Chapter 11
Credit Derivatives

Credit derivatives are option contracts that can be used as a protection against default risk in a creditor/debtor relationship, by transferring risk to a third party. This chapter reviews the construction and properties of several credit derivatives such as Collateralized Debt Obligations (CDOs) and Credit Default Swaps (CDSs). We also address the issue of counterparty default risk via the computation of Credit Valuation Adjustments (CVAs).

11.1 Credit Default Swaps (CDS)

Detailed information on the status of credit default swap (CDS) contracts can be obtained from the Bank for International Settlements. We note in particular that the outstanding notional amount of CDS contracts has decreased from its historical high of \$61.2 trillion at year-end 2007 to \$7.6 trillion at year-end 2019.

In this chapter, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$ that represents a sequence of possible payment dates. We also let τ be a default time, and given a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we consider the enlarged filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ given by $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau)$, $t \geqslant 0$, which contains the additional information given by τ, see Definition 10.2.

Definition 11.1. *A Credit Default Swap (CDS) is a contract consisting in*

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- A premium leg*: the buyer is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, and has to make a fixed spread payment $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation.*
- A protection leg*: the seller or issuer of the contract makes a compensation payment $1 - \xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$, where ξ_{k+1} is the* recovery rate.

In the sequel, we let

$$P(t, T_k) := \mathbb{1}_{\{\tau > t\}} \mathbb{E}\left[\exp\left(-\int_t^{T_k} (r_s + \lambda_s) ds\right) \Big| \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T_k,$$

denote the default bond price with maturity T_k, $k = i, \ldots, j-1$, see Lemma 10.3 and Proposition 10.4.

Proposition 11.2. *The discounted value at time t of the premium leg is given by*

$$V^p(t, T) = S_t^{i,j} P(t, T_i, T_j), \tag{11.1}$$

where $\delta_k := T_{k+1} - T_k$, $k = i, \ldots, j-1$, and

$$P(t, T_i, T_j) := \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1})$$

is the (default) annuity numéraire, cf. e.g. *Relation (19.27) in Privault (2022).*

Proof. We have

$$\begin{aligned}
V^p(t, T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Big| \mathcal{G}_t\right] \\
&= \sum_{k=i}^{j-1} S_t^{i,j} \delta_k \mathbb{E}\left[\mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Big| \mathcal{G}_t\right] \\
&= S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t, T_{k+1}) \\
&= S_t^{i,j} P(t, T_i, T_j).
\end{aligned}$$

□

For simplicity, in the above proof we have ignored a possible accrual interest term over the time interval $[T_k, \tau]$ when $\tau \in [T_k, T_{k+1}]$ in the above value of the premium leg. Similarly, we have the following result.

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Proposition 11.3. *The value at time t of the protection leg is given by*

$$V^d(t,T) := \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right], \tag{11.2}$$

where ξ_{k+1} is the recovery rate associated with the maturity T_{k+1}, $k = i,\ldots,j-1$.

In the case of a non-random recovery rate ξ_k, the value of the protection leg becomes

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right].$$

The spread $S_t^{i,j}$ is computed by equating the values of the premium (11.1) and protection (11.2) legs as $V^p(t,T) = V^d(t,T)$, *i.e.* from the relation

$$\begin{aligned}
V^p(t,T) &= S_t^{i,j} P(t,T_i,T_j) \\
&= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right] \\
&= V^d(t,T),
\end{aligned}$$

which yields

$$S_t^{i,j} = \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right]. \tag{11.3}$$

The spread $S_t^{i,j}$, which is quoted in basis points per year and paid at regular time intervals, gives protection against defaults on payments of \$1. For a notional amount N the premium payment will become $N \times S_t^{i,j}$.

Fig. 11.1: CDS price evolution on Credit Suisse, 2023.

In the case of a constant recovery rate ξ, we find

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t \right],$$

and if τ is constrained to take values in the tenor structure $\{t = T_i, \ldots, T_j\}$, we get

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)} \mathbb{E}\left[\mathbb{1}_{(t,T]}(\tau) \exp\left(-\int_t^{\tau} r_s ds\right) \Bigm| \mathcal{G}_t \right].$$

The buyer of a Credit Default Swap (CDS) is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making a fixed payment $S_t^{i,j}$ (the premium leg) at times $T_{i+1}, \ldots, T_j$. On the other hand, the issuer of the contract makes a payment $1 - \xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$.

The contract is priced in terms of the swap rate $S_t^{i,j}$ (or spread) computed by equating the values $V^d(t,T)$ and $V^p(t,T)$ of the protection and premium legs, and acts as a compensation that makes the deal fair to both parties. Recall that from (11.3) and Lemma 10.3, we have

$$\begin{aligned} S_t^{i,j} &= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t \right] \\ &= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}})(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t \right] \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(-\int_t^{T_k} \lambda_s ds\right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds\right)\right) \right. \\ &\qquad \left. \times \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{F}_t \right]. \end{aligned}$$

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Estimating a deterministic failure rate

In case the rates $r(s)$, $\lambda(s)$ and the recovery rate ξ_{k+1} are deterministic, the above spread can be written as

$$S_t^{i,j} P(t, T_i, T_j) = \mathbb{1}_{\{\tau > t\}} \sum_{k=i}^{j-1} (1 - \xi_k) \exp\left(- \int_t^{T_{k+1}} r(s) ds \right) \times \left(\exp\left(- \int_t^{T_k} \lambda_s ds \right) - \exp\left(- \int_t^{T_{k+1}} \lambda_s ds \right) \right).$$

Given that

$$P(t, T_i, T_j) = \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}), \qquad T_i \leqslant t \leqslant T_{i+1},$$

we can write

$$S_t^{i,j} \sum_{k=i}^{j-1} (T_{k+1} - T_k) \exp\left(- \int_t^{T_{k+1}} (r(s) + \lambda(s)) ds \right)$$
$$= \mathbb{1}_{\{\tau > t\}} \sum_{k=i}^{j-1} (1 - \xi_k) \exp\left(- \int_t^{T_{k+1}} r(s) ds \right) \left(\exp\left(- \int_t^{T_k} \lambda(s) ds \right) - \exp\left(- \int_t^{T_{k+1}} \lambda(s) ds \right) \right).$$

In particular, when $r(t)$ and $\lambda(t)$ are written as in (10.7) and assuming that $\xi_k = \xi$ is constant, $k = i, \ldots, j$, we get, with $t = T_i$ and writing $\delta_k = T_{k+1} - T_k$, $k = i, \ldots, j-1$,

$$S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right)$$
$$= \mathbb{1}_{\{\tau > t\}} (1 - \xi) \sum_{k=i}^{j-1} \exp\left(- \sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) \left(\mathrm{e}^{\delta_k \lambda_k} - 1 \right).$$

Assuming further that $\lambda_k = \lambda$ is constant, $k = i, \ldots, j$, we have

$$S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right) \tag{11.4}$$
$$= (1 - \xi) \sum_{k=i}^{j-1} \left(\mathrm{e}^{-\lambda \delta_k} - 1 \right) \exp\left(- \sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right),$$

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https://personal.ntu.edu.sg/nprivault/indext.html

which can be solved numerically for λ, cf. Sections 4 and 5 of Castellacci (2008) for the JP Morgan model, and Exercises 11.1 and 11.2.

11.2 Collateralized Debt Obligations (CDO)

Consider a portfolio consisting of $N = j - i$ bonds with default times $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, and recovery rates $\xi_k \in [0,1]$, $k = i+1, \ldots, j$.

A synthetic CDO is a structured investment product constructed by splitting the above portfolio into n ordered tranches numbered $i = 1, 2, \ldots, n$, where tranche n°i represents a percentage $p_i\%$ of the total portfolio value. We let

$$\alpha_l := p_1 + p_2 + \cdots + p_l, \qquad l = 1, 2, \ldots, n, \tag{11.5}$$

denote the corresponding cumulative percentages, with $\alpha_0 = 0$ and $\alpha_n = p_1 + p_2 + \cdots + p_n = 100\%$.

The tranches are ordered according to increasing default risk, tranche n°1 being the riskiest one ("equity tranche"), and tranche n°n being the safest one ("senior tranche"), while the intermediate tranches are referred to as "mezzanine tranches". In practice, losses occur first to the "equity" tranches, next to the "mezzanine" tranche holders, and finally to "senior" tranches.

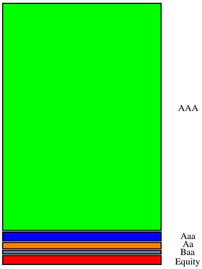

Fig. 11.2: A representation of CDO tranches.

CDOs can attract different types of investors.

- Unfunded investors (usually for the higher tranches) are receiving premiums and make payments in case of default.
- Funded investors (usually in the lower tranches) are investing in risky bonds to receive principal payments at maturity, and they are the first in line to incur losses.

- A CDO can also be used as a Credit Default Swap (CDS) for the "short investors" who make premium payments in exchange for credit protection in case of default.

The market for synthetic CDOs has been significantly reduced since the 2006-2008 subprime crisis.

Synthetic CDOs are based on $N = j - i$ bonds that can potentially generate a cumulative loss

$$L_t := \sum_{l=i}^{j-1} (1 - \xi_{l+1}) \mathbb{1}_{\{\tau_l \leqslant t\}} \in [0, N],$$

at time $t \in [T_i, T_j]$, based on the default time τ_l and recovery rate ξ_{l+1} of each involved CDS, $k = i, \ldots, j-1$, with $N = j - i$.

When the first loss occurs, tranche n°1 is the first in line, and it loses the amount

$$L_t^1 = L_t \mathbb{1}_{\{L_t \leqslant p_1 N\}} + N p_1 \mathbb{1}_{\{L_t > p_1 N\}} = N \min(L_t / N, p_1).$$

In case $L_t > p_1 N$, then tranche n°2 takes the remaining loss up to the amount $N p_2$, that means the loss L_t^2 of tranche n°2 is

$$\begin{aligned}
L_t^2 &= (L_t - N p_1) \mathbb{1}_{\{p_1 N < L_t \leqslant (p_1 + p_2) N\}} + N p_2 \mathbb{1}_{\{L_t > (p_1 + p_2) N\}} \\
&= (L_t - N p_1) \mathbb{1}_{\{p_1 N < L_t \leqslant \alpha_2 N\}} + N p_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\
&= (L_t - N p_1)^+ \mathbb{1}_{\{L_t \leqslant \alpha_2 N\}} + N p_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\
&= \min((L_t - N p_1)^+, N p_2) \\
&= \mathrm{Max}(\min(L_t, N p_1 + N p_2) - N p_1, 0) \\
&= \mathrm{Max}(\min(L_t, N \alpha_2) - N p_1, 0).
\end{aligned}$$

By induction, the potential loss taken by tranche n°i is given by

$$\begin{aligned}
L_t^i &= (L_t - N \alpha_{i-1}) \mathbb{1}_{\{\alpha_{i-1} N < L_t \leqslant \alpha_i N\}} + N p_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\
&= (L_t - N \alpha_{i-1})^+ \mathbb{1}_{\{L_t \leqslant \alpha_i N\}} + N p_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\
&= \min((L_t - N \alpha_{i-1})^+, N p_i) \\
&= \mathrm{Max}(\min(L_t, N \alpha_i) - N \alpha_{i-1}, 0),
\end{aligned}$$

where $\alpha_i := p_1 + p_2 + \cdots + p_i$, $i = 1, 2, \ldots, n$.

In the end, tranche n°n will take the loss

$$L_t^n = (L_t - N \alpha_{n-1}) \mathbb{1}_{\{\alpha_{n-1} N < L_t\}} = (L_t - N \alpha_{n-1})^+.$$

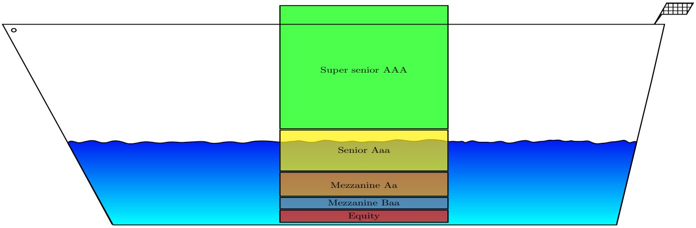

Fig. 11.3: A Titanic-style representation of cumulative tranche losses.

The CDO tranche n°l, $l = 1, 2, \ldots, n$, can be decomposed into:

- A premium leg: the short investor in tranche n°l is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making fixed payments $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation. This premium can also be received by the unfunded investor.

  The discounted value at time t of the premium leg for the tranche n°l is

$$V_l^p(t,T) = \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^l \delta_k (Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \;\Big|\; \mathcal{G}_t\right]$$
$$= S_t^l \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \;\Big|\; \mathcal{G}_t\right] \quad (11.6)$$

  $l = 1, 2, \ldots, N$, where the premium spread S_t^l is quoted as a proportion of the compensation $Np_l - L_{T_{k+1}}^l$ and is paid at each time T_{k+1} until $k = j-1$ or $L_{T_{k+1}} = 100\%$, whichever comes first.

- A protection leg: the short investor receives protection against default from the premium leg, which can also be paid by the unfunded investors. Noting that at each default time $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, the loss L_t^l taken by tranche n°l jumps by the amount $\Delta L_{\tau_k}^l = L_{\tau_k}^l - L_{\tau_k^-}^l$, the value at time t of the protection leg for tranche n°l can be written as

$$V_l^d(t,T) = \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{[T_i,T_j]}(\tau_k) \Delta L_{\tau_k}^l \exp\left(-\int_t^{\tau_k} r_u du\right) \;\Big|\; \mathcal{G}_t\right]$$
$$= \mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL_s^l \;\Big|\; \mathcal{G}_t\right] \quad (11.7)$$
$$= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l - \exp\left(-\int_t^{T_i} r_u du\right) L_{T_i}^l \;\Big|\; \mathcal{G}_t\right]$$

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$$+\mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right]$$
$$= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right],$$

$l = 1, 2, \ldots, n$, where we applied integration by parts on $[T_i, T_j]$ and used the fact that $L_{T_i} = 0$.

The spread S_t^l paid by tranche n°l is computed by equating the values $V_l^p(t,T) = V_l^d(t,T)$ of the protection and premium legs in (11.6) and (11.7), which yields

$$S_t^l = \frac{\mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL_s^l \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]}$$
$$= \frac{\mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]}$$
$$\geqslant 0,$$

$l = 1, 2, \ldots, n$.

Expected tranche loss

The expected cumulative loss given the parameter M can be computed by linearity in the multiple default time model (9.5) of Chapter 9 as

$$\begin{aligned}
\mathbb{E}[L_t \mid M = m] &= \sum_{l=i}^{j-1} \mathbb{E}\left[(1 - \xi_{l+1}) \mathbb{1}_{\{\tau_l \leqslant t\}} \mid M = m\right] \\
&= \sum_{l=i}^{j-1} (1 - \xi_{l+1}) \mathbb{P}\left(\tau_l \leqslant t \mid M = m\right) \\
&= \sum_{l=i}^{j-1} (1 - \xi_{l+1}) \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_l \leqslant t)) + a_k m}{\sqrt{1 - a_k^2}}\right),
\end{aligned}$$

by (9.5), and the expected cumulative loss can be written as

$$\mathbb{E}[L_t] = \int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right] \phi(m) dm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}\left[L_t \mid M = m\right] \mathrm{e}^{-m^2/2} dm.$$

The situation is different for the expected loss of tranche n°k is written as the expected value

$$\mathbb{E}[L_t^k] = \mathbb{E}\left[\min((L_t - N\alpha_{k-1})^+, Np_k)\right], \qquad k = 1, 2, \ldots, n,$$

of the *nonlinear* function $f_k(x) := \min((x - N\alpha_{k-1})^+, Np_k)$ of L_t, where α_{k-1} is defined in (11.5).

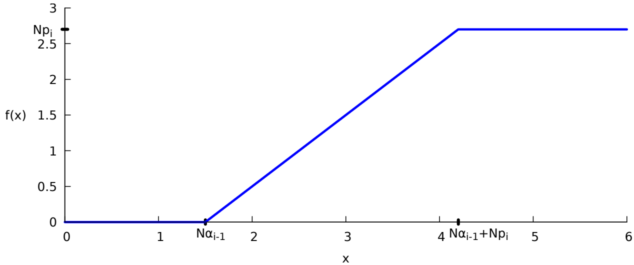

Fig. 11.4: Function $f_k(x) = \min((x - N\alpha_{k-1})^+, Np_k)$.

The expected tranche loss $\mathbb{E}[L_t^k]$ n°k can be estimated by the Monte Carlo method when the default times are generated according to (9.8).

In order to compute expected tranche losses we can use the fact that the cumulative loss L_t is a discrete random variable, with for example

$$\mathbb{P}\left(L_t = N - \sum_{k=i}^{j-1} \xi_{k+1}\right) = \mathbb{P}(\tau_i \leqslant t, \ldots, \tau_{j-1} \leqslant t),$$

and

$$\mathbb{P}(L_t = 0) = \mathbb{P}(\tau_i > t, \ldots, \tau_{j-1} > t),$$

which require the knowledge of the joint distribution of the default times $\tau_i, \ldots, \tau_{j-1}$.

If the $\tau_k' s$ are independent and identically distributed with common cumulative distribution function F_τ and $a_k = a$, $\xi_k = \xi$, $k = i+1, \ldots, j$, then the cumulative loss L_t has a binomial distribution given M, given by

$$\mathbb{P}(L_t = (1-\xi)k \mid M) = \binom{N}{k} (1 - \mathbb{P}(\tau \leqslant T \mid M))^{N-k} (\mathbb{P}(\tau \leqslant T \mid M))^k$$
$$= \binom{N}{k} \left(1 - \Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^{N-k} \left(\Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^k,$$

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$k = 0, 1, \ldots, N$. The expected loss of tranche n°k can then be expressed as

$$\begin{aligned}\mathbb{E}[L_t^k] &= \int_{-\infty}^{\infty} \mathbb{E}\left[f_k(L_t) \mid M = m\right] \phi(m) dm \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}\left[f_k(L_t) \mid M = m\right] \mathrm{e}^{-m^2/2} dm,\end{aligned}$$

$k = 1, 2, \ldots, n$, where $\mathbb{E}\left[f_k(L_t) \mid M = m\right]$ is computed either by the Monte Carlo method, from the distribution of L_t.

In Vašiček (2002), the tranche loss has been approximated by a Gaussian random variable for very large portfolios with $N \to \infty$.

The α-percentile loss of the portfolio can be estimated as

$$\mathbb{E}[L_t \mid M = m] = \sum_{k=i}^{j-1} (1 - \xi_{k+1}) \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k \Phi^{-1}(\alpha)}{\sqrt{1 - a_k^2}}\right),$$

where $m = \Phi^{-1}(\alpha)$.

Such (Gaussian) Merton (1974) and Vašiček (2002) type models have been implemented in the Basel II recommendations on Banking Supervision (2005). Namely in Basel II, banks are expected to hold capital in prevision of unexpected losses in a worst case scenario, according to the Internal Ratings-Based (IRB) formula

$$\sum_{k=i}^{j-1} (1 - \xi_{k+1}) \left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k \Phi^{-1}(\alpha)}{\sqrt{1 - a_k^2}}\right) - \mathbb{P}(\tau_k \leqslant T)\right),$$

with confidence level set at $\alpha = 0.999$ *i.e.* $m = \Phi^{-1}(0.999) = 3.09$, cf. Relation (2.4) page 10 of Aas (2005). Recall that the function

$$x \longmapsto \Phi\left(\frac{\Phi^{-1}(x) + a_k \Phi^{-1}(\alpha)}{\sqrt{1 - a_k^2}}\right)$$

always lies above the graph of x when $a_k < 0$, as in the next figure.

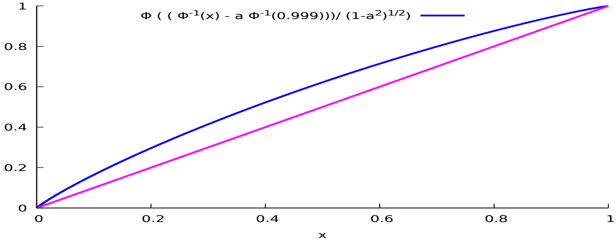

Fig. 11.5: Internal Ratings-Based (IRB) formula.

11.3 Credit Valuation Adjustment (CVA)

Credit Valuation Adjustments (CVA) aim at estimating the amount of capital required in the event of counterparty default, and are specially relevant to the Basel III regulatory framework. Other credit value adjustments (XVA) include the Funding Valuation Adjustments (FVA), Debit Valuation Adjustments (DVA), Capital Valuation Adjustments (KVA), and Margin Valuation Adjustments (MVA). The purpose of XVAs is also to take into account the future value of trades and their associated risks. The real-time estimation of XVA measures is generally highly demanding from a computational point of view.

Net Present Value (NPV) of a CDS

As above, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$. Let

$$\begin{aligned}
\Pi(T_l, T_j) &:= \text{protection_leg} - \text{premium_leg} \\
&= \sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&\quad - \sum_{k=l}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&= (1-\xi) \sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\
&\quad - \sum_{k=l}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(-\int_t^{T_{k+1}} r_s ds\right)
\end{aligned}$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$= \sum_{k=l}^{j-1} \left((1-\xi) \mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \right.$$
$$\left. - \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \right)$$

denote the difference between the remaining protection and premium legs from time T_l until time T_j. Note that by definition of the spread $S_t^{i,j}$ we have $\Pi(t, T_j) = 0$, $0 \leqslant t \leqslant T_i$.

Definition 11.4. *The Net Present Value (NPV) at time T_l of the CDS is the conditional expected value*

$$\text{NPV}(T_l, T_j) := \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{G}_{T_l}\big]$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j, where $(\mathcal{G}_t)_{t \in \mathbb{R}_+}$ is the filtration (10.4) enlarged as with the additional information on the default time τ.

The Net Present Value (NPV) at time T_l of the CDS satisfies

$$\begin{aligned}
&\text{NPV}(T_l, T_j) := \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{G}_{T_l}\big] \\
&= \mathbb{E}\left[\sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \;\middle|\; \mathcal{G}_{T_l} \right] \qquad (11.8) \\
&\quad - \mathbb{E}\left[\sum_{k=l}^{j-1} S_t^{i,j} \delta_k \mathbb{1}_{\{\tau > T_{k+1}\}} \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \;\middle|\; \mathcal{G}_{T_l} \right] \\
&= (1-\xi) \sum_{k=l}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \;\middle|\; \mathcal{G}_t \right] - S_t^{i,j} \sum_{k=l}^{j-1} \delta_k P(t, T_{k+1}) \\
&= \sum_{k=l}^{j-1} \left((1-\xi) \mathbb{E}\left[\mathbb{1}_{(T_k, T_{k+1}]}(\tau) \exp\left(- \int_t^{T_{k+1}} r_s ds \right) \;\middle|\; \mathcal{G}_t \right] - S_t^{i,j} \delta_k P(t, T_{k+1}) \right)
\end{aligned}$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j.

In addition to the credit default time τ we introduce a second stopping time $\nu \in [T_l, T_j]$ representing the possible default time of the party providing the protection leg.

The Net Present Value $\text{NPV}(\nu, T_j)$ is estimated when default occurs at time ν.

i) If $\text{NPV}(\nu, T_j) > 0$ then a payment is due from the party providing the protection leg, and only a fraction $\eta \text{NPV}(\nu, T_j)$ of this payment may be recovered, where $\eta \in [0,1]$ is the recovery rate of the party providing protection in the CDS.

ii) On the other hand, if $\mathrm{NPV}(\nu, T_j) < 0$ then the original fee payment $-\mathrm{NPV}(\nu, T_j)$ is still due.

As a consequence, in the event of default at time $\nu \in [T_l, T_j]$, the net present value of the CDS at time ν is

$$\begin{aligned}
&\eta \mathrm{NPV}(\nu, T_j) \mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)>0\}} + \mathrm{NPV}(\nu, T_j) \mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)<0\}} \\
&\qquad = \eta \big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(\mathrm{NPV}(\nu, T_j)\big)^- \\
&\qquad = \eta \big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(-\mathrm{NPV}(\nu, T_j)\big)^+ \\
&\qquad = \eta \big(\mathrm{NPV}(\nu, T_j)\big)^+ + \big(\mathrm{NPV}(\nu, T_j) - \big(\mathrm{NPV}(\nu, T_j)\big)^+\big) \\
&\qquad = \mathrm{NPV}(\nu, T_j) - (1-\eta)\big(\mathrm{NPV}(\nu, T_j)\big)^+. \qquad (11.9)
\end{aligned}$$

Credit Valuation Adjustment (CVA)

Under the event of counterparty default at a time $\nu \in [T_l, T_j]$, the discounted payment estimated at time T_l becomes

$$\begin{aligned}
&\Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \left(\eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(-\mathrm{NPV}(\nu, T_j)\big)^+\right) \\
&= \Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \left(\mathrm{NPV}(\nu, T_j) - (1-\eta)\big(\mathrm{NPV}(\nu, T_j)\big)^+\right) \\
&= \Pi(T_l, T_j) - (1-\eta) \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \big(\mathrm{NPV}(\nu, T_j)\big)^+,
\end{aligned}$$

since

$$\Pi(T_l, T_j) = \Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \mathrm{NPV}(\nu, T_j).$$

More generally, the total discounted payment due at time T_l under counterparty risk rewrites as

$$\begin{aligned}
&\Pi^D(T_l, T_j) = \mathbb{1}_{\{T_j<\nu\}} \Pi(T_l, T_j) \\
&+ \mathbb{1}_{\{T_l<\nu\leqslant T_j\}} \left(\Pi(T_l, \nu) + \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \left(\eta\big(\mathrm{NPV}(\nu, T_j)\big)^+ - \big(-\mathrm{NPV}(\nu, T_j)\big)^+\right)\right) \\
&= \mathbb{1}_{\{T_j<\nu\}} \Pi(T_l, T_j) \\
&\quad + \mathbb{1}_{\{T_l<\nu\leqslant T_j\}} \left(\Pi(T_l, T_j) - (1-\eta) \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \big(\mathrm{NPV}(\nu, T_j)\big)^+\right)
\end{aligned}$$

This version: January 10, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

$$= \Pi(T_l, T_j) - \mathbb{1}_{\{T_l < \nu \leqslant T_j\}}(1-\eta) \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \big(\mathrm{NPV}(\nu, T_j)\big)^+, \quad (11.10)$$

see Brigo and Masetti (2006), Brigo and Chourdakis (2009). As a consequence of (11.10), we derive the following result.

Proposition 11.5. *The price at time T_l of the payoff $\Pi^D(T_l, T_j)$ under counterparty risk is given by*

$$\mathbb{E}\big[\Pi^D(T_l, T_j) \mid \mathcal{F}_{T_l}\big] = \mathbb{E}\big[\Pi(T_l, T_j) \mid \mathcal{F}_{T_l}\big] \\ -(1-\eta)\mathbb{E}\left[\mathbb{1}_{\{T_l < \nu \leqslant T_j\}} \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \big(\mathrm{NPV}(\nu, T_j)\big)^+ \;\Big|\; \mathcal{F}_{T_l}\right].$$

The quantity

$$(1-\eta)\mathbb{E}\left[\mathbb{1}_{\{T_l < \nu \leqslant T_j\}} \exp\left(-\int_{T_l}^{\nu} r_s ds\right) \big(\mathrm{NPV}(\nu, T_j)\big)^+ \;\Big|\; \mathcal{F}_{T_l}\right]$$

is called the (positive) Counterparty Risk (CR) Credit Valuation Adjustment (CVA).

Exercises

Exercise 11.1 Show that the equation (11.4) admits a numerical solution $\lambda > 0$.

Exercise 11.2 Credit default swaps. From the CDS market data of Figure 11.6 on McDonald's Corp, estimate the first default rate λ_1 and the associated default probability in the framework of (11.4), cf. also Castellacci (2008).

Fig. 11.6: Cashflow data.

Exercise 11.3 Consider a tenor structure $\{t = T_i < \cdots < T_j = T\}$, a sequence

$$P(t, T_k) = \exp\left(-\int_t^{T_k} r(s)ds\right) = \mathrm{e}^{-(T_k - t)r_k}, \qquad k = i, \ldots, j,$$

of *deterministic* discount factors, and a family

$$Q(t, T_k) = \mathbb{E}\left[\exp\left(-\int_t^{T_k} \lambda_s ds\right) \;\Big|\; \mathcal{F}_t\right]$$

of survival probabilities.

a) Show that the discounted value at time t of the protection leg equals

$$\sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1 - \xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \;\Big|\; \mathcal{G}_t\right]$$
$$= \mathbb{1}_{\{\tau > t\}}(1 - \xi) \sum_{k=i}^{j-1} P(t, T_{k+1}) \left(Q(t, T_k) - Q(t, T_{k+1})\right).$$

b) Letting $\delta_k := T_{k+1} - T_k$, $k = i, \ldots, j-1$, show that the discounted value at time t of the premium leg, equals

$$V^p(t,T) = \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}).$$

c) By equating the protection and premium legs, find the value of $Q(t,T_{i+1})$ with $Q(t,T_i) = 1$, and derive a recurrence relation between $Q(t,T_{j+1})$ and $Q(t,T_i),\ldots,Q(t,T_j)$.

Exercise 11.4 (Exercise 11.3 continued). From the spread data and survival probabilities data of Figure 11.7 on the Coca-Cola Company, retrieve the corresponding CDS spreads $S_t^{i,j}$ and discount factors $P(t,T_i),\ldots,P(t,T_n)$, and estimate the corresponding survival probabilities $Q(t,T_i),\ldots,Q(t,T_n)$.

Fig. 11.7: CDS Market data.