbb{P}. \tag{7.8}$$

In this case, F is called the Radon-Nikodym density *of $\mathbb{Q}$ with respect to $\mathbb{P}$.*

Relation (7.8) is equivalent to the relation

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[G] &= \int_\Omega G(\omega) d\mathbb{Q}(\omega) \\ &= \int_\Omega G(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) \\ &= \int_\Omega G(\omega) F(\omega) d\mathbb{P}(\omega) \\ &= \mathbb{E}[FG], \end{aligned}$$

for any random variable G integrable under $\mathbb{Q}$.

Coming back to Brownian motion considered as a discrete random walk with independent increments $\pm\sqrt{\Delta t}$, we try to construct a new probability measure denoted $\mathbb{P}^*$, under which the drifted process $\widehat{B}_t := \nu t + B_t$ will be a standard Brownian motion. This probability measure will be defined through its Radon-Nikodym density

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} &:= \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})} \\ &= \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})} \\ &= \frac{1}{(1/2)^N} \mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t}), \end{aligned} \tag{7.9}$$

$\epsilon_1, \epsilon_2, \ldots, \epsilon_N \in \{-1, 1\}$, with respect to the historical probability measure $\mathbb{P}$, obtained by taking the product of the above probabilities divided by the reference probability $1/2^N$ corresponding to the symmetric random walk.

Interpreting $N = T/\Delta t$ as an (infinitely large) number of discrete time steps and under the identification $[0, T] \simeq \{0 = t_0, t_1, \ldots, t_N = T\}$, this Radon-

Nikodym density (7.9) can be rewritten as

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t} \right) \tag{7.10}$$

where 2^N becomes a normalization factor.

The following R code is rescaling probabilities as in (7.7) based on the value of the drift μ.

```
nsim <- 100; N=12; t <- 0:N; T<-1.0; dt <- T/N; nu=3; p=0.5*(1-nu*(dt)^0.5);
dB <- matrix((dt)^0.5*(rbinom( nsim * N, 1, p)-0.5)*2, nsim, N)
X <- cbind(rep(0, nsim), t(apply(dB, 1, cumsum)))
plot(t, X[1, ], xlab = "Time", ylab = "", type = "l", ylim = c(-2*N*dt,2*N*dt), col =
       0,cex.axis=1.4,cex.lab=1.4,xaxs="i", mgp = c(1, 2, 0), las=1)
for (i in 1:nsim){if (N<20) {points(t,t*nu*dt+X[i,],pch=20,cex=0.6, col=i+1,lwd=2)}
lines(t,t*nu*dt+X[i,],type="l",col=i+1,lwd=2)}
```

The discretized illustration in Figure 7.2 displays the drifted Brownian motion $\widehat{B}_t := \nu t + B_t$ under the shifted probability measure $\mathbb{P}^*$ in (7.10) using the above R code with $N = 100$. The code makes big transitions less frequent than small transitions, resulting into a standard, centered Brownian motion under $\mathbb{P}^*$.

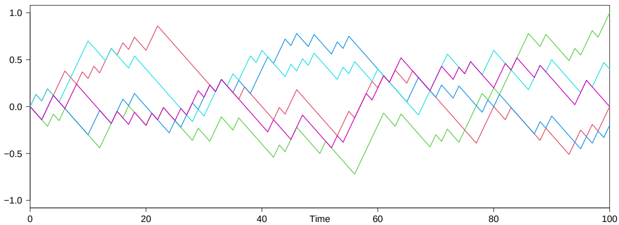


Fig. 7.2: Drifted Brownian motion paths under a shifted Girsanov measure.

Next, using the expansion

$$\begin{aligned}\log\left(1 \pm \nu\sqrt{\Delta t}\right) &= \pm\nu\sqrt{\Delta t} - \frac{1}{2}(\pm\nu\sqrt{\Delta t})^2 + o(\Delta t) \\ &= \pm\nu\sqrt{\Delta t} - \frac{\nu^2}{2}\Delta t + o(\Delta t),\end{aligned}$$

for small values of Δt, this Radon-Nikodym density can be informally shown to converge as follows as N tends to infinity, *i.e.* as the time step $\Delta t = T/N$ tends to zero:

$$\begin{aligned}
\frac{d\mathbb{P}^*}{d\mathbb{P}} &= 2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) \\
&= \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right) \\
&= \exp\left(\log \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{\Delta t} - \frac{1}{2}\sum_{0<t<T} (\mp\nu\sqrt{\Delta t})^2\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \pm\sqrt{\Delta t} - \frac{\nu^2}{2}\sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \Delta B_t - \frac{\nu^2}{2}\sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu B_T - \frac{\nu^2}{2} T\right),
\end{aligned}$$

based on the identifications

$$B_T \simeq \sum_{0<t<T} \pm\sqrt{\Delta t} \quad \text{and} \quad T \simeq \sum_{0<t<T} \Delta t.$$

Informally, the drifted process $(\widehat{B}_t)_{t\in[0,T]} = (\nu t + B_t)_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\nu B_T - \frac{\nu^2}{2} T\right).$$

7.3 Change of Measure and the Girsanov Theorem

In this section we restate the Girsanov Theorem in a more rigorous way, using changes of probability measures. The Girsanov Theorem can actually be extended to shifts by adapted processes $(\psi_t)_{t\in[0,T]}$ as follows, cf. *e.g.* Theorem III-42, page 141 of Protter (2004). An extension of the Girsanov Theorem to jump processes will be covered in Section 20.5. Recall also that here, $\Omega := \mathcal{C}_0([0,T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0,T]$ starting at 0 in $t = 0$. The Girsanov Theorem 7.3 will be used in Section 7.4 for the construction of a unique risk-neutral probability measure $\mathbb{P}^*$, showing absence of arbitrage and completeness in the Black-Scholes market, see Theorems 5.7 and 5.11.

Theorem 7.3. *Let* $(\psi_t)_{t\in[0,T]}$ *be an adapted process satisfying the Novikov integrability condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty, \tag{7.11}$$

and let $\mathbb{Q}$ *denote the probability measure defined by the Radon-Nikodym density*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right).$$

Then

$$\widehat{B}_t := B_t + \int_0^t \psi_s ds, \qquad 0 \leqslant t \leqslant T,$$

is a standard Brownian motion under $\mathbb{Q}$.

In the case of the simple shift

$$\widehat{B}_t := B_t + \nu t, \qquad 0 \leqslant t \leqslant T,$$

by a drift νt with constant $\psi_s = \nu \in \mathbb{R}$, the process $\big(\widehat{B}_t\big)_{t\in\mathbb{R}_+}$ is a standard (centered) Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$\mathrm{d}\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right)\mathrm{d}\mathbb{P}(\omega).$$

For example, the fact that $\widehat{B}_T$ has a centered Gaussian distribution under $\mathbb{Q}$ can be recovered as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\big[f\big(\widehat{B}_T\big)\big] &= \mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] \\
&= \int_\Omega f(\nu T + B_T)\mathrm{d}\mathbb{Q} \\
&= \int_\Omega f(\nu T + B_T)\exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right)\mathrm{d}\mathbb{P} \\
&= \int_{-\infty}^{\infty} f(\nu T + x)\exp\left(-\nu x - \frac{1}{2}\nu^2 T\right)\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(\nu T + x)\,\mathrm{e}^{-(\nu T + x)^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-y^2/(2T)}\frac{dy}{\sqrt{2\pi T}} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)],
\end{aligned}$$

i.e.

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] &= \int_\Omega f(\nu T + B_T) \mathrm{d}\mathbb{Q} \qquad (7.12)\\ &= \int_\Omega f(B_T) \mathrm{d}\mathbb{P}\\ &= \mathbb{E}_{\mathbb{P}}[f(B_T)],\end{aligned}$$

showing that, under $\mathbb{Q}$, $\nu T + B_T$ has the centered $\mathcal{N}(0,T)$ Gaussian distribution with variance T. For example, taking $f(x) = x$, Relation (7.12) recovers the fact that $\widehat{B}_T$ is a centered random variable under $\mathbb{Q}$, *i.e.*

$$\mathbb{E}_{\mathbb{Q}}\big[\widehat{B}_T\big] = \mathbb{E}_{\mathbb{Q}}[\nu T + B_T)] = \mathbb{E}_{\mathbb{P}}[B_T] = 0.$$

The Girsanov Theorem 7.3 also allows us to extend (7.12) as

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[F(\cdot)] &= \mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s ds\right)\exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right)\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[F\left(B_\cdot + \int_0^\cdot \psi_s ds\right)\right], \qquad (7.13)\end{aligned}$$

for all random variables $F \in L^1(\Omega)$, see also Exercise 7.25.

When applied to the (constant) market price of risk (or Sharpe ratio)

$$\psi_t := \frac{\mu - r}{\sigma},$$

the Girsanov Theorem 7.3 shows that the process

$$\widehat{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad 0 \leqslant t \leqslant T, \qquad (7.14)$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T\right). \qquad (7.15)$$

Hence by Proposition 7.1 the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ solution of

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t = \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \geqslant 0, \qquad (7.16)$$

is a martingale under $\mathbb{P}^*$, therefore $\mathbb{P}^*$ is a risk-neutral probability measure. We also check that $\mathbb{P}^* = \mathbb{P}$ when $\mu = r$.

In the sequel, we consider probability measures $\mathbb{Q}$ that are *equivalent* to $\mathbb{P}$ in the sense that they share the same events of zero probability, see Definition 1.5. Precisely, recall that a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{Q}(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathcal{F}.$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

Note that when $\mathbb{Q}$ is defined by (7.8), it is *equivalent* to $\mathbb{P}$ if and only if $F > 0$ with $\mathbb{P}$-probability one.

7.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that according to the first fundamental theorem of asset pricing Theorem 5.7, a continuous market is without arbitrage opportunities if and only if there exists (at least) an equivalent risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be *complete*.

The equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \geqslant 0,$$

satisfied by the price process $(S_t)_{t\in\mathbb{R}_+}$ can be rewritten using (7.14) as

$$\frac{dS_t}{S_t} = r dt + \sigma d\widehat{B}_t, \qquad t \geqslant 0, \tag{7.17}$$

with the solution

$$S_t = S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} = S_0 \,\mathrm{e}^{r t + \sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \geqslant 0. \tag{7.18}$$

By the discounting Lemma 5.13, we have

$$\begin{aligned} d\widetilde{S}_t &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t \\ &= \sigma \widetilde{S}_t \left(\frac{\mu - r}{\sigma} dt + dB_t \right) \\ &= \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \geqslant 0, \end{aligned} \tag{7.19}$$

hence the discounted price process

$$\begin{aligned} \widetilde{S}_t &:= \mathrm{e}^{-rt} S_t \\ &= S_0 \,\mathrm{e}^{(\mu - r)t + \sigma B_t - \sigma^2 t/2} \\ &= S_0 \,\mathrm{e}^{\sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \geqslant 0, \end{aligned}$$

is a martingale under the probability measure $\mathbb{P}^*$ defined by (7.15). We note that $\mathbb{P}^*$ is a risk-neutral probability measure equivalent to $\mathbb{P}$, also called martingale measure, whose existence and uniqueness ensure absence of arbitrage and completeness according to Theorems 5.7 and 5.11.

Therefore, by Lemma 5.14 the discounted value $\widetilde{V}_t$ of a self-financing portfolio can be written as

$$\begin{aligned}\widetilde{V}_t &= \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u \\ &= \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \geqslant 0,\end{aligned}$$

and by Proposition 7.1 it becomes a martingale under $\mathbb{P}^*$.

As in Chapter 3, the value V_t at time t of a self-financing portfolio strategy $(\xi_t)_{t\in[0,T]}$ hedging an attainable claim payoff C will be called an *arbitrage-free price* of the claim payoff C at time t and denoted by $\pi_t(C)$, $t \in [0,T]$. Arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value ("mark to market").

Theorem 7.4. *Let $(\xi_t, \eta_t)_{t\in[0,T]}$ be a portfolio strategy with value*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T,$$

and let C be a contingent claim payoff, such that

(i) $(\xi_t, \eta_t)_{t\in[0,T]}$ is a self-financing portfolio, and

(ii) $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, i.e. we have $V_T = C$.

Then, the arbitrage-free price of the claim payoff C is given by the portfolio value

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{7.20}$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral probability measure $\mathbb{P}^*$.

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing, by Lemma 5.14 and (7.19) the discounted portfolio value $\widetilde{V}_t = \mathrm{e}^{-rt}V_t$ satisfies

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \geqslant 0,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 7.1, hence

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\big[\widetilde{V}_T \bigm| \mathcal{F}_t\big] \\ &= \mathbb{E}^*\big[\mathrm{e}^{-rT}V_T \mid \mathcal{F}_t\big] \\ &= \mathbb{E}^*\big[\mathrm{e}^{-rT}C \mid \mathcal{F}_t\big] \\ &= \mathrm{e}^{-rT}\mathbb{E}^*[C \mid \mathcal{F}_t],\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

□

Black-Scholes PDE for vanilla options by the martingale method

The martingale method can be used to recover the Black-Scholes PDE of Proposition 6.1. As the process $(S_t)_{t\in\mathbb{R}_+}$ has the Markov property, see Section 4.5, § V-6 of Protter (2004) and Definition 7.14 below, the value

$$\begin{aligned} V_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(S_T) \mid S_t], \quad 0 \leqslant t \leqslant T, \end{aligned}$$

of the portfolio at time $t \in [0,T]$ can be written from (7.20) as a function

$$V_t = g(t, S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(S_T) \mid S_t] \tag{7.21}$$

of t and S_t, $0 \leqslant t \leqslant T$.

Proposition 7.5. *Assume that ϕ is a Lipschitz payoff function, and that $(S_t)_{t\in\mathbb{R}_+}$ is the geometric Brownian motion*

$$(S_t)_{t\in\mathbb{R}_+} = \left(S_0\,\mathrm{e}^{\sigma\widehat{B}_t+(r-\sigma^2/2)t}\right)_{t\in\mathbb{R}_+}$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Then, the function $g(t,x)$ defined in (7.21) is in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R}_+)$ and solves the Black-Scholes PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x) \\ \\ g(T,x) = \phi(x), \quad x > 0. \end{cases}$$

Proof. It can be checked similarly to the proof of Proposition 6.10 that the function $g(t,x)$ defined by

$$g(t,S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(S_T) \mid S_t] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(xS_T/S_t)]_{|x=S_t}$$

is in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R}_+)$ when ϕ is a Lipschitz function, by differentiation of the lognormal distribution of S_T/S_t. We note that by (4.24), the application of Itô's formula Theorem 4.24 to $V_t = g(t,S_t)$ and (7.17) with $u_t = \sigma S_t$ and $v_t = rS_t$ leads to

$$\begin{aligned} &d\big(\mathrm{e}^{-rt}g(t,S_t)\big) = -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}dg(t,S_t) \\ &= -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial g}{\partial t}(t,S_t)dt \\ &\quad + \mathrm{e}^{-rt}\frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\,\mathrm{e}^{-rt}(dS_t)^2\frac{\partial^2 g}{\partial x^2}(t,S_t) \end{aligned}$$

$$
\begin{aligned}
&= -r\,\mathrm{e}^{-rt} g(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial g}{\partial t}(t,S_t)dt \\
&\quad + v_t\,\mathrm{e}^{-rt}\frac{\partial g}{\partial x}(t,S_t)dt + u_t\,\mathrm{e}^{-rt}\frac{\partial g}{\partial x}(t,S_t)d\widehat{B}_t + \frac{1}{2}\,\mathrm{e}^{-rt}|u_t|^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\
&= -r\,\mathrm{e}^{-rt} g(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial g}{\partial t}(t,S_t)dt \qquad (7.22)\\
&\quad + rS_t\,\mathrm{e}^{-rt}\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\,\mathrm{e}^{-rt}\sigma^2 S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma\,\mathrm{e}^{-rt}S_t\frac{\partial g}{\partial x}(t,S_t)d\widehat{B}_t.
\end{aligned}
$$

By Lemma 5.14 and Proposition 7.1, the discounted price $\widetilde{V}_t = \mathrm{e}^{-rt}g(t,S_t)$ of a self-financing hedging portfolio is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, therefore from *e.g.* Corollary II-6-1, page 72 of Protter (2004), all terms in dt should vanish in the above expression of $d\big(\mathrm{e}^{-rt}g(t,S_t)\big)$, showing that

$$
-rg(t,S_t) + \frac{\partial g}{\partial t}(t,S_t) + rS_t\frac{\partial g}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t) = 0,
$$

and leads to the Black-Scholes PDE

$$
rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 g}{\partial x^2}(t,x), \qquad x > 0.
$$

□

From (7.22) in the proof of Proposition 7.5, we also obtain the stochastic integral expression

$$
\begin{aligned}
\mathrm{e}^{-rT}\phi(S_T) &= \mathrm{e}^{-rT}g(T,S_T) \\
&= g(0,S_0) + \int_0^T d\big(\mathrm{e}^{-rt}g(t,S_t)\big) \\
&= g(0,S_0) + \sigma\int_0^T \mathrm{e}^{-rt}S_t\frac{\partial g}{\partial x}(t,S_t)d\widehat{B}_t,
\end{aligned}
$$

see also Proposition 5.14, and Proposition 7.11 below.

Forward contracts

The long forward contract with payoff $C = S_T - K$ is priced as

$$
\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\
&= S_t - K\,\mathrm{e}^{-(T-t)r}, \quad 0 \leqslant t \leqslant T,
\end{aligned}
$$

which recovers the Black-Scholes PDE solution (6.9), *i.e.*

$$
g(t,x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T.
$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

European call options

In the case of European call options with payoff function $\phi(x) = (x-K)^+$ we recover the Black-Scholes formula (6.11), cf. Proposition 6.11, by a probabilistic argument.

Proposition 7.6. *The price at time $t \in [0,T]$ of the European call option with strike price K and maturity T is given by*

$$\begin{aligned} g(t,S_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] \qquad (7.23)\\ &= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad 0 \leqslant t \leqslant T, \end{aligned}$$

with

$$\begin{cases} d_+(T-t) := \dfrac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) := \dfrac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T, \end{cases}$$

where "log" denotes the natural logarithm *"ln" and Φ is the standard Gaussian Cumulative Distribution Function.*

Proof. The proof of Proposition 7.6 is a consequence of (7.20) and Lemma 7.7 below. Using the relation

$$\begin{aligned} S_T &= S_0\,\mathrm{e}^{rT+\sigma\widehat{B}_T-\sigma^2T/2} \\ &= S_t\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

that follows from (7.18), by Theorem 7.4 the value at time $t \in [0,T]$ of the portfolio hedging C is given by

$$\begin{aligned} V_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T-K)^+ \,\big|\, \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\big(S_t\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K\big)^+ \,\big|\, \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\big(x\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K\big)^+\big]_{|x=S_t} \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\big(\mathrm{e}^{m(x)+X}-K\big)^+\big]_{|x=S_t}, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := (\widehat{B}_T - \widehat{B}_t)\sigma \simeq \mathcal{N}(0, (T-t)\sigma^2)$$

is a centered Gaussian random variable with variance

$$\mathrm{Var}[X] = \mathrm{Var}\big[(\widehat{B}_T - \widehat{B}_t)\sigma\big] = \sigma^2 \mathrm{Var}\big[\widehat{B}_T - \widehat{B}_t\big] = (T-t)\sigma^2$$

under $\mathbb{P}^*$. Hence by Lemma 7.7 below we have

$$\begin{aligned}
g(t, S_t) &= V_t \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(\mathrm{e}^{m(x)+X} - K)^+\big]_{|x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\,\mathrm{e}^{m(S_t)+\sigma^2(T-t)/2}\Phi\left(v + \frac{m(S_t) - \log K}{v}\right) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t) - \log K}{v}\right) \\
&= S_t\Phi\left(v + \frac{m(S_t) - \log K}{v}\right) - K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t) - \log K}{v}\right) \\
&= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),
\end{aligned}$$

$0 \leqslant t \leqslant T$. □

Relation (7.23) can also be written as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T - K)^+ \,\big|\, \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T - K)^+ \,\big|\, S_t\big] \qquad (7.24) \\
&\quad = S_t\Phi\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\qquad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Lemma 7.7. *Let $X \simeq \mathcal{N}(0, v^2)$ be a centered Gaussian random variable with variance $v^2 > 0$. We have*

$$\mathbb{E}\big[(\mathrm{e}^{m+X} - K)^+\big] = \mathrm{e}^{m+v^2/2}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v).$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}\big[(\mathrm{e}^{m+X} - K)^+\big] = \frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}(\mathrm{e}^{m+x} - K)^+\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}(\mathrm{e}^{m+x} - K)\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{\mathrm{e}^m}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{x-x^2/(2v^2)}dx - \frac{K}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-x^2/(2v^2)}dx
\end{aligned}$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} e^{-(v^2-x)^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi}} \int_{(-m+\log K)/v}^{\infty} e^{-y^2/2} dy$$
$$= \frac{e^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{\infty} e^{-y^2/(2v^2)} dy - K\Phi((m - \log K)/v)$$
$$= e^{m+v^2/2}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v),$$

where we used Relation (6.15). □

Call-put parity

Let

$$g_{\rm p}(t, S_t) := e^{-(T-t)r}\mathbb{E}^*\big[(K - S_T)^+ \,\big|\, \mathcal{F}_t\big], \qquad 0 \leqslant t \leqslant T,$$

denote the price of the put option with strike price K and maturity T.

Proposition 7.8. Call-put parity. *We have the relation*

$$g_{\rm c}(t, S_t) - g_{\rm p}(t, S_t) = S_t - e^{-(T-t)r}K \tag{7.25}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K\,e^{-(T-t)r}$.

Proof. From the relation

$$S_T - K = (S_T - K)^+ - (K - S_T)^+,$$

see https://optioncreator.com/stijwns, and Theorem 7.4, we have

$$\begin{aligned}
&g_{\rm c}(t, S_t) - g_{\rm p}(t, S_t) \\
&= e^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] - e^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] \\
&= e^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ - (K - S_T)^+ \mid \mathcal{F}_t] \\
&= e^{-(T-t)r}\mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \\
&= e^{-(T-t)r}\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\,e^{-(T-t)r} \\
&= S_t - e^{-(T-t)r}K, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

as we have $\mathbb{E}^*[S_T \mid \mathcal{F}_t] = e^{(T-t)r}S_t$, $t \in [0, T]$, under the risk-neutral probability measure $\mathbb{P}^*$. □

European put options

Using the *call-put parity* Relation (7.25) we can recover the European put option price (6.11) from the European call option price (6.11)-(7.23).

Proposition 7.9. *The price at time $t \in [0,T]$ of the European put option with strike price K and maturity T is given by*

$$\begin{aligned} g_{\mathrm{p}}(t,S_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t] \\ &= K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T, \end{aligned}$$

with

$$\begin{cases} d_+(T-t) := \dfrac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) := \dfrac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T, \end{cases}$$

where "log" denotes the natural logarithm *"ln" and Φ is the standard Gaussian Cumulative Distribution Function.*

Proof. By Relation (6.15) and the *call-put parity* (7.25), we have

$$\begin{aligned} g_{\mathrm{p}}(t,S_t) &= g_{\mathrm{c}}(t,S_t) - S_t + \mathrm{e}^{-(T-t)r}K \\ &= S_t\Phi\big(d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K - S_t - \mathrm{e}^{-(T-t)r}K\Phi\big(d_-(T-t)\big) \\ &= -S_t(1-\Phi\big(d_+(T-t)\big)) + \mathrm{e}^{-(T-t)r}K(1-\Phi\big(d_-(T-t)\big)) \\ &= -S_t\Phi\big(-d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K\Phi\big(-d_-(T-t)\big). \end{aligned}$$

□

7.5 Hedging by the Martingale Method

Hedging exotic options

In the next Proposition 7.10 we compute a self-financing hedging strategy leading to an arbitrary square-integrable random claim payoff $C \in L^2(\Omega)$ of an exotic option admitting a stochastic integral decomposition of the form

$$C = \mathbb{E}^*[C] + \int_0^T \zeta_t d\widehat{B}_t, \tag{7.26}$$

where $(\zeta_t)_{t\in[0,t]}$ is a square-integrable adapted process, see for example page 214. Consequently, the mathematical problem of finding the stochastic integral decomposition (7.26) of a given random variable has important applications in finance. The process $(\zeta_t)_{t\in[0,T]}$ can be computed using the Malliavin gradient on the Wiener space, see, *e.g.*, Di Nunno et al. (2009) or § 8.2 of Privault (2009).

Simple examples of stochastic integral decompositions include the relations

$$(B_T)^2 = T + 2\int_0^T B_t dB_t,$$

cf. Exercises 6.1 and 7.1, and

$$(B_T)^3 = 3\int_0^T (T - t + B_t^2) dB_t,$$

see Exercise 4.10. In the sequel, recall that the risky asset follows the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \geqslant 0, \qquad S_0 > 0,$$

and by (7.16), the discounted asset price $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$

$$d\widetilde{S}_t = \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \geqslant 0, \qquad \widetilde{S}_0 = S_0 > 0, \tag{7.27}$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. The following proposition applies to arbitrary square-integrable payoff functions, in particular it covers exotic and path-dependent options.

Proposition 7.10. *Consider a random claim payoff $C \in L^2(\Omega)$ and the process $(\zeta_t)_{t\in[0,T]}$ given by (7.26), and let*

$$\xi_t = \frac{\mathrm{e}^{-(T-t)r}}{\sigma S_t} \zeta_t, \tag{7.28}$$

$$\eta_t = \frac{\mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t] - \xi_t S_t}{A_t}, \qquad 0 \leqslant t \leqslant T. \tag{7.29}$$

Then the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing, and letting

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T, \tag{7.30}$$

we have

$$V_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{7.31}$$

In particular we have

$$V_T = C, \tag{7.32}$$

i.e. the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ yields a hedging strategy leading to the claim payoff C at maturity, after starting from the initial value

$$V_0 = \mathrm{e}^{-rT}\mathbb{E}^*[C].$$

Proof. Relation (7.31) follows from (7.29) and (7.30), and it implies

$$V_0 = \mathrm{e}^{-rT}\mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0$$

at $t=0$, and (7.32) at $t=T$. It remains to show that the portfolio strategy $(\xi_t,\eta_t)_{t\in[0,T]}$ is self-financing. By (7.26) and Proposition 7.1 we have

$$\begin{aligned}
V_t &= \eta_t A_t + \xi_t S_t \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{E}^*[C] + \int_0^T \zeta_u d\widehat{B}_u \;\Big|\; \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\left(\mathbb{E}^*[C] + \int_0^t \zeta_u d\widehat{B}_u\right) \\
&= \mathrm{e}^{rt}V_0 + \mathrm{e}^{-(T-t)r}\int_0^t \zeta_u d\widehat{B}_u \\
&= \mathrm{e}^{rt}V_0 + \sigma\int_0^t \xi_u S_u \,\mathrm{e}^{(t-u)r} d\widehat{B}_u \\
&= \mathrm{e}^{rt}V_0 + \sigma\,\mathrm{e}^{rt}\int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u.
\end{aligned}$$

By (7.27) this shows that the portfolio strategy $(\xi_t,\eta_t)_{t\in[0,T]}$ given by (7.28)-(7.29) and its discounted portfolio value $\widetilde{V}_t := \mathrm{e}^{-rt}V_t$ satisfy

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad 0 \leqslant t \leqslant T,$$

which implies that $(\xi_t,\eta_t)_{t\in[0,T]}$ is self-financing by Lemma 5.14. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}^*[C]\,\mathrm{e}^{-rT}.$$

In addition, since there exists a hedging strategy leading to

$$\widetilde{V}_T = \mathrm{e}^{-rT}C,$$

then $(\widetilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale, with

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{V}_T \mid \mathcal{F}_t\big] = \mathrm{e}^{-rT}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T,$$

and initial value

$$\widetilde{V}_0 = \mathbb{E}^*\big[\widetilde{V}_T\big] = \mathrm{e}^{-rT}\mathbb{E}^*[C].$$

Hedging vanilla options

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t\in[0,T]}$ appearing in (7.26). This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property, see *e.g.* Protter (2001). The next lemma allows us to compute the process $(\zeta_t)_{t\in[0,T]}$ in case the payoff C is of the form $C=\phi(S_T)$ for some function ϕ.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

Proposition 7.11. *Assume that ϕ is a Lipschitz payoff function. Then, the function $g_c(t,x)$ defined from the Markov property of $(S_t)_{t\in[0,T]}$ by*

$$g_c(t,S_t) := \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t] = \mathbb{E}^*[\phi(S_T) \mid S_t]$$

is in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$, and the stochastic integral decomposition

$$\phi(S_T) = \mathbb{E}^*\big[\phi(S_T)\big] + \int_0^T \zeta_t d\widehat{B}_t \tag{7.33}$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{7.34}$$

In addition, the self-financing hedging strategy $(\xi_t)_{t\in[0,T]}$ satisfies

$$\xi_t = \mathrm{e}^{-(T-t)r}\frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{7.35}$$

Proof. It can be checked as in the proof of Proposition 7.5 that the function $g_c(t,x)$ is in $\mathcal{C}^{1,2}([0,T)\times\mathbb{R})$. Therefore, we can apply the Itô formula to the process

$$t \mapsto g_c(t,S_t) = \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

which is a martingale from the tower property (A.33) of conditional expectations as in (7.42) or Example (1) page 265. From the fact that the finite variation term in the Itô formula vanishes when $(g_c(t,S_t))_{t\in[0,T]}$ is a martingale, (see *e.g.* Corollary II-6-1 page 72 of Protter (2004)), we obtain:

$$g_c(t,S_t) = g_c(0,S_0) + \sigma\int_0^t S_u \frac{\partial C}{\partial x}(u,S_u)d\widehat{B}_u, \quad 0 \leqslant t \leqslant T, \tag{7.36}$$

with $g_c(0,S_0) = \mathbb{E}^*[\phi(S_T)]$. Letting $t := T$, we have

$$\phi(S_T) = g_c(T,S_T) = g_c(0,S_0) + \sigma\int_0^T S_t \frac{\partial C}{\partial x}(t,S_t)d\widehat{B}_t$$

which yields (7.34) by uniqueness of the stochastic integral decomposition (7.33) of $C = \phi(S_T)$. Finally, (7.35) follows from (7.28) and (7.34) by applying Proposition 7.10. □

In the case of European options, the process ζ can be computed via the next proposition which recovers the formula (6.3) for the Delta of a vanilla option, and follows from Proposition 7.11 and the relation

$$g_c(t,x) = \mathbb{E}^*\big[f(S_{t,T}^x)\big], \qquad 0 \leqslant t \leqslant T, \ x > 0.$$

In particular, we have $\xi_t \geqslant 0$ and there is no short selling when the payoff function ϕ is non-decreasing.

Corollary 7.12. *Assume that* $C = (S_T - K)^+$. *Then, for* $0 \leqslant t \leqslant T$ *we have*

$$\zeta_t = \sigma S_t \mathbb{E}^* \left[\frac{S_T}{S_t} \mathbb{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{|x=S_t}, \qquad 0 \leqslant t \leqslant T, \tag{7.37}$$

and

$$\xi_t = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\frac{S_T}{S_t} \mathbb{1}_{[K,\infty)} \left(x \frac{S_T}{S_t} \right) \right]_{|x=S_t}, \qquad 0 \leqslant t \leqslant T. \tag{7.38}$$

Proof. By (7.34) and the relation

$$S_T = S_0 \, \mathrm{e}^{\sigma B_T + \mu T - \sigma^2 T/2} = S_t \, \mathrm{e}^{(B_T - B_t)\sigma + (T-t)\mu - (T-t)\sigma^2/2},$$

we have

$$\begin{aligned} \zeta_t &= \sigma S_t \left(\frac{\partial}{\partial x} \mathbb{E}^*[\phi(S_T) \mid S_t = x] \right)_{x=S_t} \\ &= \sigma S_t \left(\frac{\partial}{\partial x} \mathbb{E}^* \left[\phi \left(x \frac{S_T}{S_t} \right) \right] \right)_{x=S_t}, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

as in Relation (A.49), hence by (7.28) we have

$$\begin{aligned} \xi_t &= \frac{1}{\sigma S_t} \mathrm{e}^{-(T-t)r} \zeta_t \\ &= \mathrm{e}^{-(T-t)r} \left(\frac{\partial}{\partial x} \mathbb{E}^* \left[\phi \left(x \frac{S_T}{S_t} \right) \right] \right)_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\frac{S_T}{S_t} \phi' \left(x \frac{S_T}{S_t} \right) \right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T. \end{aligned} \tag{7.39}$$

The above derivation can be checked for $\phi(x) = (x-K)^+$ and $\phi'(x) = \mathbb{1}_{[K,\infty)}(x)$ *e.g.* by writing expected values as integrals. □

By evaluating the expectation (7.37) in Corollary 7.12 we can recover the formula (6.16) in Proposition 6.4 for the Delta of the European call option in the Black-Scholes model. In that sense, the next proposition provides another proof of the result of Proposition 6.4.

Proposition 7.13. *The Delta of the European call option with payoff function* $f(x) = (x-K)^+$ *is given by*

$$\xi_t = \Phi\big(d_+(T-t)\big) = \Phi\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right), \quad 0 \leqslant t \leqslant T.$$

Proof. By Proposition 7.10 and Corollary 7.12, we have

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$\begin{aligned}
\xi_t &= \frac{1}{\sigma S_t}\,\mathrm{e}^{-(T-t)r}\zeta_t \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\frac{S_T}{S_t}\mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{|x=S_t} \\
&= \mathrm{e}^{-(T-t)r} \\
&\quad\times\mathbb{E}^*\left[\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\mathbb{1}_{[K,\infty)}\big(x\,\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\big)\right]_{|x=S_t} \\
&= \frac{1}{\sqrt{2(T-t)\pi}}\int_{(T-t)\sigma/2-(T-t)r/\sigma+\sigma^{-1}\log(K/S_t)}^{\infty}\mathrm{e}^{\sigma y-(T-t)\sigma^2/2-y^2/(2(T-t))}dy \\
&= \frac{1}{\sqrt{2(T-t)\pi}}\int_{-d_-(T-t)/\sqrt{T-t}}^{\infty}\mathrm{e}^{-(y-(T-t)\sigma)^2/(2(T-t))}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_-(T-t)}^{\infty}\mathrm{e}^{-(y-(T-t)\sigma)^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_+(T-t)}^{\infty}\mathrm{e}^{-y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+(T-t)}\mathrm{e}^{-y^2/2}dy \\
&= \Phi\big(d_+(T-t)\big).
\end{aligned}$$

□

The *Delta* of the Black-Scholes put option can be obtained as in Proposition 6.7 from (6.3), by differentiation of the call-put parity relation (7.8), and application of Proposition 7.13.

Proposition 7.13, combined with Proposition 7.6, shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

$$\xi_t = \Phi\big(d_+(T-t)\big) = \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{7.40}$$

of the risky asset, and to borrow a quantity

$$-\eta_t = K\,\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_t/K)+(r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{7.41}$$

of the riskless (savings) account, see also Corollary 16.18 in Chapter 16.

As noted above, the result of Proposition 7.13 recovers (6.17) which is obtained by a direct differentiation of the Black-Scholes function as in (6.3) or (7.39).

Markovian semi-groups

For completeness, we provide the definition of Markovian semi-groups which can be used to reformulate the proofs of this section.

Definition 7.14. *The Markov semi-group $(P_t)_{0\leqslant t\leqslant T}$ associated to $(S_t)_{t\in[0,T]}$ is the mapping P_t defined on functions $f \in \mathcal{C}_b^2(\mathbb{R})$ as*

$$P_t f(x) := \mathbb{E}^*[f(S_t) \mid S_0 = x], \qquad t \geqslant 0.$$

By the Markov property and time homogeneity of $(S_t)_{t\in[0,T]}$ we also have

$$P_t f(S_u) := \mathbb{E}^*[f(S_{t+u}) \mid \mathcal{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t, u \geqslant 0,$$

and the semi-group $(P_t)_{0\leqslant t\leqslant T}$ satisfies the composition property

$$P_s P_t = P_t P_s = P_{s+t} = P_{t+s}, \qquad s, t \geqslant 0,$$

as we have, using the Markov property and the tower property (A.33) of conditional expectations as in (7.42),

$$\begin{aligned} P_s P_t f(x) &= \mathbb{E}^*[P_t f(S_s) \mid S_0 = x] \\ &= \mathbb{E}^*\big[\mathbb{E}^*[f(S_t) \mid S_0 = y]_{y=S_s} \,\big|\, S_0 = x\big] \\ &= \mathbb{E}^*\big[\mathbb{E}^*[f(S_{t+s}) \mid S_s = y]_{y=S_s} \,\big|\, S_0 = x\big] \\ &= \mathbb{E}^*\big[\mathbb{E}^*[f(S_{t+s}) \mid \mathcal{F}_s] \,\big|\, S_0 = x\big] \\ &= \mathbb{E}^*[f(S_{t+s}) \mid S_0 = x] \\ &= P_{t+s} f(x), \qquad s, t \geqslant 0. \end{aligned}$$

Similarly, we can show that the process $(P_{T-t} f(S_t))_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale as in Example (i) above, see (7.1), *i.e.*:

$$\begin{aligned} \mathbb{E}^*[P_{T-t} f(S_t) \mid \mathcal{F}_u] &= \mathbb{E}^*\big[\mathbb{E}^*[f(S_T) \mid \mathcal{F}_t] \,\big|\, \mathcal{F}_u\big] \\ &= \mathbb{E}^*[f(S_T) \mid \mathcal{F}_u] \\ &= P_{T-u} f(S_u), \qquad 0 \leqslant u \leqslant t \leqslant T, \end{aligned} \tag{7.42}$$

and we have

$$P_{t-u} f(x) = \mathbb{E}^*[f(S_t) \mid S_u = x] = \mathbb{E}^*\left[f\left(x\frac{S_t}{S_u}\right)\right], \quad 0 \leqslant u \leqslant t. \tag{7.43}$$

Exercises

Exercise 7.1 (Bachelier (1900) model, Exercise 6.1 continued). Consider a market made of a riskless asset priced $A_t = A_0$ with zero interest rate, $t \geqslant 0$, and a risky asset whose price modeled by a standard Brownian motion as $S_t = B_t$, $t \geqslant 0$. Price the vanilla option with payoff $C = (B_T)^2$, and recover the solution of the Black-Scholes PDE of Exercise 6.1.

Exercise 7.2 Given the price process $(S_t)_{t\in\mathbb{R}_+}$ defined as the geometric Brownian motion

$$S_t := S_0 \,\mathrm{e}^{\sigma B_t + (r-\sigma^2/2)t}, \qquad t \geqslant 0,$$

price the option with payoff function $\phi(S_T)$ by writing $\mathrm{e}^{-rT}\mathbb{E}^*[\phi(S_T)]$ as an integral with respect to the lognormal probability density function, see Exercise 5.1.

Exercise 7.3 (See Exercise 7.29). Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{7.44}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \in \mathbb{R}$ and $\sigma > 0$.

a) Find the stochastic differential equation satisfied by the power $\left(S_t^p\right)_{t\in\mathbb{R}_+}$ of order $p \in \mathbb{R}$ of $(S_t)_{t\in\mathbb{R}_+}$.
b) Using the Girsanov Theorem 7.3 and the discounting Lemma 5.13, construct a probability measure under which the discounted process $\left(\mathrm{e}^{-rt}S_t^p\right)_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 7.4 Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$ in a market with interest rate $r = 0$, and let ϕ be a convex payoff function. Show that, for any two maturities $T_1 < T_2$ and $p, q \in [0,1]$ such that $p + q = 1$ we have

$$\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] \leqslant \mathbb{E}^*[\phi(S_{T_2})],$$

i.e. the price of the basket option with payoff $\phi(pS_{T_1} + qS_{T_2})$ is upper bounded by the price of the option with payoff $\phi(S_{T_2})$.

Hints:

i) For ϕ a convex function we have $\phi(px + qy) \leqslant p\phi(x) + q\phi(y)$ for any $x, y \in \mathbb{R}$ and $p, q \in [0,1]$ such that $p + q = 1$.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

ii) Any convex function $(\phi(S_t))_{t\in\mathbb{R}_+}$ of a martingale $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale.

Exercise 7.5 Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ under a risk-neutral measure $\mathbb{P}^*$ with risk-free interest rate r.

a) Does the European *call* option price $C(K) := \mathrm{e}^{-rT}\mathbb{E}^*[(S_T - K)^+]$ increase or decrease with the strike price K? Justify your answer.
b) Does the European *put* option price $C(K) := \mathrm{e}^{-rT}\mathbb{E}^*[(K - S_T)^+]$ increase or decrease with the strike price K? Justify your answer.

Exercise 7.6 Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ under a risk-neutral measure $\mathbb{P}^*$ with risk-free interest rate r.

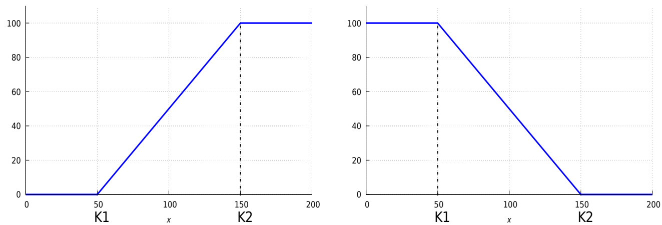
a) Show that the price at time t of the European call option with strike price K and maturity T is lower bounded by the positive part $\big(S_t - K\,\mathrm{e}^{-(T-t)r}\big)^+$ of the corresponding forward contract price, *i.e.* we have the *model-free* bound

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] \geqslant \big(S_t - K\,\mathrm{e}^{-(T-t)r}\big)^+, \quad 0 \leqslant t \leqslant T.$$

b) Show that the price at time t of the European put option with strike price K and maturity T is lower bounded by $K\,\mathrm{e}^{-(T-t)r} - S_t$, *i.e.* we have the *model-free* bound

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] \geqslant \big(K\,\mathrm{e}^{-(T-t)r} - S_t\big)^+, \quad 0 \leqslant t \leqslant T.$$

Exercise 7.7 The following two graphs describe the payoff functions ϕ of *bull spread* and *bear spread* options with payoff $\phi(S_N)$ on an underlying asset priced S_N at maturity time N.



(i) Bull spread payoff. (ii) Bear spread payoff.

Fig. 7.3: Payoff functions of bull spread and bear spread options.

a) Show that in each case (i) and (ii) the corresponding option can be realized by purchasing and/or short selling standard European call and put options with strike prices to be specified.
b) Price the bull spread option in cases (i) and (ii) using the Black-Scholes formula.

Hint: An option with payoff $\phi(S_T)$ is priced $\mathrm{e}^{-rT}\mathbb{E}^*[\phi(S_T)]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_T - K)^+$, resp. $(K - S_T)^+$.

Exercise 7.8 Given two strike prices $K_1 < K_2$, we consider a long box spread option realized as the combination of four legs having the same maturity time $N \geqslant 1$:

- One *long* call with strike price K_1 and payoff function $(x - K_1)^+$,
- One *short* put with strike price K_1 and payoff function $-(K_1 - x)^+$,
- One *short* call with strike price K_2 and payoff function $-(x - K_2)^+$,
- One *long* put with strike price K_2 and payoff function $(K_2 - x)^+$.

The risk-free interest rate is denoted by $r \geqslant 0$.

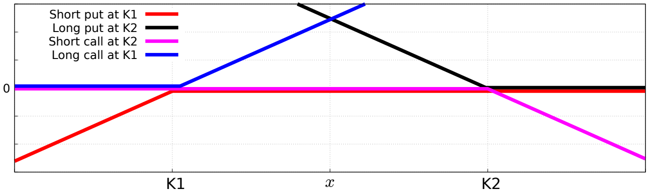


Fig. 7.4: Graphs of call/put payoff functions.

a) Find the payoff of the long box spread option in terms of K_1 and K_2.
b) Price the long box spread option at times $k = 0, 1, \ldots, N$ using K_1, K_2 and the interest rate r.
c) From Table 7.1 below, find a choice of strike prices $K_1 < K_2$ that can be used to build a long box spread option on the Hang Seng Index (HSI).
d) Price the option built in part (c) in index points, and then in HK$.

Hints.

i) The closing prices in Table 7.1 are warrant prices quoted in index points.
ii) Warrant prices are converted to option prices by multiplication by the number given in the "Entitlement Ratio" column.
iii) The conversion from index points to HK$ is given in Table 7.2.

e) Would you buy the option priced in part (d) ? Here we can take $r = 0$ for simplicity.

DERIVATIVE WARRANT SEARCH

Link to Relevant Exchange Traded Options

Updated: 2 March 2021

Basic Data										Market Data								
DW Code	Issuer	UL	Call/ Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike Currency	Strike	Entitle- ment Ratio^	Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Trading Currency	Day High	Day Low	Closing Price	T/O ('000)
17334	HT	HSI	Put	Standard	12-11-2020	28-05-2021	-	24600	10000	400,000,000	5.11	(0.001)	33.268	HKD	0.034	0.025	0.034	314
17535	UB	HSI	Put	Standard	13-11-2020	28-05-2021	-	24600	10000	300,000,000	22.07	(0.001)	29.507	HKD	0.025	0.017	0.023	132
17589	CS	HSI	Put	Standard	13-11-2020	28-05-2021	-	24900	9500	425,000,000	8.61	(0.001)	30.838	HKD	0.033	0.028	0.033	80
18242	UB	HSI	Put	Standard	19-11-2020	28-05-2021	-	25000	9500	300,000,000	18.51	(0.001)	29.028	HKD	0.030	0.023	0.029	265
18606	SG	HSI	Put	Standard	23-11-2020	29-06-2021	-	25088	10000	300,000,000	8.01	(0.002)	30.968	HKD	0.054	0.042	0.053	459
19399	HT	HSI	Put	Standard	02-12-2020	29-06-2021	-	25200	10000	400,000,000	0.06	(0.002)	32.190	HKD	0.000	0.000	0.061	0
19485	BI	HSI	Put	Standard	03-12-2020	29-06-2021	-	25200	10000	150,000,000	21.41	(0.002)	28.154	HKD	0.044	0.037	0.044	59
22857	VT	HSI	Put	Standard	27-02-2020	29-06-2021	-	25000	8000	80,000,000	22.45	(0.002)	30.905	HKD	0.065	0.043	0.064	1,165
26601	BI	HSI	Call	Standard	28-12-2020	29-06-2021	-	25200	11000	150,000,000	0.00	0.018	25.347	HKD	0.390	0.360	0.370	84
27489	BP	HSI	Call	Standard	18-09-2020	29-06-2021	-	25000	7500	80,000,000	2.95	0.009	28.392	HKD	0.590	0.540	0.540	6
28231	HS	HSI	Call	Standard	30-09-2020	29-06-2021	-	25118	7500	200,000,000	0.00	0.012	24.897	HKD	0.000	0.000	0.570	0

^ The entitlement ratio in general represents the number of derivative warrants required to be exercised into one share or one unit of the underlying asset (subject to any adjustments as may be necessary to reflect any capitalization, rights issue, distribution or the like).

Delayed data on Delta and Implied Volatility of Derivative Warrants are provided by Reuters.
Users should not use such data provided by Reuters for commercial purposes without its prior written consent.

For underlying stock price, please refer to Securities Prices of Market Data.

Table 7.1: Call and put options on the Hang Seng Index (HSI).

CONTRACT SUMMARY

Item	Standard Options	Flexible Options
Underlying Index	Hang Seng Index	
HKATS Code	HSI	XHS
Contract Multiplier	HK$50 per index point	
Minimum Fluctuation	One index point	
Contract Months	Short-dated Options:- Spot, next three calendar months & next three calendar quarter months and Long-dated Options:- the next 3 months of June & December and the following 3 December months	Any calendar month not further out than the longest term of expiry months that are available for trading
Exercise Style	European Style	
Option Premium	Quoted in whole index points	

Table 7.2: Contract summary.

Exercise 7.9 Butterfly options. A long call butterfly option is designed to deliver a limited payoff when the future volatility of the underlying asset is expected to be low. The payoff function of a long call butterfly option is plotted in Figure 7.5, with $K_1 := 50$ and $K_2 := 150$.

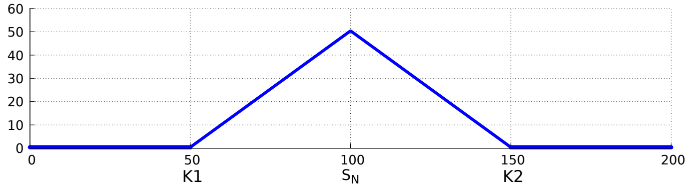


Fig. 7.5: Long call butterfly payoff function.

a) Show that the long call butterfly option can be realized by purchasing and/or issuing standard European call or put options with strike prices to be specified.
b) Price the long call butterfly option using the Black-Scholes formula.
c) Does the hedging strategy of the long call butterfly option involve holding or shorting the underlying stock?

Hints: Recall that an option with payoff $\phi(S_N)$ is priced in discrete time as $(1+r)^{-N}\mathbb{E}^*[\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 7.10 Forward contracts revisited. Consider a risky asset whose price S_t is given by $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Consider a forward contract with maturity T and payoff $S_T - \kappa$.

a) Compute the price C_t of this claim at any time $t \in [0,T]$.

b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 7.11 Option pricing with dividends (Exercise 6.3 continued). Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ paying dividends at the continuous-time rate $\delta > 0$, and modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Show that as in Lemma 5.14, if $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \geqslant 0,$$

where the dividend yield δS_t per share is continuously reinvested in the portfolio, then the discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic integral

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \geqslant 0,$$

b) Show that, as in Theorem 7.4, if $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, *i.e.* if $V_T = C$, then the arbitrage-free price of the claim payoff C is given by

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T,$$

where $\mathbb{E}^*$ denotes expectation under a risk-neutral probability measure $\mathbb{P}^*$.

c) Compute the price at time $t \in [0, T]$ of a European call option in a market with dividend rate δ by the martingale method.

d) Compute the Delta of the option.

Exercise 7.12 Forward start options (Rubinstein (1991)). A *forward start* European call option is an option whose holder receives at time T_1 (*e.g.* your birthday) the value of a standard European call option *at the money* and with maturity $T_2 > T_1$. Price this birthday present at any time $t \in [0, T_1]$, *i.e.* compute the price

$$\mathrm{e}^{-(T_1-t)r}\mathbb{E}^*\big[\mathrm{e}^{-(T_2-T_1)r}\mathbb{E}^*\big[(S_{T_2} - S_{T_1})^+ \,\big|\, \mathcal{F}_{T_1}\big] \,\big|\, \mathcal{F}_t\big]$$

at time $t \in [0, T_1]$, of the *forward start* European call option using the Black-Scholes formula

$$\mathrm{Bl}(x, K, \sigma, r, T-t) = x\Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) - K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right),$$

$0 \leqslant t < T$.

Exercise 7.13 Cliquet option. Let $0 = T_0 < T_1 < \cdots < T_n$ denote a sequence of financial settlement dates, and consider a risky asset priced as the geometric Brownian motion $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{P}^*$. Compute the price at time $t = 0$ of the cliquet option whose payoff consists in the sum of n payments $(S_{T_k}/S_{T_{k-1}} - K)^+$ made at times T_k, $k = 1, \ldots, n$. For this, use the Black-Scholes formula

$$\mathrm{e}^{-rT}\mathbb{E}^*[(S_T - \kappa)^+] = S_0\Phi\left(\frac{\log(S_0/\kappa) + (r+\sigma^2/2)T}{|\sigma|\sqrt{T}}\right) - \kappa\,\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_0/\kappa) + (r-\sigma^2/2)T}{|\sigma|\sqrt{T}}\right), \qquad T > 0.$$

Exercise 7.14 Log contracts. (Exercise 6.10 continued), see also Exercise 8.6. Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset valued $A_t = A_0 \mathrm{e}^{rt}$, $t \in [0, T]$, with $r > 0$. Compute the arbitrage-free price

$$C(t, S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^*[\log S_T \mid \mathcal{F}_t],$$

at time $t \in [0, T]$, of the log contract with payoff $\log S_T$.

Exercise 7.15 Power option. (Exercise 6.5 continued). Consider the price process $(S_t)_{t \in [0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset valued $A_t = A_0 \mathrm{e}^{rt}$, $t \in [0, T]$, with $r > 0$. In this problem, $(\eta_t, \xi_t)_{t \in [0,T]}$ denotes a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T.$$

a) Compute the arbitrage-free price

$$C(t, S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[|S_T|^2 \,\middle|\, \mathcal{F}_t \right],$$

at time $t \in [0, T]$, of the power option with payoff $C = |S_T|^2$.

b) Compute a self-financing hedging strategy $(\eta_t, \xi_t)_{t \in [0,T]}$ hedging the claim payoff $|S_T|^2$.

Exercise 7.16 (Bachelier (1900) model, Exercise 6.12 continued).

a) Consider the solution $(S_t)_{t \in \mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t.$$

For which value α_M of α is the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, a martingale under $\mathbb{P}$?

b) For each value of α, build a probability measure $\mathbb{P}_\alpha$ under which the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, is a martingale.

c) Compute the arbitrage-free price

$$C(t, S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha \left[\mathrm{e}^{S_T} \,\middle|\, \mathcal{F}_t \right]$$

at time $t \in [0, T]$ of the contingent claim with payoff $\exp(S_T)$, and recover the result of Exercise 6.12.

d) Explicitly compute the portfolio strategy $(\eta_t, \xi_t)_{t \in [0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

e) Check that this strategy is self-financing.

Exercise 7.17 Compute the arbitrage-free price

$$C(t, S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}_\alpha\big[(S_T)^2 \,\big|\, \mathcal{F}_t\big]$$

at time $t \in [0, T]$ of the power option with payoff $(S_T)^2$ in the framework of the Bachelier (1900) model of Exercise 7.16.

Exercise 7.18 (Exercise 5.8 continued, see Proposition 4.1 in Carmona and Durrleman (2003)). Consider two assets whose prices $S_t^{(1)}$, $S_t^{(2)}$ at time $t \in [0, T]$ follow the geometric Brownian dynamics

$$dS_t^{(1)} = rS_t^{(1)}dt + \sigma_1 S_t^{(1)}dW_t^{(1)} \quad dS_t^{(2)} = rS_t^{(2)}dt + \sigma_2 S_t^{(2)}dW_t^{(2)} \quad t \in [0, T],$$

where $\big(W_t^{(1)}\big)_{t\in[0,T]}$, $\big(W_t^{(2)}\big)_{t\in[0,T]}$ are two standard Brownian motions with correlation $\rho \in [-1, 1]$ under a risk-neutral probability measure $\mathbb{P}^*$, with $dW_t^{(1)} \bullet dW_t^{(2)} = \rho dt$.

Estimate the price $\mathrm{e}^{-rT}\mathbb{E}^*[(S_T - K)^+]$ of the spread option on $S_T := S_T^{(2)} - S_T^{(1)}$ with maturity $T > 0$ and strike price $K > 0$ by matching the first two moments of S_T to those of a Gaussian random variable.

Exercise 7.19 (Exercise 6.2 continued). Price the option with vanilla payoff $C = \phi(S_T)$ using the noncentral Chi square probability density function (17.5) of the Cox et al. (1985) (CIR) model.

Exercise 7.20 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Recall that for $f \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$, Itô's formula for $(B_t)_{t\in\mathbb{R}_+}$ reads

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds.$$

a) Let $r \in \mathbb{R}$, $\sigma > 0$, $f(x, t) = \mathrm{e}^{rt+\sigma x-\sigma^2 t/2}$, and $S_t = f(t, B_t)$. Compute $df(t, B_t)$ by Itô's formula, and show that S_t solves the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where $r > 0$ and $\sigma > 0$.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

b) Show that

$$\mathbb{E}\left[e^{\sigma B_T} \,\middle|\, \mathcal{F}_t\right] = e^{\sigma B_t + (T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T.$$

Hint: Use the independence of increments of $(B_t)_{t\in[0,T]}$ in the time splitting decomposition

$$B_T = (B_t - B_0) + (B_T - B_t),$$

and the Gaussian moment generating function $\mathbb{E}\left[e^{\alpha X}\right] = e^{\alpha^2\eta^2/2}$ when $X \simeq \mathcal{N}(0, \eta^2)$.

c) Show that the process $(S_t)_{t\in\mathbb{R}_+}$ satisfies

$$\mathbb{E}[S_T \mid \mathcal{F}_t] = e^{(T-t)r} S_t, \qquad 0 \leqslant t \leqslant T.$$

d) Let $C = S_T - K$ denote the payoff of a forward contract with exercise price K and maturity T. Compute the discounted expected payoff

$$V_t := e^{-(T-t)r}\mathbb{E}[C \mid \mathcal{F}_t].$$

e) Find a self-financing portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ such that

$$V_t = \xi_t S_t + \eta_t A_t, \qquad 0 \leqslant t \leqslant T,$$

where $A_t = A_0 e^{rt}$ is the price of a riskless asset with fixed interest rate $r > 0$. Show that it recovers the result of Exercise 6.7-(c).

f) Show that the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ found in Question (e) *hedges* the payoff $C = S_T - K$ at time T, *i.e.* show that $V_T = C$.

Exercise 7.21 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. A binary (or digital) *call*, *resp. put*, option is a contract with maturity T, strike price K, and payoff

$$C_d := \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K, \end{cases} \qquad \textit{resp.} \quad P_d := \begin{cases} \$1 & \text{if } S_T \leqslant K, \\ 0 & \text{if } S_T > K. \end{cases}$$

Recall that the prices $\pi_t(C_d)$ and $\pi_t(P_d)$ at time t of the binary call and put options are given by the discounted expected payoffs

$$\pi_t(C_d) = e^{-(T-t)r}\mathbb{E}[C_d \mid \mathcal{F}_t] \quad \text{and} \quad \pi_t(P_d) = e^{-(T-t)r}\mathbb{E}[P_d \mid \mathcal{F}_t]. \quad (7.45)$$

a) Show that the payoffs C_d and P_d can be rewritten as

$$C_d = \mathbb{1}_{[K,\infty)}(S_T) \qquad \text{and} \qquad P_d = \mathbb{1}_{[0,K]}(S_T).$$

b) Using Relation (7.45), Question (a), and the relation

$$\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \,\middle|\, S_t = x\right] = \mathbb{P}^*(S_T \geqslant K \mid S_t = x),$$

show that the price $\pi_t(C_d)$ is given by

$$\pi_t(C_d) = C_d(t, S_t),$$

where $C_d(t,x)$ is the function defined by

$$C_d(t,x) := \mathrm{e}^{-(T-t)r}\mathbb{P}^*(S_T \geqslant K \mid S_t = x).$$

c) Using the results of Exercise 5.10-(d) and of Question (b), show that the price $\pi_t(C_d) = C_d(t, S_t)$ of the binary call option is given by the function

$$\begin{aligned} C_d(t,x) &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

where

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) Assume that the binary option holder is entitled to receive a "return amount" $\alpha \in [0,1]$ in case the underlying asset price ends out of the money at maturity. Compute the price at time $t \in [0,T]$ of this modified contract.

e) Using Relation (7.45) and Question (a), prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = \mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T. \tag{7.46}$$

If needed, you may use the fact that $\mathbb{P}^*(S_T = K) = 0$.

f) Using the results of Questions (e) and (c), show that the price $\pi_t(P_d)$ of the binary put option is given as

$$\pi_t(P_d) = \mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big).$$

g) Using the result of Question (c), compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t, S_t)$$

of the binary call option. Does the Black-Scholes hedging strategy of such a call option involve short selling? Why?

h) Using the result of Question (f), compute the Delta

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$\xi_t := \frac{\partial P_d}{\partial x}(t, S_t)$$

of the binary put option. Does the Black-Scholes hedging strategy of such a put option involve short selling? Why?

Exercise 7.22 Computation of Greeks. Consider an underlying asset whose price $(S_t)_{t\in\mathbb{R}_+}$ is given by a stochastic differential equation of the form

$$dS_t = rS_t dt + \sigma(S_t)dB_t,$$

where $\sigma(x)$ is a Lipschitz coefficient, and an option with payoff function ϕ and price

$$C(x,T) = \mathrm{e}^{-rT}\mathbb{E}[\phi(S_T) \mid S_0 = x],$$

where $\phi(x)$ is a twice continuously differentiable ($\mathcal{C}^2$) function, with $S_0 = x$. Using the Itô formula, show that the sensitivity

$$\text{Theta}_T = \frac{\partial}{\partial T}\big(\mathrm{e}^{-rT}\mathbb{E}[\phi(S_T) \mid S_0 = x]\big)$$

of the option price with respect to maturity T can be expressed as

$$\text{Theta}_T = -r\,\mathrm{e}^{-rT}\mathbb{E}[\phi(S_T) \mid S_0 = x] + r\,\mathrm{e}^{-rT}\mathbb{E}\big[S_t\phi'(S_T)\,\big|\,S_0 = x\big] + \frac{1}{2}\,\mathrm{e}^{-rT}\mathbb{E}\big[\phi''(S_T)\sigma^2(S_T)\,\big|\,S_0 = x\big].$$

Problem 7.23 Chooser options. In this problem we denote by $C(t, S_t, K, T)$, resp. $P(t, S_t, K, T)$, the price at time t of the European call, resp. put, option with strike price K and maturity T, on an underlying asset priced $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure

a) Prove the call-put parity formula

$$C(t, S_t, K, T) - P(t, S_t, K, T) = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T. \quad (7.47)$$

b) Consider an option contract with maturity T, which entitles its holder to receive at time T the value of the European put option with strike price K and maturity $U > T$.
   Write down the price this contract at time $t \in [0, T]$ using a conditional expectation under the risk-neutral probability measure $\mathbb{P}^*$.
c) Consider now an option contract with maturity T, which entitles its holder to receive at time T either the value of a European call option or a European put option, whichever is higher. The European call and put options have same strike price K and same maturity $U > T$.

   Show that at maturity T, the payoff of this contract can be written as

$$P(T, S_T, K, U) + \text{Max}\left(0, S_T - K\,\mathrm{e}^{-(U-T)r}\right).$$

Hint: Use the call-put parity formula (7.47).

d) Price the contract of Question (c) at any time $t \in [0, T]$ using the call and put option pricing functions $C(t, x, K, T)$ and $P(t, x, K, U)$.

e) Using the Black-Scholes formula, compute the self-financing hedging strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ with portfolio value

$$V_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt}, \qquad 0 \leqslant t \leqslant T,$$

for the option contract of Question (c).

f) Consider now an option contract with maturity T, which entitles its holder to receive at time T the value of either a European call or a European put option, whichever is *lower*. The two options have same strike price K and same maturity $U > T$.

Show that the payoff of this contract at maturity T can be written as

$$C(T, S_T, K, U) - \text{Max}\left(0, S_T - K\,\mathrm{e}^{-(U-T)r}\right).$$

g) Price the contract of Question (f) at any time $t \in [0, T]$.

h) Using the Black-Scholes formula, compute the self-financing hedging strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ with portfolio value

$$V_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt}, \qquad 0 \leqslant t \leqslant T,$$

for the option contract of Question (f).

i) Give the price and hedging strategy of the contract that yields the sum of the payoffs of Questions (c) and (f).

j) What happens when $U = T$? Give the payoffs of the contracts of Questions (c), (f) and (i).

Problem 7.24 (Peng (2010)). Consider a risky asset priced

$$S_t = S_0\,\mathrm{e}^{\sigma B_t + \mu t - \sigma^2 t/2}, \quad i.e. \quad dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \geqslant 0,$$

a riskless asset valued $A_t = A_0\,\mathrm{e}^{rt}$, and a self-financing portfolio allocation $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ with value

$$V_t := \eta_t A_t + \xi_t S_t, \qquad t \geqslant 0.$$

a) Using the portfolio self-financing condition $dV_t = \eta_t dA_t + \xi_t dS_t$, show that we have

$$V_T = V_t + \int_t^T (rV_s + (\mu - r)\xi_s S_s)ds + \sigma \int_t^T \xi_s S_s dB_s.$$

b) Show that under the risk-neutral probability measure $\mathbb{P}^*$ the portfolio value V_t satisfies the *Backward Stochastic Differential Equation* (BSDE)

$$V_t = V_T - \int_t^T rV_s ds - \int_t^T \pi_s d\widehat{B}_s, \tag{7.48}$$

where $\pi_t := \sigma\xi_t S_t$ is the risky amount invested on the asset S_t, multiplied by σ, and $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

Hint: the Girsanov Theorem 7.3 states that

$$\widehat{B}_t := B_t + \frac{(\mu - r)t}{\sigma}, \qquad t \geqslant 0,$$

is a standard Brownian motion under $\mathbb{P}^*$.

c) Show that under the risk-neutral probability measure $\mathbb{P}^*$, the discounted portfolio value $\widetilde{V}_t := \mathrm{e}^{-rt}V_t$ can be rewritten as

$$\widetilde{V}_T = \widetilde{V}_0 + \int_0^T \mathrm{e}^{-rs}\pi_s d\widehat{B}_s. \tag{7.49}$$

d) Express $dv(t, S_t)$ by the Itô formula, where $v(t, x)$ is a $\mathcal{C}^2$ function of t and x.

e) Consider now a more general BSDE of the form

$$V_t = V_T - \int_t^T f(s, S_s, V_s, \pi_s)ds - \int_t^T \pi_s dB_s, \tag{7.50}$$

with terminal condition $V_T = g(S_T)$. By matching (7.50) to the Itô formula of Question (d), find the PDE satisfied by the function $v(t, x)$ defined as $V_t = v(t, S_t)$.

f) Show that when

$$f(t, x, v, z) = rv + \frac{\mu - r}{\sigma}z,$$

the PDE of Question (e) recovers the standard Black-Scholes PDE.

g) Assuming again $f(t, x, v, z) = rv + \frac{\mu - r}{\sigma}z$ and taking the terminal condition

$$V_T = (S_0\,\mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} - K)^+,$$

give the process $(\pi_t)_{t\in[0,T]}$ appearing in the stochastic integral representation (7.49) of the discounted claim payoff $\mathrm{e}^{-rT}(S_0\,\mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} - K)^+$.*

h) From now on we assume that short selling is penalized† at a rate $\gamma > 0$, *i.e.* $\gamma S_t|\xi_t|dt$ is subtracted from the portfolio value change dV_t whenever

* General Black-Scholes knowledge can be used for this question.

† SGX started to penalize naked short sales with an interim measure in September 2008.

$\xi_t < 0$ over the time interval $[t, t+dt]$. Rewrite the self-financing condition using $(\xi_t)^- := -\min(\xi_t, 0)$.

i) Find the BSDE of the form (7.50) satisfied by $(V_t)_{t\in\mathbb{R}_+}$, and the corresponding function $f(t,x,v,z)$.

j) Under the above penalty on short selling, find the PDE satisfied by the function $u(t,x)$ when the portfolio value V_t is given as $V_t = u(t, S_t)$.

k) *Differential interest rate.* Assume that one can borrow only at a rate R which is higher* than the risk-free interest rate $r > 0$, *i.e.* we have

$$dV_t = R\eta_t A_t dt + \xi_t dS_t$$

when $\eta_t < 0$, and

$$dV_t = r\eta_t A_t dt + \xi_t dS_t$$

when $\eta_t > 0$. Find the PDE satisfied by the function $u(t,x)$ when the portfolio value V_t is given as $V_t = u(t, S_t)$.

l) Assume that the portfolio differential reads

$$dV_t = \eta_t dA_t + \xi_t dS_t - dU_t,$$

where $(U_t)_{t\in\mathbb{R}_+}$ is a non-decreasing process. Show that the corresponding portfolio strategy $(\xi_t)_{t\in\mathbb{R}_+}$ is superhedging the claim payoff $V_T = C$.

Exercise 7.25 Girsanov Theorem. Assume that the Novikov integrability condition (7.11) is not satisfied. How does this modify the statement (7.13) of the Girsanov Theorem 7.3?

Problem 7.26 The Capital Asset Pricing Model (CAPM) of W.F. Sharpe (1990 Nobel Prize in Economics) is based on a linear decomposition

$$\frac{dS_t}{S_t} = (r+\alpha)dt + \beta \times \left(\frac{dM_t}{M_t} - rdt\right)$$

of stock returns dS_t/S_t into:

- a risk-free interest rate† r,
- an excess return α,
- a risk premium given by the difference between a benchmark market index return dM_t/M_t and the risk free rate r.

The coefficient β measures the sensitivity of the stock return dS_t/S_t with respect to the market index returns dM_t/M_t. In other words, β is the relative volatility of dS_t/S_t with respect to dM_t/M_t, and it measures the risk of $(S_t)_{t\in\mathbb{R}_+}$ in comparison to the market index $(M_t)_{t\in\mathbb{R}_+}$.

* Regular savings account usually pays r=0.05% per year. Effective Interest Rates (EIR) for borrowing could be as high as R=20.61% per year.

† The risk-free interest rate r is typically the yield of the 10-year Treasury bond.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

If $\beta > 1$, resp. $\beta < 1$, then the stock price S_t is more volatile (*i.e.* more risky), resp. less volatile (*i.e.* less risky), than the benchmark market index M_t. For example, if $\beta = 2$, then S_t goes up (or down) twice as much as the index M_t. Inverse Exchange-Traded Funds (IETFs) have a negative value of β. On the other hand, a fund which has a $\beta = 1$ can track the index M_t.

Vanguard 500 Index Fund (VFINX) has a $\beta = 1$ and can be considered as replicating the variations of the S&P 500 index M_t, while Invesco S&P 500 (SPHB) has a $\beta = 1.42$, and Xtrackers Low Beta High Yield Bond ETF (HYDW) has a β close to 0.36 and $\alpha = 6.36$.

In what follows, we assume that the benchmark market is represented by an index fund $(M_t)_{t\in\mathbb{R}_+}$ whose value is modeled according to

$$\frac{dM_t}{M_t} = \mu dt + \sigma_M dB_t, \tag{7.51}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. The asset price $(S_t)_{t\in\mathbb{R}_+}$ is modeled in a stochastic version of the CAPM as

$$\frac{dS_t}{S_t} = rdt + \alpha dt + \beta\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t, \tag{7.52}$$

with an additional stock volatility term $\sigma_S dW_t$, where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion independent of $(B_t)_{t\in\mathbb{R}_+}$, with

$$\operatorname{Cov}(B_t, W_t) = 0 \quad \text{and} \quad dB_t \bullet dW_t = 0, \quad t \geqslant 0.$$

The following 10 questions are interdependent and should be treated in sequence.

a) Show that β coincides with the regression coefficient

$$\beta = \frac{\operatorname{Cov}(dS_t/S_t, dM_t/M_t)}{\operatorname{Var}[dM_t/M_t]}.$$

Hint: We have

$$\operatorname{Cov}(dW_t, dW_t) = dt, \quad \operatorname{Cov}(dB_t, dB_t) = dt, \quad \text{and} \quad \operatorname{Cov}(dW_t, dB_t) = 0.$$

b) Show that the evolution of $(S_t)_{t\in\mathbb{R}_+}$ can be written as

$$dS_t = (r + \alpha + \beta(\mu - r))S_t dt + S_t\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}dZ_t$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Hint: The standard Brownian motion $(Z_t)_{t\in\mathbb{R}_+}$ can be characterized as the only continuous (local) martingale such that $(dZ_t)^2 = dt$, see *e.g.* Theorem 7.36 page 203 of Klebaner (2005).

From now on, we assume that β is allowed to depend locally on the state of the benchmark market index M_t, as $\beta(M_t)$, $t \geqslant 0$.

c) Rewrite the equations (7.51)-(7.52) into the system

$$\begin{cases} \dfrac{dM_t}{M_t} = rdt + \sigma_M dB_t^*, \\ \\ \dfrac{dS_t}{S_t} = rdt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^*, \end{cases}$$

where $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ have to be determined explicitly.

d) Using the Girsanov Theorem 7.3, construct a probability measure $\mathbb{P}^*$ under which $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ are independent standard Brownian motions.

Hint: Only the expression of the Radon-Nikodym density $d\mathbb{P}^*/d\mathbb{P}$ is needed here.

e) Show that the market based on the assets S_t and M_t is without arbitrage opportunities.

f) Consider a portfolio strategy $(\xi_t, \zeta_t, \eta_t)_{t\in[0,T]}$ based on the three assets $(S_t, M_t, A_t)_{t\in[0,T]}$, with value

$$V_t = \xi_t S_t + \zeta_t M_t + \eta_t A_t, \qquad t \in [0, T],$$

where $(A_t)_{t\in\mathbb{R}_+}$ is a riskless asset given by $A_t = A_0 \, e^{rt}$. Write down the self-financing condition for the portfolio strategy $(\xi_t, \zeta_t, \eta_t)_{t\in[0,T]}$.

g) Consider an option with payoff $C = h(S_T, M_T)$, priced as

$$f(t, S_t, M_t) = e^{-(T-t)r} \mathbb{E}^*[h(S_T, M_T) \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

Assuming that the portfolio $(V_t)_{t\in[0,T]}$ replicates the option price process $(f(t, S_t, M_t))_{t\in[0,T]}$, derive the pricing PDE satisfied by the function $f(t, x, y)$ and its terminal condition.

Hint: The following version of the Itô formula with two variables can be used for the function $f(t, x, y)$, see (4.26):

$$df(t, S_t, M_t) = \frac{\partial f}{\partial t}(t, S_t, M_t)dt + \frac{\partial f}{\partial x}(t, S_t, M_t)dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t)$$
$$+ \frac{\partial f}{\partial y}(t, S_t, M_t)dM_t + \frac{1}{2}(dM_t)^2 \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) + dS_t \bullet dM_t \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t).$$

h) Find the self-financing hedging portfolio strategy $(\xi_t, \zeta_t, \eta_t)_{t\in[0,T]}$ replicating the vanilla payoff $h(S_T, M_T)$.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

i) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when $\beta(M_t) = \beta$ is a constant and C is the European *call option* payoff on S_T with strike price K.
j) Solve the PDE of Question (g) and compute the replicating portfolio of Question (h) when $\beta(M_t) = \beta$ is a constant and C is the European *put option* payoff on S_T with strike price K.

Problem 7.27 Market bubbles occur when a financial asset becomes overvalued for various reasons, for example in the Dutch tulip bubble (1636-1637), Japan's stock market bubble (1986), dotcom bubble (2000), or US housing bubble (2009). Local martingales are used for the modeling of *market bubbles* and market crashes, see Cox and Hobson (2005), Heston et al. (2007), Jarrow et al. (2007), in which case the option call-put parity does not hold in general. In what follows we let $T > 0$ and we consider a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on $[0, T]$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$.

An $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $(M_t)_{t\in[0,T]}$ is called a (true) *martingale* on $[0, T]$ if

i) $\mathbb{E}[|M_t|] < \infty$ for all $t \in [0, T]$,
ii) $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$, for all $0 \leqslant s \leqslant t$.

An $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $(M_t)_{t\in[0,T]}$ is called a *supermartingale* on $[0, T]$ if

i) $\mathbb{E}[|M_t|] < \infty$ for all $t \in [0, T]$,
ii) $\mathbb{E}[M_t \mid \mathcal{F}_s] \leqslant M_s$, for all $0 \leqslant s \leqslant t$.

An $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $(M_t)_{t\in[0,T]}$ is called a *local martingale* on $[0, T]$ if there exists a nondecreasing sequence $(\tau_n)_{n\geqslant 1}$ of $[0, T]$-valued stopping times such that

i) $\lim_{n\to\infty} \tau_n = T$ almost surely,
ii) for all $n \geqslant 1$ the stopped process $(M_{\tau_n\wedge t})_{t\in[0,T]}$ is a (true) martingale under $\mathbb{P}$.

A local martingale on $[0, T]$ which is not a true martingale is called a *strict* local martingale.

1) a) Show that any martingale $(M_t)_{t\in[0,T]}$ on $[0, T]$ is a local martingale in $[0, T]$.
   b) Show that any *non-negative* local martingale $(M_t)_{t\in[0,T]}$ is a *supermartingale*.

   Hint: Use Fatou's lemma.
   c) Show that if $(M_t)_{t\in[0,T]}$ is a non-negative and *strict* local martingale on $[0, T]$ we have $\mathbb{E}[M_T] < M_0$.

Hint: Do the proof by contradiction using the tower property, the answer to Question (1b), and the fact that if a random variable X satisfies $X \leqslant 0$ *a.s.* and $\mathbb{E}[X] = 0$, then $X = 0$ a.s..

d) Show that the call-put parity

$$C(0, M_0) - P(0, M_0) = \mathbb{E}[M_0] - \mathrm{e}^{-rT} K$$

between $C(0, M_0)$ and $P(0, M_0)$ fails when the discounted asset price process $(M_t)_{t\in[0,T]}$ is a **strict** local martingale.
Hint: See Relation (7.8) in Proposition 7.25.

2) Let $(S_t)_{t\in[0,T]}$ be the solution of the stochastic differential equation

$$dS_t = \frac{S_t}{\sqrt{T-t}} dB_t \tag{7.53}$$

with $S_0 > 0$.

a) Show that $(S_t)_{t\in[0,T-\varepsilon]}$ is a martingale on $[0, T-\varepsilon]$ for every $\varepsilon \in (0, T)$.

Hint: Solve the stochastic differential equation (7.53) by the method of Proposition 6.16-*a*), and use Exercise 5.11-*b*).

b) Find the value of S_T by a simple argument.

c) Show that $(S_t)_{t\in[0,T]}$ is a strict local martingale on $[0, T]$.

Hint: Consider the stopping times

$$\tau_n := \left(\left(1 - \frac{1}{n}\right) T\right) \wedge \inf\{t \in [0, T] \ : \ |S_t| \geqslant n\}, \qquad n \geqslant 1,$$

and use Proposition 8.1.

d) Plot a sample graph of $(S_t)_{t\in[0,T]}$ with $T = 1$, and attach or upload it with your submission.

3) CEV model. Consider the positive strict local martingale $(S_t)_{t\in[0,T]}$ solution of $dS_t = S_t^2 dB_t$ with $S_0 > 0$, where S_t has the probability density function

$$\varphi_t(x) = \frac{S_0}{x^3\sqrt{2\pi t}} \left(\exp\left(-\frac{(1/x - 1/S_0)^2}{2t} \right) - \exp\left(-\frac{(1/x + 1/S_0)^2}{2t} \right) \right),$$

$x > 0$, $t \in (0, T]$, see § 2.1.2 in Jacquier (2017).

a) Plot a sample graph of $(S_t)_{t\in[0,T]}$ with $T = 1$, and attach or upload it with your submission.

b) Compute $\mathbb{E}[S_T]$ and check that the condition of Question (1c) is satisfied.
Hint: Use the change of variable $y = 1/x$ and the standard normal CDF Φ.

c) Compute the limit of $\mathbb{E}[S_T]$ as S_0 tends to infinity.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

d) Compute the price $\mathbb{E}[(S_T - K)^+]$ of a European call option with strike price $K > 0$ in this model, assuming a risk-free interest rate $r = 0$.
Hint: The final answer should be written in terms of the standard normal CDF Φ and of the normal PDF φ.

e) Show that $\mathbb{E}[(S_T - K)^+]$ is bounded uniformly in $S_0 > 0$ and $K > 0$ by a constant depending on $T > 0$.

Problem 7.28 Quantile hedging (Föllmer and Leukert (1999), §6.2 of Mel′nikov et al. (2002)). Recall that given two probability measures $\mathbb{P}$ and $\mathbb{Q}$, the Radon-Nikodym density $d\mathbb{P}/d\mathbb{Q}$ links the expectations of random variables F under $\mathbb{P}$ and under $\mathbb{Q}$ via the relation

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[F] &= \int_\Omega F(\omega) d\mathbb{Q}(\omega) \\ &= \int_\Omega F(\omega)\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) \\ &= \mathbb{E}_{\mathbb{P}}\left[F\frac{d\mathbb{Q}}{d\mathbb{P}}\right].\end{aligned}$$

a) Neyman-Pearson Lemma. Given $\mathbb{P}$ and $\mathbb{Q}$ two probability measures, consider the event

$$A_\alpha := \left\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \alpha\right\}, \qquad \alpha \geqslant 0.$$

Show that for A any event, $\mathbb{Q}(A) \leqslant \mathbb{Q}(A_\alpha)$ implies $\mathbb{P}(A) \leqslant \mathbb{P}(A_\alpha)$.

Hint: Start by *proving* that we always have

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}} - \alpha\right)(2\mathbb{1}_{A_\alpha} - 1) \geqslant \left(\frac{d\mathbb{P}}{d\mathbb{Q}} - \alpha\right)(2\mathbb{1}_A - 1). \tag{7.54}$$

b) Let $C \geqslant 0$ denote a nonnegative claim payoff on a financial market with risk-neutral measure $\mathbb{P}^*$. Show that the Radon-Nikodym density

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}^*} := \frac{C}{\mathbb{E}_{\mathbb{P}^*}[C]} \tag{7.55}$$

defines a probability measure $\mathbb{Q}^*$.

Hint: Check first that $d\mathbb{Q}^*/d\mathbb{P}^* \geqslant 0$, and then that $\mathbb{Q}^*(\Omega) = 1$. In the following questions we consider a nonnegative contingent claim with payoff $C \geqslant 0$ and maturity $T > 0$, priced $e^{-rT}\mathbb{E}_{\mathbb{P}^*}[C]$ at time 0 under the risk-neutral measure $\mathbb{P}^*$.

Budget constraint. *In what follows we will assume that no more than a certain fraction $\beta \in (0, 1]$ of the claim price $e^{-rT}\mathbb{E}_{\mathbb{P}^*}[C]$ is available to construct the initial hedging portfolio V_0 at time* 0.

Since a self-financing portfolio process $(V_t)_{t\in\mathbb{R}_+}$ *started at* $V_0 := \beta\,\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C]$ *may fall short of hedging the claim* C *when* $\beta < 1$*, we will attempt to maximize the probability* $\mathbb{P}(V_T \geqslant C)$ *of successful hedging, or, equivalently, to minimize the shortfall probability* $\mathbb{P}(V_T < C)$*.*

For this, given A *an event we consider the self-financing portfolio process* $(V_t^A)_{t\in[0,T]}$ *hedging the claim* $C\mathbb{1}_A$*, priced* $V_0^A = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_A]$ *at time* 0*, and such that* $V_T^A = C\mathbb{1}_A$ *at maturity* T*.*

c) Show that if α satisfies $\mathbb{Q}^*(A_\alpha) = \beta$, the event

$$A_\alpha = \left\{ \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^*} > \alpha \right\} = \left\{ \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} > \alpha \frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}^*} \right\} = \left\{ \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} > \frac{\alpha C}{\mathbb{E}_{\mathbb{P}^*}[C]} \right\}$$

maximizes $\mathbb{P}(A)$ over all possible events A, under the condition

$$\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}\left[V_T^A\right] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_A] \leqslant \beta\,\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C]. \tag{7.56}$$

Hint: Rewrite Condition (7.56) using the probability measure $\mathbb{Q}^*$, and apply the *Neyman-Pearson Lemma* of Question (a) to $\mathbb{P}$ and $\mathbb{Q}^*$.

d) Show that $\mathbb{P}(A_\alpha)$ coincides with the successful hedging probability

$$\mathbb{P}\big(V_T^{A_\alpha} \geqslant C\big) = \mathbb{P}(C\mathbb{1}_{A_\alpha} \geqslant C),$$

i.e. show that

$$\mathbb{P}(A_\alpha) = \mathbb{P}\big(V_T^{A_\alpha} \geqslant C\big) = \mathbb{P}(C\mathbb{1}_{A_\alpha} \geqslant C).$$

Hint: To prove an equality $x = y$ we can show first that $x \leqslant y$, and then that $x \geqslant y$. One inequality is obvious, and the other one follows from Question (c).

e) Check that the self-financing portfolio process $\big(V_t^{A_\alpha}\big)_{t\in[0,T]}$ hedging the claim with payoff $C\mathbb{1}_{A_\alpha}$ uses only the initial budget $\beta\,\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C]$, and that $\mathbb{P}\big(V_T^{A_\alpha} \geqslant C\big)$ maximizes the successful hedging probability.

In the next Questions (f)-(j) we assume that $C = (S_T - K)^+$ is the payoff of a European option in the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{7.57}$$

with $\mathbb{P} = \mathbb{P}^*$, $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{P}^* = 1$, and

$$S_0 := 1 \quad \text{and} \quad r = \frac{\sigma^2}{2} := \frac{1}{2}. \tag{7.58}$$

f) Solve the stochastic differential equation (7.57) with the parameters (7.58).

g) Compute the successful hedging probability

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$\mathbb{P}\big(V_T^{A_\alpha} \geqslant C\big) = \mathbb{P}(C\mathbb{1}_{A_\alpha} \geqslant C) = \mathbb{P}(A_\alpha)$$

for the claim $C =: (S_T - K)^+$ in terms of K, T, $\mathbb{E}_{\mathbb{P}^*}[C]$ and the parameter $\alpha > 0$.

h) From the result of Question (g), express the parameter α using K, T, $\mathbb{E}_{\mathbb{P}^*}[C]$, and the successful hedging probability $\mathbb{P}\big(V_T^{A_\alpha} \geqslant C\big)$ for the claim $C =: (S_T - K)^+$.

i) Compute the minimal initial budget $\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}]$ required to hedge the claim $C = (S_T - K)^+$ in terms of $\alpha > 0$, K, T and $\mathbb{E}_{\mathbb{P}^*}[C]$.

j) Taking $K := 1$, $T := 1$ and assuming a successful hedging probability of 90%, compute numerically:

   i) The European call price $\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+]$ from the Black-Scholes formula.

   ii) The value of $\alpha > 0$ obtained from Question (h).

   iii) The minimal initial budget needed to successfully hedge the European claim $C = (S_T - K)^+$ with probability 90% from Question (i).

   iv) The value of β, *i.e.* the budget reduction ratio which suffices to successfully hedge the claim $C =: (S_T - K)^+$ with 90% probability.

Problem 7.29 (Leung and Sircar (2015)) ProShares Ultra S&P500 and ProShares UltraShort S&P500 are leveraged investment funds that seek daily investment results, before fees and expenses, that correspond to β times (βx) the daily performance of the S&P500,® with respectively $\beta = 2$ for ProShares Ultra and $\beta = -2$ for ProShares UltraShort. Here, *leveraging* with a factor $\beta : 1$ aims at multiplying the potential return of an investment by a factor β. The following ten questions are interdependent and should be treated in sequence.

a) Consider a risky asset priced $S_0 := \$4$ at time $t = 0$ and taking two possible values $S_1 = \$5$ and $S_1 = \$2$ at time $t = 1$. Compute the two possible returns (in %) achieved when investing \$4 in one share of the asset S, and the expected return under the risk-neutral probability measure, assuming that the risk-free interest rate is zero.

b) Leveraging. Still based on an initial \$4 investment, we decide to leverage by a factor $\beta = 3$ by borrowing another $(\beta - 1) \times \$4 = 2 \times \$4$ at rate zero to purchase a total of $\beta = 3$ shares of the asset S. Compute the two returns (in %) possibly achieved in this case, and the expected return under the risk-neutral probability measure, assuming that the risk-free interest rate is zero.

c) Denoting by F_t the ProShares value at time t, how much should the fund invest in the underlying asset priced S_t, and how much \$ should it borrow or save on the risk-free market at any time t in order to leverage with a factor $\beta : 1$?

d) Find the portfolio allocation (ξ_t, η_t) for the fund value

$$F_t = \xi_t S_t + \eta_t A_t, \qquad t \geqslant 0,$$

according to Question (c), where $A_t := A_0 \, \mathrm{e}^{rt}$ is the riskless money market account.

e) We choose to model the S&P500 index S_t as the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \geqslant 0,$$

under the risk-neutral probability measure $\mathbb{P}^*$. Find the stochastic differential equation satisfied by $(F_t)_{t\in\mathbb{R}_+}$ under the self-financing condition $dF_t = \xi_t dS_t + \eta_t dA_t$, and show that the discounted fund value is a martingale.

f) Is the discounted fund value $(\mathrm{e}^{-rt}F_t)_{t\in\mathbb{R}_+}$ a martingale under the risk-neutral probability measure $\mathbb{P}^*$?

g) Find the relation between the fund value F_t and the index S_t by solving the stochastic differential equation obtained for F_t in Question (e). For simplicity we normalize $F_0 := S_0^\beta$.

h) Write the price at time $t = 0$ of the call option with claim payoff $C = (F_T - K)^+$ on the ProShares index using the Black-Scholes formula.

i) Show that when $\beta > 0$, the Delta at time $t \in [0, T)$ of the call option with claim payoff $C = (F_T - K)^+$ on ProShares Ultra is equal to the Delta of the call option with claim payoff $C = (S_T - K_\beta(t))^+$ on the S&P500, for a certain strike price $K_\beta(t)$ to be determined explicitly.

j) When $\beta < 0$, find the relation between the Delta at time $t \in [0, T)$ of the call option with claim payoff $C = (F_T - K)^+$ on ProShares UltraShort and the Delta of the *put* option with claim payoff $C = (K_\beta(t) - S_T)^+$ on the S&P500.

Problem 7.30 Log options. Log options can be used for the pricing of realized variance swaps, see § 8.2.

a) Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 4.22-(d) and a riskless asset with price $A_t = \$1 \times \mathrm{e}^{rt}$ and risk-free interest rate $r = \sigma^2/2$. From the answer to Exercise 4.22-(b), show that the arbitrage-free price

$$V_t = \mathrm{e}^{-(T-t)r} \mathbb{E}\big[(\log S_T)^+ \,\big|\, \mathcal{F}_t\big]$$

at time $t \in [0, T]$ of a log call option with payoff $(\log S_T)^+$ is equal to

$$V_t = \sigma \, \mathrm{e}^{-(T-t)r} \sqrt{\frac{T-t}{2\pi}} \, \mathrm{e}^{-B_t^2/(2(T-t))} + \sigma \, \mathrm{e}^{-(T-t)r} B_t \Phi\left(\frac{B_t}{\sqrt{T-t}}\right).$$

b) Show that V_t can be written as

$$V_t = g(T-t, S_t),$$

where $g(\tau, x) = \mathrm{e}^{-r\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma\sqrt{\frac{\tau}{2\pi}}\,\mathrm{e}^{-y^2/(2\sigma^2\tau)} + y\Phi\left(\frac{y}{\sigma\sqrt{\tau}}\right).$$

c) Figure 7.6 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.05 = 5\%$ *per year* and $\sigma = 0.1$. Assume that the current underlying asset price is \$1 and there remains 700 days to maturity. What is the price of the option?

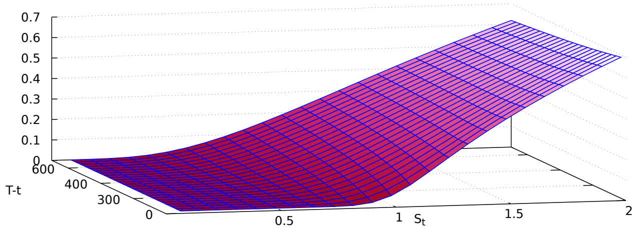


Fig. 7.6: Option price as a function of underlying asset price and time to maturity.

d) Show* that the (possibly fractional) quantity $\xi_t = \frac{\partial g}{\partial x}(T-t, S_t)$ of S_t at time t in a portfolio hedging the payoff $(\log S_T)^+$ is equal to

$$\xi_t = \mathrm{e}^{-(T-t)r}\frac{1}{S_t}\Phi\left(\frac{\log S_t}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant t \leqslant T.$$

e) Figure 7.7 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying asset price is \$1 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

* Recall the chain rule of derivation $\frac{\partial}{\partial x} f(\tau, \log x) = \frac{1}{x}\frac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

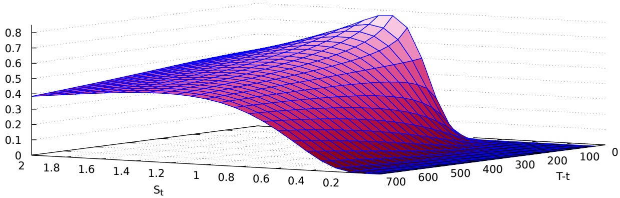


Fig. 7.7: Delta as a function of underlying asset price and time to maturity.

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset $A_t = \$1 \times \mathrm{e}^{rt}$, and for what amount?

g) Show that the Gamma of the portfolio, defined as $\Gamma_t = \frac{\partial^2 g}{\partial x^2}(T-t, S_t)$, equals

$$\Gamma_t = \mathrm{e}^{-(T-t)r}\frac{1}{S_t^2}\left(\frac{1}{\sigma\sqrt{2(T-t)\pi}}\mathrm{e}^{-(\log S_t)^2/(2(T-t)\sigma^2)} - \Phi\left(\frac{\log S_t}{\sigma\sqrt{T-t}}\right)\right),$$

$0 \leqslant t < T$.

h) Figure 7.8 represents the graph of Gamma. Assume that there remains 60 days to maturity and that S_t, currently at \$1, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

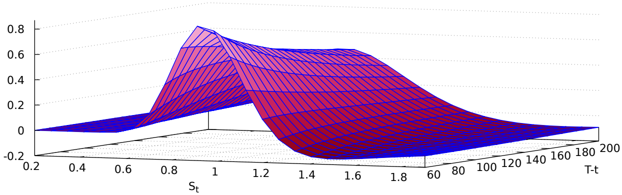


Fig. 7.8: Gamma as a function of underlying asset price and time to maturity.

i) Let now $\sigma = 1$. Show that the function $f(\tau, y)$ of Question (b) solves the *heat equation*

$$\begin{cases} \frac{\partial f}{\partial \tau}(\tau, y) = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\tau, y) \\ \\ f(0, y) = (y)^+. \end{cases}$$

Problem 7.31 Log put options with a given strike price.

a) Consider a market model made of a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ as in Exercise 5.10, a riskless asset valued $A_t = \$1 \times \mathrm{e}^{rt}$, risk-free interest rate $r = \sigma^2/2$ and $S_0 = 1$. From the answer to Exercise A.4-(c), show that the arbitrage-free price

$$V_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K - \log S_T)^+ \,\middle|\, \mathcal{F}_t\right]$$

at time $t \in [0, T]$ of a log call option with strike price K and payoff $(K - \log S_T)^+$ is equal to

$$V_t = \sigma\,\mathrm{e}^{-(T-t)r}\sqrt{\frac{T-t}{2\pi}}\,\mathrm{e}^{-(B_t-K/\sigma)^2/(2(T-t))} + \mathrm{e}^{-(T-t)r}(K-\sigma B_t)\Phi\left(\frac{K/\sigma - B_t}{\sqrt{T-t}}\right).$$

b) Show that V_t can be written as

$$V_t = g(T-t, S_t),$$

where $g(\tau, x) = \mathrm{e}^{-r\tau} f(\tau, \log x)$, and

$$f(\tau, y) = \sigma\sqrt{\frac{\tau}{2\pi}}\,\mathrm{e}^{-(K-y)^2/(2\sigma^2\tau)} + (K-y)\Phi\left(\frac{K-y}{\sigma\sqrt{\tau}}\right).$$

c) Figure 7.9 represents the graph of $(\tau, x) \mapsto g(\tau, x)$, with $r = 0.125$ *per year* and $\sigma = 0.5$. Assume that the current underlying asset price is \$3, that $K = 1$, and that there remains 700 days to maturity. What is the price of the option?

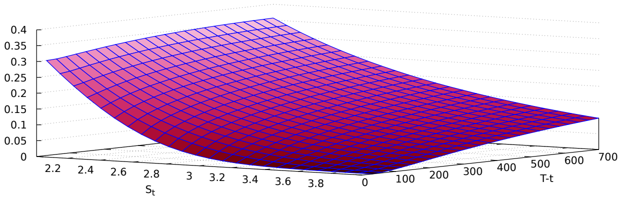


Fig. 7.9: Option price as a function of underlying asset price and time to maturity.

d) Show* that the quantity $\xi_t = \dfrac{\partial g}{\partial x}(T-t, S_t)$ of S_t at time t in a portfolio hedging the payoff $(K - \log S_T)^+$ is equal to

$$\xi_t = -\,\mathrm{e}^{-(T-t)r}\frac{1}{S_t}\Phi\left(\frac{K - \log S_t}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant t \leqslant T.$$

* Recall the chain rule of derivation $\dfrac{\partial}{\partial x} f(\tau, \log x) = \dfrac{1}{x}\dfrac{\partial f}{\partial y}(\tau, y)_{|y=\log x}$.

e) Figure 7.10 represents the graph of $(\tau, x) \mapsto \frac{\partial g}{\partial x}(\tau, x)$. Assuming that the current underlying asset price is \$3 and that there remains 700 days to maturity, how much of the risky asset should you hold in your portfolio in order to hedge one log option?

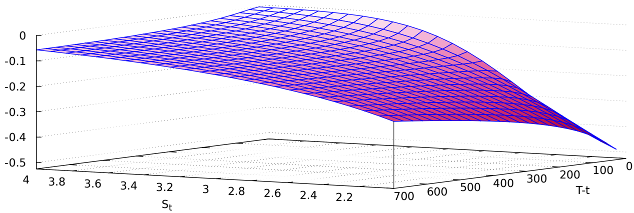


Fig. 7.10: Delta as a function of underlying asset price and time to maturity.

f) Based on the framework and answers of Questions (c) and (e), should you borrow or lend the riskless asset $A_t = \$1 \times e^{rt}$, and for what amount?

g) Show that the Gamma of the portfolio, defined as $\Gamma_t = \dfrac{\partial^2 g}{\partial x^2}(T - t, S_t)$, equals

$$\Gamma_t = e^{-(T-t)r} \frac{1}{S_t^2} \left(\frac{1}{\sigma\sqrt{2(T-t)\pi}} e^{-(K-\log S_t)^2/(2(T-t)\sigma^2)} + \Phi\left(\frac{K - \log S_t}{\sigma\sqrt{T-t}} \right) \right),$$

$0 \leqslant t \leqslant T$.

h) Figure 7.11 represents the graph of Gamma. Assume that there remains 10 days to maturity and that S_t, currently at \$3, is expected to increase. Should you buy or (short) sell the underlying asset in order to hedge the option?

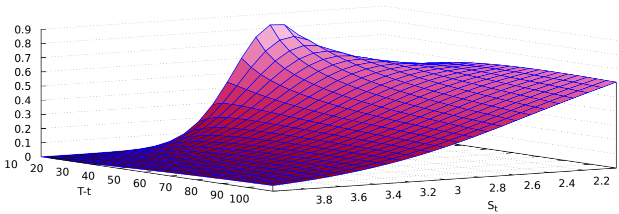


Fig. 7.11: Gamma as a function of underlying asset price and time to maturity.

i) Show that the function $f(\tau, y)$ of Question (b) solves the *heat equation*

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$
\begin{cases}
\dfrac{\partial f}{\partial \tau}(\tau, y) = \dfrac{\sigma^2}{2}\dfrac{\partial^2 f}{\partial y^2}(\tau, y) \\
\\
f(0, y) = (K - y)^+.
\end{cases}
$$