Chapter 19
Pricing of Interest Rate Derivatives

Interest rate derivatives are option contracts whose payoffs can be based on fixed-income securities such as bonds, or on cash flows exchanged in *e.g.* interest rate swaps. In this chapter we consider the pricing and hedging of interest rate and fixed income derivatives such as bond options, caplets, caps and swaptions, using the change of numéraire technique and forward measures.

19.1 Forward Measures and Tenor Structure

The maturity dates are arranged according to a discrete *tenor structure*

$$\{0 = T_0 < T_1 < T_2 < \cdots < T_n\}.$$

A sample of forward interest rate curve data is given in Table 19.1, which contains the values of $(T_1, T_2, \ldots, T_{23})$ and of $\{f(t, t+T_i, t+T_i+\delta)\}_{i=1,2,\ldots,23}$, with $t = 07/05/2003$ and $\delta =$ six months.

Maturity	2D	1W	1M	2M	3M	1Y	2Y	3Y	4Y	5Y	6Y	7Y
Rate (%)	2.55	2.53	2.56	2.52	2.48	2.34	2.49	2.79	3.07	3.31	3.52	3.71
Maturity	8Y	9Y	10Y	11Y	12Y	13Y	14Y	15Y	20Y	25Y	30Y	
Rate (%)	3.88	4.02	4.14	4.23	4.33	4.40	4.47	4.54	4.74	4.83	4.86	

Table 19.1: Forward rates arranged according to a tenor structure.

Recall that by definition of $P(t,T_i)$ and absence of arbitrage the discounted bond price process

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_i), \qquad 0 \leqslant t \leqslant T_i,$$

is an $\mathcal{F}_t$-martingale under the probability measure $\mathbb{P}^* = \mathbb{P}$, hence it satisfies the Assumption (A) on page 556 for $i = 1,2,\ldots,n$. As a consequence the bond price process can be taken as a numéraire

$$N_t^{(i)} := P(t,T_i), \qquad 0 \leqslant t \leqslant T_i,$$

in the definition

$$\frac{d\widehat{\mathbb{P}}_i}{d\mathbb{P}^*} = \frac{1}{P(0,T_i)} \mathrm{e}^{-\int_0^{T_i} r_s ds} \tag{19.1}$$

of the *forward measure* $\widehat{\mathbb{P}}_i$, see Definition 16.1. The following proposition will allow us to price contingent claims using the forward measure $\widehat{\mathbb{P}}_i$, it is a direct consequence of Proposition 16.5, noting that here we have $P(T_i,T_i) = 1$.

Proposition 19.1. *For all sufficiently integrable random variables C we have*

$$\mathbb{E}^*\left[C \,\mathrm{e}^{-\int_t^{T_i} r_s ds} \,\Big|\, \mathcal{F}_t \right] = P(t,T_i) \widehat{\mathbb{E}}_i[C \mid \mathcal{F}_t], \quad 0 \leqslant t \leqslant T_i, \quad i = 1,2,\ldots,n. \tag{19.2}$$

Recall that by Proposition 16.4, the deflated process

$$t \mapsto \frac{P(t,T_j)}{P(t,T_i)}, \qquad 0 \leqslant t \leqslant \min(T_i,T_j),$$

is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}_i$ for all $T_i, T_j \geqslant 0$, $i,j = 1,2,\ldots,n$.

In the sequel we assume as in (17.25) that the dynamics of the bond price $P(t,T_i)$ is given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t) dW_t, \qquad i = 1,2,\ldots,n, \tag{19.3}$$

see *e.g.* (17.28) in the Vasicek case, where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$ and $(r_t)_{t\in\mathbb{R}_+}$ and $(\zeta_i(t))_{t\in\mathbb{R}_+}$ are adapted processes with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(W_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$P(t,T_i) = P(0,T_i) \exp\left(\int_0^t r_s ds + \int_0^t \zeta_i(s) dW_s - \frac{1}{2}\int_0^t |\zeta_i(s)|^2 ds \right),$$

$0 \leqslant t \leqslant T_i$, $i = 1,2,\ldots,n$.

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Forward Brownian motions

Proposition 19.2. *For all $i = 1, 2, \ldots, n$, the process*

$$\widehat{W}_t^{(i)} := W_t - \int_0^t \zeta_i(s)ds, \qquad 0 \leqslant t \leqslant T_i, \tag{19.4}$$

is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}_i$.

Proof. The Girsanov Proposition 16.7 applied to the numéraire

$$N_t^{(i)} := P(t, T_i), \qquad 0 \leqslant t \leqslant T_i,$$

as in (16.13), shows that

$$\begin{aligned} d\widehat{W}_t^{(i)} &:= dW_t - \frac{1}{N_t^{(i)}} dN_t^{(i)} \bullet dW_t \\ &= dW_t - \frac{1}{P(t, T_i)} dP(t, T_i) \bullet dW_t \\ &= dW_t - \frac{1}{P(t, T_i)} (P(t, T_i) r_t dt + \zeta_i(t) P(t, T_i) dW_t) \bullet dW_t \\ &= dW_t - \zeta_i(t) dt, \end{aligned}$$

is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}_i$ for all $i = 1, 2, \ldots, n$. □

We have

$$d\widehat{W}_t^{(i)} = dW_t - \zeta_i(t) dt, \qquad i = 1, 2, \ldots, n, \tag{19.5}$$

and

$$d\widehat{W}_t^{(j)} = dW_t - \zeta_j(t) dt = d\widehat{W}_t^{(i)} + (\zeta_i(t) - \zeta_j(t)) dt, \quad i, j = 1, 2, \ldots, n,$$

which shows that $(\widehat{W}_t^{(j)})_{t \in \mathbb{R}_+}$ has drift $(\zeta_i(t) - \zeta_j(t))_{t \in \mathbb{R}_+}$ under $\widehat{\mathbb{P}}_i$.

Bond price dynamics under the forward measure

In order to apply Proposition 19.1 and to compute the price

$$\mathbb{E}^*\left[e^{-\int_t^{T_i} r_s ds} C \,\Big|\, \mathcal{F}_t \right] = P(t, T_i) \widehat{\mathbb{E}}_i[C \mid \mathcal{F}_t],$$

of a random claim payoff C, it can be useful to determine the dynamics of the underlying variables r_t, $f(t,T,S)$, and $P(t,T)$ via their stochastic differential equations written under the forward measure $\widehat{\mathbb{P}}_i$.

As a consequence of Proposition 19.2 and (19.3), the dynamics of $t \mapsto P(t,T_j)$ under $\widehat{\mathbb{P}}_i$ is given by

$$\frac{dP(t,T_j)}{P(t,T_j)} = r_t dt + \zeta_i(t)\zeta_j(t)dt + \zeta_j(t)d\widehat{W}_t^{(i)}, \quad i,j = 1,2,\ldots,n, \tag{19.6}$$

where $(\widehat{W}_t^{(i)})_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}_i$, and we have

$$\begin{aligned}
&P(t,T_j)\\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)dW_s - \frac{1}{2}\int_0^t |\zeta_j(s)|^2 ds\right) \qquad [\text{under } \mathbb{P}^*]\\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^{(j)} + \frac{1}{2}\int_0^t |\zeta_j(s)|^2 ds\right) \qquad [\text{under } \widehat{\mathbb{P}}_j]\\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^{(i)} + \int_0^t \zeta_j(s)\zeta_i(s)ds - \frac{1}{2}\int_0^t |\zeta_j(s)|^2 ds\right) \ [\text{under } \widehat{\mathbb{P}}_i]\\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^{(i)} - \frac{1}{2}\int_0^t |\zeta_j(s)-\zeta_i(s)|^2 ds + \frac{1}{2}\int_0^t |\zeta_i(s)|^2 ds\right),
\end{aligned}$$

$t \in [0,T_j]$, $i,j = 1,2,\ldots,n$. Consequently, the forward price $P(t,T_j)/P(t,T_i)$ can be written as

$$\begin{aligned}
&\frac{P(t,T_j)}{P(t,T_i)}\\
&= \frac{P(0,T_j)}{P(0,T_i)}\exp\left(\int_0^t (\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^{(j)} + \frac{1}{2}\int_0^t |\zeta_j(s)-\zeta_i(s)|^2 ds\right) \quad [\text{under } \widehat{\mathbb{P}}_j]\\
&= \frac{P(0,T_j)}{P(0,T_i)}\exp\left(\int_0^t (\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^{(i)} - \frac{1}{2}\int_0^t |\zeta_i(s)-\zeta_j(s)|^2 ds\right), \quad [\text{under } \widehat{\mathbb{P}}_i]
\end{aligned} \tag{19.7}$$

$t \in [0,\min(T_i,T_j)]$, $i,j = 1,2,\ldots,n$, which also follows from Proposition 16.8.

Short rate dynamics under the forward measure

In case the short rate process $(r_t)_{t\in\mathbb{R}_+}$ is given as the (Markovian) solution to the stochastic differential equation

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t,$$

by (19.5) its dynamics will be given under $\widehat{\mathbb{P}}_i$ by

$$\begin{aligned} dr_t &= \mu(t,r_t)dt + \sigma(t,r_t)\big(\zeta_i(t)dt + d\widehat{W}_t^{(i)}\big) \\ &= \mu(t,r_t)dt + \sigma(t,r_t)\zeta_i(t)dt + \sigma(t,r_t)d\widehat{W}_t^{(i)}. \end{aligned} \tag{19.8}$$

In the case of the Vašíček (1977) model, by (17.28) we have

$$dr_t = (a - br_t)dt + \sigma dW_t,$$

and

$$\zeta_i(t) = -\frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big), \qquad 0 \leqslant t \leqslant T_i,$$

hence from (19.8) we have

$$d\widehat{W}_t^{(i)} = dW_t - \zeta_i(t)dt = dW_t + \frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)dt, \tag{19.9}$$

and

$$dr_t = (a - br_t)dt - \frac{\sigma^2}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)dt + \sigma d\widehat{W}_t^{(i)} \tag{19.10}$$

and we obtain

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \frac{\sigma^2}{b^2}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)^2 dt - \frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)d\widehat{W}_t^{(i)},$$

from (17.28).

19.2 Bond Options

The next proposition can be obtained as an application of the Margrabe formula (16.30) of Proposition 16.15 by taking $X_t = P(t,T_j)$, $N_t^{(i)} = P(t,T_i)$, and $\widehat{X}_t = X_t/N_t^{(i)} = P(t,T_j)/P(t,T_i)$. In the Vasicek model, this formula has been first obtained in Jamshidian (1989).

We work with a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$, generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted short rate process $(r_t)_{t\in\mathbb{R}_+}$.

Proposition 19.3. *Let $0 \leqslant T_i \leqslant T_j$ and assume as in (17.25) that the dynamics of the bond prices $P(t,T_i)$, $P(t,T_j)$ under $\mathbb{P}^*$ are given by*

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t)dW_t, \qquad \frac{dP(t,T_j)}{P(t,T_j)} = r_t dt + \zeta_j(t)dW_t,$$

where $(\zeta_i(t))_{t\in\mathbb{R}_+}$ and $(\zeta_j(t))_{t\in\mathbb{R}_+}$ are deterministic *volatility functions. Then, the price of a bond call option on $P(T_i,T_j)$ with payoff*

$$C := (P(T_i,T_j) - \kappa)^+$$

can be written as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}(P(T_i,T_j)-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \tag{19.11}$$
$$= P(t,T_j)\Phi\left(\frac{v(t,T_i)}{2}+\frac{1}{v(t,T_i)}\log\frac{P(t,T_j)}{\kappa P(t,T_i)}\right)$$
$$-\kappa P(t,T_i)\Phi\left(-\frac{v(t,T_i)}{2}+\frac{1}{v(t,T_i)}\log\frac{P(t,T_j)}{\kappa P(t,T_i)}\right),$$

where $v^2(t,T_i) := \int_t^{T_i} |\zeta_i(s)-\zeta_j(s)|^2 ds$ *and*

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathrm{e}^{-y^2/2}dy, \qquad x\in\mathbb{R},$$

is the Gaussian cumulative distribution function.

Proof. First, we note that using $N_t^{(i)} := P(t,T_i)$ as a numéraire the price of a bond call option on $P(T_i,T_j)$ with payoff $F=(P(T_i,T_j)-\kappa)^+$ can be written from Proposition 16.5 using the forward measure $\widehat{\mathbb{P}}_i$, or directly by (16.9), as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}(P(T_i,T_j)-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] = P(t,T_i)\widehat{\mathbb{E}}_i\big[(P(T_i,T_j)-\kappa)^+ \,\big|\, \mathcal{F}_t\big]. \tag{19.12}$$

Next, by (19.7) or by solving (16.15) in Proposition 16.8 we can write $P(T_i,T_j)$ as the geometric Brownian motion

$$\begin{aligned} P(T_i,T_j) &= \frac{P(T_i,T_j)}{P(T_i,T_i)} \\ &= \frac{P(t,T_j)}{P(t,T_i)}\exp\left(\int_t^{T_i}(\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^{(i)} - \frac{1}{2}\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds\right), \end{aligned}$$

under the forward measure $\widehat{\mathbb{P}}_i$, and rewrite (19.12) as

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}(P(T_i,T_j)-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \\ &= P(t,T_i)\widehat{\mathbb{E}}_i\left[\left(\frac{P(t,T_j)}{P(t,T_i)}\mathrm{e}^{\int_t^{T_i}(\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^{(i)}-\frac{1}{2}\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}-\kappa\right)^+ \,\Bigg|\, \mathcal{F}_t\right] \\ &= \widehat{\mathbb{E}}_i\left[\left(P(t,T_j)\,\mathrm{e}^{\int_t^{T_i}(\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^{(i)}-\frac{1}{2}\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}-\kappa P(t,T_i)\right)^+ \,\Bigg|\, \mathcal{F}_t\right]. \end{aligned}$$

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Since $(\zeta_i(s))_{s\in[0,T_i]}$ and $(\zeta_j(s))_{s\in[0,T_j]}$ in (19.3) are deterministic volatility functions, $P(T_i,T_j)$ is a lognormal random variable given $\mathcal{F}_t$ under $\widehat{\mathbb{P}}_i$ and as in Proposition 16.15 we can use Lemma 7.7 to price the bond option by the zero-rate Black-Scholes formula

$$\mathrm{Bl}\big(P(t,T_j),\kappa P(t,T_i),v(t,T_i)/\sqrt{T_i-t},0,T_i-t\big)$$

with underlying asset price $P(t,T_j)$, strike level $\kappa P(t,T_i)$, volatility parameter

$$\frac{v(t,T_i)}{\sqrt{T_i-t}}=\sqrt{\frac{\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}{T_i-t}},$$

time to maturity T_i-t, and zero interest rate, which yields (19.11). $\square$

Note that from Corollary 16.17 the decomposition (19.11) gives the self-financing portfolio in the assets $P(t,T_i)$ and $P(t,T_j)$ for the claim with payoff $(P(T_i,T_j)-\kappa)^+$.

In the Vasicek case the above bond option price could also be computed from the joint distribution of $\left(r_T,\int_t^T r_s ds\right)$, which is Gaussian, or from the dynamics (19.6)-(19.10) of $P(t,T)$ and r_t under $\widehat{\mathbb{P}}_i$, see Kim (2002) and § 8.3 of Privault (2021b).

19.3 Caplet Pricing

An interest rate caplet is an option contract that offers protection against the fluctuations of a variable (or floating) rate with respect to a fixed rate κ. The payoff of a LIBOR caplet on the yield (or spot forward rate) $L(T_i,T_i,T_{i+1})$ with strike level κ can be written as

$$(L(T_i,T_i,T_{i+1})-\kappa)^+,$$

and priced at time $t\in[0,T_i]$ from Proposition 16.5 using the forward measure $\widehat{\mathbb{P}}_{i+1}$ as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}}r_sds}(L(T_i,T_i,T_{i+1})-\kappa)^+\,\middle|\,\mathcal{F}_t\right] \\ =P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(L(T_i,T_i,T_{i+1})-\kappa)^+\mid\mathcal{F}_t\right],\end{aligned}\tag{19.13}$$

by taking $N_t^{(i+1)}=P(t,T_{i+1})$ as a numéraire.

Proposition 19.4. *The LIBOR rate*

$$L(t,T_i,T_{i+1}):=\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right),\quad 0\leqslant t\leqslant T_i<T_{i+1},$$

is a martingale under the forward measure $\widehat{\mathbb{P}}_{i+1}$ *defined in* (19.1).

Proof. The LIBOR rate $L(t,T_i,T_{i+1})$ is a deflated process according to the forward numéraire process $(P(t,T_{i+1}))_{t\in[0,T_{i+1}]}$. Therefore, by Proposition 16.4 it is a martingale under $\widehat{\mathbb{P}}_{i+1}$. □

The caplet on $L(T_i,T_i,T_{i+1})$ can be priced at time $t\in[0,T_i]$ as

$$\begin{aligned}&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathcal{F}_t\right] \qquad (19.14)\\ &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}\left(\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right)-\kappa\right)^+ \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

where the discount factor is counted from the settlement date T_{i+1}. The next pricing formula (19.16) allows us to price and hedge a caplet using a portfolio based on the bonds $P(t,T_i)$ and $P(t,T_{i+1})$, cf. (19.20) below, when $L(t,T_i,T_{i+1})$ is modeled in the BGM model of Section 18.6.

Proposition 19.5. *(Black LIBOR caplet formula). Assume that* $L(t,T_i,T_{i+1})$ *is modeled in the BGM model as*

$$\frac{dL(t,T_i,T_{i+1})}{L(t,T_i,T_{i+1})} = \gamma_i(t)d\widehat{W}_t^{i+1}, \qquad (19.15)$$

$0\leqslant t\leqslant T_i$, $i=1,2,\ldots,n-1$, *where* $\gamma_i(t)$ *is a deterministic volatility function of time* $t\in[0,T_i]$, $i=1,2,\ldots,n-1$. *The caplet on* $L(T_i,T_i,T_{i+1})$ *with strike level* κ *is priced at time* $t\in[0,T_i]$ *as*

$$\begin{aligned}&(T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathcal{F}_t\right] \qquad (19.16)\\ &= (P(t,T_i)-P(t,T_{i+1}))\Phi(d_+(t,T_i)) - \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_i)),\end{aligned}$$

$0\leqslant t\leqslant T_i$, *where*

$$d_+(t,T_i) = \frac{\log(L(t,T_i,T_{i+1})/\kappa)+(T_i-t)\sigma_i^2(t,T_i)/2}{\sigma_i(t,T_i)\sqrt{T_i-t}}, \qquad (19.17)$$

and

$$d_-(t,T_i) = \frac{\log(L(t,T_i,T_{i+1})/\kappa)-(T_i-t)\sigma_i^2(t,T_i)/2}{\sigma_i(t,T_i)\sqrt{T_i-t}}, \qquad (19.18)$$

and

$$|\sigma_i(t,T_i)|^2 = \frac{1}{T_i-t}\int_t^{T_i}|\gamma_i|^2(s)ds. \qquad (19.19)$$

Proof. Taking $P(t,T_{i+1})$ as a numéraire, the forward price

$$\widehat{X}_t := \frac{P(t,T_i)}{P(t,T_{i+1})} = 1 + (T_{i+1}-T_i)L(T_i,T_i,T_{i+1})$$

and the forward LIBOR rate process $(L(t,T_i,T_{i+1})_{t\in[0,T_i]}$ are martingales under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 19.4, $i = 1,2,\ldots,n-1$. More precisely, by (19.15) we have

$$L(T_i,T_i,T_{i+1}) = L(t,T_i,T_{i+1}) \exp\left(\int_t^{T_i} \gamma_i(s) d\widehat{W}_s^{i+1} - \frac{1}{2}\int_t^{T_i} |\gamma_i(s)|^2 ds\right),$$

$0 \leqslant t \leqslant T_i$, *i.e.* $t \mapsto L(t,T_i,T_{i+1})$ is a geometric Brownian motion with time-dependent volatility $\gamma_i(t)$ under $\widehat{\mathbb{P}}_{i+1}$. Hence by (19.13), since $N_{T_{i+1}}^{(i+1)} = 1$, we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\big[(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\big|\, \mathcal{F}_t\big] \\
&= P(t,T_{i+1})\left(L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - \kappa\Phi(d_-(t,T_i))\right) \\
&= P(t,T_{i+1})\mathrm{Bl}(L(t,T_i,T_{i+1}),\kappa,\sigma_i(t,T_i),0,T_i-t),
\end{aligned}$$

$t \in [0,T_i]$, where

$$\mathrm{Bl}(x,\kappa,\sigma,0,\tau) = x\Phi(d_+(t,T_i)) - \kappa\Phi(d_-(t,T_i))$$

is the zero-interest rate Black-Scholes function, with

$$|\sigma_i(t,T_i)|^2 = \frac{1}{T_i-t}\int_t^{T_i} |\gamma_i|^2(s) ds.$$

Therefore, we obtain

$$\begin{aligned}
&(T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= (T_{i+1}-T_i)P(t,T_{i+1})L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - (T_{i+1}-T_i)\kappa P(t,T_{i+1})\Phi(d_-(t,T_i)) \\
&= P(t,T_{i+1})\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right)\Phi(d_+(t,T_i)) \\
&\quad -\kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_i)),
\end{aligned}$$

which yields (19.16). □

In addition, from Corollary 16.17 we obtain the self-financing portfolio strategy

$$(\Phi(d_+(t,T_i)), -\Phi(d_+(t,T_i)) - \kappa(T_{i+1}-T_i)\Phi(d_-(t,T_i))) \tag{19.20}$$

in the bonds priced $(P(t,T_i),P(t,T_{i+1}))$ with maturities T_i and T_{i+1}, cf. Corollary 16.18 and Privault and Teng (2012).

The formula (19.16) can be applied to options on underlying futures or forward contracts on commodities whose prices are modeled according to (19.15), as in the next corollary.

Corollary 19.6. *(Black (1976) formula). Let $L(t,T_i,T_{i+1})$ be modeled as in (19.15) and let the bond price $P(t,T_{i+1})$ be given as $P(t,T_{i+1}) = \mathrm{e}^{-(T_{i+1}-t)r}$. Then, (19.16) becomes*

$$\mathrm{e}^{-(T_{i+1}-t)r} L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - \kappa\,\mathrm{e}^{-(T_{i+1}-t)r}\Phi(d_-(t,T_i)),$$

$0 \leqslant t \leqslant T_i$.

Floorlet pricing

The floorlet on $L(T_i,T_i,T_{i+1})$ with strike level κ is a contract with payoff $(\kappa - L(T_i,T_i,T_{i+1}))^+$. Floorlets are analog to put options and can be similarly priced by the call/put parity in the Black-Scholes formula.

Proposition 19.7. *Assume that $L(t,T_i,T_{i+1})$ is modeled in the BGM model as in (19.15). The floorlet on $L(T_i,T_i,T_{i+1})$ with strike level κ is priced at time $t \in [0,T_i]$ as*

$$(T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(\kappa - L(T_i,T_i,T_{i+1}))^+ \,\Big|\, \mathcal{F}_t\right] \tag{19.21}$$
$$= \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi\big(-d_-(t,T_i)\big) - (P(t,T_i)-P(t,T_{i+1}))\Phi\big(-d_+(t,T_i)\big),$$

$0 \leqslant t \leqslant T_i$, where $d_+(t,T_i)$, $d_-(t,T_i)$ and $|\sigma_i(t,T_i)|^2$ are defined in (19.17)-(19.19).

Proof. We have

$$\begin{aligned}
&(T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(\kappa - L(T_i,T_i,T_{i+1}))^+ \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(\kappa - L(T_i,T_i,T_{i+1}))^+ \mid \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\left(\kappa\Phi\big(-d_-(t,T_i)\big) - L(t,T_i,T_{i+1})\Phi\big(-d_+(t,T_i)\big)\right)\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\kappa\Phi\big(-d_-(t,T_i)\big) - (P(t,T_i)-P(t,T_{i+1}))\Phi\big(-d_+(t,T_i)\big),
\end{aligned}$$

$0 \leqslant t \leqslant T_i$. □

Cap pricing

More generally, one can consider interest rate caps that are relative to a given tenor structure $\{T_1, T_2, \ldots, T_n\}$, with discounted payoff

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k) \, \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+.$$

Pricing formulas for interest rate caps are easily deduced from analog formulas for caplets, since the payoff of a cap can be decomposed into a sum of caplet payoffs. Thus, the cap price at time $t \in [0, T_i]$ is given by

$$\begin{aligned}
&\mathbb{E}^* \left[\sum_{k=i}^{j-1} (T_{k+1} - T_k) \, \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+ \, \Big| \, \mathcal{F}_t \right] \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (L(T_k, T_k, T_{k+1}) - \kappa)^+ \, \Big| \, \mathcal{F}_t \right] \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \widehat{\mathbb{E}}_{k+1} \big[(L(T_k, T_k, T_{k+1}) - \kappa)^+ \, \big| \, \mathcal{F}_t \big].
\end{aligned} \tag{19.22}$$

In the BGM model (19.15) the interest rate cap with payoff

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k)(L(T_k, T_k, T_{k+1}) - \kappa)^+$$

can be priced at time $t \in [0, T_1]$ by the Black formula

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \mathrm{Bl}(L(t, T_k, T_{k+1}), \kappa, \sigma_k(t, T_k), 0, T_k - t),$$

where

$$|\sigma_k(t, T_k)|^2 = \frac{1}{T_k - t} \int_t^{T_k} |\gamma_k|^2(s) ds.$$

SOFR Caplets

The backward-looking SOFR caplet has payoff $(R(S, T, S) - K)^+$, which is known only at time S. By the Jensen (1906) inequality we note the relation

$$\mathbb{E}_S[(R(S, T, S) - K)^+ \mid \mathcal{F}_t] = \mathbb{E}_S[\mathbb{E}_S[(R(S, T, S) - K)^+ \mid \mathcal{F}_T] \mid \mathcal{F}_t]$$

$$\geqslant \mathbb{E}_S[(\mathbb{E}_S[R(S,T,S) \mid \mathcal{F}_T] - K)^+ \mid \mathcal{F}_t]$$
$$= \mathbb{E}_S[(R(T,T,S) - K)^+ \mid \mathcal{F}_t]$$
$$= \mathbb{E}_S[(L(T,T,S) - K)^+ \mid \mathcal{F}_t],$$

hence the backward-looking SOFR caplet is more expensive than the forward-looking LIBOR caplet. The caplet on the SOFR rate $R(T_{i+1}, T_i, T_{i+1})$ with payoff $(R(T_{i+1}, T_i, T_{i+1}) - \kappa)^+$ and strike level κ can be priced at time $t \in [0, T_i]$ with a discount factor counted from the settlement date T_{i+1} from Proposition 16.5 as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1}, T_i, T_{i+1}) - \kappa)^+ \,\Big|\, \mathcal{F}_t\right] \tag{19.23}$$
$$= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}\left(\frac{1}{T_{i+1} - T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right) - \kappa\right)^+ \,\Bigg|\, \mathcal{F}_t\right]$$
$$= P(t, T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(R(T_{i+1}, T_i, T_{i+1}) - \kappa)^+ \mid \mathcal{F}_t\right],$$

by taking $N_t^{(i+1)} := P(t, T_{i+1})$ as a numéraire and using the forward measure $\widehat{\mathbb{P}}_{i+1}$.

Proposition 19.8. *The SOFR rate*

$$R(t, T_i, T_{i+1}) := \frac{1}{T_{i+1} - T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right), \quad 0 \leqslant T_i \leqslant t \leqslant T_{i+1},$$

is a martingale under the forward measure $\widehat{\mathbb{P}}_{i+1}$.

Proof. The SOFR rate $R(t, T_i, T_{i+1})$ is a deflated process according to the forward numéraire process $(P(t, T_{i+1}))_{t \in [0, T_{i+1}]}$. Therefore, it is a martingale under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 16.4. □

The next pricing formula (19.25) allows us to price and hedge a caplet using a portfolio based on the bonds $P(t, T_i)$ and $P(t, T_{i+1})$, cf. (19.26) below, when $R(t, T_i, T_{i+1})$ is modeled in the BGM model.

Proposition 19.9. *(Black SOFR caplet formula). Assume that* $R(t, T_i, T_{i+1})$ *is modeled in the BGM model as*

$$\frac{dR(t, T_i, T_{i+1})}{R(t, T_i, T_{i+1})} = \gamma_i(t) d\widehat{W}_t^{i+1}, \tag{19.24}$$

$0 \leqslant t \leqslant T_{i+1}$, $i = 1, 2, \ldots, n-1$, *where* $\gamma_i(t)$ *is a deterministic volatility function of time* $t \in [0, T_{i+1}]$, $i = 1, 2, \ldots, n-1$. *The caplet on* $R(T_{i+1}, T_i, T_{i+1})$ *with strike level* $\kappa > 0$ *is priced at time* $t \in [0, T_{i+1}]$ *as*

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$$(T_{i+1}-T_i)\mathbb{E}^*\left[e^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \;\Big|\; \mathcal{F}_t\right] \tag{19.25}$$
$$= (P(t,T_i)-P(t,T_{i+1}))\Phi(d_+(t,T_{i+1})) - \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_{i+1})),$$

$0 \leqslant t \leqslant T_{i+1}$, *where*

$$d_+(t,T_{i+1}) = \frac{\log(R(t,T_i,T_{i+1})/\kappa) + (T_{i+1}-t)\sigma_i^2(t,T_{i+1})/2}{\sigma_i(t,T_{i+1})\sqrt{T_{i+1}-t}},$$

and

$$d_-(t,T_{i+1}) = \frac{\log(R(t,T_i,T_{i+1})/\kappa) - (T_{i+1}-t)\sigma_i^2(t,T_{i+1})/2}{\sigma_i(t,T_{i+1})\sqrt{T_{i+1}-t}},$$

and

$$|\sigma_i(t,T_{i+1})|^2 = \frac{1}{T_{i+1}-t}\int_t^{T_{i+1}} |\gamma_i|^2(s)ds.$$

Proof. The forward price

$$\widehat{X}_t := \frac{P(t,T_i)}{P(t,T_{i+1})} = 1 + (T_{i+1}-T_i)R(T_{i+1},T_i,T_{i+1})$$

and the SOFR rate process $(R(t,T_i,T_{i+1})_{t\in[0,T_{i+1}]}$ are martingales under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 19.8, $i = 1,2,\ldots,n-1$, and

$$R(T_{i+1},T_i,T_{i+1}) = R(t,T_i,T_{i+1})\exp\left(\int_t^{T_{i+1}}\gamma_i(s)d\widehat{W}_s^{i+1} - \frac{1}{2}\int_t^{T_{i+1}}|\gamma_i(s)|^2 ds\right),$$

$0 \leqslant t \leqslant T_{i+1}$, where $t \mapsto R(t,T_i,T_{i+1})$ is a geometric Brownian motion under $\widehat{\mathbb{P}}_{i+1}$ (19.24). Hence by (19.23) we have

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \;\Big|\; \mathcal{F}_t\right]\\
&= P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \,\big|\, \mathcal{F}_t\right]\\
&= P(t,T_{i+1})\left(R(t,T_i,T_{i+1})\Phi(d_+(t,T_{i+1})) - \kappa\Phi(d_-(t,T_{i+1}))\right)\\
&= P(t,T_{i+1})\mathrm{Bl}(R(t,T_i,T_{i+1}),\kappa,\sigma_i(t,T_{i+1}),0,T_{i+1}-t),
\end{aligned}$$

$t \in [0,T_{i+1}]$, with

$$|\sigma_i(t,T_{i+1})|^2 = \frac{1}{T_{i+1}-t}\int_t^{T_{i+1}} |\gamma_i|^2(s)ds.$$

□

In addition, we obtain the self-financing portfolio strategy

$$(\Phi(d_+(t,T_{i+1})), -\Phi(d_+(t,T_{i+1})) - \kappa(T_{i+1}-T_i)\Phi(d_-(t,T_{i+1}))) \quad (19.26)$$

in the bonds priced $(P(t,T_i), P(t,T_{i+1}))$, $t \in [0,T_{i+1}]$, with maturities T_i and T_{i+1}.

19.4 Forward Swap Measures

In this section we introduce the forward swap (or annuity) measures, or annuity measures, to be used for the pricing of swaptions, and we study their properties. We start with the definition of the *annuity numéraire*

$$N_t^{(i,j)} := P(t,T_i,T_j) = \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(t,T_{k+1}), \qquad 0 \leqslant t \leqslant T_i, \quad (19.27)$$

with in particular, when $j = i+1$,

$$P(t,T_i,T_{i+1}) = (T_{i+1}-T_i)P(t,T_{i+1}), \qquad 0 \leqslant t \leqslant T_i.$$

$1 \leqslant i < n$. The annuity numéraire can be also used to price a *bond ladder*. It satisfies the following martingale property, which can be proved by linearity and the fact that $t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_k)$ is a martingale for all $k = 1,2,\ldots,n$, under Assumption (A).

Remark 19.10. *The discounted annuity numéraire*

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_i,T_j) = \mathrm{e}^{-\int_0^t r_s ds} \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(t,T_{k+1}), \quad 0 \leqslant t \leqslant T_i,$$

is a martingale under $\mathbb{P}^*$.

The forward swap measure $\widehat{\mathbb{P}}_{i,j}$ is defined, according to Definition 16.1, by

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_{i,j}}{\mathrm{d}\mathbb{P}^*} := \mathrm{e}^{-\int_0^{T_i} r_s ds} \frac{N_{T_i}^{(i,j)}}{N_0^{(i,j)}} = \mathrm{e}^{-\int_0^{T_i} r_s ds} \frac{P(T_i,T_i,T_j)}{P(0,T_i,T_j)}, \quad (19.28)$$

$1 \leqslant i < j \leqslant n$.

Remark 19.11. *By* (16.2) *we have*

$$\mathbb{E}^*\left[\frac{\mathrm{d}\widehat{\mathbb{P}}_{i,j}}{\mathrm{d}\mathbb{P}^*} \,\Big|\, \mathcal{F}_t\right] = \frac{1}{P(0,T_i,T_j)} \mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_i} r_s ds} P(T_i,T_i,T_j) \,\Big|\, \mathcal{F}_t\right]$$

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$$
\begin{aligned}
&= \frac{1}{P(0, T_i, T_j)} \mathbb{E}^* \left[\mathrm{e}^{-\int_0^{T_i} r_s ds} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) \, \Big| \, \mathcal{F}_t \right] \\
&= \frac{1}{P(0, T_i, T_j)} \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[\mathrm{e}^{-\int_0^{T_i} r_s ds} P(T_i, T_{k+1}) \, \Big| \, \mathcal{F}_t \right] \\
&= \frac{1}{P(0, T_i, T_j)} \mathrm{e}^{-\int_0^t r_s ds} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \\
&= \mathrm{e}^{-\int_0^t r_s ds} \frac{P(t, T_i, T_j)}{P(0, T_i, T_j)},
\end{aligned}
$$

$0 \leqslant t \leqslant T_i$, *see Remark 19.10, and*

$$
\frac{d\widehat{\mathbb{P}}_{i,j|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} = \mathrm{e}^{-\int_t^{T_i} r_s ds} \frac{P(T_i, T_i, T_j)}{P(t, T_i, T_j)}, \qquad 0 \leqslant t \leqslant T_{i+1}, \tag{19.29}
$$

by Relation (16.3) in Lemma 16.2.

Proposition 19.12. *The LIBOR swap rate*

$$
S(t, T_i, T_j) = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)}, \qquad 0 \leqslant t \leqslant T_i,
$$

see Corollary 18.12, is a martingale under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$.

Proof. We use the fact that the deflated process

$$
t \mapsto \frac{P(t, T_k)}{P(t, T_i, T_j)}, \qquad i, j, k = 1, 2, \ldots, n,
$$

is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}_{i,j}$ by Proposition 16.4. □

The following pricing formula is then stated for a given integrable claim with payoff of the form $P(T_i, T_i, T_j)F$, using the forward swap measure $\widehat{\mathbb{P}}_{i,j}$:

$$
\begin{aligned}
\mathbb{E}^* \left[\mathrm{e}^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j) F \, \Big| \, \mathcal{F}_t \right] &= P(t, T_i, T_j) \mathbb{E}^* \left[F \frac{d\widehat{\mathbb{P}}_{i,j|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \, \Big| \, \mathcal{F}_t \right] \\
&= P(t, T_i, T_j) \widehat{\mathbb{E}}_{i,j}[F \mid \mathcal{F}_t],
\end{aligned} \tag{19.30}
$$

after applying (19.28) and (19.29) on the last line, or Proposition 16.5.

19.5 Swaption Pricing

Definition 19.13. *A payer (or call) swaption gives the option, but not the obligation, to enter an interest rate swap as payer of a fixed rate κ and as* *receiver of floating LIBOR rates* $L(T_i, T_k, T_{k+1})$ *at time* T_{k+1}, $k = i, \ldots, j - 1$, *and has the payoff*

$$\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{k+1}} r_s ds} \,\Big|\, \mathcal{F}_{T_i}\right](L(T_i,T_k,T_{k+1})-\kappa)\right)^+$$
$$= \left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \qquad (19.31)$$

at time T_i.

This swaption can be priced at time $t \in [0, T_i]$ under the risk-neutral probability measure $\mathbb{P}^*$ as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \,\Bigg|\, \mathcal{F}_t\right], \qquad (19.32)$$

$t \in [0, T_i]$. When $j = i + 1$, the swaption price (19.32) coincides with the price at time t of a caplet on $[T_i, T_{i+1}]$ up to a factor $\delta_i := T_{i+1} - T_i$, since

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left((T_{i+1}-T_i)P(T_i,T_{i+1})(L(T_i,T_i,T_{i+1})-\kappa)\right)^+ \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_{i+1})\left(L(T_i,T_i,T_{i+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{i+1}} r_s ds} \,\Big|\, \mathcal{F}_{T_i}\right]\left(L(T_i,T_i,T_{i+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)\mathbb{E}^*\left[\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\,\mathrm{e}^{-\int_{T_i}^{T_{i+1}} r_s ds}\left(L(T_i,T_i,T_{i+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_{T_i}\right] \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}\left(L(T_i,T_i,T_{i+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right], \qquad (19.33)
\end{aligned}$$

$0 \leqslant t \leqslant T_i$, which coincides with the caplet price (19.13) up to the factor $T_{i+1} - T_i$. Unlike in the case of interest rate caps, the sum in (19.32) cannot be taken out of the positive part. Nevertheless, the price of the swaption can be bounded as in the next proposition.

Proposition 19.14. *The payer swaption price* (19.32) *can be upper bounded by the interest rate cap price* (19.22) *as*

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \,\Bigg|\, \mathcal{F}_t\right]$$

$$\leqslant \sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right],$$

$0 \leqslant t \leqslant T_i$.

Proof. Due to the inequality

$$(x_1+x_2+\cdots+x_m)^+ \leqslant x_1^+ + x_2^+ + \cdots + x_m^+, \quad x_1,x_2,\ldots,x_m \in \mathbb{R},$$

we have

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \,\Bigg|\, \mathcal{F}_t\right]$$
$$\leqslant \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]$$
$$= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_{k+1})\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]$$
$$= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{k+1}} r_s ds} \,\Big|\, \mathcal{F}_{T_i}\right]\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]$$
$$= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_{T_i}\right] \,\Big|\, \mathcal{F}_t\right]$$
$$= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right]$$
$$= \mathbb{E}^*\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right],$$

$0 \leqslant t \leqslant T_i$. □

The payoff of the payer swaption can be rewritten as in the following lemma which is a direct consequence of the definition of the swap rate $S(T_i, T_i, T_j)$, see Proposition 18.11 and Corollary 18.12.

Lemma 19.15. *The payer swaption payoff* (19.31) *at time* T_i *with swap rate* $\kappa = S(t, T_j, T_j)$ *can be rewritten as*

$$\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+$$
$$= \left(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j)\right)^+ \qquad (19.34)$$

$$= P(T_i, T_i, T_j)\,(S(T_i, T_i, T_j) - \kappa)^+ . \tag{19.35}$$

Proof. The relation

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) (L(t, T_k, T_{k+1}) - S(t, T_i, T_j)) = 0$$

that defines the forward swap rate $S(t, T_i, T_j)$ shows that

$$\begin{aligned}
&\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) L(t, T_k, T_{k+1}) \\
&\qquad = S(t, T_i, T_j) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \\
&\qquad = P(t, T_i, T_j) S(t, T_i, T_j) \\
&\qquad = P(t, T_i) - P(t, T_j)
\end{aligned}$$

as in the proof of Corollary 18.12, hence by the definition (19.27) of $P(t, T_i, T_j)$ we have

$$\begin{aligned}
&\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) (L(t, T_k, T_{k+1}) - \kappa) \\
&\qquad = P(t, T_i) - P(t, T_j) - \kappa P(t, T_i, T_j) \\
&\qquad = P(t, T_i, T_j)\,(S(t, T_i, T_j) - \kappa),
\end{aligned}$$

and for $t = T_i$ we get

$$\begin{aligned}
&\left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \\
&\qquad = P(T_i, T_i, T_j)\,(S(T_i, T_i, T_j) - \kappa)^+ .
\end{aligned}$$

□

The next proposition simply states that a payer swaption on the LIBOR rate can be priced as a European call option on the swap rate $S(T_i, T_i, T_j)$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$.

Proposition 19.16. *The price* (19.32) *of the payer swaption with payoff*

$$\left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \tag{19.36}$$

on the LIBOR market can be written under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$ *as the European call price*

$$P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\big[(S(T_i,T_i,T_j)-\kappa)^+\big|\mathcal{F}_t\big], \quad 0 \leqslant t \leqslant T_i,$$

on the swap rate $S(T_i,T_i,T_j)$*.*

Proof. As a consequence of (19.30) and Lemma 19.15, we find

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j))^+\middle|\mathcal{F}_t\right] \qquad (19.37)\\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+\middle|\mathcal{F}_t\right]\\
&= P(t,T_i,T_j)\mathbb{E}^*\left[\frac{\mathrm{d}\widehat{\mathbb{P}}_{i,j|\mathcal{F}_t}}{\mathrm{d}\mathbb{P}^*_{|\mathcal{F}_t}}\,(S(T_i,T_i,T_j)-\kappa)^+\middle|\mathcal{F}_t\right]\\
&= P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\big[\big(S(T_i,T_i,T_j)-\kappa\big)^+\,\big|\,\mathcal{F}_t\big]. \qquad (19.38)
\end{aligned}$$

□

In the next Proposition 19.17 we price the payer swaption with payoff (19.36) or equivalently (19.35), by modeling the swap rate $(S(t,T_i,T_j))_{0\leqslant t\leqslant T_i}$ using standard Brownian motion $(\widehat{W}_t^{i,j})_{0\leqslant t\leqslant T_i}$ under the swap forward measure $\widehat{\mathbb{P}}_{i,j}$. See Exercise 19 for swaption pricing without the Black-Scholes formula.

Proposition 19.17. *(Black swaption formula for payer swaptions). Assume that the LIBOR swap rate* (18.21) *is modeled as a geometric Brownian motion under* $\widehat{\mathbb{P}}_{i,j}$*, i.e.*

$$dS(t,T_i,T_j) = S(t,T_i,T_j)\widehat{\sigma}_{i,j}(t)d\widehat{W}_t^{i,j}, \qquad (19.39)$$

where $\big(\widehat{\sigma}_{i,j}(t)\big)_{t\in\mathbb{R}_+}$ *is a deterministic volatility function of time. Then, the payer swaption with payoff*

$$(P(T,T_i)-P(T,T_j)-\kappa P(T_i,T_i,T_j))^+ = P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+$$

can be priced using the Black-Scholes call formula as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+\middle|\mathcal{F}_t\right]\\
&\qquad = (P(t,T_i)-P(t,T_j))\Phi(d_+(t,T_i))
\end{aligned}$$

$$-\kappa\Phi(d_-(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}),$$

$t\in[0,T_i]$, *where*

$$d_+(t,T_i)=\frac{\log(S(t,T_i,T_j)/\kappa)+\sigma_{i,j}^2(t,T_i)(T_i-t)/2}{\sigma_{i,j}(t,T_i)\sqrt{T_i-t}}, \tag{19.40}$$

and

$$d_-(t,T_i)=\frac{\log(S(t,T_i,T_j)/\kappa)-\sigma_{i,j}^2(t,T_i)(T_i-t)/2}{\sigma_{i,j}(t,T_i)\sqrt{T_i-t}}, \tag{19.41}$$

and

$$|\sigma_{i,j}(t,T_i)|^2=\frac{1}{T_i-t}\int_t^{T_i}|\widehat{\sigma}_{i,j}(s)|^2ds, \qquad 0\leqslant t\leqslant T_i. \tag{19.42}$$

Proof. Since $S(t,T_i,T_j)$ is a geometric Brownian motion with volatility function $(\widehat{\sigma}(t))_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}_{i,j}$, by (19.34)-(19.35) in Lemma 19.15 or (19.37)-(19.38) we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i}r_sds}P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j)-\kappa\right)^+\Big|\mathcal{F}_t\right]\\
&=\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T}r_sds}(P(T,T_i)-P(T,T_j)-\kappa P(T_i,T_i,T_j))^+\ \Big|\ \mathcal{F}_t\right]\\
&=P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+\,\big|\,\mathcal{F}_t\right]\\
&=P(t,T_i,T_j)\mathrm{Bl}(S(t,T_i,T_j),\kappa,\sigma_{i,j}(t,T_i),0,T_i-t)\\
&=P(t,T_i,T_j)\left(S(t,T_i,T_j)\Phi_+(t,S(t,T_i,T_j))-\kappa\Phi_-(t,S(t,T_i,T_j))\right)\\
&=\big(P(t,T_i)-P(t,T_j)\big)\Phi_+(t,S(t,T_i,T_j))-\kappa P(t,T_i,T_j)\Phi_-(t,S(t,T_i,T_j))\\
&=\big(P(t,T_i)-P(t,T_j)\big)\Phi_+(t,S(t,T_i,T_j))\\
&\quad-\kappa\Phi_-(t,S(t,T_i,T_j))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}).
\end{aligned}$$

□

In addition, the hedging strategy

$$\begin{aligned}
&(\Phi_+(t,S(t,T_i,T_j)),-\kappa\Phi_-(t,S(t,T_i,T_j))(T_{i+1}-T_i),\ldots\\
&\ldots,-\kappa\Phi_-(t,S(t,T_i,T_j))(T_{j-1}-T_{j-2}),-\Phi_+(t,S(t,T_i,T_j)))
\end{aligned}$$

based on the assets $(P(t,T_i),\ldots,P(t,T_j))$ is self-financing by Corollary 16.18, see also Privault and Teng (2012). Similarly to the above, a receiver (or put) swaption gives the option, but not the obligation, to enter an interest rate swap as receiver of a fixed rate κ and as payer of floating LIBOR rates

$L(T_i, T_k, T_{k+1})$ at times $T_{i+1}, \ldots, T_j$, and can be priced as in the next proposition.

Proposition 19.18. *(Black swaption formula for receiver swaptions). Assume that the LIBOR swap rate (18.21) is modeled as the geometric Brownian motion (19.39) under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$. Then, the receiver swaption with payoff*

$$\left(\kappa P(T_i, T_i, T_j) - \left(P(T, T_i) - P(T, T_j)\right)\right)^+ = P(T_i, T_i, T_j)\left(\kappa - S(T_i, T_i, T_j)\right)^+$$

can be priced using the Black-Scholes put formula as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j)\left(\kappa - S(T_i, T_i, T_j)\right)^+ \Big| \mathcal{F}_t\right] \\ = \kappa \Phi(-d_-(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \\ -(P(t, T_i) - P(t, T_j))\Phi(-d_+(t, T_i)),\end{aligned}$$

where $d_+(t, T_i)$, and $d_-(t, T_i)$ and $|\sigma_{i,j}(t, T_i)|^2$ are defined in (19.40)-(19.42).

When the SOFR swap rate (18.25) is modeled as a geometric Brownian motion under $\widehat{\mathbb{P}}_{i,j}$ as in (19.39), SOFR swaptions are priced in the same way as LIBOR swaptions.

Swaption prices can also be computed by an approximation formula, from the exact dynamics of the swap rate $S(t, T_i, T_j)$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$, based on the bond price dynamics of the form (19.3), cf. Schoenmakers (2005), page 17.

Swaption volatilities can be estimated from swaption prices as implied volatilities from the Black pricing formula:

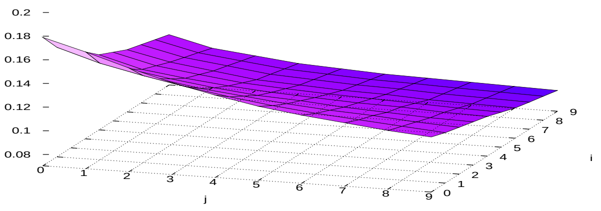

Fig. 19.1: Implied swaption volatilities.

Implied swaption volatilities can then be used to calibrate the BGM model, cf. Schoenmakers (2005), Privault and Wei (2009), or § 9.5 of Privault (2021b).

LIBOR-SOFR Swaps

We consider the swap contract with payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})),$$

for the exchange of a backward-looking SOFR rate $R(T_{k+1},T_k,T_{k+1})$ with the forward-looking LIBOR rate $L(T_k,T_k,T_{k+1})$ over the time period $[T_k,T_{k+1}]$. The price of this interest rate swap vanishes at any time $t\in[0,T_1]$, as

$$\begin{aligned}
&(T_{k+1}-T_k)\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1}))\,\Big|\,\mathcal{F}_t\right]\\
&=(T_{k+1}-T_k)P(t,T_{k+1})\mathbb{E}_{k+1}[R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})\mid\mathcal{F}_t]\\
&=(T_{k+1}-T_k)P(t,T_{k+1})(R(t,T_k,T_{k+1})-L(t,T_k,T_{k+1})\\
&=0,\qquad 0\leqslant t\leqslant T_k.
\end{aligned}$$

see Mercurio (2018). On the other hand, for any $i=1,\ldots,n$, we also have

$$\begin{aligned}
&(T_{k+1}-T_k)\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1}))\,\Big|\,\mathcal{F}_{T_i}\right]\\
&=(T_{k+1}-T_k)P(T_i,T_{k+1})\mathbb{E}_{k+1}[R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})\mid\mathcal{F}_{T_k}]\\
&=(T_{k+1}-T_k)P(T_i,T_{k+1})(R(T_i,T_k,T_{k+1})-L(T_i,T_k,T_{k+1})\\
&=0.
\end{aligned}$$

Bermudan swaption pricing in Quantlib

The Bermudan swaption on the tenor structure $\{T_i,\ldots,T_j\}$ is priced as the supremum

$$\begin{aligned}
&\operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}\left(\sum_{k=l}^{j-1}\delta_k P(T_l,T_{k+1})(L(T_l,T_k,T_{k+1})-\kappa)\right)^+\,\Bigg|\,\mathcal{F}_t\right]\\
&=\operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}\left(P(T_l,T_l)-P(T_l,T_j)-\kappa P(T_l,T_l,T_j)\right)^+\,\Big|\,\mathcal{F}_t\right]\\
&=\operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}P(T_l,T_l,T_j)(S(T_l,T_l,T_j)-\kappa)^+\,\Big|\,\mathcal{F}_t\right],
\end{aligned}$$

where the supremum is over all stopping times taking values in $\{T_i,\ldots,T_j\}$.

Bermudan swaptions can be priced using this **Rcode**[*] in (R)quantlib, with the following output:

```
Summary of pricing results for Bermudan Swaption

Price (in bp) of Bermudan swaption is  24.92137
Strike is NULL (ATM strike is  0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a =  0.04641
sigma =  0.005869
```

This modified code[†] can be used in particular the pricing of ordinary swaptions, with the output:

```
Summary of pricing results for Bermudan Swaption

Price (in bp) of Bermudan swaption is  22.45436
Strike is NULL (ATM strike is  0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a =  0.07107
sigma =  0.006018
```

Table 19.2 summarizes some possible uses of change of numéraire in option pricing.

[*] Click to open or download.

[†] Click to open or download.

Application	Asset price	Numéraire process	Option payoff	Forward measure $\widehat{\mathbb{P}}$	Deflated process	Option price	Change of numéraire formula
Risk-neutral pricing	S_t	$N_t = \mathrm{e}^{\int_0^t r_s ds}$	C	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = 1$	$\widetilde{S}_t = \mathrm{e}^{-\int_0^t r_s ds} S_t$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle\vert\, \mathcal{F}_t\right]$	$\mathrm{e}^{\int_0^t r_s ds} \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} C \,\middle\vert\, \mathcal{F}_t\right]$
Exchange option	S_t	N_t	$(S_T - \kappa N_T)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0}$	$\widehat{X}_t = \frac{S_t}{N_t}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} (S_T - \kappa N_T)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$N_t \widehat{\mathbb{E}}\big[(\widehat{X}_T - \kappa)^+ \mid \mathcal{F}_t\big]$
Exotic	$S_t = S_0 \mathrm{e}^{rt + \sigma W_t - \sigma^2 t/2}$	$N_t = S_t$	$S_T(S_T - K)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-(T-t)r} \frac{S_T}{S_0}$	$\widehat{X}_t = 1$	$\mathbb{E}^*\big[\mathrm{e}^{-(T-t)r} S_T (S_T - K)^+ \mid \mathcal{F}_t\big]$	$S_t \widehat{\mathbb{E}}[(S_T - K)^+ \mid \mathcal{F}_t]$
Foreign exchange	$\mathrm{e}^{r^f t} R_t$	$N_t = \mathrm{e}^{r^f t} R_t$	$(R_T - \kappa)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{(r^f - r)T} \frac{R_T}{R_0}$	$\widehat{X}_t = 1$	$\mathrm{e}^{r^f T} \mathbb{E}^*\big[\mathrm{e}^{-(T-t)r} (R_T - \kappa) \mid \mathcal{F}_t\big]$	$\mathrm{e}^{r^f t} R_t \widehat{\mathbb{E}}\left[\left(1 - \frac{\kappa}{R_T}\right)^+ \,\middle\vert\, R_t\right]$
Bond option	$P(t,S)$	$N_t = P(t,T)$	$(P(T,S) - K)^+$	$\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^T r_s ds}}{P(0,T)}$	$\widehat{X}_t = \frac{P(t,S)}{P(t,T)}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} (P(T,S) - K)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$P(t,T) \widehat{\mathbb{E}}_T[(P(T,S) - K)^+ \mid \mathcal{F}_t]$
Caplets and caps	$P(t,T)$	$N_t = P(t,S)$	$(S-T)(L(T,T,S) - \kappa)^+$	$\frac{d\widehat{\mathbb{P}}_S}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^S r_s ds}}{P(0,S)}$	$L(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$	$(S-T)\mathbb{E}^*\left[\mathrm{e}^{-\int_t^S r_s ds} (L(T,T,S) - K)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$(S-T)P(t,S)\widehat{\mathbb{E}}_S[(L(T,T,S) - K)^+ \mid \mathcal{F}_t]$
Swaption	$P(t,T_1)$, $P(t,T_n)$	$N_t = P(t,T_1,T_n)$	$(P(T_1,T_1) - P(T_1,T_n) - \kappa P(T_1,T_1,T_n))^+$	$\frac{d\widehat{\mathbb{P}}_{1,n}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^{T_1} r_s ds} \frac{P(T_1,T_1,T_n)}{P(0,T_1,T_n)}$	$S(t,T_1,T_n) = \frac{P(t,T_1) - P(t,T_n)}{P(t,T_1,T_n)}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} (P(T_1,T_1) - P(T_1,T_n) - \kappa P(T_1,T_1,T_n))^+ \,\middle\vert\, \mathcal{F}_t\right]$	$P(t,T_1,T_n)\widehat{\mathbb{E}}_{1,n}\big[(S(T_1,T_1,T_n) - \kappa)^+ \mid \mathcal{F}_t\big]$
Power options	$S_t = S_0 \mathrm{e}^{rt + \sigma W_t - \sigma^2 t/2}$	$N_t = S_t^n \mathrm{e}^{-(n-1)n\sigma^2 t/2 - (n-1)rt}$	$(S_T^n - K^n)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{n\sigma W_T - (n\sigma)^2 T/2}$	$\widehat{X}_t = S_t^{n-1} \mathrm{e}^{(n-1)n\sigma^2 t/2 + (n-1)rt}$	$\mathrm{e}^{-(T-t)r} \mathbb{E}^*\big[S_T^n \mathbb{1}_{\{S_T \geqslant K\}} \mid \mathcal{F}_t\big] - K^n \mathrm{e}^{-(T-t)r} \mathbb{E}^*\big[\mathbb{1}_{\{S_T \geqslant K\}} \mid \mathcal{F}_t\big]$	$\mathrm{e}^{(n-1)n\sigma^2 T/2 + (n-1)rT} N_t \widehat{\mathbb{E}}\big[\mathbb{1}_{\{S_T \geqslant K\}} \mid \mathcal{F}_t\big]$ $- K^n \mathrm{e}^{-(T-t)r} \mathbb{E}^*\big[\mathbb{1}_{\{S_T \geqslant K\}} \mid \mathcal{F}_t\big]$

Table 19.2: A list of numéraire processes and their applications.

Exercises

Exercise 19.1 Consider a floorlet on a three-month LIBOR rate in nine month's time, with a notional principal amount of $\$10,000$ per interest rate percentage point. The term structure is flat at 3.95% per year with discrete compounding, the volatility of the forward LIBOR rate in nine months is 10%, and the floor rate is 4.5%.

a) What are the key assumptions on the LIBOR rate in nine month in order to apply Black's formula to price this floorlet?
b) Compute the price of this floorlet using Black's formula as an application of Proposition 19.7 and (19.21), using the functions $\Phi(d_+)$ and $\Phi(d_-)$.

Exercise 19.2 Consider a payer swaption giving its holder the right, but not the obligation, to enter into a 3-year annual pay swap in four years, where a fixed rate of 5% will be paid and the LIBOR rate will be received. Assume that the yield curve is flat at 5% with continuous annual compounding and the volatility of the swap rate is 20%. The notional principal is \$100,000 per interest rate percentage point.

a) What are the key assumptions in order to apply Black's formula to value this swaption?
b) Compute the price of this swaption using Black's formula as an application of Proposition 19.17.

Exercise 19.3 Consider a *receiver* swaption which is giving its holder the right, but not the obligation, to enter into a 2-year annual pay swap in three years, where a fixed rate of 5% will be received and the LIBOR rate will be paid. Assume that the yield curve is flat at 2% with continuous annual compounding and the volatility of the swap rate is 10%. The notional principal is \$10,000 per percentage point, and the swaption price is quoted in basis points. Write down the expression of the price of this swaption using Black's formula.

Exercise 19.4 Consider two bonds with maturities T_1 and T_2, $T_1 < T_2$, which follow the stochastic differential equations

$$dP(t, T_1) = r_t P(t, T_1)dt + \zeta_1(t)P(t, T_1)dW_t$$

and

$$dP(t, T_2) = r_t P(t, T_2)dt + \zeta_2(t)P(t, T_2)dW_t.$$

a) Using Itô calculus, show that the forward process $P(t, T_2)/P(t, T_1)$ is a driftless geometric Brownian motion driven by $d\widehat{W}_t := dW_t - \zeta_1(t)dt$ under the T_1-forward measure $\widehat{\mathbb{P}}$.

b) Compute the price $\mathbb{E}^*\left[e^{-\int_t^{T_1} r_s ds}(K - P(T_1, T_2))^+ \;\Big|\; \mathcal{F}_t\right]$ of a bond put option at time $t \in [0, T_1]$ using change of numéraire and the Black-Scholes formula.

Hint: Given X a Gaussian random variable with mean m and variance v^2 given $\mathcal{F}_t$, we have:

$$\mathbb{E}\big[(\kappa - e^X)^+ \mid \mathcal{F}_t\big] = \kappa\Phi\left(-\frac{1}{v}(m - \log\kappa)\right) - e^{m+v^2/2}\Phi\left(-\frac{1}{v}(m + v^2 - \log\kappa)\right). \tag{19.43}$$

Exercise 19.5 Given two bonds with maturities T, S and prices $P(t, T)$, $P(t, S)$, consider the LIBOR rate

$$L(t, T, S) := \frac{P(t, T) - P(t, S)}{(S - T)P(t, S)}$$

at time $t \in [0, T]$, modeled as

$$dL(t, T, S) = \mu_t L(t, T, S)dt + \sigma L(t, T, S)dW_t, \quad 0 \leqslant t \leqslant T, \tag{19.44}$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $\sigma > 0$ is a constant, and $(\mu_t)_{t\in[0,T]}$ is an adapted process. Let

$$F(t) := \mathbb{E}^*\left[e^{-\int_t^S r_s ds}(\kappa - L(T, T, S))^+ \;\Big|\; \mathcal{F}_t\right]$$

denote the price at time t of a floorlet option with strike level κ, maturity T, and payment date S.

a) Rewrite the value of $F(t)$ using the forward measure $\widehat{\mathbb{P}}_S$ with maturity S.
b) What is the dynamics of $L(t, T, S)$ under the forward measure $\widehat{\mathbb{P}}_S$?
c) Write down the value of $F(t)$ using the Black-Scholes formula.

Hint: Given X a centered Gaussian random variable with variance v^2, we have

$$\mathbb{E}^*[(\kappa - e^{m+X})^+] = \kappa\Phi(-(m - \log\kappa)/v) - e^{m+v^2/2}\Phi(-v - (m - \log\kappa)/v),$$

where Φ denotes the Gaussian cumulative distribution function.

Exercise 19.6 Jamshidian's trick (Jamshidian (1989)). Consider a family $(P(t, T_l))_{l=i,\ldots,j}$ of bond prices defined from a short rate process $(r_t)_{t\in\mathbb{R}_+}$. We assume that the bond prices are functions $P(T_i, T_{l+1}) = F_{l+1}(T_i, r_{T_i})$ of r_{T_i} that are *increasing* in the variable r_{T_i}, for all $l = i, i + 1, \ldots, j - 1$.

a) Compute the price $P(t,T_i,T_j)$ of the annuity numéraire paying coupons $c_{i+1},\ldots,c_j$ at times $T_{i+1},\ldots,T_j$ in terms of the bond prices

$$P(t,T_{i+1}),\ldots,P(t,T_j).$$

b) Show that the payoff

$$\big(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j)\big)^+$$

of a European swaption can be rewrittten as

$$\left(1-\kappa\sum_{l=i}^{j-1}\tilde{c}_{l+1}P(T_i,T_{l+1})\right)^+,$$

by writing $\tilde{c}_l$ in terms of c_l, $l=i+1,\ldots,j$.

c) Assuming that the bond prices are functions $P(T_i,T_{l+1})=F_l(T_i,r_{T_i})$ of r_{T_i} that are *increasing* in the variable r_{T_i}, for all $l=i,\ldots,j-1$, show, choosing γ_κ such that

$$\kappa\sum_{l=i}^{j-1}c_{l+1}F_{l+1}(T_i,\gamma_\kappa)=1,$$

that the European swaption with payoff

$$\big(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j)\big)^+=\left(1-\kappa\sum_{l=i}^{j-1}c_{l+1}P(T_i,T_{l+1})\right)^+,$$

where c_j contains the final coupon payment, can be priced as a weighted sum of bond put options under the forward measure $\widehat{\mathbb{P}}_i$ with numéraire $N_t^{(i)}:=P(t,T_i)$.

Exercise 19.7 Path freezing. Consider n bonds with prices $(P(t,T_i))_{i=1,\ldots,n}$ and the bond option with payoff

$$\left(\sum_{i=2}^{n}c_iP(T_0,T_i)-\kappa P(T_0,T_1)\right)^+=P(T_0,T_1)\left(X_{T_0}-\kappa\right)^+,$$

where $N_t:=P(t,T_1)$ is taken as numéraire and

$$X_t:=\frac{1}{P(t,T_1)}\sum_{i=2}^{n}c_iP(t,T_i)=\sum_{i=2}^{n}c_i\widehat{P}(t,T_i),\qquad 0\leqslant t\leqslant T_1.$$

with $\widehat{P}(t,T_i):=P(t,T_i)/P(t,T_1)$, $i=2,3,\ldots,n$.

a) Assuming that the deflated bond price $(\widehat{P}(t,T_i))_{t\in[0,T_i]}$ has the (martingale) dynamics $d\widehat{P}(t,T_i) = \sigma_i(t)\widehat{P}(t,T_i)d\widehat{W}_t$ under the forward measure $\widehat{\mathbb{P}}_1$, where $(\sigma_i(t))_{t\in\mathbb{R}_+}$ is a deterministic function, write down the dynamics of X_t as $dX_t = \sigma_t X_t d\widehat{W}_t$, where σ_t is to be computed explicitly.
b) Approximating $(\widehat{P}(t,T_i))_{t\in[0,T_i]}$ by $\widehat{P}(0,T_i)$ and $(P(t,T_2,T_n))_{t\in[0,T_2]}$ by $P(0,T_2,T_n)$, find a deterministic approximation $\widehat{\sigma}(t)$ of σ_t, and deduce an expression of the option price

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_1} r_s ds}\left(\sum_{i=2}^n c_i P(T_0,T_i) - \kappa P(T_0,T_1)\right)^+\right] = P(0,T_1)\widehat{\mathbb{E}}\big[(X_{T_0}-\kappa)^+\big]$$

using the Black-Scholes formula.

Hint: Given X a centered Gaussian random variable with variance v^2, we have:

$$\mathbb{E}\big[\big(x\,\mathrm{e}^{X-v^2/2}-\kappa\big)^+\big] = x\Phi(v/2+(\log(x/\kappa))/v) - \kappa\Phi(-v/2+(\log(x/\kappa))/v).$$

Exercise 19.8 (Exercise 17.4 continued). We work in the short rate model

$$dr_t = \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, and $\widehat{\mathbb{P}}_2$ is the forward measure defined by

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_2}{\mathrm{d}\mathbb{P}^*} = \frac{1}{P(0,T_2)}\,\mathrm{e}^{-\int_0^{T_2} r_s ds}.$$

a) State the expressions of $\zeta_1(t)$ and $\zeta_2(t)$ in

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t)dB_t, \qquad i=1,2,$$

and the dynamics of the $P(t,T_1)/P(t,T_2)$ under $\widehat{\mathbb{P}}_2$, where $P(t,T_1)$ and $P(t,T_2)$ are bond prices with maturities T_1 and T_2.
Hint: Use Exercise 17.4 and the relation (17.25).
b) State the expression of the forward rate $f(t,T_1,T_2)$.
c) Compute the dynamics of $f(t,T_1,T_2)$ under the forward measure $\widehat{\mathbb{P}}_2$ with

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_2}{\mathrm{d}\mathbb{P}^*} = \frac{1}{P(0,T_2)}\,\mathrm{e}^{-\int_0^{T_2} r_s ds}.$$

d) Compute the price

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$(T_2 - T_1)\mathbb{E}^*\left[e^{-\int_t^{T_2} r_s ds}(f(T_1,T_1,T_2)-\kappa)^+ \;\middle|\; \mathcal{F}_t \right]$$

of an interest rate cap at time $t \in [0, T_1]$, using the expectation under the forward measure $\widehat{\mathbb{P}}_2$.

e) Compute the dynamics of the swap rate process

$$S(t,T_1,T_2) = \frac{P(t,T_1)-P(t,T_2)}{(T_2-T_1)P(t,T_2)}, \qquad t \in [0,T_1],$$

under $\widehat{\mathbb{P}}_2$.

f) Using (19.33), compute the swaption price

$$(T_2 - T_1)\mathbb{E}^*\left[e^{-\int_t^{T_1} r_s ds}P(T_1,T_2)(S(T_1,T_1,T_2)-\kappa)^+ \;\middle|\; \mathcal{F}_t \right]$$

on the swap rate $S(T_1,T_1,T_2)$ using the expectation under the forward swap measure $\widehat{\mathbb{P}}_{1,2}$.

Exercise 19.9 Consider three zero-coupon bonds $P(t,T_1)$, $P(t,T_2)$ and $P(t,T_3)$ with maturities $T_1 = \delta$, $T_2 = 2\delta$ and $T_3 = 3\delta$ respectively, and the forward LIBOR $L(t,T_1,T_2)$ and $L(t,T_2,T_3)$ defined by

$$L(t,T_i,T_{i+1}) = \frac{1}{\delta}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right), \qquad i = 1,2.$$

Assume that $L(t,T_1,T_2)$ and $L(t,T_2,T_3)$ are modeled in the BGM model by

$$\frac{dL(t,T_1,T_2)}{L(t,T_1,T_2)} = e^{-at}d\widehat{W}_t^{(2)}, \qquad 0 \leqslant t \leqslant T_1, \tag{19.45}$$

and $L(t,T_2,T_3) = b$, $0 \leqslant t \leqslant T_2$, for some constants $a,b>0$, where $\widehat{W}_t^{(2)}$ is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}_2$ defined by

$$\frac{d\widehat{\mathbb{P}}_2}{d\mathbb{P}^*} = \frac{e^{-\int_0^{T_2} r_s ds}}{P(0,T_2)}.$$

a) Compute $L(t,T_1,T_2)$, $0 \leqslant t \leqslant T_2$ by solving Equation (19.45).

b) Show that the price at time $t \in [0,T_1]$ of the caplet with strike level κ can be written as

$$\mathbb{E}^*\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+ \;\middle|\; \mathcal{F}_t \right] = P(t,T_2)\widehat{\mathbb{E}}_2\left[(L(T_1,T_1,T_2)-\kappa)^+ \mid \mathcal{F}_t \right],$$

where $\widehat{\mathbb{E}}_2$ denotes the expectation under the forward measure $\widehat{\mathbb{P}}_2$.

c) Using the hint below, compute the price at time t of the caplet with strike level κ on $L(T_1, T_1, T_2)$.

d) Compute

$$\frac{P(t,T_1)}{P(t,T_1,T_3)}, \quad 0 \leqslant t \leqslant T_1, \quad \text{and} \quad \frac{P(t,T_3)}{P(t,T_1,T_3)}, \qquad 0 \leqslant t \leqslant T_2,$$

in terms of b and $L(t, T_1, T_2)$, where $P(t, T_1, T_3)$ is the annuity numéraire

$$P(t,T_1,T_3) = \delta P(t,T_2) + \delta P(t,T_3), \qquad 0 \leqslant t \leqslant T_2.$$

e) Compute the dynamics of the swap rate

$$t \mapsto S(t,T_1,T_3) = \frac{P(t,T_1) - P(t,T_3)}{P(t,T_1,T_3)}, \qquad 0 \leqslant t \leqslant T_1,$$

i.e. show that we have

$$dS(t,T_1,T_3) = \sigma_{1,3}(t) S(t,T_1,T_3) d\widehat{W}_t^{(2)},$$

where $\sigma_{1,3}(t)$ is a stochastic process to be determined.

f) Using the Black-Scholes formula, compute an approximation of the swaption price

$$\mathbb{E}^* \left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1,T_1,T_3)(S(T_1,T_1,T_3) - \kappa)^+ \;\middle|\; \mathcal{F}_t \right]$$
$$= P(t,T_1,T_3) \widehat{\mathbb{E}}_2 \left[(S(T_1,T_1,T_3) - \kappa)^+ \mid \mathcal{F}_t \right],$$

at time $t \in [0, T_1]$. You will need to approximate $\sigma_{1,3}(s)$, $s \geqslant t$, by "freezing" all random terms at time t.

Hint: Given X a centered Gaussian random variable with variance v^2, we have

$$\mathbb{E}^* \left[(\mathrm{e}^{m+X} - \kappa)^+ \right] = \mathrm{e}^{m+v^2/2} \Phi(v + (m - \log \kappa)/v) - \kappa \Phi((m - \log \kappa)/v),$$

where Φ denotes the Gaussian cumulative distribution function.

Exercise 19.10 Bond option hedging. Consider a portfolio allocation $(\xi_t^T, \xi_t^S)_{t \in [0,T]}$ made of two bonds with maturities T, S, and value

$$V_t = \xi_t^T P(t,T) + \xi_t^S P(t,S), \qquad 0 \leqslant t \leqslant T,$$

at time t. We assume that the portfolio is self-financing, *i.e.*

$$dV_t = \xi_t^T dP(t,T) + \xi_t^S dP(t,S), \qquad 0 \leqslant t \leqslant T, \tag{19.46}$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

and that it *hedges* the claim payoff $(P(T,S)-\kappa)^+$, so that

$$\begin{aligned} V_t &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \;\Big|\; \mathcal{F}_t \right] \\ &= P(t,T)\mathbb{E}_T\big[(P(T,S)-\kappa)^+ \,\big|\, \mathcal{F}_t\big], \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

a) Show that we have

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-K)^+ \;\Big|\; \mathcal{F}_t \right] \\ &= P(0,T)\mathbb{E}_T\big[(P(T,S)-K)^+\big] + \int_0^t \xi_s^T dP(s,T) + \int_0^t \xi_s^S dP(s,S). \end{aligned}$$

b) Show that under the self-financing condition (19.46), the deflated portfolio value $\widetilde{V}_t = \mathrm{e}^{-\int_0^t r_s ds} V_t$ satisfies

$$d\widetilde{V}_t = \xi_t^T d\widetilde{P}(t,T) + \xi_t^S d\widetilde{P}(t,S),$$

where

$$\widetilde{P}(t,T) := \mathrm{e}^{-\int_0^t r_s ds} P(t,T), \quad t \in [0,T],$$

and

$$\widetilde{P}(t,S) := \mathrm{e}^{-\int_0^t r_s ds} P(t,S), \quad t \in [0,S],$$

denote the discounted bond prices.

c) From now on we work in the framework of Proposition 19.3, and we let the function $C(x,v)$ be defined by

$$C(X_t, v(t,T)) := \mathbb{E}_T\big[(P(T,S)-K)^+ \,\big|\, \mathcal{F}_t\big],$$

where X_t is the forward price $X_t := P(t,S)/P(t,T)$, $t \in [0,T]$, and

$$v^2(t,T) := \int_t^T \big|\sigma_s^S - \sigma_s^T\big|^2 ds.$$

Show that

$$\begin{aligned} \mathbb{E}_T\big[(P(T,S)-K)^+ \,\big|\, \mathcal{F}_t\big] &= \mathbb{E}_T\big[(P(T,S)-K)^+\big] \\ &\quad + \int_0^t \frac{\partial C}{\partial x}(X_u, v(u,T)) dX_u, \quad t \geqslant 0. \end{aligned}$$

Hint: Use the martingale property and the Itô formula.

d) Show that the deflated portfolio value $\widehat{V}_t = V_t / P(t,T)$ satisfies

$$\begin{aligned} d\widehat{V}_t &= \frac{\partial C}{\partial x}(X_t, v(t,T)) dX_t \\ &= \frac{P(t,S)}{P(t,T)} \frac{\partial C}{\partial x}(X_t, v(t,T))(\sigma_t^S - \sigma_t^T) d\widehat{B}_t^T. \end{aligned}$$

e) Show that

$$dV_t = P(t,S)\frac{\partial C}{\partial x}(X_t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \widehat{V}_t dP(t,T).$$

f) Show that

$$d\widetilde{V}_t = \widetilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, v(t,T))(\sigma_t^S - \sigma_t^T)dB_t + \widehat{V}_t d\widetilde{P}(t,T).$$

g) Compute the hedging strategy $(\xi_t^T, \xi_t^S)_{t\in[0,T]}$ of this bond option.

h) Show that

$$\frac{\partial C}{\partial x}(x,v) = \Phi\left(\frac{\log(x/K) + \tau v^2/2}{\sqrt{\tau}v}\right),$$

and compute the hedging strategy $(\xi_t^T, \xi_t^S)_{t\in[0,T]}$ in terms of the normal cumulative distribution function Φ.

Exercise 19.11 Consider a LIBOR rate $L(t,T,S)$, $t \in [0,T]$, modeled as $dL(t,T,S) = \mu_t L(t,T,S)dt + \sigma(t)L(t,T,S)dW_t$, $0 \leqslant t \leqslant T$, where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $(\mu_t)_{t\in[0,T]}$ is an adapted process, and $\sigma(t) > 0$ is a deterministic volatility function of time t.

a) What is the dynamics of $L(t,T,S)$ under the forward measure $\widehat{\mathbb{P}}$ with numéraire $N_t := P(t,S)$?

b) Rewrite the price

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^S r_s ds}\phi(L(T,T,S))\big|\mathcal{F}_t\right] \qquad (19.47)$$

at time $t \in [0,T]$ of an option with payoff function ϕ using the forward measure $\widehat{\mathbb{P}}$.

c) Write down the above option price (19.47) using an integral.

Exercise 19.12 Given n bonds with maturities $T_1, T_2, \ldots, T_n$, consider the annuity numéraire

$$P(t,T_i,T_j) = \sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t,T_{k+1})$$

and the swap rate

$$S(t,T_i,T_j) = \frac{P(t,T_i) - P(t,T_j)}{P(t,T_i,T_j)}$$

at time $t \in [0,T_i]$, modeled as

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$dS(t, T_i, T_j) = \mu_t S(t, T_i, T_j)dt + \sigma S(t, T_i, T_j)dW_t, \qquad 0 \leqslant t \leqslant T_i, \quad (19.48)$$

where $(W_t)_{t\in[0,T_i]}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $(\mu_t)_{t\in[0,T]}$ is an adapted process and $\sigma > 0$ is a constant. Let

$$\mathbb{E}^* \left[e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j)\phi(S(T_i, T_i, T_j)) \;\Big|\; \mathcal{F}_t \right] \qquad (19.49)$$

at time $t \in [0, T_i]$ of an option with payoff function ϕ.

a) Rewrite the option price (19.49) at time $t \in [0, T_i]$ using the forward swap measure $\widehat{\mathbb{P}}_{i,j}$ defined from the annuity numéraire $P(t, T_i, T_j)$.
b) What is the dynamics of $S(t, T_i, T_j)$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$?
c) Write down the above option price (19.47) using a Gaussian integral.
d) Apply the above to the computation at time $t \in [0, T_i]$ of the put swaption price

$$\mathbb{E}^* \left[e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j)(\kappa - S(T_i, T_i, T_j))^+ \;\Big|\; \mathcal{F}_t \right]$$

with strike level κ, using the Black-Scholes formula.

Hint: Given X a centered Gaussian random variable with variance v^2, we have

$$\mathbb{E}[(\kappa - e^{m+X})^+] = \kappa\Phi(-(m - \log\kappa)/v) - e^{m+v^2/2}\Phi(-v - (m - \log\kappa)/v),$$

where Φ denotes the Gaussian cumulative distribution function.

Exercise 19.13 Consider a bond market with two bonds with maturities T_1, T_2, whose prices $P(t, T_1), P(t, T_2)$ at time t are given by

$$\frac{dP(t, T_1)}{P(t, T_1)} = r_t dt + \zeta_1(t)dB_t, \qquad \frac{dP(t, T_2)}{P(t, T_2)} = r_t dt + \zeta_2(t)dB_t,$$

where $(r_t)_{t\in\mathbb{R}_+}$ is a short-term interest rate process, $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and $\zeta_1(t), \zeta_2(t)$ are volatility processes. The LIBOR rate $L(t, T_1, T_2)$ is defined by

$$L(t, T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)}.$$

Recall that a caplet on the LIBOR market can be priced at time $t \in [0, T_1]$ as

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds} (L(T_1, T_1, T_2) - \kappa)^+ \Big| \mathcal{F}_t \right] \qquad (19.50) \\ &= P(t, T_2)\widehat{\mathbb{E}}\big[(L(T_1, T_1, T_2) - \kappa)^+ \;\big|\; \mathcal{F}_t \big], \end{aligned}$$

under the forward measure $\widehat{\mathbb{P}}$ defined by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = e^{-\int_0^{T_1} r_s ds} \frac{P(T_1, T_2)}{P(0, T_2)},$$

under which

$$\widehat{B}_t := B_t - \int_0^t \zeta_2(s) ds, \qquad t \in \mathbb{R}_+, \tag{19.51}$$

is a standard Brownian motion.

In what follows we let $\underline{L_t = L(t, T_1, T_2)}$ for simplicity of notation.

a) Using Itô calculus, show that the LIBOR rate satisfies

$$dL_t = L_t \sigma(t) d\widehat{B}_t, \qquad 0 \leqslant t \leqslant T_1, \tag{19.52}$$

where the LIBOR rate volatility is given by

$$\sigma(t) = \frac{P(t, T_1)(\zeta_1(t) - \zeta_2(t))}{P(t, T_1) - P(t, T_2)}.$$

b) Solve the equation (19.52) on the interval $[t, T_1]$, and compute L_{T_1} from the initial condition L_t.

c) Assuming that $\sigma(t)$ in (19.52) is a deterministic volatility function of time $t \in [0, T_1]$, show that the price

$$P(t, T_2) \widehat{\mathbb{E}}\big[(L_{T_1} - \kappa)^+ \,\big|\, \mathcal{F}_t\big]$$

of the caplet can be written as $P(t, T_2) C(L_t, v(t, T_1))$, where $v^2(t, T_1) = \int_t^{T_1} |\sigma(s)|^2 ds$, and $C(t, v(t, T_1))$ is a function of L_t and $v(t, T_1)$.

d) Consider a portfolio allocation $(\xi_t^{(1)}, \xi_t^{(2)})_{t \in [0, T_1]}$ made of bonds with maturities T_1, T_2 and value

$$V_t = \xi_t^{(1)} P(t, T_1) + \xi_t^{(2)} P(t, T_2),$$

at time $t \in [0, T_1]$. We assume that the portfolio is self-financing, *i.e.*

$$dV_t = \xi_t^{(1)} dP(t, T_1) + \xi_t^{(2)} dP(t, T_2), \qquad 0 \leqslant t \leqslant T_1, \tag{19.53}$$

and that it *hedges* the claim payoff $(L_{T_1} - \kappa)^+$, so that

$$\begin{aligned} V_t &= \mathbb{E}\left[e^{-\int_t^{T_1} r_s ds} (P(T_1, T_2)(L_{T_1} - \kappa))^+ \middle| \mathcal{F}_t \right] \\ &= P(t, T_2) \widehat{\mathbb{E}}\big[(L_{T_1} - \kappa)^+ \mid \mathcal{F}_t\big], \end{aligned}$$

$0 \leqslant t \leqslant T_1$. Show that we have

$$\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds} \big(P(T_1, T_2)(L_{T_1} - \kappa) \big)^+ \,\Big|\, \mathcal{F}_t \right]$$
$$= P(0, T_2)\widehat{\mathbb{E}}\big[\big(L_{T_1} - \kappa\big)^+ \big] + \int_0^t \xi_s^{(1)} dP(s, T_1) + \int_0^t \xi_s^{(2)} dP(s, T_1),$$

$0 \leqslant t \leqslant T_1$.

e) Show that under the self-financing condition (19.53), the discounted portfolio value $\widetilde{V}_t = e^{-\int_0^t r_s ds} V_t$ satisfies

$$d\widetilde{V}_t = \xi_t^{(1)} d\widetilde{P}(t, T_1) + \xi_t^{(2)} d\widetilde{P}(t, T_2),$$

where $\widetilde{P}(t, T_1) := e^{-\int_0^t r_s ds} P(t, T_1)$ and $\widetilde{P}(t, T_2) := e^{-\int_0^t r_s ds} P(t, T_2)$ denote the discounted bond prices.

f) Show that

$$\widehat{\mathbb{E}}\big[\big(L_{T_1} - \kappa\big)^+ \mid \mathcal{F}_t \big] = \widehat{\mathbb{E}}\big[\big(L_{T_1} - \kappa\big)^+ \big] + \int_0^t \frac{\partial C}{\partial x}(L_u, v(u, T_1)) dL_u,$$

and that the deflated portfolio value $\widehat{V}_t = V_t / P(t, T_2)$ satisfies

$$d\widehat{V}_t = \frac{\partial C}{\partial x}(L_t, v(t, T_1)) dL_t = \sigma(t) L_t \frac{\partial C}{\partial x}(L_t, v(t, T_1)) d\widehat{B}_t.$$

Hint: use the martingale property and the Itô formula.

g) Show that

$$dV_t = (P(t, T_1) - P(t, T_2)) \frac{\partial C}{\partial x}(L_t, v(t, T_1)) \sigma(t) dB_t + \widehat{V}_t dP(t, T_2).$$

h) Show that

$$d\widetilde{V}_t = \frac{\partial C}{\partial x}(L_t, v(t, T_1)) d(\widetilde{P}(t, T_1) - \widetilde{P}(t, T_2)) + \left(\widehat{V}_t - L_t \frac{\partial C}{\partial x}(L_t, v(t, T_1)) \right) d\widetilde{P}(t, T_2),$$

and deduce the values of the hedging portfolio allocation $(\xi_t^{(1)}, \xi_t^{(2)})_{t \in \mathbb{R}_+}$.

Problem 19.14 Consider a bond market with tenor structure $\{T_i, \ldots, T_j\}$ and $j - i + 1$ bonds with maturities $T_i, \ldots, T_j$, whose prices $P(t, T_i), \ldots P(t, T_j)$ at time t are given by

$$\frac{dP(t, T_k)}{P(t, T_k)} = r_t dt + \zeta_k(t) dB_t, \qquad k = i, \ldots, j,$$

where $(r_t)_{t\in\mathbb{R}_+}$ is a short-term interest rate process and $(B_t)_{t\in\mathbb{R}_+}$ denotes a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and $\zeta_i(t),\ldots,\zeta_j(t)$ are volatility processes.

The swap rate $S(t,T_i,T_j)$ is defined by

$$S(t,T_i,T_j)=\frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)},$$

where

$$P(t,T_i,T_j)=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})$$

is the annuity numéraire. Recall that a swaption on the LIBOR market can be priced at time $t\in[0,T_i]$ as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(S(T_i,T_k,T_{k+1})-\kappa)\right)^+\,\middle|\,\mathcal{F}_t\right]\\ = P(t,T_i,T_j)\mathbb{E}_{i,j}\big[(S(T_i,T_i,T_j)-\kappa)^+\,\big|\,\mathcal{F}_t\big],\end{aligned}\tag{19.54}$$

under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$ defined by

$$\frac{d\widehat{\mathbb{P}}_{i,j}}{d\mathbb{P}^*}=\mathrm{e}^{-\int_0^{T_i} r_s ds}\frac{P(T_i,T_i,T_j)}{P(0,T_i,T_j)},\qquad 1\leqslant i<j\leqslant n,$$

under which

$$\widehat{B}_t^{i,j}:=B_t-\sum_{k=i}^{j-1}(T_{k+1}-T_k)\frac{P(t,T_{k+1})}{P(t,T_i,T_j)}\zeta_{k+1}(t)dt\tag{19.55}$$

is a standard Brownian motion. Recall that the swap rate can be modeled as

$$dS(t,T_i,T_j)=S(t,T_i,T_j)\sigma_{i,j}(t)d\widehat{B}_t^{i,j},\qquad 0\leqslant t\leqslant T_i,\tag{19.56}$$

where the swap rate volatilities are given by

$$\begin{aligned}\sigma_{i,j}(t)=\sum_{l=i}^{j-1}(T_{l+1}-T_l)\frac{P(t,T_{l+1})}{P(t,T_i,T_j)}(\zeta_i(t)-\zeta_{l+1}(t))\\ +\frac{P(t,T_j)}{P(t,T_i)-P(t,T_j)}(\zeta_i(t)-\zeta_j(t))\end{aligned}\tag{19.57}$$

$1\leqslant i,j\leqslant n$, cf. *e.g.* Proposition 8.12 in Privault (2021b). In what follows we denote $S_t=S(t,T_i,T_j)$ for simplicity of notation.

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

a) Solve the equation (19.56) on the interval $[t, T_i]$, and compute $S(T_i, T_i, T_j)$ from the initial condition $S(t, T_i, T_j)$.

b) Assuming that $\sigma_{i,j}(t)$ is a deterministic volatility function of time $t \in [0, T_i]$ for $1 \leqslant i, j \leqslant n$, show that the price (19.38) of the swaption can be written as

$$P(t, T_i, T_j)C(S_t, v(t, T_i)),$$

where

$$v^2(t, T_i) := \int_t^{T_i} |\sigma_{i,j}(s)|^2 ds,$$

and $C(x, v)$ is a function to be specified using the Black-Scholes formula $\mathrm{Bl}(x, K, \sigma, r, \tau)$, with the relation

$$\mathbb{E}[(x\,\mathrm{e}^{m+X} - K)^+] = \Phi(v + (m + \log(x/K))/v) - K\Phi((m + \log(x/K))/v),$$

where X is a centered Gaussian random variable with variance v^2.

c) Consider a portfolio allocation $(\xi_t^{(i)}, \ldots, \xi_t^{(j)})_{t \in [0,T_i]}$ made of bonds with maturities $T_i, \ldots, T_j$ and value

$$V_t = \sum_{k=i}^{j} \xi_t^{(k)} P(t, T_k),$$

at time $t \in [0, T_i]$. We assume that the portfolio is self-financing, *i.e.*

$$dV_t = \sum_{k=i}^{j} \xi_t^{(k)} dP(t, T_k), \qquad 0 \leqslant t \leqslant T_i, \tag{19.58}$$

and that it *hedges* the claim payoff $(S(T_i, T_i, T_j) - \kappa)^+$, so that

$$\begin{aligned} V_t &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \,\middle|\, \mathcal{F}_t \right] \\ &= P(t, T_i, T_j) \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \,\middle|\, \mathcal{F}_t \right], \end{aligned}$$

$0 \leqslant t \leqslant T_i$. Show that

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(T_i, T_{k+1}) (L(T_i, T_k, T_{k+1}) - \kappa) \right)^+ \,\middle|\, \mathcal{F}_t \right] \\ &\quad = P(0, T_i, T_j) \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \right] + \sum_{k=i}^{j} \int_0^t \xi_s^{(k)} dP(s, T_i), \end{aligned}$$

$0 \leqslant t \leqslant T_i$.

d) Show that under the self-financing condition (19.58), the discounted portfolio value $\widetilde{V}_t = e^{-\int_0^t r_s ds} V_t$ satisfies

$$d\widetilde{V}_t = \sum_{k=i}^{j} \xi_t^{(k)} d\widetilde{P}(t, T_k),$$

where $\widetilde{P}(t, T_k) = e^{-\int_0^t r_s ds} P(t, T_k)$, $k = i, i+1 \ldots, j$, denote the discounted bond prices.

e) Show that

$$\begin{aligned} &\mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+\right] + \int_0^t \frac{\partial C}{\partial x}(S_u, v(u, T_i)) dS_u. \end{aligned}$$

Hint: use the martingale property and the Itô formula.

f) Show that the deflated portfolio value $\widehat{V}_t = V_t / P(t, T_i, T_j)$ satisfies

$$d\widehat{V}_t = \frac{\partial C}{\partial x}(S_t, v(t, T_i)) dS_t = S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sigma_t^{i,j} d\widehat{B}_t^{i,j}.$$

g) Show that

$$dV_t = (P(t, T_i) - P(t, T_j)) \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sigma_t^{i,j} dB_t + \widehat{V}_t dP(t, T_i, T_j).$$

h) Show that

$$\begin{aligned} dV_t = {} & S_t \zeta_i(t) \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) dB_t \\ & + (\widehat{V}_t - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i))) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \zeta_{k+1}(t) dB_t \\ & + \frac{\partial C}{\partial x}(S_t, v(t, T_i)) P(t, T_j) (\zeta_i(t) - \zeta_j(t)) dB_t. \end{aligned}$$

i) Show that

$$\begin{aligned} d\widetilde{V}_t = {} & \frac{\partial C}{\partial x}(S_t, v(t, T_i)) d(\widetilde{P}(t, T_i) - \widetilde{P}(t, T_j)) \\ & + (\widehat{V}_t - S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i))) d\widetilde{P}(t, T_i, T_j). \end{aligned}$$

j) Show that

$$\frac{\partial C}{\partial x}(x, v(t, T_i)) = \Phi\left(\frac{\log(x/K)}{v(t, T_i)} + \frac{v(t, T_i)}{2}\right).$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

k) Show that we have

$$d\widetilde{V}_t = \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right) d(\widetilde{P}(t,T_i) - \widetilde{P}(t,T_j))$$
$$-\kappa\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right) d\widetilde{P}(t,T_i,T_j).$$

l) Show that the hedging strategy is given by

$$\xi_t^{(i)} = \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right),$$

$$\xi_t^{(j)} = -\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right) - \kappa(T_j - T_{j-1})\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right),$$

and

$$\xi_t^{(k)} = -\kappa(T_{k+1} - T_k)\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right), \qquad i \leqslant k \leqslant j-2.$$