

Chapter 4

Correlation and Dependence

Correlation and dependence play a central role in risk management, particularly when assessing or preventing any potential “domino effects” arising from interactions between entities exposed to uncertainty.* This chapter presents several standard models for the statistical interactions that occur in correlated risk modeling. In particular, we use copulas to capture the uncertainty and dependence structures observed between random variables or data samples.

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4.1 Joint Bernoulli Distribution

Our study of dependence structures starts with the simplest case of two correlated random variables X and Y , each of them taking only two possible values. For this, let X and Y be two Bernoulli random variables, with

$$p_X = \mathbb{P}(X = 1) = \mathbb{E}[\mathbb{1}_{\{X=1\}}] \quad \text{and} \quad p_Y = \mathbb{P}(Y = 1) = \mathbb{E}[\mathbb{1}_{\{Y=1\}}]$$

and correlation coefficient

$$\rho := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\text{Var}[Y]}} \tag{4.1}$$

* Correlation does not imply causation. Try “[Spurious Correlations](#)”.

$$\begin{aligned}
&= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} \\
&= \frac{\mathbb{P}(X = 1 \text{ and } Y = 1) - p_X p_Y}{\sqrt{p_X(1-p_X)p_Y(1-p_Y)}},
\end{aligned}$$

with $\rho \in [-1, 1]$, from the Cauchy-Schwarz inequality. We note that in this case, the joint distribution $\mathbb{P}(X = i \text{ and } Y = j)$, $i, j = 0, 1$, is fully determined by the data of $p_X = \mathbb{P}(X = 1)$, $p_Y = \mathbb{P}(Y = 1)$ and the correlation coefficient $\rho \in [-1, 1]$, as

$$\left\{ \begin{array}{l}
\mathbb{P}(X = 1 \text{ and } Y = 1) = \mathbb{E}[XY] \\
\qquad\qquad\qquad = p_X p_Y + \rho \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\
\mathbb{P}(X = 0 \text{ and } Y = 1) = \mathbb{E}[(1-X)Y] = \mathbb{P}(Y = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\
\qquad\qquad\qquad = (1-p_X)p_Y - \rho \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\
\mathbb{P}(X = 1 \text{ and } Y = 0) = \mathbb{E}[X(1-Y)] = \mathbb{P}(X = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\
\qquad\qquad\qquad = p_X(1-p_Y) - \rho \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\
\mathbb{P}(X = 0 \text{ and } Y = 0) = \mathbb{E}[(1-X)(1-Y)] \\
\qquad\qquad\qquad = (1-p_X)(1-p_Y) + \rho \sqrt{p_X p_Y (1-p_X)(1-p_Y)},
\end{array} \right.$$

see Exercise 4.2.

4.2 Joint Gaussian Distribution

Consider now two *centered* Gaussian random variables $X \simeq \mathcal{N}(0, \sigma^2)$ and $Y \simeq \mathcal{N}(0, \eta^2)$ with probability density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)} \quad \text{and} \quad f_Y(x) = \frac{1}{\sqrt{2\pi\eta^2}} e^{-x^2/(2\eta^2)}, \quad x \in \mathbb{R}.$$

Let

$$\rho = \text{corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}}.$$

When the covariance matrix

$$\Sigma := \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho\sigma\eta \\ \rho\sigma\eta & \eta^2 \end{bmatrix} \quad (4.2)$$

with determinant

$$\begin{aligned}\det \Sigma &= \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[XY])^2 \\ &= \mathbb{E}[X^2]\mathbb{E}[Y^2](1 - (\text{corr}(X, Y))^2) \\ &\geq 0,\end{aligned}$$

is invertible, there exists a probability density function

$$\begin{aligned}f_{\Sigma}(x, y) &= \frac{1}{\sqrt{2\pi \det \Sigma}} \exp\left(-\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^{\top} \Sigma^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \frac{1}{\sqrt{2\pi \det \Sigma}} \exp\left(-\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^{\top} \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right),\end{aligned}\quad (4.3)$$

with respective marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$.

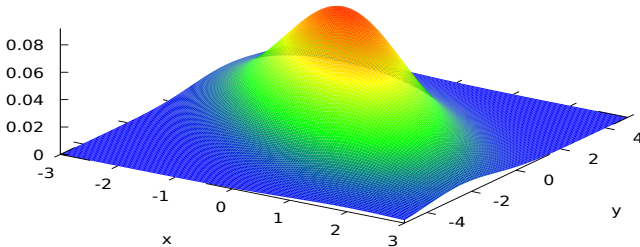


Fig. 4.1: Joint Gaussian probability density.

Definition 4.1. The probability density function (4.3) is called the centered joint (bivariate) Gaussian probability density with covariance matrix Σ .

Note that when $\rho = \text{corr}(X, Y) = \pm 1$ we have $\det \Sigma = 0$ and the joint probability density function $f_{\Sigma}(x, y)$ is not defined. Remark 4.2 plays an important role in the modeling of joint default probabilities, see [here](#) for a detailed discussion.

Remark 4.2. There exist couples (X, Y) of random variables with Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$, such that

- i) (X, Y) does not have the bivariate Gaussian distribution with probability density function $f_{\Sigma}(x, y)$, where Σ is the covariance matrix (4.2) of (X, Y) .
- ii) the random variable $X + Y$ is not even Gaussian.

Proof. See Exercise 4.5. □

More generally, we can consider random vectors having a joint (or multivariate) distribution in any dimension.

Definition 4.3. A random vector (X_1, \dots, X_n) is said to have a multivariate centered Gaussian distribution if every linear combination

$$a_1 X_1 + \dots + a_n X_n, \quad a_1, \dots, a_n \in \mathbb{R}, \quad n \geq 1,$$

has a centered Gaussian distribution.

Recall that if (X_1, \dots, X_n) has a multivariate centered Gaussian distribution, then its probability density function takes the form

$$f_{\Sigma}(x_1, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x_1, \dots, x_n)^T \Sigma^{-1} (x_1, \dots, x_n)\right),$$

$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, where Σ is the covariance matrix

$$\Sigma = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_{n-1}) & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Var}[X_2] & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \text{Var}[X_{n-1}] & \text{Cov}(X_{n-1}, X_n) \\ \text{Cov}(X_1, X_n) & \text{Cov}(X_2, X_n) & \cdots & \text{Cov}(X_{n-1}, X_n) & \text{Var}[X_n] \end{bmatrix},$$

which extends (4.3) to $n \geq 3$.

4.3 Copulas and Dependence Structures

The word copula derives from the Latin noun for a “link” or “tie” that connects two different objects or concepts.

Definition 4.4. A two-dimensional copula is any joint cumulative distribution function

$$\begin{aligned} C : [0, 1] \times [0, 1] &\longrightarrow [0, 1] \\ (u, v) &\longmapsto C(u, v) \end{aligned}$$

with uniform $[0, 1]$ -valued marginals.

In other words, any copula function $C(u, v)$ can be written as

$$C(u, v) = \mathbb{P}(U \leq u \text{ and } V \leq v), \quad 0 \leq u, v \leq 1,$$

where U and V are uniform $[0, 1]$ -valued random variables.



Examples.

- a) The copula function corresponding to the fully correlated case $U = V$ is given by

$$\begin{aligned} C(u, v) &= \mathbb{P}(U \leq u \text{ and } V \leq v) \\ &= \mathbb{P}(U \leq \min(u, v)) \\ &= \min(u, v), \quad 0 \leq u, v \leq 1. \end{aligned}$$

- b) The copula function corresponding to independent uniform random variables (U, V) is given by

$$\begin{aligned} C(u, v) &= \mathbb{P}(U \leq u \text{ and } V \leq v) \\ &= \mathbb{P}(U \leq u)\mathbb{P}(V \leq v) \\ &= uv, \quad 0 \leq u, v \leq 1. \end{aligned}$$

- c) The copula function corresponding to the fully anticorrelated case $U = 1 - V$ is given by

$$\begin{aligned} C(u, v) &:= \mathbb{P}(U \leq u \text{ and } V \leq v) \\ &= \mathbb{P}(U \leq u \text{ and } 1 - U \leq v) \\ &= \mathbb{P}(1 - v \leq U \leq u) \\ &= (u + v - 1)^+, \quad 0 \leq u, v \leq 1. \end{aligned}$$

The above copulas are plotted in Figure 4.2.

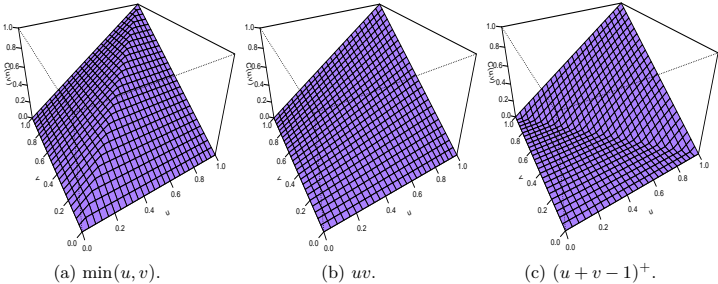


Fig. 4.2: Copula graphs $C(u, v) = \min(u, v)$, $C(u, v) = uv$, $C(u, v) = (u + v - 1)^+$.

In what follows, F_X^{-1} denotes the inverse of the *Cumulative Distribution Function* F_X of X .

Lemma 4.5. *Let X be a random variable having a continuous and strictly increasing cumulative distribution function $F_X(x) := \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. Then, the random variable Y defined as $Y := F_X(X)$ is uniformly distributed on $[0, 1]$.*

Proof. Denoting by F_X^{-1} the continuous inverse of F_X , we have

$$\begin{aligned} \mathbb{P}(Y \leq u) &= \mathbb{P}(F_X(X) \leq u) \\ &= \mathbb{P}(X \leq F_X^{-1}(u)) \\ &= F_X(F_X^{-1}(u)) \\ &= u, \quad 0 \leq u \leq 1. \end{aligned}$$

□

```
1 X=rexp(1e6);plot(density(X),col='blue',lwd=3);plot(density(pexp(X)),col='blue',lwd=3);
2 X=rnorm(1e6);plot(density(X),col='blue',lwd=3);plot(density(pnorm(X)),col='blue',lwd=3);
```

Listing 4.1:  code - CDF Composition examples.

```
1 import numpy as np, matplotlib.pyplot as plt;X=np.random.exponential(size=10**5);
2 from scipy.stats import gaussian_kde, expon, norm;ls2=np.linspace(0,1,1000)
3 ls3=np.linspace(np.percentile(X,[1,99.9]),1000);ls=np.linspace(0,np.percentile(X,99.9),1000)
4 plt.plot(*(lambda x:(x.gaussian_kde(X(x)))(ls),lw=3); plt.show(); plt.figure();
5 plt.plot(*(lambda u:(u.gaussian_kde(expon.cdf(X))(u))(ls2),lw=3);plt.show()
6 X=np.random.normal(size=10**5); plt.figure();
7 plt.plot(*(lambda x:(x.gaussian_kde(X(x)))(ls3),lw=3);plt.show(); plt.figure();
8 plt.plot(*(lambda u:(u.gaussian_kde(norm.cdf(X))(u))(ls2),lw=3);plt.show()
```

Listing 4.2: Python code - CDF Composition examples.

As in Lemma 4.5, given (X, Y) a couple of random variables with joint cumulative distribution function

$$F_{(X,Y)}(x, y) := \mathbb{P}(X \leq x \text{ and } Y \leq y), \quad x, y \in \mathbb{R},$$

and continuous strictly increasing marginal cumulative distribution functions

$$F_X(x) = F_{(X,Y)}(x, \infty) = \mathbb{P}(X \leq x)$$

and

$$F_Y(y) = F_{(X,Y)}(\infty, y) = \mathbb{P}(Y \leq y),$$

we note the following points.

- i) By Lemma 4.5, the random variables U, V defined as

$$U := F_X(X) \quad \text{and} \quad V := F_Y(Y)$$

are uniformly distributed on $[0, 1]$.

ii) The copula function

$$C_{(X,Y)}(u, v) := \mathbb{P}(U \leq u \text{ and } V \leq v), \quad 0 \leq u, v \leq 1,$$

satisfies

$$\begin{aligned} C_{(X,Y)}(u, v) &= \mathbb{P}(U \leq u \text{ and } V \leq v) \\ &= \mathbb{P}(F_X(X) \leq u \text{ and } F_Y(Y) \leq v) \\ &= \mathbb{P}(X \leq F_X^{-1}(u) \text{ and } Y \leq F_Y^{-1}(v)) \\ &= F_{(X,Y)}(F_X^{-1}(u), F_Y^{-1}(v)), \quad 0 \leq u, v \leq 1. \end{aligned}$$

iii) The joint cumulative distribution function $F_{(X,Y)}$ of (X, Y) can be recovered from the copula function $C_{(X,Y)}$ and the marginal cumulative distribution functions F_X, F_Y as

$$\begin{aligned} F_{(X,Y)}(x, y) &= \mathbb{P}(X \leq x \text{ and } Y \leq y) \\ &= \mathbb{P}(F_X(X) \leq F_X(x) \text{ and } F_Y(Y) \leq F_Y(y)) \\ &= \mathbb{P}(U \leq F_X(x) \text{ and } V \leq F_Y(y)) \\ &= C_{(X,Y)}(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}. \end{aligned}$$

Higher-dimensional copulas

Definition 4.6. An n -dimensional copula is any joint cumulative distribution function

$$\begin{aligned} C : [0, 1] \times \cdots \times [0, 1] &\longrightarrow [0, 1] \\ (u_1, \dots, u_n) &\longmapsto C(u_1, \dots, u_n) \end{aligned}$$

of n uniform $[0, 1]$ -valued random variables.

Consider the joint cumulative distribution function

$$F_{(X_1, \dots, X_n)}(x_1, \dots, x_n) := \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n)$$

of a family (X_1, \dots, X_n) of random variables with marginal cumulative distribution functions

$$F_{X_i}(x) = F_{(X_1, \dots, X_n)}(\infty, \dots, \infty, x, \infty, \dots, \infty), \quad x \in \mathbb{R},$$

$i = 1, 2, \dots, n$. The copula defined in the next Sklar's Theorem 4.7 encodes the dependence structure of the vector (X_1, \dots, X_n) .

Theorem 4.7. [Sklar's theorem* (Sklar (1959; 2010))] Given a joint cumulative distribution function $F_{(X_1, \dots, X_n)}$, there exists an n -dimensional copula function $C(u_1, \dots, u_n)$ such that

$$F_{(X_1, \dots, X_n)}(x_1, x_2, \dots, x_n) = C(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)),$$

$x_1, x_2, \dots, x_n \in \mathbb{R}$.

Corollary 4.8 is a consequence of Sklar's Theorem 4.7.

Corollary 4.8. Assume that the marginal distribution functions F_{X_i} are continuous and strictly increasing. Then the joint cumulative distribution function $F_{(X_1, \dots, X_n)}$ defines a n -dimensional copula function

$$C(u_1, \dots, u_n) := F_{(X_1, \dots, X_n)}(F_{X_1}^{-1}(u_1), \dots, F_{X_n}^{-1}(u_n)), \quad (4.4)$$

$u_1, u_2, \dots, u_n \in [0, 1]$, which encodes the dependence structure of the vector (X_1, \dots, X_n) .

It can also be checked as in Lemma 4.5 that $C(u_1, \dots, u_n)$ defined in (4.4) has uniform marginal distributions on $[0, 1]$, as

$$\begin{aligned} & C(1, \dots, 1, u, 1, \dots, 1) \\ &= F_{(X_1, \dots, X_n)}(F_{X_1}^{-1}(1), \dots, F_{X_{i-1}}^{-1}(1), F_{X_i}^{-1}(u), F_{X_{i+1}}^{-1}(1), \dots, F_{X_n}^{-1}(1)) \\ &= F_{(X_1, \dots, X_n)}(\infty, \dots, \infty, F_{X_i}^{-1}(u), \infty, \dots, \infty) \\ &= F_{X_i}(F_{X_i}^{-1}(u)) \\ &= u, \quad 0 \leq u \leq 1. \end{aligned}$$

In Proposition 4.9 we construct a vector of random variables from the data of a copula and a family of marginal distributions.

Proposition 4.9. Given a family of continuous strictly increasing cumulative distribution functions F_1, \dots, F_n and a multidimensional copula function $C(u_1, \dots, u_n)$, the function

$$F_{(X_1, \dots, X_n)}^C(x_1, \dots, x_n) := C(F_1(x_1), \dots, F_n(x_n)), \quad (4.5)$$

$x_1, x_2, \dots, x_n \in \mathbb{R}$, defines a joint cumulative distribution function with marginals F_1, \dots, F_n .

* "The author considers continuous non-decreasing functions C_n on the n -dimensional cube $[0, 1]^n$ with $C_n(0, \dots, 0) = 0$, $C_n(1, \dots, 1, \alpha, 1, \dots, 1) = \alpha$. Several theorems are stated relating n -dimensional distribution functions and their marginals in terms of functions C_n . No proofs are given." M. Loève, Math. Reviews MR0125600.

Proof. Given (U_1, \dots, U_n) a vector of n uniform random variables having the copula function $C(u_1, \dots, u_n)$ for cumulative distribution function, we let

$$X_1 := F_1^{-1}(U_1), \dots, X_n := F_n^{-1}(U_n).$$

Then, we have

$$\begin{aligned} F_{(X_1, \dots, X_n)}^C(x_1, \dots, x_n) &= C(F_1(x_1), \dots, F_n(x_n)) \\ &= \mathbf{P}(U_1 \leq F_1(x_1), \dots, U_n \leq F_n(x_n)) \\ &= \mathbf{P}(F_1^{-1}(U_1) \leq x_1, \dots, F_n^{-1}(U_n) \leq x_n) \\ &= \mathbf{P}(X_1 \leq x_1, \dots, X_n \leq x_n), \quad x_1, x_2, \dots, x_n \in \mathbb{R}. \end{aligned}$$

Finally, we verify that the marginal distributions generated by $F_{(X_1, \dots, X_n)}^C$ coincide with the respective marginals of (X_1, \dots, X_n) , as we have

$$\begin{aligned} F_{(X_1, \dots, X_n)}^C(\infty, \dots, \infty, u, \infty, \dots, \infty) &= C(F_1(\infty), \dots, F_{i-1}(\infty), F_i(u), F_{i+1}(\infty), \dots, F_n(\infty)) \\ &= C(1, \dots, 1, F_i(u), 1, \dots, 1) \\ &= F_i(u), \quad 0 \leq u \leq 1. \end{aligned}$$

□

4.4 Examples of Copulas

Gaussian copulas

The choice of (4.3) above as joint probability density function, see Figure 4.1, actually induces a particular dependence structure between the Gaussian random variables X and Y , and corresponding to the joint cumulative distribution function

$$\begin{aligned} \Phi_{\Sigma}(x, y) &:= \mathbf{P}(X \leq x \text{ and } Y \leq y) \\ &= \frac{1}{\sqrt{2\pi \det \Sigma}} \int_{-\infty}^x \int_{-\infty}^y \exp\left(-\frac{1}{2} \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \Sigma^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle\right) dudv, \end{aligned}$$

$x, y \in \mathbb{R}$. In case the random variables X, Y are normalized centered Gaussian random variables with unit variance, Σ is given by

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with correlation coefficient $\rho \in (-1, 1)$. Letting

$$\Phi_X(x) := \mathbb{P}(X \leq x) \quad \text{and} \quad \Phi_Y(y) := \mathbb{P}(Y \leq y),$$

denote the cumulative distribution functions of X and Y , the random variables $\Phi_X(X)$ and $\Phi_Y(Y)$ are known to be uniformly distributed on $[0, 1]$ by Lemma 4.5, and $(\Phi_X(X), \Phi_Y(Y))$ is a $[0, 1] \times [0, 1]$ -valued random variable with joint cumulative distribution function

$$\begin{aligned} C_\Sigma(u, v) &:= \mathbb{P}(\Phi_X(X) \leq u \text{ and } \Phi_Y(Y) \leq v) \\ &= \mathbb{P}(X \leq \Phi_X^{-1}(u) \text{ and } Y \leq \Phi_Y^{-1}(v)) \\ &= \Phi_\Sigma(\Phi_X^{-1}(u), \Phi_Y^{-1}(v)), \quad 0 \leq u, v \leq 1. \end{aligned} \quad (4.6)$$

According to Proposition 4.9, the function $C_\Sigma(u, v)$, which is the joint cumulative distribution function of a couple of uniformly distributed $[0, 1]$ -valued random variables, is the *Gaussian copula* generated by the jointly Gaussian distribution of (X, Y) with covariance matrix Σ .

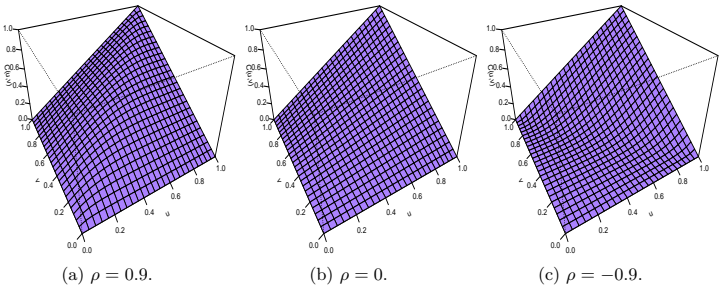


Fig. 4.3: Gaussian copula graphs for $\rho = -0.9$, $\rho = 0$, and $\rho = 0.9$.

The graphs of Figures 4.3-(c) and 4.3-(a) correspond to intermediate dependence levels given by Gaussian copulas, cf. (4.6).

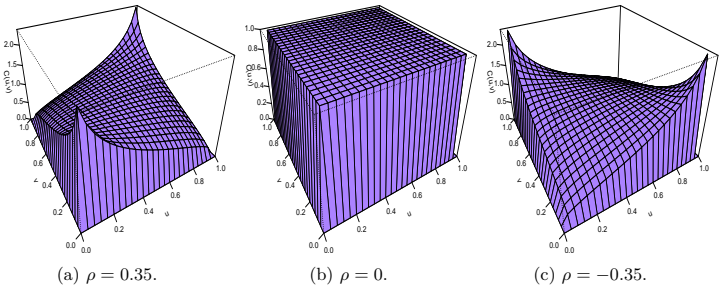


Fig. 4.4: Different Gaussian copula *density* graphs for $\rho = -0.35$, $\rho = 0$ and $\rho = 0.35$.

Figures 4.4 and 4.5 present the corresponding copula density graphs.

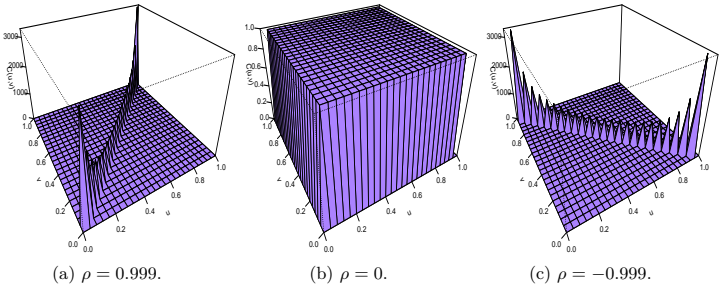


Fig. 4.5: Different Gaussian copula *density* graphs for $\rho = -0.999$, $\rho = 0$ and $\rho = 0.999$.

Figure 4.4-(c) represents a uniform (product) probability density function on the square $[0, 1] \times [0, 1]$, which corresponds to two independent uniformly distributed $[0, 1]$ -valued random variables U, V . Figure 4.4-(a) shows the probability distribution of the fully correlated couple (U, U) , which does not admit a probability density on the square $[0, 1] \times [0, 1]$.

The Gaussian copula function $C_{\Sigma}(u, v)$ admits a probability density function on $[0, 1] \times [0, 1]$ given by

$$\begin{aligned} c_{\Sigma}(u, v) &= \frac{\partial^2 C_{\Sigma}}{\partial u \partial v}(u, v) \\ &= \frac{\partial^2}{\partial u \partial v} \Phi_{\Sigma}(\Phi_X^{-1}(u), \Phi_Y^{-1}(v)) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial u} \left(\frac{1}{\Phi'_Y(\Phi_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y} (\Phi_X^{-1}(u), \Phi_Y^{-1}(v)) \right) \\
 &= \frac{\partial}{\partial u} \left(\frac{1}{f_Y(\Phi_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y} (\Phi_X^{-1}(u), \Phi_Y^{-1}(v)) \right) \\
 &= \frac{1}{f_X(\Phi_X^{-1}(u)) f_Y(\Phi_Y^{-1}(v))} \frac{\partial^2 \Phi_\Sigma}{\partial x \partial y} (\Phi_X^{-1}(u), \Phi_Y^{-1}(v)) \\
 &= \frac{f_\Sigma(\Phi_X^{-1}(u), \Phi_Y^{-1}(v))}{f_X(\Phi_X^{-1}(u)) f_Y(\Phi_Y^{-1}(v))},
 \end{aligned}$$

hence the Gaussian copula function $C_\Sigma(u, v)$ can be expressed as

$$\begin{aligned}
 C_\Sigma(u, v) &= \int_0^u \int_0^v c_\Sigma(a, b) da db \\
 &= \int_0^u \int_0^v \frac{f_\Sigma(\Phi_X^{-1}(a), \Phi_Y^{-1}(b))}{f_X(\Phi_X^{-1}(a)) f_Y(\Phi_Y^{-1}(b))} da db, \quad 0 \leq u, v \leq 1.
 \end{aligned}$$

Using Corollary 4.8, the joint cumulative distribution function $\Phi_{(X,Y)}(x, y)$ of (X, Y) can be recovered from the Gaussian copula function $C_\Sigma(x, y)$ and the respective cumulative distribution functions $\Phi_X(x)$, $\Phi_Y(y)$ of X and Y as

$$\Phi_{(X,Y)}(x, y) = C_\Sigma(\Phi_X(x), \Phi_Y(y)), \quad x, y \in \mathbb{R}. \quad (4.7)$$

In that sense, the Gaussian copula function $C_\Sigma(x, y)$ encodes the Gaussian dependence structure of the covariance matrix Σ .

Moreover, the Gaussian copula function $C_\Sigma(x, y)$ can be used as in Proposition 4.9, to generate a joint distribution function $F_{(X,Y)}^C(x, y)$ by letting

$$F_{(X,Y)}^C(x, y) := C_\Sigma(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R}, \quad (4.8)$$

based on other, possibly non-Gaussian cumulative distribution functions $F_X(x)$, $F_Y(y)$ of two random variables X and Y .

We note that the marginals of the joint cumulative distribution function $F_{(X,Y)}^C(x, y)$ are $F_X(x)$ and $F_Y(y)$ because $C_\Sigma(x, y)$ has uniform marginals on $[0, 1]$.

Example - Gumbel copula

The Gumbel copula function is given by

$$C(u, v) = \exp\left(-\left((-\log u)^\theta + (-\log v)^\theta\right)^{1/\theta}\right), \quad 0 \leq u, v \leq 1,$$

with $\theta \geq 1$, and $C(u, v) = uv$ when $\theta = 1$.

Uniform marginals with given copulas

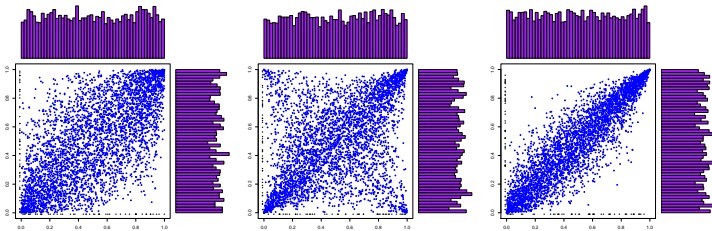
The following `R` code generates random samples according to the Gaussian, Student, and Gumbel copulas with uniform marginals, as illustrated in Figure 4.6.

```

1  install.packages("copula"); install.packages("gumbel")
2  library(copula);library(gumbel);norm.cop<-normalCopula(.35)
3  persp(norm.cop,pCopula,51,xlab="u",ylab="v",zlab="C(u,v)",col='mediumpurple1')
4  persp(norm.cop,dCopula,51,xlab="u",ylab="v",zlab="c(u,v)",col='mediumpurple1')
5  norm<-rCopula(4000,normalCopula(.7));plot(norm,cex=3,pch='.',col='blue')
6  stud<-rCopula(4000,tCopula(.5,df=1));points(stud,cex=3,pch='.',col='red')
7  gumb<-rgumbel(4000,4);points(gumb,cex=3,pch='.',col='green')

```

Listing 4.3: `R` code - Copulas with random samples generation.



(a) Gaussian copula.

(b) Student copula.

(c) Gumbel copula.

Fig. 4.6: Samples with uniform marginals and given copulas.

```

1  import numpy as np, matplotlib.pyplot as plt
2  from copulae import GaussianCopula, StudentCopula, GumbelCopula
3  from scipy.stats import gumbel_r; np.random.seed(123);corr_g= np.array([[1.,.7], [.7,1.]]);
4  norm_cop=GaussianCopula(dim=2);norm_cop[:]=corr_g;norm_s= norm_cop.random(4000);
5  corr_student=np.array([[1.,.5],[.5,1.]]);st_c=StudentCopula(dim=2,df=1)
6  st_c[:]= corr_student; st_s= st_c.random(4000); gb_cop= GumbelCopula(dim=2, theta=4);
7  gb_s= gb_cop.random(4000); plt.figure(figsize=(10, 8))
8  plt.scatter(norm_s[:,0],norm_s[:,1],alpha=.5,s=1,c='b',label='Gaussian Copula rho=.7')
9  plt.scatter(st_s[:,0],st_s[:,1],alpha=.5,s=1,c='r',label='Student-t Copula rho=.5,df=1')
10 plt.scatter(gb_s[:,0],gb_s[:,1],alpha=.5,s=1,c='g',label='Gumbel Copula theta=4')
11 plt.xlabel('u'); plt.ylabel('v'); plt.title('Copula Samples Comparison')
12 plt.xlim(0, 1); plt.ylim(0, 1); plt.grid(True, alpha=.3); plt.legend(); plt.show()

```

Listing 4.4: Python code - Copulas with random samples generation.

The following **R** code is plotting the histograms of Figure 4.6.

```

1 joint_hist <- function(u) {x <- u[,1]; y <- u[,2]; xhist <- hist(x, breaks=40,plot=FALSE)
2 yhist <- hist(y, breaks=40,plot=FALSE); top <- max(c(xhist$counts, yhist$counts))
3 nf <- layout(matrix(c(2,0,1,3),2,2,byrow=TRUE), c(3,1), c(1,3), TRUE); par(mar=c(3,3,1,1))
4 plot(x, y, xlab="", ylab="", col="blue",pch=19,cex=4)
5 points(x,-.01+rep(min(y),length(x)), xlab="", ylab="",pch=18,cex=8)
6 points(-.01+rep(min(x),length(y)), y, xlab="", ylab="",pch=18,cex=8); par(mar=c(0,3,1,1))
7 barplot(xhist$counts,axes=F,ylim=c(0,top),space=0,col="purple");par(mar=c(3,0,1,1))
8 barplot(yhist$counts, axes=F, xlim=c(0, top), space=0, horiz=TRUE,col="purple")
9 joint_hist(norm);joint_hist(stud);joint_hist(gumb)

```

Listing 4.5: **R** code - Copulas with histograms.

```

1 import seaborn as sns, numpy as np, matplotlib.pyplot as plt;
2 from copulae import GaussianCopula, StudentCopula, GumbelCopula;
3 from scipy.stats import norm;
4 exec("""joint_hist = lambda u: (x := u[:, 0], y := u[:, 1], fig := plt.figure(figsize=(10, 10)), gs :=
5 plt.GridSpec(2, 2, width_ratios=[3, 1], height_ratios=[1, 3], wspace=.05, hspace=.05),
6 ax_scatter := fig.add_subplot(gs[1, 0]), ax_scatter.scatter(x, y, color='b', s=1, alpha=.6), x_r
7 := x.max() - x.min(), y_r := y.max() - y.min(), ax_scatter.set_xlim(x.min() - .05 * x_r,
8 x.max() + .05 * x_r), ax_scatter.set_ylim(y.min() - .05 * y_r, y.max() + .05 * y_r),
9 ax_scatter.set_xlabel('x'), ax_scatter.set_ylabel('y'), ax_scatter.scatter(x, np.full_like(x,
10 y.min() - .01 * y_r), color='k', s=8, alpha=.5, marker='o'), ax_scatter.scatter(np.full_like(y,
11 x.min() - .01 * x_r), y, color='k', s=8, alpha=.5, marker='o'), ax_histx :=
12 fig.add_subplot(gs[0, 0], sharex=ax_scatter), ax_histx.hist(x, bins=40, color='m', alpha=.7,
13 ec='k'), ax_histx.set_ylim(0, max(np.histogram(x, bins=40)[0].max(), np.histogram(y,
14 bins=40)[0].max())), ax_histx.axis('off'), ax_histy := fig.add_subplot(gs[1, 1],
15 sharey=ax_scatter), ax_histy.hist(y, bins=40, orientation='horizontal', color='m', alpha=.7,
16 ec='k'), ax_histy.set_xlim(0, max(np.histogram(x, bins=40)[0].max(), np.histogram(y,
17 bins=40)[0].max())), ax_histy.axis('off'), plt.tight_layout(), plt.show())""")
18 joint_hist(norm_s); joint_hist(st_s); joint_hist(gb_s)

```

Listing 4.6: Python code - Copulas with histograms.

Gaussian marginals with given copulas

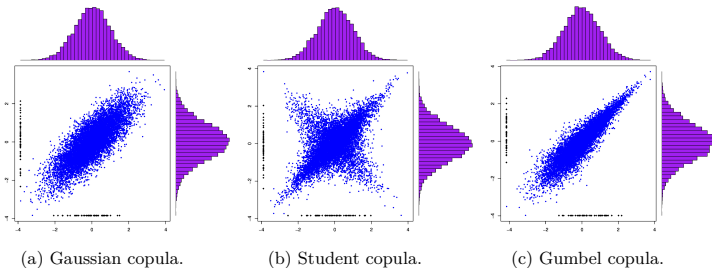


Fig. 4.7: Samples with Gaussian marginals and given copulas.

The next **R** code generates random samples according to the Gaussian, Student, and Gumbel copulas with Gaussian marginals, as illustrated in Figure 4.7.

```

1 set.seed(100);N=10000;m=c("norm","norm");p=list(list(mean=0,sd=1),list(mean=0,sd=1))
2 gaussM<-mvdC(normalCopula(8),m,p);norm<-rMvdc(N,gaussM)
3 studentM<-mvdC(tCopula(.5,df=1),m,p);stud<-rMvdc(N,studentM)
4 gumbelM<-mvdC(gumbelCopula(4),m,p);gmb<-rMvdc(N,gumbelM)
5 plot(norm,pch='.',cex=3,col='blue');points(norm[,1],-.01+rep(min(norm[,2]),N),pch=18,cex=.8);
6 points(-.01+rep(min(norm[,1]),N),norm[,2],pch=18,cex=.8);joint_hist(norm)
7 plot(stud,pch='.',cex=3,col='blue');points(stud[,1],-.01+rep(min(stud[,2]),N),pch=18,cex=.8);
8 points(-.01+rep(min(stud[,1]),N),stud[,2],pch=18,cex=.8);joint_hist(stud)
9 plot(gmb,pch='.',cex=3,col='blue');points(gmb[,1],-.01+rep(min(gmb[,2]),N),pch=18,cex=.8);
10 points(-.01+rep(min(gmb[,1]),N),gmb[,2],pch=18,cex=.8);joint_hist(gmb)

```

Listing 4.7: **R** code - Copulas with Gaussian marginals.

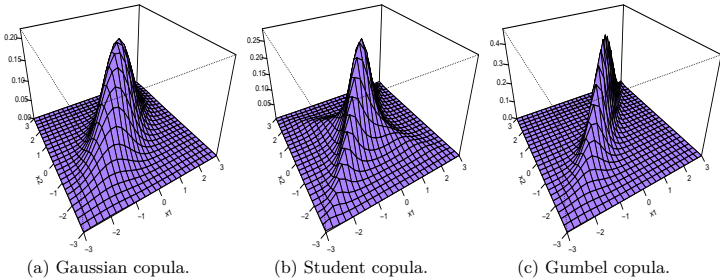


Fig. 4.8: Joint densities with Gaussian marginals and given copulas.

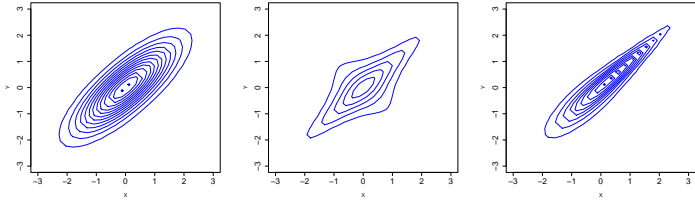
```

1 import numpy as np, matplotlib.pyplot as plt, seaborn as sns;from scipy.stats import norm
2 from copulae import GaussianCopula, StudentCopula, GumbelCopula
3 exec("""plot_s= lambda data, title: (plt.figure(figsize=(8, 8)), x := data[:, 0], y := data[:, 1],
4 plt.scatter(x, y, c='b', s=1, alpha=.6), x_r := x.max() - x.min(), y_r := y.max() - y.min(),
5 plt.scatter(x, np.full_like(x, x.min() - .01*y_r), c='k', s=8, alpha=.5, marker='o'),
6 plt.scatter(np.full_like(y, x.min() - .01*x_r), y, c='k', s=8, alpha=.5, marker='o'),
7 plt.title(title), plt.xlabel('x'), plt.ylabel('y'), plt.grid(True, alpha=.3), plt.show())""")
8 np.random.seed(100);N=10000;gs_c=GaussianCopula(dim=2);gs_cf:=np.array([[1.,.8],[.8,1.]])
9 gs_u=gs_c.random(N);st_c=StudentCopula(dim=2, df=1);st_cf:=np.array([[1.,.5],[.5,1.]])
10 st_u=st_c.random(N);gb_cop=GumbelCopula(dim=2,theta=4);gb_u=gb_cop.random(N)
11 norm_s= norm.ppf(gs_u); st_s= norm.ppf(st_u); gb_s= norm.ppf(gb_u)
12 plot_s(norm_s,"Gaussian Copula (rho=.8) with N(0,1) Marginals"); joint_hist(norm_s)
13 plot_s(st_s,"Student-t Copula (rho=.5, df=1) with N(0,1) Marginals"); joint_hist(st_s)
14 plot_s(gb_s,"Gumbel Copula (theta=4) with N(0,1) Marginals"); joint_hist(gb_s)

```

Listing 4.8: Python code - Copulas with Gaussian marginals.

Figure 4.8 plots joint densities with Gaussian marginals and given copulas.



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.9: Joint density contour plots with Gaussian marginals and given copulas.

Figure 4.9 shows countour plots with Gaussian marginals and given copulas.

Exponential marginals with given copulas

The following **R** code generates random samples with exponential marginals according to the Gaussian, Student, and Gumbel copulas as illustrated in Figure 4.10.

```

1 library(copula);set.seed(100);N=4000;m=c("exp","exp");p=list(list(rate=1),list(rate=1))
2 gaussM<-mvdc(normalCopula(.7),m,p);studentM<-mvdc(tCopula(.5,df=1),m,p);
3 gumbelM<-mvdc(gumbelCopula(4),m,p); norm<-rMvdc(N,gaussM);
4 stud<-rMvdc(N,studentM);gumb<-rMvdc(N,gumbelM); plot(norm,pch='.',cex=3,col='blue');
5 plot(stud,pch='.',cex=3,col='blue'); plot(gumb,pch='.',cex=3,col='blue');
6 persp(gaussM,dMvdc,0:1,0:1,col='mediumpurple1')
7 persp(studentM,dMvdc,0:1,0:1,col='mediumpurple1');
8 persp(gumbelM,dMvdc,0:1,0:1,col='mediumpurple1')
9 contour(gaussM,dMvdc,0:1,0:1,nlevels=10,xlab="X",ylab="Y",cex.axis=1.5); joint_hist(norm)
10 contour(studentM,dMvdc,0:1,0:1,nlevels=10,xlab="X",ylab="Y",cex.axis=1.5); joint_hist(stud)
11 contour(gumbelM,dMvdc,0:1,0:1,nlevels=10,xlab="X",ylab="Y",cex.axis=1.5); joint_hist(gumb)

```

Listing 4.9: **R** code - Copulas with exponential marginals.

```

1 import seaborn as sns, numpy as np, matplotlib.pyplot as plt;
2 from copulae import GaussianCopula, StudentCopula, GumbelCopula;
3 from scipy.stats import norm; np.random.seed(100); from scipy.stats import expon; N= 4000
4 gs_c= GaussianCopula(dim=2); gs_c.:= [[1,.7], [.7,1]]; u_g= gs_c.random(N)
5 st_c= StudentCopula(dim=2, df=1); st_c.:= [[1,.5], [.5,1]]; u_stud= st_c.random(N)
6 gb_cop= GumbelCopula(dim=2, theta=4); u_g= gb_cop.random(N)
7 norm_s= expon(scale=1).ppf(u_g); st_s= expon(scale=1).ppf(u_stud)
8 gb_s=expon(scale=1).ppf(u_g); joint_hist(norm_s);joint_hist(st_s);joint_hist(gb_s)

```

Listing 4.10: Python code - Copulas with exponential marginals.

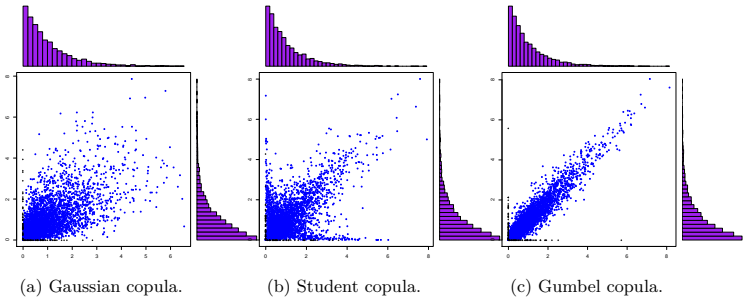


Fig. 4.10: Samples with exponential marginals and given copulas.

Exercises

Exercise 4.1 Copulas. In what follows, U denotes a uniformly distributed $[0, 1]$ -valued random variable.

- a) To which couple (U, V) of uniformly distributed $[0, 1]$ -valued random variables does the copula function

$$\bar{C}(u, v) = \min(u, v), \quad 0 \leq u, v \leq 1,$$

correspond?

- b) Show that the function

$$\underline{C}(u, v) := (u + v - 1)^+, \quad 0 \leq u, v \leq 1,$$

is the copula on $[0, 1] \times [0, 1]$ corresponding to $(U, V) = (U, 1 - U)$.

- c) Show that for any copula function $C(u, v)$ on $[0, 1] \times [0, 1]$ we have

$$C(u, v) \leq \bar{C}(u, v), \quad 0 \leq u, v \leq 1. \quad (4.9)$$

- d) Show that for any copula function $C(u, v)$ on $[0, 1] \times [0, 1]$ we also have

$$\underline{C}(u, v) \leq C(u, v), \quad 0 \leq u, v \leq 1. \quad (4.10)$$

Hint: For fixed $v \in [0, 1]$, let $h(u) := C(u, v) - (u + v - 1)$ and show that $h(1) = 0$ and $h'(u) \leq 0$.

Exercise 4.2 Consider two Bernoulli random variables X and Y , with $p_X = \mathbb{P}(X = 1)$, $p_Y = \mathbb{P}(Y = 1)$, correlation coefficient $\rho \in [-1, 1]$, and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X p_Y + \rho \sqrt{p_X p_Y (1 - p_X)(1 - p_Y)}, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = (1 - p_X) p_Y - \rho \sqrt{p_X p_Y (1 - p_X)(1 - p_Y)}, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X (1 - p_Y) - \rho \sqrt{p_X p_Y (1 - p_X)(1 - p_Y)}, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = (1 - p_X)(1 - p_Y) + \rho \sqrt{p_X p_Y (1 - p_X)(1 - p_Y)}. \end{cases}$$

a) Is it possible to have $\rho = 1$ *without* having $p_X = p_Y$ and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X = p_Y, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = 0, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = 1 - p_X = 1 - p_Y ? \end{cases}$$

b) Similarly, is it possible to have $\rho = 1$ *without* having $p_X = 1 - p_Y$ and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = 0, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = p_Y, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = 0 ? \end{cases}$$

Exercise 4.3 Exponential copulas. Consider the random vector (X, Y) of nonnegative random variables, whose joint distribution is given by the survival function

$$\mathbb{P}(X \geq x \text{ and } Y \geq y) := e^{-\lambda x - \mu y - \nu \text{Max}(x, y)}, \quad x, y \in \mathbb{R}_+,$$

where $\lambda, \mu, \nu > 0$.

- Find the marginal distributions of X and Y .
- Find the joint cumulative distribution function

$$F(x, y) := \mathbb{P}(X \leq x \text{ and } Y \leq y)$$

of (X, Y) .

- Construct an “exponential copula” based on the joint cumulative distribution function of (X, Y) .

Exercise 4.4 Gumbel bivariate logistic distribution. Consider the random vector (X, Y) of nonnegative random variables, whose joint distribution is given by the joint cumulative distribution function (CDF)

$$F_{(X,Y)}(x, y) := \mathbb{P}(X \leq x \text{ and } Y \leq y) := \frac{1}{1 + e^{-x} + e^{-y}}, \quad x, y \in \mathbb{R}.$$

- Find the marginal distributions of X and Y .
- Construct the copula based on the joint CDF of (X, Y) .

Exercise 4.5 Consider the random vector (X, Y) with the joint probability density function

$$\tilde{f}(x, y) := \frac{1}{\pi\sigma\eta} \mathbb{1}_{\mathbb{R}_-^2}(x, y) e^{-x^2/(2\sigma^2) - y^2/(2\eta^2)} + \frac{1}{\pi\sigma\eta} \mathbb{1}_{\mathbb{R}_+^2}(x, y) e^{-x^2/(2\sigma^2) - y^2/(2\eta^2)},$$

plotted as a heat map in Figure 4.11b.

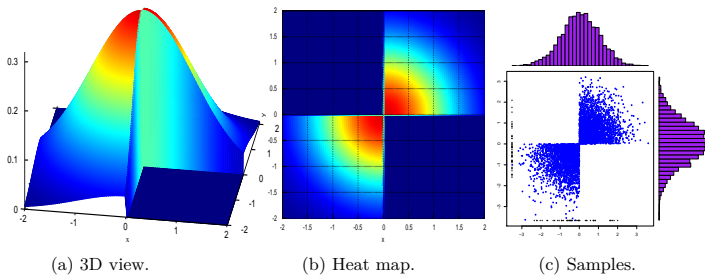



Fig. 4.11: Truncated two-dimensional Gaussian density.

```

1 library(MASS); Sigma <- matrix(c(1,0,0,1),2,2);N=10000
2 u<-mvrnorm(N,rep(0,2),Sigma);j=1;for (i in 1:N){
3   if (u[i,1]>0 &&& u[i,2]>0) {j<-j+1;}
4   if (u[i,1]<0 &&& u[i,2]<0) {j<-j+1;}
5   v<-matrix(nrow=j-1, ncol=2);j=1;for (i in 1:N){
6     if (u[i,1]>0 &&& u[i,2]>0) {v[j,]=u[i,];j<-j+1;}
7     if (u[i,1]<0 &&& u[i,2]<0) {v[j,]=u[i,];j<-j+1;}
8   }
9   joint_hist(v) # Function defined the previous section

```

Listing 4.11:  code - Truncated Gaussian density.

```

1 import numpy as np, matplotlib.pyplot as plt; from scipy.stats import multivariate_normal
2 np.random.seed(100);N=10000;Sigma=np.array([[1,0],[0,1]])
3 u=multivariate_normal.rvs(mean=[0,0],cov=Sigma,size=N)
4 v= u[(u[:, 0] > 0) & (u[:, 1] > 0)] | ((u[:, 0] < 0) & (u[:, 1] < 0)); joint_hist(v)

```

Listing 4.12: Python code - Truncated Gaussian density.

- Show that (X, Y) has the Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$.
- Does the couple (X, Y) have the bivariate Gaussian distribution with probability density function $f_{\Sigma}(x, y)$, where Σ is the covariance matrix (4.2) of (X, Y) ?
- Show that the random variable $X + Y$ is not Gaussian (take $\sigma = \eta = 1$ for simplicity).
- Show that under the rotation

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

of angle $\theta \in [0, 2\pi]$ the random vector $(X \cos \theta - Y \sin \theta, X \sin \theta + Y \cos \theta)$ can have an arbitrary covariance depending on the value of $\theta \in [0, 2\pi]$.

- Find the probability distribution of $X^2 + Y^2$ (take $\sigma = \eta = 1$ for simplicity).

Exercise 4.6 Let τ_1, τ_2 and τ denote three independent exponentially distributed random times with respective parameters $\lambda_1, \lambda_2, \lambda > 0$. Consider two firms with respective default times $\tau_1 \wedge \tau = \min(\tau_1, \tau)$ and $\tau_2 \wedge \tau = \min(\tau_2, \tau)$, where τ represents the time of a macro-economic shock.

- Find the tail (or survival) distribution functions of $\tau_1 \wedge \tau$ and $\tau_2 \wedge \tau$.
- Compute the joint survival probability

$$\mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t), \quad s, t \geq 0.$$

Hint: Use the relation

$$\text{Max}(s, t) = s + t - \min(s, t), \quad s, t \geq 0.$$

- Compute the joint cumulative distribution function

$$\mathbb{P}(\tau_1 \wedge \tau \leq s \text{ and } \tau_2 \wedge \tau \leq t), \quad s, t \geq 0.$$

- Compute the resulting copula function

$$C(u, v) := F_{X, Y}(F_X^{-1}(u), F_Y^{-1}(v)), \quad 0 \leq u, v \leq 1.$$

- Compute the resulting copula density function $\frac{\partial^2 C}{\partial u \partial v}(u, v)$, $u, v \in [0, 1]$.