Chapter 14
Optimal Stopping Theorem

Stopping times are random times whose value can be determined by the historical behavior of a stochastic process modeling market data. This chapter presents additional material on optimal stopping and martingales, for use in the pricing and optimal exercise of American options in Chapter 15. Applications are given to hitting probabilities for Brownian motion.

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14.1 Filtrations and Information Flow

Let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ denote the *filtration* generated by a stochastic process $(X_t)_{t\in\mathbb{R}_+}$. In other words, $\mathcal{F}_t$ denotes the collection of all events possibly generated by $\{X_s \ : \ 0 \leqslant s \leqslant t\}$ up to time t. Examples of such events include the event

$$\{X_{t_0} \leqslant a_0, \ X_{t_1} \leqslant a_1, \ \ldots, \ X_{t_n} \leqslant a_n\}$$

for $a_0, a_1, \ldots, a_n$ a given fixed sequence of real numbers and $0 \leqslant t_1 < \cdots < t_n < t$, and $\mathcal{F}_t$ is said to represent the *information* generated by $(X_s)_{s\in[0,t]}$ up to time $t \geqslant 0$.

By construction, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a *non-decreasing* family of σ-algebras in the sense that we have $\mathcal{F}_s \subset \mathcal{F}_t$ (information known at time s is contained in the information known at time t) when $0 < s < t$.

One refers sometimes to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ as the non-decreasing flow of information generated by $(X_t)_{t\in\mathbb{R}_+}$.

14.2 Submartingales and Supermartingales

Let us recall the definition of *martingale* (cf. Definition 4.2), and introduce in addition the definitions of *super*martingale and *sub*martingale.*

Definition 14.1. *An integrable*† *stochastic process* $(Z_t)_{t\in\mathbb{R}_+}$ *is a martingale (resp. a* super*martingale, resp. a* sub*martingale) with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *if it satisfies the property*

$$Z_s = \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t, \qquad \text{(martingale)}$$

resp.

$$Z_s \geqslant \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t, \qquad (\textit{super}\text{martingale})$$

resp.

$$Z_s \leqslant \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t. \qquad (\textit{sub}\text{martingale})$$

Clearly, a stochastic process $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale if and only if it is both a *super*martingale and a *sub*martingale.

A particular property of martingales is that their expectation is constant over time $t \in \mathbb{R}_+$. In the next proposition we also check that *super*martingales have *non-increasing* expectation over time, while *sub*martingales have a *non-decreasing* expectation.

Proposition 14.2. *Let* $(Z_t)_{t\in\mathbb{R}_+}$ *denote an adapted integrable process.*

a) If $(Z_t)_{t\in\mathbb{R}_+}$ *is a* super*martingale, we have*

$$\mathbb{E}[Z_s] \geqslant \mathbb{E}[Z_t], \qquad 0 \leqslant s \leqslant t. \qquad (\textit{super}\text{martingale})$$

b) If $(Z_t)_{t\in\mathbb{R}_+}$ *is a* sub*martingale, we have*

$$\mathbb{E}[Z_s] \leqslant \mathbb{E}[Z_t], \qquad 0 \leqslant s \leqslant t. \qquad (\textit{sub}\text{martingale})$$

c) If $(Z_t)_{t\in\mathbb{R}_+}$ *be a martingale, we have*

$$\mathbb{E}[Z_s] = \mathbb{E}[Z_t], \qquad 0 \leqslant s \leqslant t. \qquad \text{(martingale)}$$

Proof. The case where $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale follows from the tower property (A.33) of conditional expectations, which shows that

$$\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t \mid \mathcal{F}_s]] = \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t. \tag{14.1}$$

Regarding *super*martingales, similarly to (14.1) we have

* "This obviously inappropriate nomenclature was chosen under the malign influence of the noise level of radio's SUPERman program, a favorite supper-time program of Doob's son during the writing of Doob (1953)", cf. Doob (1984), historical notes, page 808.

† This condition means that $\mathbb{E}[|Z_t|] < \infty$ for all $t \geqslant 0$.

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$$\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t \mid \mathcal{F}_s]] \leqslant \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t.$$

The proof is similar in the *sub*martingale case. □

Independent increments processes whose increments have negative expectation give examples of *super*martingales. For example, if $(Z_t)_{t\in\mathbb{R}_+}$ is such a stochastic process, then we have

$$\begin{aligned}\mathbb{E}[Z_t \mid \mathcal{F}_s] &= \mathbb{E}[Z_s \mid \mathcal{F}_s] + \mathbb{E}\big[Z_t - Z_s \,\big|\, \mathcal{F}_s\big] \\ &= \mathbb{E}[Z_s \mid \mathcal{F}_s] + \mathbb{E}[Z_t - Z_s] \\ &\leqslant \mathbb{E}[Z_s \mid \mathcal{F}_s] \\ &= Z_s, \qquad 0 \leqslant s \leqslant t.\end{aligned}$$

Similarly, a stochastic process with independent increments which have positive expectation will be a *sub*martingale. Brownian motion $B_t + \mu t$ with positive drift $\mu > 0$ is such an example, as in Figure 14.1 below.

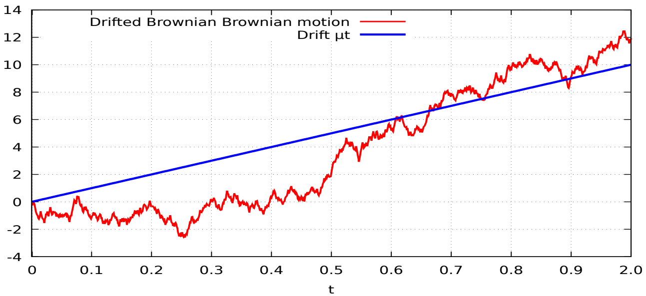

Fig. 14.1: Drifted Brownian path.

The following example comes from gambling.

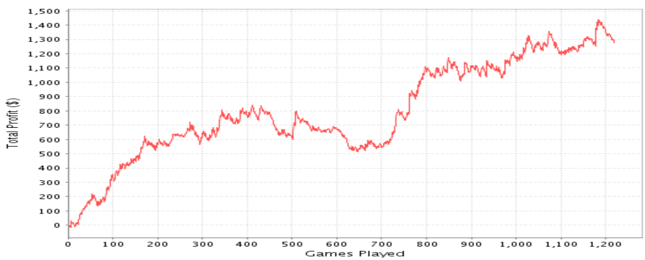

Fig. 14.2: Evolution of the fortune of a poker player *vs.* number of games played.

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Definition 14.3. *A function* $\phi : \mathbb{R} \to \mathbb{R}$ *is said to be* convex *if*

$$\phi(px + qy) \leqslant p\phi(x) + q\phi(y)$$

for any $p, q \in [0, 1]$ *such that* $p + q = 1$ *and* $x, y \in \mathbb{R}$*.*

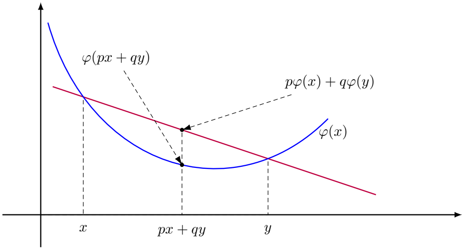

Fig. 14.3: Convex function.

Proposition 14.4. Jensen (1906) inequality*. Jensen's inequality states that for any sufficiently integrable random variable* X *and convex function* $\phi : \mathbb{R} \to \mathbb{R}$ *we have*

$$\phi(\mathbb{E}[X]) \leqslant \mathbb{E}[\phi(X)].$$

Proof. See *e.g.* (3.7.1) in Hardy et al. (1988). We only consider the case where X is a discrete random variable taking values in a finite set $\{x_1, \ldots, x_n\}$, with $\mathbb{P}(X = x_i) = p_i$, $i = 1, \ldots, n$, and show by induction on $n \geqslant 1$ that

$$\phi(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \leqslant p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_n\phi(x_n), \quad (14.2)$$

$x_1, \ldots, x_n \in \mathbb{R}$, for any sequence of coefficients $p_1, p_2, \ldots, p_n \geqslant 0$ such that $p_1 + p_2 + \cdots + p_n = 1$. The inequality (14.2) clearly holds for $n = 1$, and for $n = 2$ it coincides with the convexity property of ϕ, *i.e.*

$$\phi(p_1 x_1 + p_2 x_2) \leqslant p_1\phi(x_1) + p_2\phi(x_2), \qquad x_1, x_2 \in \mathbb{R}.$$

Assuming that (14.2) holds for some $n \geqslant 1$ and taking $p_1, p_2, \ldots, p_{n+1} \geqslant 0$ such that $p_1 + p_2 + \cdots + p_{n+1} = 1$ and $0 < p_{n+1} < 1$ and applying (14.2) at the second order, we have

$$\begin{aligned}
&\phi(p_1 x_1 + p_2 x_2 + \cdots + p_{n+1} x_{n+1}) \\
&= \phi\left((1 - p_{n+1})\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{1 - p_{n+1}} + p_{n+1} x_{n+1}\right) \\
&\leqslant (1 - p_{n+1})\phi\left(\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{1 - p_{n+1}}\right) + p_{n+1}\phi(x_{n+1})
\end{aligned}$$

$$\leqslant (1-p_{n+1})\left(\frac{p_1\phi(x_1)+p_2\phi(x_2)+\cdots+p_n\phi(x_n)}{1-p_{n+1}}\right)+p_{n+1}\phi(x_{n+1})$$
$$= p_1\phi(x_1)+p_2\phi(x_2)+\cdots+p_{n+1}\phi(x_{n+1}),$$

and we conclude by induction. □

A natural way to construct *sub*martingales is to take convex functions of martingales and to apply Jensen's inequality Proposition 14.4.

Proposition 14.5. *a) Given $(M_t)_{t\in\mathbb{R}_+}$ a martingale and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function, the process $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a* sub*martingale.*
b) Given $(M_t)_{t\in\mathbb{R}_+}$ a sub*martingale and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ a* non-decreasing *convex function, the process $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a* sub*martingale.*

Proof. *a*) By Jensen's inequality Proposition 14.4 we have

$$\phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t, \tag{14.3}$$

which shows that

$$\phi(M_s) = \phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t.$$

b) If ϕ is convex non-decreasing and $(M_t)_{\in\mathbb{R}_+}$ is a *sub*martingale, the above rewrites as

$$\phi(M_s) \leqslant \phi(\mathbb{E}[M_t \mid \mathcal{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t,$$

showing that $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a *sub*martingale. □

Similarly, $(\phi(M_t))_{t\in\mathbb{R}_+}$ will be a *super*martingale when $(M_t)_{\in\mathbb{R}_+}$ is a martingale and the function ϕ is *concave*.

As a direct application of Proposition 14.5, the process $(B_t^2)_{t\in\mathbb{R}_+}$ is a *sub*martingale as $\phi(x) = x^2$ is a convex function. Other examples of (*super*, *sub*)-martingales include geometric Brownian motion

$$S_t = S_0\,\mathrm{e}^{rt+\sigma B_t-\sigma^2 t/2}, \qquad t \geqslant 0,$$

which is a martingale for $r = 0$, a *super*martingale for $r \leqslant 0$, and a *sub*martingale for $r \geqslant 0$.

14.3 Optimal Stopping Theorem

Next, we turn to the definition of *stopping time*, which is based on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+} \subset \mathcal{F}$.

Definition 14.6. *An $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-*stopping time *is a random variable $\tau : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that*

$$\{\tau > t\} \in \mathcal{F}_t, \qquad t \geqslant 0. \tag{14.4}$$

The meaning of Relation (14.4) is that the knowledge of the event $\{\tau > t\}$ depends only on the information present in $\mathcal{F}_t$ up to time t, *i.e.* on the knowledge of $(X_s)_{0\leqslant s\leqslant t}$.

In other words, an event occurs at a *stopping time* τ if at any time t it can be decided whether the event has already occurred ($\tau \leqslant t$) or not ($\tau > t$) based on the information $\mathcal{F}_t$ generated by $(X_s)_{s\in\mathbb{R}_+}$ up to time t.

For example, the day you bought your first car is a stopping time (one can always answer the question "did I ever buy a car"), whereas the day you will buy your *last* car may not be a stopping time (one may not be able to answer the question "will I ever buy another car").

Proposition 14.7. *Every constant time is a stopping time. In addition, if τ and θ are stopping times, then*

i) the minimum $\tau \wedge \theta := \min(\tau, \theta)$ of τ and θ is also a stopping time,

ii) the maximum $\tau \vee \theta := \mathrm{Max}(\tau, \theta)$ of τ and θ is also a stopping time.

Proof. Point (i) is easily checked. Regarding (ii), we have

$$\{\tau \wedge \theta > t\} = \{\tau > t \ \text{ and } \ \theta > t\} = \{\tau > t\} \cap \{\theta > t\} \in \mathcal{F}_t, \quad t \geqslant 0.$$

On the other hand, we have

$$\{\tau \vee \theta \leqslant t\} = \{\tau \leqslant t \ \text{ and } \ \theta \leqslant t\} = \{\tau > t\}^c \cap \{\theta > t\}^c \in \mathcal{F}_t, \quad t \geqslant 0,$$

which implies

$$\{\tau \vee \theta > t\} = \{\tau \vee \theta \leqslant t\}^c \in \mathcal{F}_t, \qquad t \geqslant 0.$$

□

Hitting times

Hitting times provide natural examples of stopping times. The hitting time of level x by the process $(X_t)_{t\in\mathbb{R}_+}$, defined as

$$\tau_x = \inf\{t \in \mathbb{R}_+ \ : \ X_t = x\},$$

is a stopping time,* as we have (here in discrete time)

$$\begin{aligned}\{\tau_x > t\} &= \{X_s \neq x \text{ for all } s \in [0, t]\} \\ &= \{X_0 \neq x\} \cap \{X_1 \neq x\} \cap \cdots \cap \{X_t \neq x\} \in \mathcal{F}_t, \qquad t \in \mathbb{N}.\end{aligned}$$

* As a convention we let $\tau = +\infty$ in case there exists no $t \geqslant 0$ such that $X_t = x$.

In gambling, a hitting time can be used as an exit strategy from the game. For example, letting

$$\tau_{x,y} := \inf\{t \in \mathbb{R}_+ \ : \ X_t = x \text{ or } X_t = y\} \tag{14.5}$$

defines a hitting time (hence a stopping time) which allows a gambler to exit the game as soon as losses become equal to $x = -10$, or gains become equal to $y = +100$, whichever comes first. Hitting times can be used to trigger for "buy limit" or "sell stop" orders in finance.

However, not every $\mathbb{R}_+$-valued random variable is a stopping time. For example the random time

$$\tau = \inf \left\{ t \in [0,T] \ : \ X_t = \operatorname*{Sup}_{s\in[0,T]} X_s \right\},$$

which represents the first time the process $(X_t)_{t\in[0,T]}$ reaches its maximum over $[0,T]$, is not a stopping time with respect to the filtration generated by $(X_t)_{t\in[0,T]}$. Indeed, the information known at time $t \in (0,T)$ is not sufficient to determine whether $\{\tau > t\}$.

Stopped process

Given $(Z_t)_{t\in\mathbb{R}_+}$ a stochastic process and $\tau : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ a stopping time, the *stopped process* $(Z_{t\wedge\tau})_{t\in\mathbb{R}_+}$ is defined as

$$Z_{t\wedge\tau} := Z_{\min(t,\tau)} = \begin{cases} Z_t \ \text{ if } t < \tau, \\ \\ Z_\tau \ \text{ if } t \geqslant \tau, \end{cases}$$

Using indicator functions, we may also write

$$Z_{t\wedge\tau} = Z_t \mathbb{1}_{\{t<\tau\}} + Z_\tau \mathbb{1}_{\{t\geqslant\tau\}}, \qquad t \geqslant 0.$$

The following Figure 14.4 is an illustration of the path of a stopped process.

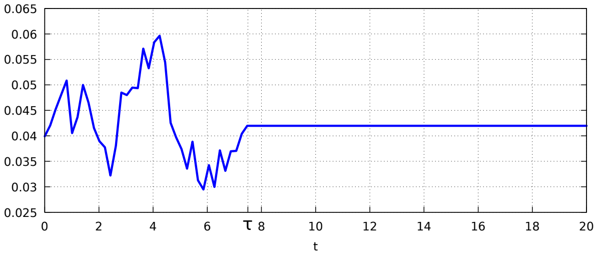

Fig. 14.4: Stopped process.

Theorem 14.8 below is called the *Stopping Time* (or *Optional Sampling*, or *Optional Stopping*) Theorem, it is due to the mathematician J.L. Doob (1910-2004). It is also used in Exercise 14.6 below.

Theorem 14.8. *Assume that* $(M_t)_{t\in\mathbb{R}_+}$ *is a martingale with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*, and that* τ *is an* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*-stopping time. Then, the* stopped process $(M_{t\wedge\tau})_{t\in\mathbb{R}_+}$ *is also a martingale with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Proof. We only give the proof in discrete time by applying the martingale transform argument of Theorem 2.11. Writing the telescoping sum

$$M_n = M_0 + \sum_{l=1}^{n}(M_l - M_{l-1}),$$

we have

$$M_{\tau\wedge n} = M_0 + \sum_{l=1}^{\tau\wedge n}(M_l - M_{l-1}) = M_0 + \sum_{l=1}^{n} \mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}),$$

and for $k \leqslant n$,

$$\begin{aligned}
\mathbb{E}[M_{\tau\wedge n} \mid \mathcal{F}_k] &= \mathbb{E}\left[M_0 + \sum_{l=1}^{n} \mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \,\Big|\, \mathcal{F}_k\right] \\
&= M_0 + \sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{k} \mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathcal{F}_k] \\
&\quad + \sum_{l=k+1}^{n} \mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathcal{F}_k]
\end{aligned}$$

$$
\begin{aligned}
&= M_0 + \sum_{l=1}^{k} (M_l - M_{l-1}) \mathbb{E}[\mathbb{1}_{\{l \leqslant \tau\}} \mid \mathcal{F}_k] \\
&\quad + \sum_{l=k+1}^{n} \mathbb{E}[\mathbb{E}[(M_l - M_{l-1}) \mathbb{1}_{\{l \leqslant \tau\}} \mid \mathcal{F}_{l-1}] \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{k} (M_l - M_{l-1}) \mathbb{1}_{\{l \leqslant \tau\}} \\
&\quad + \sum_{l=k+1}^{n} \mathbb{E}[\mathbb{1}_{\{l \leqslant \tau\}} \underbrace{\mathbb{E}[(M_l - M_{l-1}) \mid \mathcal{F}_{l-1}]}_{=0} \mid \mathcal{F}_k] \\
&= M_0 + \sum_{l=1}^{k} (M_l - M_{l-1}) \mathbb{1}_{\{l \leqslant \tau\}} \\
&= M_0 + \sum_{l=1}^{\tau \wedge k} (M_l - M_{l-1}) \\
&= M_{\tau \wedge k}, \qquad k = 0, 1, \ldots, n,
\end{aligned}
$$

as by the martingale property of $(M_l)_{l \in \mathbb{N}}$, we have

$$
\begin{aligned}
\mathbb{E}[(M_l - M_{l-1}) \mid \mathcal{F}_{l-1}] &= \mathbb{E}[M_l \mid \mathcal{F}_{l-1}] - \mathbb{E}[M_{l-1} \mid \mathcal{F}_{l-1}] \\
&= \mathbb{E}[M_l \mid \mathcal{F}_{l-1}] - M_{l-1} \\
&= 0, \qquad l \geqslant 1.
\end{aligned}
$$

□

Remarks.

a) More generally, if $(M_t)_{t \in \mathbb{R}_+}$ is a *super* (resp. *sub*)-martingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, then the *stopped process* $(M_{t \wedge \tau})_{t \in \mathbb{R}_+}$ remains a *super* (resp. *sub*)-martingale with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, see *e.g.* Exercise 14.6 below for the case of *sub*martingales in discrete time.

b) Since by Theorem 14.8 the stopped process $(M_{\tau \wedge t})_{t \in \mathbb{R}_+}$ is a martingale, we find that its expected value $\mathbb{E}[M_{\tau \wedge t}]$ is constant over time $t \in \mathbb{R}_+$ by Proposition 14.2-*c*).

   As a consequence, if $(M_t)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-martingale and τ is a stopping time bounded by a constant $T > 0$, *i.e.* $\tau \leqslant T$ almost surely,* we have

$$
\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau \wedge T}] = \mathbb{E}[M_{\tau \wedge 0}] = \mathbb{E}[M_0] = \mathbb{E}[M_T]. \tag{14.6}
$$

* "$\tau \leqslant T$ almost surely" means $\mathbb{P}(\tau \leqslant T) = 1$, *i.e.* $\mathbb{P}(\tau > T) = 0$.

c) From (14.6), if τ, ν are two stopping times *a.s. bounded* by a constant $T > 0$ and $(M_t)_{t\in\mathbb{R}_+}$ is a martingale, we have

$$\mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}[M_\nu] = \mathbb{E}[M_T]. \tag{14.7}$$

d) If τ, ν are two stopping times *a.s. bounded* by a constant $T > 0$ and such that $\tau \leqslant \nu$ *a.s.*, then, by Theorem 14.8,

 (i) when $(M_t)_{t\in\mathbb{R}_+}$ is a *super*martingale, we have

$$\mathbb{E}[M_0] \geqslant \mathbb{E}[M_\tau] \geqslant \mathbb{E}[M_\nu] \geqslant \mathbb{E}[M_T], \tag{14.8}$$

 (ii) when $(M_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, we have

$$\mathbb{E}[M_0] \leqslant \mathbb{E}[M_\tau] \leqslant \mathbb{E}[M_\nu] \leqslant \mathbb{E}[M_T], \tag{14.9}$$

 see Exercise 14.6 below for a proof in discrete time.

e) In case τ is finite with probability one (but not bounded by a constant), we may also write

$$\mathbb{E}[M_\tau] = \mathbb{E}\Big[\lim_{t\to\infty} M_{\tau\wedge t}\Big] = \lim_{t\to\infty} \mathbb{E}[M_{\tau\wedge t}] = \mathbb{E}[M_0], \tag{14.10}$$

provided that

$$|M_{\tau\wedge t}| \leqslant C, \qquad a.s., \quad t \geqslant 0+. \tag{14.11}$$

More generally, (14.10) holds provided that the limit and expectation signs can be exchanged, and this can be done using *e.g.* the *Dominated Convergence Theorem*. In some situations the exchange of limit and expectation signs may not be valid.*

In case $\mathbb{P}(\tau = +\infty) > 0$, (14.10) holds under the above conditions, provided that

$$M_\infty := \lim_{t\to\infty} M_t \tag{14.12}$$

exists with probability one.

Relations (14.8), (14.9) and (14.7) can be extended to unbounded stopping times along the same lines and conditions as (14.10), such as (14.11) applied to both τ and ν. Dealing with unbounded stopping times can be necessary in the case of hitting times.

f) In general, for all *a.s.* finite (bounded or unbounded) stopping times τ it remains true that

$$\mathbb{E}[M_\tau] = \mathbb{E}\Big[\lim_{t\to\infty} M_{\tau\wedge t}\Big] \leqslant \lim_{t\to\infty} \mathbb{E}\big[M_{\tau\wedge t}\big] \leqslant \lim_{t\to\infty} \mathbb{E}[M_0] = \mathbb{E}[M_0], \tag{14.13}$$

* Consider for example the sequence $M_n = n\mathbb{1}_{\{X<1/n\}}$, $n \geqslant 1$, where $X \simeq U(0,1]$ is a uniformly distributed random variable on $(0,1]$.

provided that $(M_t)_{t\in\mathbb{R}_+}$ is a nonnegative *super*martingale, where we used Fatou's Lemma A.12.* As in (14.10), the limit (14.12) is required to exist with probability one if $\mathbb{P}(\tau = +\infty) > 0$.

g) As a counterexample to (14.7), the random time

$$\tau := \inf\left\{t \in [0,T] \ : \ M_t = \operatorname*{Sup}_{s\in[0,T]} M_s\right\},$$

which is not a stopping time, will satisfy

$$\mathbb{E}[M_\tau] > \mathbb{E}[M_T],$$

although $\tau \leqslant T$ almost surely. Similarly,

$$\tau := \inf\left\{t \in [0,T] \ : \ M_t = \inf_{s\in[0,T]} M_s\right\},$$

is not a stopping time and satisfies

$$\mathbb{E}[M_\tau] < \mathbb{E}[M_T].$$

Martingales and stopping times as gambling strategies

When $(M_t)_{t\in[0,T]}$ is a martingale, *e.g.* a centered random walk with independent increments, the message of the Stopping Time Theorem 14.8 is that the expected gain of the exit strategy $\tau_{x,y}$ of (14.5) remains zero on average since

$$\mathbb{E}\big[M_{\tau_{x,y}}\big] = \mathbb{E}[M_0] = 0,$$

if $M_0 = 0$. Therefore, on average, this exit strategy does not increase the average gain of the player. More precisely we have

$$\begin{aligned} 0 = M_0 = \mathbb{E}[M_{\tau_{x,y}}] &= x\mathbb{P}(M_{\tau_{x,y}} = x) + y\mathbb{P}(M_{\tau_{x,y}} = y) \\ &= -10 \times \mathbb{P}(M_{\tau_{x,y}} = -10) + 100 \times \mathbb{P}(M_{\tau_{x,y}} = 100), \end{aligned}$$

which shows that

$$\mathbb{P}(M_{\tau_{x,y}} = -10) = \frac{10}{11} \quad \text{and} \quad \mathbb{P}(M_{\tau_{x,y}} = 100) = \frac{1}{11},$$

provided that the relation $\mathbb{P}(M_{\tau_{x,y}} = x) + \mathbb{P}(M_{\tau_{x,y}} = y) = 1$ is satisfied, see below for further applications to Brownian motion.

* $\mathbb{E}[\lim_{n\to\infty} F_n] \leqslant \lim_{n\to\infty} \mathbb{E}[F_n]$ for any sequence $(F_n)_{n\in\mathbb{N}}$ of *nonnegative* random variables, provided that the limits exist.

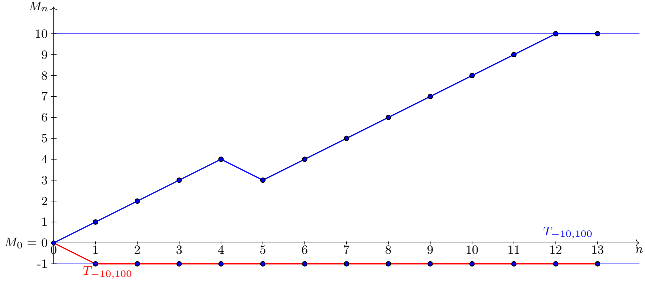

Fig. 14.5: Sample paths of a gambling process $(M_n)_{n\in\mathbb{N}}$.

In Table 14.1 we summarize some of the results obtained in this section for bounded stopping times.

		Bounded stopping times $\tau,\ \nu$
$(M_t)_{t\in\mathbb{R}_+}$	*super*martingale	$\mathbb{E}[M_\tau] \geqslant \mathbb{E}[M_\nu]$ if $\tau \leqslant \nu$.
	martingale	$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu]$.
	*sub*martingale	$\mathbb{E}[M_\tau] \leqslant \mathbb{E}[M_\nu]$ if $\tau \leqslant \nu$.

Table 14.1: Martingales and stopping times.

In the sequel we note that, as an application of the Stopping Time Theorem 14.8, a number of expectations can be computed in a simple and elegant way.

14.4 Drifted Brownian Motion

Brownian motion hitting a barrier

Given $a, b \in \mathbb{R}$, $a < b$, let the hitting* time $\tau_{a,b} : \Omega \longrightarrow \mathbb{R}_+$ be defined by

$$\tau_{a,b} = \inf\{t \geqslant 0 \ : \ B_t = a \text{ or } B_t = b\},$$

which is the hitting time of the boundary $\{a, b\}$ of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, $a < b \in \mathbb{R}$.

* Hitting times are stopping times.

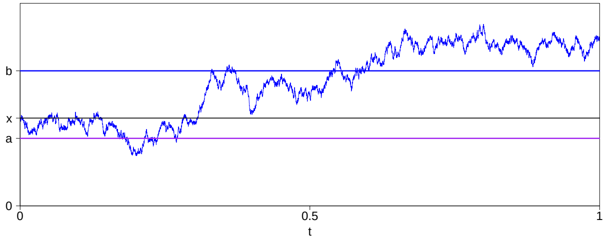

Fig. 14.6: Brownian motion hitting a barrier.

Recall that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale since it has independent increments, and those increments are centered:

$$\mathbb{E}[B_t - B_s] = 0, \qquad 0 \leqslant s \leqslant t.$$

Consequently, $(B_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is still a martingale, and by (14.10) we have

$$\mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = \mathbb{E}[B_0 \mid B_0 = x] = x,$$

as the exchange between limit and expectation in (14.10) can be justified since

$$|B_{t\wedge\tau_{a,b}}| \leqslant \mathrm{Max}(|a|, |b|), \qquad t \geqslant 0.$$

Hence we have

$$\begin{cases} x = \mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = a \times \mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) + b \times \mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x), \\ \\ \mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) + \mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) = 1, \end{cases}$$

which yields

$$\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) = \frac{x-a}{b-a}, \qquad a \leqslant x \leqslant b,$$

and also shows that

$$\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) = \frac{b-x}{b-a}, \qquad a \leqslant x \leqslant b.$$

Note that the above result and its proof actually apply to any continuous martingale, and not only to Brownian motion.

Drifted Brownian motion hitting a barrier

Next, let us turn to the case of drifted Brownian motion

$$X_t = x + B_t + \mu t, \qquad t \geqslant 0.$$

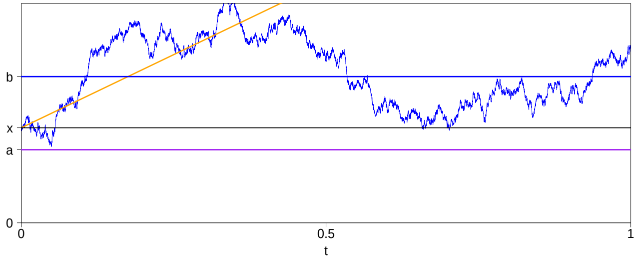

Fig. 14.7: Drifted Brownian motion hitting a barrier.

In this case, the process $(X_t)_{t\in\mathbb{R}_+}$ is no longer a martingale and in order to use Theorem 14.8 we need to construct a martingale of a different type. Here we note that the process

$$M_t := \mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

is a martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Indeed, we have

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\big[\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \,\big|\, \mathcal{F}_s\big] = \mathrm{e}^{\sigma B_s - \sigma^2 s/2}, \qquad 0 \leqslant s \leqslant t,$$

cf. *e.g.* Example 3 page 266.

By Theorem 14.8, we know that the stopped process $(M_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is a martingale, hence its expected value is constant over time $t \in \mathbb{R}_+$ Proposition 14.2-*c*), and (14.10) yields

$$1 = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_{a,b}}],$$

as the exchange between limit and expectation in (14.10) can be justified since

$$|M_{t\wedge\tau_{a,b}}| \leqslant \mathrm{Max}\left(\mathrm{e}^{\sigma|a|}, \mathrm{e}^{\sigma|b|}\right), \qquad t \geqslant 0.$$

Next, we note that taking $\mu = -\sigma/2$, *i.e.* $\sigma = -2\mu$, we have $M_t = \mathrm{e}^{-\sigma x}\,\mathrm{e}^{\sigma X_t}$, and

$$\mathrm{e}^{\sigma X_t} = \mathrm{e}^{\sigma x + \sigma B_t + \sigma \mu t} = \mathrm{e}^{\sigma x + \sigma B_t - \sigma^2 t/2} = \mathrm{e}^{\sigma x} M_t,$$

hence

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$$
\begin{aligned}
1 &= \mathbb{E}[M_{\tau_{a,b}}] \\
&= \mathrm{e}^{-\sigma x}\mathbb{E}[\mathrm{e}^{\sigma X_{\tau_{a,b}}}] \\
&= \mathrm{e}^{(a-x)\sigma}\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathrm{e}^{(b-x)\sigma}\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) \\
&= \mathrm{e}^{-2(a-x)\mu}\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathrm{e}^{-2(b-x)\mu}\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x),
\end{aligned}
$$

under the additional condition

$$
\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1.
$$

Finally, this gives

$$
\begin{cases}
\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) = \dfrac{\mathrm{e}^{\sigma x} - \mathrm{e}^{\sigma b}}{\mathrm{e}^{\sigma a} - \mathrm{e}^{\sigma b}} = \dfrac{\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} & \text{(14.14a)} \\[2em]
\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = \dfrac{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}}, & \text{(14.14b)}
\end{cases}
$$

$a \leqslant x \leqslant b$, see Figure 14.8 for an illustration with $a = 1$, $b = 2$, $x = 1.3$, $\mu = 2.0$, and $(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x})/(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}) = 0.7118437$.

```
nsim <- 1000;a=1;b=2;x=1.3;mu=2.0;N=10001; T<-2.0; t <- 0:(N-1); dt <- T/N; prob=0; time=0;
dev.new(width=16,height=8); for (i in 1:nsim){signal=0;colour="blue";Z <-
      rnorm(N,mean=0,sd= sqrt(dt));
X <- c(1,N);X[1]=x;for (j in 2:N){X[j]=X[j-1]+Z[j]+mu*dt
if (X[j]<=a && signal==0) {signal=-1;colour="purple";time=time+j}
if (X[j]>=b && signal==0) {signal=1;colour="blue";prob=prob+1;time=time+j}}
plot(t, X, xlab = "t", ylab = "", type = "l", ylim = c(-0,3), col =
      "blue",main=paste("Prob=",prob,"/",i,"=",round(prob/i, digits=5),"
      Time=",round(time*dt, digits=3),"/",i,"=",round(time*dt/i, digits=5)), xaxs="i",
      yaxs="i", xaxt="n", yaxt="n",cex.axis=1.8,cex.lab=1.8,cex.main=2)
lines(t, x+mu*t*dt, type = "l", col = "orange",lwd=3);yticks<-c(0,a,x,b);
axis(side=2, at=yticks,labels = c(0,"a","x","b"), las = 2,cex.axis=1.8)
xticks<-c(0,5000,10000); axis(side=1, at=xticks,labels = c(0,"0.5","1"), las = 1,cex.axis=1.8)
abline(h=x,lw=2); abline(h=a,col="purple",lwd=3);
abline(h=b,col="blue",lwd=3) # Sys.sleep(0.5)
readline(prompt = "Pause. Press <Enter> to continue...")}
(exp(-2*mu*a)-exp(-2*mu*x))/(exp(-2*mu*a)-exp(-2*mu*b))
(b*(exp(-2*mu*a)-exp(-2*mu*x))+a*(exp(-2*mu*x)-exp(-2*mu*b))
      -x*(exp(-2*mu*a)-exp(-2*mu*b)))/mu/(exp(-2*mu*a)-exp(-2*mu*b))
```

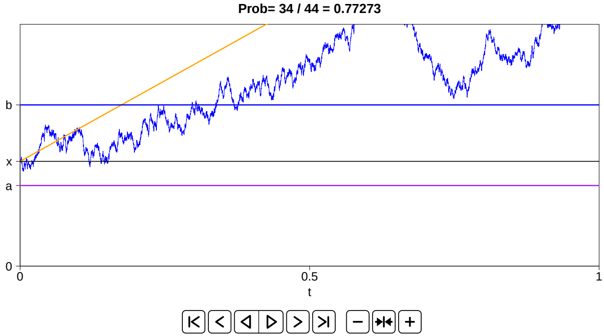

Fig. 14.8: Hitting probabilities of drifted Brownian motion.*

Escape to infinity

Letting b tend to $+\infty$ in the above equalities shows by (14.14a)-(14.14b) that the probability $\mathbb{P}(\tau_a = +\infty)$ of escape to $+\infty$ of Brownian motion started from $x \in (a, \infty)$ is equal to

$$\mathbb{P}(\tau_a = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = 1 - \mathrm{e}^{-2(x-a)\mu} > 0, & \mu > 0, \\ \\ 0, \quad \mu \leqslant 0, \end{cases}$$

i.e.

$$\mathbb{P}(\tau_a < +\infty) = \begin{cases} \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = \mathrm{e}^{-2(x-a)\mu} < 1, & \mu > 0, \\ \\ 1, \quad \mu \leqslant 0. \end{cases} \tag{14.15}$$

(14.16)

Similarly, letting a tend to $-\infty$ shows that the probability $\mathbb{P}(\tau_b = +\infty)$ of escape to $-\infty$ of Brownian motion started from $x \in (-\infty, b)$ is equal to

$$\mathbb{P}(\tau_b = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = 1 - \mathrm{e}^{-2(x-b)\mu} > 0, & \mu < 0, \\ \\ 0, \quad \mu \geqslant 0, \end{cases}$$

i.e.

* The animation works in Acrobat Reader on the entire pdf file.

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$$\mathbb{P}(\tau_b < +\infty) = \begin{cases} \mathbb{P}(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = \mathrm{e}^{-2(x-b)\mu} < 1, & \mu < 0, \\ \\ 1, & \mu \geqslant 0. \end{cases} \tag{14.17}$$

Mean hitting times for Brownian motion

The martingale method also allows us to compute the expected time $\mathbb{E}[\tau_{a,b}]$, after rechecking that

$$B_t^2 - t = 2\int_0^t B_s dB_S, \qquad t \geqslant 0,$$

is also a martingale. Indeed, we have

$$\begin{aligned}
&\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = \mathbb{E}[(B_s + (B_t - B_s))^2 - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s) - t \mid \mathcal{F}_s] \\
&= \mathbb{E}[B_s^2 - s \mid \mathcal{F}_s] - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2\mathbb{E}[B_s(B_t - B_s) \mid \mathcal{F}_s] \\
&= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2B_s\mathbb{E}[B_t - B_s \mid \mathcal{F}_s] \\
&= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[B_t - B_s] \\
&= B_s^2 - s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

Consequently the stopped process $(B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t)_{t\in\mathbb{R}_+}$ is still a martingale by Theorem 14.8 hence the expectation $\mathbb{E}[B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t]$ is constant over time $t \in \mathbb{R}_+$, hence by (14.10) we get*

$$\begin{aligned}
x^2 &= \mathbb{E}[B_0^2 - 0 \mid B_0 = x] \\
&= \mathbb{E}[B^2_{\tau_{a,b}} - \tau_{a,b} \mid B_0 = x] \\
&= \mathbb{E}[B^2_{\tau_{a,b}} \mid B_0 = x] - \mathbb{E}[\tau_{a,b} \mid B_0 = x] \\
&= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - \mathbb{E}[\tau_{a,b} \mid B_0 = x],
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathbb{E}[\tau_{a,b} \mid B_0 = x] &= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - x^2 \\
&= b^2\frac{x-a}{b-a} + a^2\frac{b-x}{b-a} - x^2 \\
&= (x-a)(b-x), \qquad a \leqslant x \leqslant b.
\end{aligned}$$

Mean hitting time for drifted Brownian motion

Finally we show how to recover the value of the mean hitting time $\mathbb{E}[\tau_{a,b} \mid X_0 = x]$ of drifted Brownian motion $X_t = x + B_t + \mu t$. As above, the process

* Here we note that it can be showed that $\mathbb{E}[\tau_{a,b}] < \infty$ in order to apply (14.10).

$X_t - \mu t$ is a martingale the stopped process $(X_{\tau_{a,b}\wedge t} - (\tau_{a,b} \wedge t)\mu)_{t\in\mathbb{R}_+}$ is still a martingale by Theorem 14.8. Hence the expectation $\mathbb{E}[X_{\tau_{a,b}\wedge t} - (\tau_{a,b} \wedge t)\mu]$ is constant over time $t \geqslant 0$.

Since the stopped process $\big(X_{\tau_{a,b}\wedge t} - (\tau_{a,b} \wedge t)\mu\big)_{t\in\mathbb{R}_+}$ is a martingale, we have

$$x = \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x],$$

which gives

$$\begin{aligned} x &= \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x] \\ &= \mathbb{E}[X_{\tau_{a,b}} \mid X_0 = x] - \mu\mathbb{E}[\tau_{a,b} \mid X_0 = x] \\ &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - \mu\mathbb{E}[\tau_{a,b} \mid X_0 = x], \end{aligned}$$

i.e. by (14.14a),

$$\begin{aligned} \mu\mathbb{E}[\tau_{a,b} \mid X_0 = x] &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - x \\ &= b\frac{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} + a\frac{\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} - x \\ &= \frac{b(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}) + a(\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}) - x(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}}, \end{aligned}$$

hence

$$\mathbb{E}[\tau_{a,b} \mid X_0 = x] = \frac{b(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}) + a(\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}) - x(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})}{(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})\mu},$$

$a \leqslant x \leqslant b$, see the code of page 509 for an illustration.

Table 14.2 presents a summary of the families of martingales used in this chapter.

Probabilities / Problem	Non drifted	Drifted
Hitting probability $\mathbb{P}(X_{\tau_{a,b}} = a, b)$	B_t	$\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$
Mean hitting time $\mathbb{E}[\tau_{a,b}]$	$B_t^2 - t$	$X_t - \mu t$

Table 14.2: List of martingales.

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Exercises

Exercise 14.1 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0, *i.e.* $B_0 = 0$.

a) Is the process $t \mapsto (2 - B_t)^+$ a *sub*martingale, a martingale or a *super*martingale?
b) Is the process $(\mathrm{e}^{B_t})_{t\in\mathbb{R}_+}$ a *sub*martingale, a martingale, or a *super*martingale?
c) Consider the random time ν defined by

$$\nu := \inf\{t \in \mathbb{R}_+ \ : \ B_t = B_{2t}\},$$

which represents the first intersection time of the curves $(B_t)_{t\in\mathbb{R}_+}$ and $(B_{2t})_{t\in\mathbb{R}_+}$.

Is ν a stopping time?
d) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \mathrm{e}^{B_t - t/2} = \alpha + \beta t\right\},$$

which represents the first time geometric Brownian motion $\mathrm{e}^{B_t - t/2}$ crosses the straight line $t \mapsto \alpha + \beta t$. Is τ a stopping time?
e) If τ is a stopping time, compute $\mathbb{E}[\tau]$ by the Doob Stopping Time Theorem 14.8 in each of the following two cases:

   i) $\alpha > 1$ and $\beta < 0$,

   ii) $\alpha < 1$ and $\beta > 0$.

Exercise 14.2 Stopping times. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0.

a) Consider the random time ν defined by

$$\nu := \inf\{t \in \mathbb{R}_+ \ : \ B_t = B_1\},$$

which represents the first time Brownian motion B_t hits the level B_1. Is ν a stopping time?
b) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \mathrm{e}^{B_t} = \alpha\,\mathrm{e}^{-t/2}\right\},$$

which represents the first time the exponential of Brownian motion B_t crosses the path of $t \mapsto \alpha\,\mathrm{e}^{-t/2}$, where $\alpha > 1$.

Is τ a stopping time? If τ is a stopping time, compute $\mathbb{E}[\mathrm{e}^{-\tau}]$ by applying the Stopping Time Theorem 14.8.

c) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ B_t^2 = 1 + \alpha t\right\},$$

which represents the first time the process $(B_t^2)_{t\in\mathbb{R}_+}$ crosses the straight line $t \mapsto 1 + \alpha t$, with $\alpha < 1$.

Is τ a stopping time? If τ is a stopping time, compute $\mathbb{E}[\tau]$ by the Doob Stopping Time Theorem 14.8.

Exercise 14.3 Consider a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ started at $B_0 = 0$, and let

$$\tau_L = \inf\{t \in \mathbb{R}_+ \ : \ B_t = L\}$$

denote the first hitting time of the level $L > 0$ by $(B_t)_{t\in\mathbb{R}_+}$.

a) Compute the Laplace transform $\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right]$ of τ_L for all $r \geqslant 0$.

   Hint: Use the Stopping Time Theorem 14.8 and the fact that $\left(\mathrm{e}^{\sqrt{2r}B_t - rt}\right)_{t\in\mathbb{R}_+}$ is a martingale when $r > 0$.
b) Find the optimal level stopping strategy depending on the value of $r > 0$ for the maximization problem

$$\operatorname*{Sup}_{L>0} \mathbb{E}\left[\mathrm{e}^{-r\tau_L} B_{\tau_L}\right].$$

Exercise 14.4 Consider $(B_t)_{t\in\mathbb{R}_+}$ a Brownian motion started at $B_0 = x \in [a, b]$ with $a < b$, and let

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ B_t = a \quad \text{or} \quad B_t = b\right\}$$

denote the first exit time of $(B_t)_{t\in\mathbb{R}_+}$ from the interval $[a, b]$.

a) Let f be a $\mathcal{C}^2$ function on $\mathbb{R}$. Show that the process $X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)ds$ is a martingale.
b) Assume that $f(x)$ solves the differential equation $f''(x) = -2$ with boundary conditions $f(a) = f(b) = 0$. Using the Doob Stopping Time Theorem, show that $f(x) = \mathbb{E}[\tau \mid B_0 = x]$, $x \in [a, b]$.
c) Find the solution $f(x)$ of the equation $f''(x) = -2$ with $f(a) = f(b) = 0$.
d) Find the value of $\mathbb{E}[\tau \mid B_0 = x]$.

Exercise 14.5 Consider a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ started at $B_0 = 0$, and let

$$\tau := \inf\{t \in \mathbb{R}_+ \ : \ B_t = \alpha + \beta t\}$$

denote the first hitting time of the straight line $t \mapsto \alpha + \beta t$ by $(B_t)_{t\in\mathbb{R}_+}$, where $\alpha, \beta \in \mathbb{R}$.

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a) Compute the Laplace transform $\mathbb{E}[\mathrm{e}^{-r\tau}]$ of τ for all $r > 0$ and $\alpha \geqslant 0$.
b) Compute the Laplace transform $\mathbb{E}[\mathrm{e}^{-r\tau}]$ of τ for all $r > 0$ and $\alpha \leqslant 0$.

Hint. Use the stopping time theorem and the fact that $\left(\mathrm{e}^{\sigma B_t - \sigma^2 t/2}\right)_{t\in\mathbb{R}_+}$ is a martingale for all $\sigma \in \mathbb{R}$.

Exercise 14.6 (Doob-Meyer decomposition in discrete time). Let $(M_n)_{n\in\mathbb{N}}$ be a discrete-time *sub*martingale with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$.

a) Show that there exists two processes $(N_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ such that

   i) $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$,

   ii) $(A_n)_{n\in\mathbb{N}}$ is non-decreasing, *i.e.* $A_n \leqslant A_{n+1}$ *a.s.*, $n \in \mathbb{N}$,

   iii) $(A_n)_{n\in\mathbb{N}}$ is predictable in the sense that A_n is $\mathcal{F}_{n-1}$-measurable, $n \in \mathbb{N}$, and

   iv) $M_n = N_n + A_n$, $n \in \mathbb{N}$.

   Hint: Let $A_0 := 0$,

$$A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n], \quad n \geqslant 0,$$

   and define $(N_n)_{n\in\mathbb{N}}$ in such a way that it satisfies the four required properties.

b) Show that for all bounded stopping times σ and τ such that $\sigma \leqslant \tau$ *a.s.*, we have

$$\mathbb{E}[M_\sigma] \leqslant \mathbb{E}[M_\tau].$$

   Hint: Use the Stopping Time Theorem 14.8 for martingales and (14.7).