Chapter 10
Credit Derivatives

Credit derivatives are option contracts that provide protection against default risk in a creditor-debtor relationship, by transferring risk to a third party. This chapter reviews the construction and properties of key credit derivatives, such as Credit Default Swaps (CDSs) and Collateralized Debt Obligations (CDOs). We also discuss counterparty default risk through the computation of Credit Valuation Adjustments (CVAs).

10.1 Credit Default Swaps (CDS)

In this chapter, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$ that represents a sequence of possible payment dates. We also let τ be a default time, and given a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we consider the enlarged filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ given by $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau)$, $t \geqslant 0$, which contains the additional information given by τ, see Definition 9.3.

Credit Default Swap (CDS) provide protection purchased at time t against potential default occurring at times T_k, $k = i+1, \ldots, j$, by making a fixed payment $S_t^{i,j}$ (the premium leg) at times $T_{i+1}, \ldots, T_j$. On the other hand, the issuer of the contract makes a compensation payment $1 - \xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, where ξ_{k+1} is the recovery rate associated with the maturity T_{k+1}, $k = i, \ldots, j-1$.

Definition 10.1. *A Credit Default Swap (CDS) is a contract consisting in:*

- A premium leg*: the buyer is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, and has to make fixed spread payments $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation.*
- A protection leg*: the seller or issuer of the contract makes a compensation payment $1 - \xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$.*

Detailed information on the notional amounts outstanding of credit default swap (CDS) contracts can be obtained from the Bank for International Settlements. We note in particular that the outstanding notional amount of CDS contracts has decreased from its historical high of \$61 trillion at year-end 2007 to \$9 trillion at year-end 2024.

In what follows, according to Lemma 9.4 and Proposition 9.5 we let

$$\begin{aligned} P(t, T_k) &:= \mathbb{E}\left[\mathbb{1}_{\{\tau > T_k\}} \exp\left(- \int_t^{T_k} r_s ds \right) \Big| \mathcal{G}_t \right] \\ &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[\exp\left(- \int_t^{T_k} (r_s + \lambda_s) ds \right) \Big| \mathcal{F}_t \right], \quad 0 \leqslant t \leqslant T_k, \end{aligned}$$

denote the zero-recovery default bond price with maturity T_k, $k = i, \ldots, j-1$, and we let

$$P(t, T_i, T_j) := \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1})$$

be the (default) annuity numéraire, cf. *e.g.* Relation (19.27) in Privault (2022).

Proposition 10.2. *The discounted value $V^{\mathrm{P}}(t, T)$ at time t of the premium leg is given by*

$$V^{\mathrm{P}}(t, T) = S_t^{i,j} P(t, T_i, T_j), \tag{10.1}$$

where $\delta_k := T_{k+1} - T_k$, $k = i, \ldots, j-1$.

Proof. We have

$$\begin{aligned} V^{\mathrm{P}}(t, T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right] \\ &= \sum_{k=i}^{j-1} S_t^{i,j} \delta_k \mathbb{E}\left[\mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right] \\ &= S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1}) \end{aligned}$$

$$= S_t^{i,j} P(t, T_i, T_j).$$

□

For simplicity, the above proof does not take into account a possible accrual interest term over the time interval $[T_k, \tau]$ when $\tau \in [T_k, T_{k+1}]$ in the value of the premium leg.

Proposition 10.3. *The value $V^{\mathrm{D}}(t,T)$ at time t of the protection leg is given by*

$$V^{\mathrm{D}}(t,T) = \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right], \tag{10.2}$$

$k = i, \ldots, j-1$.

From Lemma 9.4 and (10.4), we have the following corollary.

Corollary 10.4. *The value $V^{\mathrm{D}}(t,T)$ at time t of the protection leg is given by*

$$V^{\mathrm{D}}(t,T) = \mathbb{1}_{\{\tau > t\}} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(- \int_t^{T_k} \lambda_s ds \right) - \exp\left(- \int_t^{T_{k+1}} \lambda_s ds \right) \right) \right.$$
$$\left. \times \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{F}_t \right].$$

Proof. From Lemma 9.4 and (10.4), we have

$$\begin{aligned}
V^{\mathrm{D}}(t,T) &= \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right] \\
&= \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k < \tau\}} - \mathbb{1}_{\{T_{k+1} < \tau\}})(1-\xi_{k+1}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{G}_t \right] \\
&= \mathbb{1}_{\{\tau > t\}} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(- \int_t^{T_k} \lambda_s ds \right) - \exp\left(- \int_t^{T_{k+1}} \lambda_s ds \right) \right) \right. \\
&\qquad \left. \times \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \ \Big| \ \mathcal{F}_t \right].
\end{aligned}$$

□

In order to make the deal fair to both parties, the CDS contract is priced in terms of the swap rate $S_t^{i,j}$ (or spread) computed by equating the values $V^{\mathrm{D}}(t,T)$ and $V^{\mathrm{P}}(t,T)$ of the premium (10.1) and protection (10.2) legs, as

$$V^{\mathrm{P}}(t,T) = V^{\mathrm{D}}(t,T), \tag{10.3}$$

i.e. from the relation

$$\begin{aligned} V^{\mathrm{P}}(t,T) &= S_t^{i,j} P(t,T_i,T_j) \\ &= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{G}_t \right] \\ &= V^{\mathrm{D}}(t,T), \end{aligned}$$

which yields

$$S_t^{i,j} = \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{G}_t \right] \tag{10.4}$$

$$\begin{aligned} &= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}})(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{G}_t \right] \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(-\int_t^{T_k} \lambda_s ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \right) \right. \\ &\qquad \left. \times \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Bigm| \mathcal{F}_t \right]. \end{aligned}$$

The spread $S_t^{i,j}$, which is quoted in basis points per year and paid at regular time intervals, gives protection against defaults on payments of \$1. For a notional amount N the premium payment will become $N \times S_t^{i,j}$.

Fig. 10.1: CDS price evolution on Credit Suisse during year 2023.

As seen in Figure 10.1, in late 2022, Credit Suisse's credit default swap (CDS) spreads climbed sharply as markets grew increasingly worried about the bank's financial health.

In the case of a non-random recovery rate ξ_k, the value at time $t \in [0,T]$ of the protection leg becomes

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right].$$

In particular, in the case of a constant recovery rate ξ, we find

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right],$$

and if τ and t are constrained to take values in the tenor structure $\{T_i,\ldots,T_j\}$, we get

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)}\mathbb{E}\left[\mathbb{1}_{(t,T_j]}(\tau)\exp\left(-\int_t^{\tau} r_s ds\right) \Bigm| \mathcal{G}_t\right].$$

Estimating a deterministic failure rate

In case the rates $r(s)$, $\lambda(s)$ and the recovery rate ξ_{k+1} are deterministic, the spread $S_t^{i,j}$ satisfies

$$\begin{aligned}S_t^{i,j}P(t,T_i,T_j) = \mathbb{1}_{\{\tau>t\}}&\sum_{k=i}^{j-1}(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right)\\ &\times\left(\exp\left(-\int_t^{T_k}\lambda(s)ds\right)-\exp\left(-\int_t^{T_{k+1}}\lambda(s)ds\right)\right).\end{aligned}$$

Given that

$$P(t,T_i,T_j) = \sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}), \qquad T_i \leqslant t \leqslant T_{i+1},$$

we can write

$$\begin{aligned}&S_t^{i,j}\sum_{k=i}^{j-1}(T_{k+1}-T_k)\exp\left(-\int_t^{T_{k+1}}(r(s)+\lambda(s))ds\right)\\ &= \mathbb{1}_{\{\tau>t\}}\sum_{k=i}^{j-1}(1-\xi_k)\exp\left(-\int_t^{T_{k+1}} r(s)ds\right)\left(\exp\left(-\int_t^{T_k}\lambda(s)ds\right)-\exp\left(-\int_t^{T_{k+1}}\lambda(s)ds\right)\right).\end{aligned}$$

In particular, when $r(t)$ and $\lambda(t)$ are written as in (9.7) and assuming that $\xi_k = \xi$ is constant, $k = i, \ldots, j$, we get, with $t = T_i$ and writing $\delta_k = T_{k+1} - T_k$, $k = i, \ldots, j-1$,

$$\begin{aligned} & S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) \\ & = \mathbb{1}_{\{\tau > t\}} (1-\xi) \sum_{k=i}^{j-1} \exp\left(- \sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) \left(\mathrm{e}^{\delta_k \lambda_k} - 1 \right). \end{aligned}$$

Assuming further that $\lambda_k = \lambda$ is constant, $k = i, \ldots, j$, we have

$$\begin{aligned} & S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right) \\ & = (1-\xi) \sum_{k=i}^{j-1} \left(\mathrm{e}^{\lambda \delta_k} - 1 \right) \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right), \end{aligned} \tag{10.5}$$

which can be solved numerically for λ, cf. Sections 4 and 5 of Castellacci (2008) for the JP Morgan model, and Exercises 10.1-10.2.

10.2 Collateralized Debt Obligations (CDO)

A CDO is a structured investment product constructed by splitting the above portfolio into n ordered tranches numbered $i = 1, 2, \ldots, n$, where tranche n°i represents a percentage p_i% of the total portfolio value. We let

$$\alpha_l := p_1 + p_2 + \cdots + p_l, \qquad l = 1, 2, \ldots, n, \tag{10.6}$$

denote the corresponding cumulative percentages, with $\alpha_0 = 0$ and $\alpha_n = p_1 + p_2 + \cdots + p_n = 100\%$.

The tranches are ordered according to decreasing default risk, tranche n°1 being the riskiest one ("equity tranche"), and tranche n°n being the safest ("senior tranche"), while the intermediate tranches are referred to as "mezzanine tranches". In practice, losses occur first to the "equity" tranches, next to the "mezzanine" tranche holders, and finally to "senior" tranches.

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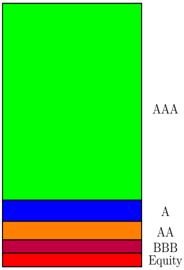

Fig. 10.2: A representation of CDO tranches.

The size of the CDO market was over \$200 billion before the 2008 financial crisis, and subsequently almost vanished. It is currently rebounding, and is expected to reach \$40 billion by 2027.

Consider a portfolio consisting of $N = j - i$ bonds with default times $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, and recovery rates $\xi_k \in [0,1]$, $k = i+1, \ldots, j$.

CDOs can attract different types of investors.

- Unfunded investors (usually in the higher tranches) are receiving premium payments, and are making payments in case of default.
- "Short" investors make premium payments and receive protection in case of default as in a Credit Default Swaps (CDS).
- Funded investors (usually in the lower tranches) are investing in risky bonds to receive principal payments at maturity, they are the first in line to incur losses.

Synthetic CDOs

Synthetic CDOs are based on $N = j - i$ bonds that can potentially generate a cumulative loss

$$L_t := \sum_{k=i}^{j-1} (1 - \xi_{k+1}) \mathbb{1}_{\{\tau_k \leqslant t\}} \in [0, N],$$

at time $t \in [T_i, T_j]$, based on the default time τ_k and recovery rate ξ_{k+1} of each involved bond, $k = i, \ldots, j-1$, with $N = j - i$.

When the first loss occurs, tranche n°1 is the first in line, and it loses the amount

$$L_t^1 = L_t \mathbb{1}_{\{L_t \leqslant p_1 N\}} + N p_1 \mathbb{1}_{\{L_t > p_1 N\}} = N \min(L_t / N, p_1).$$

In case $L_t > p_1 N$, then tranche n°2 takes the remaining loss up to the amount Np_2, that means the loss L_t^2 of tranche n°2 is

$$\begin{aligned} L_t^2 &= (L_t - Np_1)\mathbb{1}_{\{p_1 N < L_t \leqslant (p_1+p_2)N\}} + Np_2 \mathbb{1}_{\{L_t > (p_1+p_2)N\}} \\ &= (L_t - Np_1)\mathbb{1}_{\{p_1 N < L_t \leqslant \alpha_2 N\}} + Np_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\ &= (L_t - Np_1)^+ \mathbb{1}_{\{L_t \leqslant \alpha_2 N\}} + Np_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\ &= \min((L_t - Np_1)^+, Np_2) \\ &= \mathrm{Max}(\min(L_t, Np_1 + Np_2) - Np_1, 0) \\ &= \mathrm{Max}(\min(L_t, N\alpha_2) - Np_1, 0). \end{aligned}$$

By induction, the potential loss taken by tranche n°l is given by

$$\begin{aligned} L_t^l &= (L_t - N\alpha_{l-1})\mathbb{1}_{\{\alpha_{l-1} N < L_t \leqslant \alpha_l N\}} + Np_l \mathbb{1}_{\{L_t > \alpha_l N\}} \\ &= (L_t - N\alpha_{l-1})^+ \mathbb{1}_{\{L_t \leqslant \alpha_l N\}} + Np_l \mathbb{1}_{\{L_t > \alpha_l N\}} \\ &= \min((L_t - N\alpha_{l-1})^+, Np_l) \\ &= \mathrm{Max}(\min(L_t, N\alpha_l) - N\alpha_{l-1}, 0), \end{aligned} \qquad (10.7)$$

where $\alpha_l := p_1 + p_2 + \cdots + p_l$, $l = 1, 2, \ldots, n$.

Eventually, tranche n°n will take the loss

$$L_t^n = (L_t - N\alpha_{n-1})\mathbb{1}_{\{\alpha_{n-1} N < L_t\}} = (L_t - N\alpha_{n-1})^+.$$

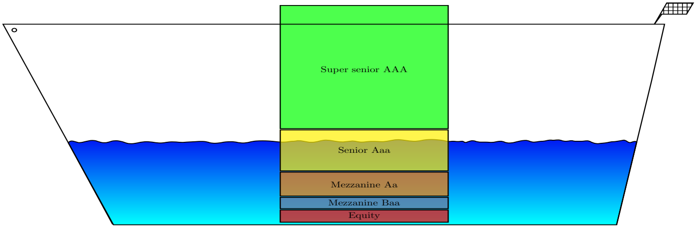

Fig. 10.3: A Titanic-style representation of cumulative tranche losses.

The CDO tranche n°l, $l = 1, 2, \ldots, n$, can be decomposed into:

- A premium leg: the short investor in tranche n°l is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making fixed payments S_t^l at times $T_{i+1}, \ldots, T_j$. Such premia can be received by the unfunded investor.

Payments are quoted as a proportion of the outstanding amount $Np_l - L^l_{T_{k+1}}$ proportionally to the rate S^l_t, and are made at times T_{k+1} until $k = j-1$ or $L_{T_{k+1}} = 100\%$, whichever comes first

Consequently, the discounted value at time t of the premium leg for the tranche n°l is

$$\begin{aligned} V_l^{\mathrm{P}}(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S^l_t \delta_k (Np_l - L^l_{T_{k+1}}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \ \Big| \ \mathcal{G}_t\right] \\ &= S^l_t \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L^l_{T_{k+1}}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \ \Big| \ \mathcal{G}_t\right], \end{aligned} \tag{10.8}$$

for $l = 1, 2, \ldots, N$.

- A protection leg: the short investor receives compensation payments against default, which can be paid by the unfunded investors. Noting that at each default time $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, the loss L^l_t taken by tranche n°l increases by the amount $\Delta L^l_{\tau_k} = L^l_{\tau_k} - L^l_{\tau_k^-}$, the value at time t of the protection leg for tranche n°l can be written as

$$V_l^{\mathrm{D}}(t,T) = \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{[T_i,T_j]}(\tau_k) \Delta L^l_{\tau_k} \exp\left(-\int_t^{\tau_k} r_u du\right) \ \Big| \ \mathcal{G}_t\right]. \tag{10.9}$$

By applying integration by parts on $[T_i, T_j]$ and using the fact that $L^l_{T_i} = 0$, $l = 1, 2, \ldots, n$, the value of the protection leg can be rewritten as

$$\begin{aligned} &V_l^{\mathrm{D}}(t,T) = \mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL^l_s \Big| \mathcal{G}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L^l_{T_j} - \exp\left(-\int_t^{T_i} r_u du\right) L^l_{T_i} \Big| \mathcal{G}_t\right] \\ &\quad + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L^l_s ds \Big| \mathcal{G}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L^l_{T_j} \Big| \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L^l_s ds \Big| \mathcal{G}_t\right]. \end{aligned}$$

The spread S^l_t paid by tranche n°l is computed by equating the values

$$V_l^{\mathrm{P}}(t,T) = V_l^{\mathrm{D}}(t,T)$$

of the protection and premium legs in (10.8) and (10.9), which yields

$$S_t^l = \frac{\mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL_s^l \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]}$$

$$= \frac{\mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]}$$

$$\geqslant 0,$$

$l = 1, 2, \ldots, n.$

Expected tranche loss

The loss of tranche n°l can be written from (10.7) using the *nonlinear* function f_l defined as

$$f_l(x) := \min((x - N\alpha_{l-1})^+, Np_l),$$

see Figure 10.4, as

$$L_t^l = \min((L_t - N\alpha_{l-1})^+, Np_l) = f_l(L_t),$$

where α_{l-1} is defined in (10.6), $l = 1, 2, \ldots, n$.

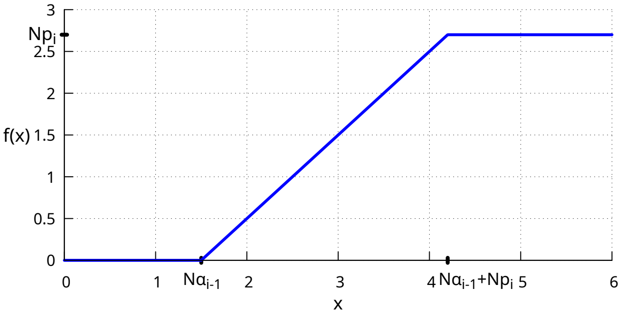

Fig. 10.4: Function $f_l(x) = \min((x - N\alpha_{l-1})^+, Np_l)$.

Hence, the expected loss of tranche n°l, can be written as

$$\mathbb{E}[L_t^l] = \mathbb{E}\left[\min((L_t - N\alpha_{l-1})^+, Np_l)\right] = \mathbb{E}[f_l(L_t)], \quad l = 1, 2, \ldots, n,$$

which can be estimated by the Monte Carlo method when the default times are generated according to (8.11).

On the other hand, in the framework of the multiple default time model of Chapter 8, the expected tranche loss $\mathbb{E}[L_t^l]$ can be computed using (8.8) given the value of the parameter M, as

$$\mathbb{E}[L_t^l] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}\left[f_l(L_t) \mid M = m \right] \mathrm{e}^{-m^2/2} dm, \quad l = 1, 2, \ldots, n.$$

The cumulative loss L_t can be modeled as a discrete random variable, with for example

$$\mathbb{P}\left(L_t = N - \sum_{k=i}^{j-1} \xi_{k+1} \right) = \mathbb{P}(\tau_i \leqslant t, \ldots, \tau_{j-1} \leqslant t),$$

and

$$\mathbb{P}(L_t = 0) = \mathbb{P}(\tau_i > t, \ldots, \tau_{j-1} > t),$$

which requires the knowledge of the joint distribution of the default times $\tau_i, \ldots, \tau_{j-1}$. Alternatively, if the $\tau_k' s$ are identically distributed with common cumulative distribution function

$$F_\tau(t) = \mathbb{P}(\tau_k \leqslant t), \quad k = i, \ldots, j-1,$$

and

$$a_k = a, \quad \xi_k = \xi, \quad k = i+1, \ldots, j,$$

we have

$$\mathbb{P}(\tau \leqslant T \mid M = m) = \Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - am}{\sqrt{1-a^2}} \right), \quad m \in \mathbb{R},$$

from (8.8). If in addition the $\tau_k' s$ are independent, then L_t has the binomial distribution

$$\mathbb{P}(L_t = (1-\xi)k \mid M) = \binom{N}{k} (1 - \mathbb{P}(\tau \leqslant T \mid M))^{N-k} (\mathbb{P}(\tau \leqslant T \mid M))^k$$

$$= \binom{N}{k} \left(1 - \Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}} \right) \right)^{N-k} \left(\Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}} \right) \right)^k,$$

$k = 0, 1, \ldots, N$, conditionally to the value of M.

Cumulative expected tranche loss

In the framework of the multiple default time model of Chapter 8, the expected cumulative loss L_t over all tranches can be computed using (8.8) given the value of the parameter M, as

$$\begin{aligned}\mathbb{E}[L_t \mid M = m] &= \sum_{k=i}^{j-1} \mathbb{E}\big[(1-\xi_{k+1})\mathbb{1}_{\{\tau_k \leqslant t\}} \mid M = m\big] \\ &= \sum_{k=i}^{j-1}(1-\xi_{k+1})\mathbb{P}\left(\tau_k \leqslant t \mid M = m\right) \\ &= \sum_{k=i}^{j-1}(1-\xi_{k+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant t)) + a_k m}{\sqrt{1-a_k^2}}\right).\end{aligned}$$

From this formula, the α-percentile loss of the portfolio can be estimated as

$$\mathbb{E}[L_t \mid M = m] = \sum_{k=i}^{j-1}(1-\xi_{k+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right),$$

where $m = \Phi^{-1}(\alpha)$, *i.e.* $\alpha = \Phi(m) = \mathbb{P}(M \leqslant m)$.

In addition, when the parameter M is normally distributed, the expected loss $\mathbb{E}[L_t]$ can be written as

$$\begin{aligned}\mathbb{E}[L_t] &= \int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right]\phi(m)dm \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right]\mathrm{e}^{-m^2/2}dm.\end{aligned}$$

Such (Gaussian) Merton (1974) and Vašiček (2002) type models have been implemented in the 2004 Basel II recommendations on Banking Supervision (2005). Namely in Basel II, banks are expected to hold capital in prevision of unexpected losses in a worst case scenario, according to the Internal Ratings-Based (IRB) formula:

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right) - \mathbb{P}(\tau_k \leqslant T)\right), \tag{10.10}$$

which represents the difference between the α-percentile loss

$$\mathbb{E}[L_t \mid M = m] = \sum_{k=i}^{j-1}(1-\xi_{k+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right),$$

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and the unconditional loss

$$\begin{aligned}\mathbb{E}[L_t] &= \sum_{k=i}^{j-1} \mathbb{E}\big[(1-\xi_{k+1})\mathbb{1}_{\{\tau_k \leqslant t\}}\big] \\ &= \sum_{k=i}^{j-1} (1-\xi_{k+1})\mathbb{E}\big[\mathbb{1}_{\{\tau_k \leqslant t\}}\big] \\ &= \sum_{k=i}^{j-1} (1-\xi_{k+1})\mathbb{P}(\tau_k \leqslant t)\end{aligned}$$

obtained when $a = 0$. The confidence level α is typically set at $\alpha = 0.999$ *i.e.* $m = \Phi^{-1}(0.999) = 3.09$, cf. Relation (2.4) page 10 of Aas (2005). Recall that the function

$$x \longmapsto \Phi\left(\frac{\Phi^{-1}(x) + a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right)$$

always lies above the graph of x when $a_k < 0$, as shown in Figure 10.5, hence (10.10) is always positive.

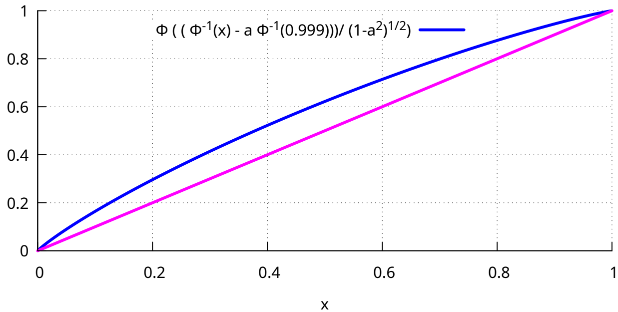

Fig. 10.5: Internal Ratings-Based (IRB) formula.

10.3 Credit Valuation Adjustment (CVA)

Credit Valuation Adjustments (CVA) aim at estimating the amount of capital required in the event of counterparty default, and are specially relevant to the 2010 Basel III regulatory framework. Other credit value adjustments (XVA) include the Funding Valuation Adjustments (FVA), Debit Valuation Adjustments (DVA), Capital Valuation Adjustments (KVA), and Margin Valuation Adjustments (MVA). The purpose of XVAs is also to take into account the future value of trades and their associated risks. The real-time estimation of XVA measures is generally highly demanding from a computational point of view.

Net Present Value (NPV) of a CDS

As above, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$, and let

$$\begin{aligned}
\Pi(t, T_j) &:= \text{protection_leg} - \text{premium_leg} \\
&= \sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&\quad - \sum_{k=l}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&= (1-\xi) \sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&\quad - \sum_{k=l}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&= \sum_{k=l}^{j-1} \left((1-\xi) \mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \right. \\
&\quad \left. - S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \right) \qquad (10.11)
\end{aligned}$$

denote the difference between the remaining protection and premium legs from time T_l until time T_j, discounted to time $t \in [0, T_i]$. We note that from the definition of the spread $S_t^{i,j}$, see (10.3), we have $\mathbb{E}[\Pi(t, T_j) \mid \mathcal{G}_t] = 0$, $0 \leqslant t \leqslant T_i$.

Definition 10.5. *The Net Present Value (NPV) at time T_l of the CDS is the conditional expected value*

$$\text{NPV}(T_l, T_j) := \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{G}_{T_l}\big]$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j, where $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$ is the filtration (9.4) enlarged with the additional information on the default time τ.

The Net Present Value (NPV) at time T_l of the CDS satisfies

$$\begin{aligned}
\mathrm{NPV}(T_l,T_j) &:= \mathbb{E}\big[\Pi(T_l,T_j) \mid \mathcal{G}_{T_l}\big] \\
&= \mathbb{E}\left[\sum_{k=l}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\middle|\, \mathcal{G}_{T_l}\right] \qquad (10.12) \\
&\quad -\mathbb{E}\left[\sum_{k=l}^{j-1} S_t^{i,j}\delta_k \mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\middle|\, \mathcal{G}_{T_l}\right] \\
&= (1-\xi)\sum_{k=l}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\middle|\, \mathcal{G}_t\right] - S_t^{i,j}\sum_{k=l}^{j-1}\delta_k P(t,T_{k+1}) \\
&= \sum_{k=l}^{j-1}\left((1-\xi)\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\middle|\, \mathcal{G}_t\right] - S_t^{i,j}\delta_k P(t,T_{k+1})\right)
\end{aligned}$$

which is the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j.

In addition to the credit default time τ we introduce a second $(\mathcal{G}_t)_{t\geqslant 0}$-stopping time $\nu \in [T_l,T_j]$ representing the possible default time of the party providing the protection leg. The Net Present Value $\mathrm{NPV}(\nu,T_j)$ is estimated when default occurs at time ν.

i) If $\mathrm{NPV}(\nu,T_j) > 0$ then a payment is due from the party providing the protection leg, and only a fraction $\eta\mathrm{NPV}(\nu,T_j)$ of this payment may be recovered, where $\eta \in [0,1]$ is the recovery rate of the party providing protection in the CDS.

ii) On the other hand, if $\mathrm{NPV}(\nu,T_j) < 0$ then the original fee $-\mathrm{NPV}(\nu,T_j)$ is still due.

As a consequence, in the event of default at time $\nu \in [T_l,T_j]$, the net present value of the CDS at time ν is

$$\begin{aligned}
&\eta\mathrm{NPV}(\nu,T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)>0\}} + \mathrm{NPV}(\nu,T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)<0\}} \\
&\qquad = \eta\big(\mathrm{NPV}(\nu,T_j)\big)^+ - \big(\mathrm{NPV}(\nu,T_j)\big)^- \\
&\qquad = \eta\big(\mathrm{NPV}(\nu,T_j)\big)^+ - \big(-\mathrm{NPV}(\nu,T_j)\big)^+ \\
&\qquad = \eta\big(\mathrm{NPV}(\nu,T_j)\big)^+ + \Big(\mathrm{NPV}(\nu,T_j) - \big(\mathrm{NPV}(\nu,T_j)\big)^+\Big) \\
&\qquad = \mathrm{NPV}(\nu,T_j) - (1-\eta)\big(\mathrm{NPV}(\nu,T_j)\big)^+, \qquad (10.13)
\end{aligned}$$

where $x^- := \mathrm{Max}(0,-x)$, $x \in \mathbb{R}$.

Credit Valuation Adjustment (CVA)

Under the event of counterparty default at a time $\nu \in [T_l, T_j]$, we estimate the corresponding discounted payment estimated at time T_l as

$$\begin{aligned}
&\Pi^{\mathrm{D}}(T_l, T_j) = \mathbb{1}_{\{T_j < \nu\}} \Pi(T_l, T_j) \\
&+ \mathbb{1}_{\{T_l < \nu \leqslant T_j\}} \left(\Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \left(\eta \big(\mathrm{NPV}(\nu, T_j) \big)^+ - \big(- \mathrm{NPV}(\nu, T_j) \big)^+ \right) \right) \\
&= \Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \left(\eta \big(\mathrm{NPV}(\nu, T_j) \big)^+ - \big(- \mathrm{NPV}(\nu, T_j) \big)^+ \right) \\
&= \Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \left(\mathrm{NPV}(\nu, T_j) - (1-\eta) \big(\mathrm{NPV}(\nu, T_j) \big)^+ \right),
\end{aligned}$$

see Brigo and Masetti (2006), Brigo and Chourdakis (2009). As a consequence, we derive the following result.

Proposition 10.6. *The price at time T_l of the payoff $\Pi^{\mathrm{D}}(T_l, T_j)$ under counterparty risk is given by*

$$\begin{aligned}
&\mathbb{E}\big[\Pi^{\mathrm{D}}(T_l, T_j) \mid \mathcal{G}_{T_l} \big] = \mathrm{NPV}(T_l, T_j) \\
&\quad -(1-\eta) \mathbb{E} \left[\mathbb{1}_{\{T_l < \nu \leqslant T_j\}} \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \big(\mathrm{NPV}(\nu, T_j) \big)^+ \ \Big| \ \mathcal{G}_{T_l} \right].
\end{aligned}$$

Proof. From (10.11) and Definition 10.5, we note the relation

$$\begin{aligned}
\mathrm{NPV}(T_l, T_j) &= \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{G}_{T_l} \big] \\
&= \mathbb{E} \left[\Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \Pi(\nu, T_j) \, \Big| \, \mathcal{G}_{T_l} \right] \\
&= \mathbb{E} \left[\Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \mathrm{NPV}(\nu, T_j) \, \Big| \, \mathcal{G}_{T_l} \right].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathbb{E}\big[\Pi^{\mathrm{D}}(T_l, T_j) \mid \mathcal{G}_{T_l} \big] = \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{G}_{T_l} \big] \\
&= \mathbb{E} \left[\Pi(T_l, \nu) + \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \left(\mathrm{NPV}(\nu, T_j) - (1-\eta) \big(\mathrm{NPV}(\nu, T_j) \big)^+ \right) \, \Big| \, \mathcal{G}_{T_l} \right] \\
&= \mathbb{E} \left[\Pi(T_l, T_j) - (1-\eta) \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \big(\mathrm{NPV}(\nu, T_j) \big)^+ \, \Big| \, \mathcal{G}_{T_l} \right] \\
&= \mathrm{NPV}(T_l, T_j) - \mathbb{E} \left[(1-\eta) \exp\left(- \int_{T_l}^{\nu} r_s ds \right) \big(\mathrm{NPV}(\nu, T_j) \big)^+ \, \Big| \, \mathcal{G}_{T_l} \right].
\end{aligned}$$

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The quantity

$$(1-\eta)\mathbb{E}\left[\mathbb{1}_{\{T_l<\nu\leqslant T_j\}}\exp\left(-\int_{T_l}^{\nu} r_s ds\right)(\mathrm{NPV}(\nu,T_j))^+ \,\Big|\, \mathcal{G}_{T_l}\right]$$

is called the (positive) Counterparty Risk (CR) Credit Valuation Adjustment (CVA).

Exercises

Exercise 10.1 Show that the equation (10.5) admits a numerical solution $\lambda_0 > 0$.

Exercise 10.2 Credit default swaps. From the market data of Figures 10.6 and 10.7 on McDonald's Corp, estimate the first default rate λ_1 and the associated default probability in the framework of (10.5), cf. also Castellacci (2008).

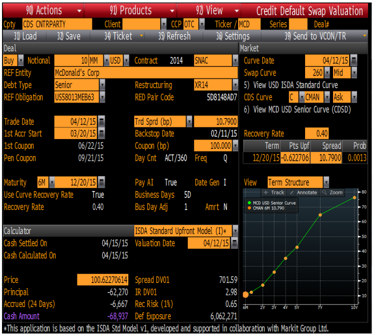

Fig. 10.6: Cashflow data.

Date	Act Cashflow	Disc Factor	Survival Prob	Disc Cashflow
06/22/2015	26,111.11	0.99952277	0.9997	26,089.53
09/21/2015	25,277.78	0.99827639	0.9992	25,213.94
12/21/2015	25,277.78	0.99607821	0.9987	25,146.99
Total	76,666.67			76,450.46

Fig. 10.7: Discount factors data.

Exercise 10.3 Consider

i) a tenor structure $\{t = T_i < \cdots < T_j = T\}$,
ii) a sequence

$$P(t, T_k) = \exp\left(-\int_t^{T_k} r(s)ds\right) = \mathrm{e}^{-(T_k - t)r_k}, \qquad k = i, \ldots, j,$$

of *deterministic* discount factors, and
iii) a family

$$Q(t, T_k) = \mathbb{E}\left[\exp\left(-\int_t^{T_k} \lambda_s ds\right) \,\Big|\, \mathcal{F}_t\right]$$

of survival probabilities.

a) Show that the discounted value at time t of the protection leg equals

$$\sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi)\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \,\Big|\, \mathcal{G}_t\right]$$
$$= \mathbb{1}_{\{\tau > t\}}(1-\xi)\sum_{k=i}^{j-1} P(t, T_{k+1})\left(Q(t, T_k) - Q(t, T_{k+1})\right).$$

b) Letting $\delta_k := T_{k+1} - T_k$, $k = i, \ldots, j-1$, show that the discounted value at time t of the premium leg, equals

$$V^{\mathrm{P}}(t,T) = \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}).$$

c) By equating the protection and premium legs, find the value of $Q(t,T_{i+1})$ with $Q(t,T_i) = 1$, and derive a recurrence relation between $Q(t,T_{j+1})$ and $Q(t,T_i),\ldots,Q(t,T_j)$.

Exercise 10.4 (Exercise 10.3 continued). From the spread data data on the Coca-Cola Company in Table 10.1 and Figure 10.8, estimate the corresponding survival probabilities $Q(t,T_i),\ldots,Q(t,T_n)$ using discount factors $P(t,T_i),\ldots,P(t,T_n)$ of your choice.

k	Maturity	T_k	$S_t^{1,k}$ (bp)
1	6M	0.5	10.97
2	1Y	1	12.25
3	2Y	2	14.32
4	3Y	3	19.91
5	4Y	4	26.48
6	5Y	5	33.29
7	7Y	7	52.91
8	10Y	10	71.91

Table 10.1: Spread market data.

Fig. 10.8: CDS Market data.