Chapter 8
Structural Approach

Credit risk is the risk of default on a debt. In this chapter, credit risk is modeled using the value of a firm's assets: a default occurs when the asset value falls below a predetermined threshold. This contrasts the reduced-form approach of Chapter 9, where stochastic processes are used to model default probabilities. We also consider the correlation and dependence between multiple default times.

8.1 Merton Model

The Merton (1974) credit risk model reframes corporate debt as an option on a firm's underlying value. Precisely the value S_t of a firm's asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

under the historical (or physical) measure $\mathbb{P}$. Recall that, using the standard Brownian motion

$$\widehat{B}_t = \frac{\mu - r}{\sigma} t + B_t, \qquad t \geqslant 0,$$

under the risk-neutral probability measure $\mathbb{P}^*$, the process $(S_t)_{t\in\mathbb{R}_+}$ is modeled as

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t.$$

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Assumption 8.1. *The company's debt is represented by an amount $K > 0$ in bonds to be paid at maturity T.*

In this setting, see *e.g.* § 4.1 of Grasselli and Hurd (2010), two events may occur at time T.

- If $S_T < K$ (default event), the bond holder will receive the recovery value S_T.
- If $S_T \geqslant K$ the bond holder receives K and the equity holder is entitled to receive the amount $S_T - K \geqslant 0$.

We note that the amount received by the equity holder can be written as $(S_T - K)^+$ in general.

- The discounted expected cash flow (or dividend) $\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t]$ received by the equity holder can be estimated at time $t \in [0, T]$ as the price of a European call option, from the Black–Scholes formula

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] = S_t\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T,$$

  see Proposition 5.4.
- Similarly, the expected loss $\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t]$ incurred by the bond holder can be estimated at time $t \in [0, T]$ as the price of a European put option, from the Black–Scholes formula

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_t)^+ \mid \mathcal{F}_t] = K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) - S_t\Phi\left(-\frac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T, \tag{8.1}$$

  see Proposition 5.6.

Proposition 8.2 is a consequence of the lognormal distribution of S_T derived in Proposition 1.9.

Proposition 8.2. *The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed as*

$$\mathbb{P}(S_T < K \mid \mathcal{F}_t) = \Phi\big(-d_-^\mu\big), \tag{8.2}$$

where Φ is the cumulative distribution function of the standard normal distribution, and

$$d_-^\mu := \frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}},$$

which can be interpreted as a distance to default*.*

Proof. The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of S_T as

$$\begin{aligned}
\mathbb{P}(S_T < K \mid \mathcal{F}_t) &= \mathbb{P}\big(S_0 \mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} < K \mid \mathcal{F}_t\big) \\
&= \mathbb{P}\left(B_T < \frac{1}{\sigma}\left(-\left(\mu - \frac{\sigma^2}{2}\right)T + \log\frac{K}{S_0}\right) \Bigm| \mathcal{F}_t\right) \\
&= \mathbb{P}\left(B_T - B_t + y < \frac{1}{\sigma}\left(-\left(\mu - \frac{\sigma^2}{2}\right)T + \log\frac{K}{S_0}\right)\right)_{|y=B_t} \\
&= \mathbb{P}\left(B_T - B_t + \frac{1}{\sigma}\left(-\left(\mu - \frac{\sigma^2}{2}\right)t + \log\frac{K}{x}\right)\right. \\
&\qquad\qquad\qquad \left. < \frac{1}{\sigma}\left(-\left(\mu - \frac{\sigma^2}{2}\right)T + \log\frac{K}{S_0}\right)\right)_{|x=S_t} \\
&= \frac{1}{\sqrt{2(T-t)\pi}} \int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/\sigma} \mathrm{e}^{-x^2/(2(T-t))} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/(\sigma\sqrt{T-t})} \mathrm{e}^{-x^2/2} dx \\
&= 1 - \Phi\left(\frac{(\mu - \sigma^2/2))(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \Phi(d_-^\mu) \\
&= \Phi(-d_-^\mu).
\end{aligned}$$

□

The formula (8.2) can be implemented as follows.

```
P <- function(S, K, mu, T, sigma)
{d1 <- (log(S/K)+(mu-sigma^2/2)*T)/(sigma*sqrt(T)); P= pnorm(-d1);P}
P(100,90,0.2,1,0.3)
```

Listing 8.1: R code - Ruin probability computation.

```
import numpy as np; from scipy.stats import norm
P=lambda S,K,mu,T,sig: norm.cdf(-(np.log(S/K)+(mu-sig**2/2)*T)/(sig*np.sqrt(T)))
P(100, 90, 0.2, 1, 0.3)
```

Listing 8.2: Python code - Ruin probability computation.

Let

$$d_-^r = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}},$$

with the relation

$$d_-^r = d_-^\mu - \frac{\mu - r}{\sigma}\sqrt{T-t}.$$

Under the risk-neutral probability measure $\mathbb{P}^*$ we have, replacing μ with r,

$$\mathbb{P}^*(S_T < K \mid \mathcal{F}_t) = \Phi(-d_-^r),$$

hence the relation

$$\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathcal{F}_t)) = -\frac{\mu - r}{\sigma}\sqrt{T-t} + \Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathcal{F}_t)).$$

where Φ^{-1} denotes the inverse function of Φ.

If the level of the firm's assets value falls below the level K at time T, default may have occurred at a random time τ such that

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) = \mathbb{P}(S_T < K \mid \mathcal{F}_t).$$

In this case, the result of Proposition 8.2 can be reinterpreted by writing the conditional distribution of the default time τ as

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) = \mathbb{P}(S_T < K \mid \mathcal{F}_t) = \Phi\left(-\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right), \tag{8.3}$$

$0 \leqslant t \leqslant T$. We also have

$$\begin{aligned}\mathbb{P}(\tau < T \mid \mathcal{F}_t) &= \mathbb{P}(S_T < K \mid \mathcal{F}_t) \\ &= \Phi\left(\Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\ &= \Phi\left(\Phi^{-1}(\mathbb{P}^*(\tau < T \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}^*(\tau < T \mid \mathcal{F}_t) &= \mathbb{P}^*(S_T < K \mid \mathcal{F}_t) \\ &= \Phi\left(-\frac{(r - \sigma^2/2))(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right) \\ &= \Phi\left(\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\ &= \Phi\left(\Phi^{-1}(\mathbb{P}(\tau < T \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right).\end{aligned} \tag{8.4}$$

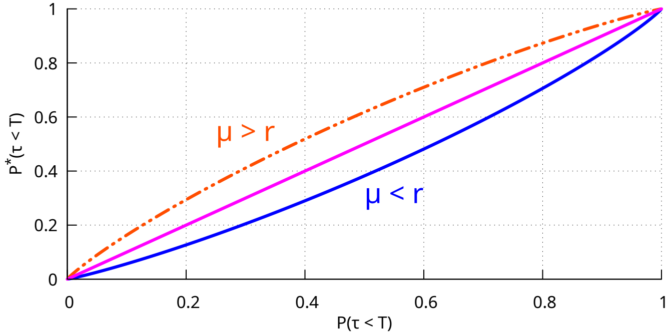

Fig. 8.1: Graph of $x \mapsto \Phi\big(\Phi^{-1}(x) - (\mu - r)\sqrt{T}/\sigma\big)$ for $\mu > r$, $\mu = r$, and $\mu < r$.

We note that when $\mu < r$,

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) > \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),$$

whereas when $\mu > r$ we get

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) < \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),$$

as illustrated in the next Figure 8.1.

8.2 Default Bonds

In the following proposition we price at time $t \in [0, T]$ the amount $\min(S_T, K)$ received by the bond holder (or junior creditor) at maturity, based on the recovery value S_T when $S_T < K$. This price can interpreted at the price $P(t, T)$ at time $t \in [0, T]$ of a default bond with face value \$1, maturity T and recovery rate $\min(S_T/K, 1)$.

Proposition 8.3. *The amount received by the bond holder (or junior creditor) at maturity is priced at time $t \in [0, T]$ as*

$$e^{-(T-t)r}\mathbb{E}^*\big[\min(S_T, K) \mid \mathcal{F}_t\big] = K e^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r), \qquad (8.5)$$

$0 \leqslant t \leqslant T$.

Proof. Using the Black–Scholes put option pricing formula (8.1) of Proposition 5.6 and the identity

$$\min(x, K) = K - (K - x)^+, \qquad x \in \mathbb{R},$$

we have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\min(S_T,K)\mid\mathcal{F}_t\big] &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[K-(K-S_T)^+\mid\mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}K - \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(K-S_T)^+\mid\mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}K + S_t\Phi(-d_+^r) - K\mathrm{e}^{-(T-t)r}\Phi(-d_-^r) \\
&= K\mathrm{e}^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r).
\end{aligned}$$

□

The formula (8.5) can be implemented as follows.

```r
DBond <- function(S, K, r, T, sigma)
{d1= (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2= d1 - sigma * sqrt(T);
DBond= K*exp(-r*T) * pnorm(d2) - S*pnorm(-d1);DBond}
DBond(100, 105, 0.05, 1, 0.2)
```

Listing 8.3: R code - Default bond pricing.

```python
from math import log, sqrt, exp; from scipy.stats import norm
d_bond= lambda S, K, r, T, sigma: (lambda d1, d2: K * exp(-r * T) * norm.cdf(d2) - S *
  norm.cdf(-d1))((log(S / K) + (r + sigma**2 / 2) * T) / (sigma * sqrt(T)), (log(S / K) + (r +
  sigma**2 / 2) * T) / (sigma * sqrt(T)) - sigma * sqrt(T))
d_bond(100, 105, 0.05, 1, 0.2)
```

Listing 8.4: Python code - Default bond pricing.

The quantity

$$\begin{aligned}
P(t,T) &:= \frac{1}{K}\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\min(S_T,K)\mid\mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}\Phi(d_-^r) - \frac{S_t}{K}\Phi(-d_+^r), \quad 0\leqslant t\leqslant T,
\end{aligned}$$

is interpreted as a default bond price paying a unit of currency in the absence of default at maturity. The default bond yield $y_{t,T}$ over the time interval $[t,T]$ is defined from the relation

$$\begin{aligned}
P(t,T) &= \mathrm{e}^{-(T-t)y_{t,T}} \\
&= \frac{1}{K}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[\min(S_T,K)\mid\mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\Phi(d_-^r) - \frac{S_t}{K}\Phi(-d_+^r),
\end{aligned}$$

which gives

$$y_{t,T} = -\frac{1}{T-t}\log(P(t,T))$$

$$\begin{aligned}
&= -\frac{1}{T-t}\log\left(\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathcal{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathcal{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\frac{1}{K}\mathbb{E}^*\left[\min\left(K,S_T\right)\Big|\mathcal{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\Phi(d_-^r)-\frac{S_t}{K}\mathrm{e}^{(T-t)r}\Phi(-d_+^r)\right),
\end{aligned}$$

which is usually higher than the risk-free yield r.

8.3 Black-Cox Model

In the Black and Cox (1976) model the firm has to maintain an account balance above the level K throughout time, therefore default occurs at the first time the process S_t hits the level K, cf. § 4.2 of Grasselli and Hurd (2010). The default time τ_K is therefore the first hitting time

$$\tau_K := \inf\left\{t \geqslant 0 \ : \ S_t := S_0\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant K\right\},$$

of the level K by

$$(S_t)_{t\in\mathbb{R}_+} = \left(S_0\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t}\right)_{t\in\mathbb{R}_+},$$

after starting from $S_0 > K$.

Proposition 8.4. *The probability distribution function of the default time τ_K is given by*

$$\mathbb{P}\big(\tau_K \leqslant T\big) = \mathbb{P}(S_T \leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{8.6}$$

with $S_0 \geqslant K$.

Proof. By *e.g.* Corollary 7.2.2 and pages 297-299 of Shreve (2004), or from Relation (10.13) in Privault (2022), we have

$$\begin{aligned}
\mathbb{P}\big(\tau_K \leqslant T\big) &= \mathbb{P}\left(\min_{t\in[0,T]} S_t \leqslant K\right)\\
&= \mathbb{P}\left(\min_{t\in[0,T]} \mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant \frac{K}{S_0}\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{P}\left(\min_{t\in[0,T]}\left(B_t + \frac{(\mu-\sigma^2/2)t}{\sigma}\right) \leqslant \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\right)\\
&= \Phi\left(\frac{\log(K/S_0)-(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&\quad + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \qquad (8.7)\\
&= \mathbb{P}(S_T\leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right),
\end{aligned}
$$

with $S_0 \geqslant K$. □

The formula (8.6) can be implemented as follows.

```r
P <- function(S, K, mu, T, sig)
{d1=(log(S/K)+(mu-sig^2/2)*T)/(sig*sqrt(T));d2=(log(K/S)+(mu-sig^2/2)*T)/(sig*sqrt(T));
P= pnorm(-d1)+(S/K)^(1-2*mu/sig^2)*pnorm(d2);P}
P(100,90,0.2,1,0.3)
```

Listing 8.5: R code - Ruin probability computation.

```python
import numpy as np; from scipy.stats import norm
P=lambda S,K,mu,T,sig: (norm.cdf(-(np.log(S/K)+(mu-sig**2/2)*T)/(sig*np.sqrt(T)))
+(S/K)**(1-2*mu/sig**2)*norm.cdf((np.log(K/S)+(mu-sig**2/2)*T)/(sig*np.sqrt(T))))
P(100, 90, 0.2, 1, 0.3)
```

Listing 8.6: Python code - Ruin probability computation.

We check that when $S_0 = K$, using (8.2), Relation (8.6) reads

$$
\begin{aligned}
\mathbb{P}(\tau_K \leqslant T) &= \Phi\left(-\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&\quad + \Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&= 1.
\end{aligned}
$$

The cash flow

$$
(S_T-K)^+\mathbb{1}_{\{\tau_K>T\}} = (S_T-K)\mathbb{1}_{\left\{\min\limits_{t\in[0,T]} S_t > K\right\}}
$$

received at maturity T by the equity holder can be priced at time $t\in[0,T]$ as a down-and-out barrier call option with strike price K and barrier level K is priced in the next proposition, in which Bl_c denotes the Black–Scholes call pricing formula (5.3).

Proposition 8.5. *We have*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t>K\right\}}\,\Big|\,\mathcal{F}_t\right]=\mathbb{1}_{\left\{\min\limits_{t\in[0,T]} S_t>B\right\}}g(t,S_t),$$

$t\in[0,T]$, *where*

$$g(t,S_t)=\mathrm{Bl}_c(S_t,K,r,T-t,\sigma)-S_t\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\mathrm{Bl}_c(K/S_t,1,r,T-t,\sigma),$$

$0\leqslant t\leqslant T$.

Proof. By *e.g.* Relation (11.10) and Exercise 11.1 in Privault (2022), we have

$$\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t>K\right\}}\,\Big|\,\mathcal{F}_t\right]=\mathbb{1}_{\left\{\min\limits_{t\in[0,T]} S_t>B\right\}}g(t,S_t),$$

$t\in[0,T]$, where

$$\begin{aligned}
g(t,S_t)&=S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\mathrm{e}^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)\\
&-K\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t}\right)\right)+\mathrm{e}^{-(T-t)r}K\left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t}\right)\right)\\
&=\mathrm{Bl}_c(S_t,K,r,T-t,\sigma)\\
&-K\left(\frac{S_t}{K}\right)^{-2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t}\right)\right)+\mathrm{e}^{-(T-t)r}S_t\left(\frac{S_t}{K}\right)^{-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t}\right)\right)\\
&=\mathrm{Bl}_c(S_t,K,r,T-t,\sigma)-S_t\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\mathrm{Bl}_c(K/S_t,1,r,T-t,\sigma),
\end{aligned}$$

$0\leqslant t\leqslant T$. □

For $t\geqslant 0$, taking now

$$\tau_K:=\inf\left\{u\in[t,\infty)\ :\ S_u:=S_0\mathrm{e}^{\sigma B_u+(\mu-\sigma^2/2)u}\leqslant K\right\},$$

the recovery value received by the bond holder at time $\min(\tau_K,T)$ is K, and it can be priced as in the next proposition.

Proposition 8.6. *After discounting from time* $\min(\tau_K, T)$ *to time* $t \in [0, T]$, *we have*

$$\begin{aligned}&\mathbb{E}^*\big[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathcal{F}_t\big]\\&= K\mathbb{1}_{\{\tau_K \geqslant t\}}\int_t^T \mathrm{e}^{-(u-t)r}\mathrm{d}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big).\end{aligned}$$

Proof. We have

$$\begin{aligned}&\mathbb{E}^*\big[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathcal{F}_t\big]\\&= \mathbb{E}^*\big[K\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} + K\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{\tau_K>T\}} \mid \mathcal{F}_t\big]\\&= K\mathbb{E}^*\big[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathcal{F}_t\big] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big)\\&= K\mathbb{1}_{\{\tau_K\geqslant t\}}\mathbb{E}^*\big[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathcal{F}_t\big] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big)\\&= K\mathbb{1}_{\{\tau_K\geqslant t\}}\int_t^T \mathrm{e}^{-(u-t)r}\mathrm{d}\mathbb{P}\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big),\end{aligned}$$

$0 \leqslant t \leqslant T$. □

The above probabilities $\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big)$ and $\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big) = 1 - \mathbb{P}^*\big(\tau_K \leqslant T \mid \mathcal{F}_t\big)$ can be computed from (8.7) as

$$\begin{aligned}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) &= \Phi\left(\frac{\log(K/S_t) - (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)\\&\quad + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)\\&= \mathbb{P}(S_u \leqslant K \mid \mathcal{F}_t) + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right),\end{aligned}$$

with $S_t \geqslant K$ and $u > t$, from which the probability density function of the hitting time τ_K can be estimated by differentiation with respect to $u > t$. Note also that we have

$$\begin{aligned}\mathbb{P}^*\big(\tau_K < \infty \mid \mathcal{F}_t\big) &= \lim_{u\to\infty}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big)\\&= \begin{cases}\left(\dfrac{K}{S_t}\right)^{-1+2r/\sigma^2} & \text{if } r > \sigma^2/2\\[2mm] 1 & \text{if } r \leqslant \sigma^2/2.\end{cases}\end{aligned}$$

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8.4 Correlated Default Times

In order to model correlated default and possible "domino effects", one can regard two given default times τ_1 and τ_2 as correlated random variables, with correlation coefficient

$$\rho := \frac{\mathrm{Cov}(\tau_1, \tau_2)}{\sqrt{\mathrm{Var}[\tau_1]\,\mathrm{Var}[\tau_2]}} \in [-1, 1].$$

Given two default events $\{\tau_1 \leqslant T\}$ and $\{\tau_2 \leqslant T\}$ with probabilities of the form

$$\mathbb{P}(\tau_1 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_1(s)ds\right) \text{ and } \mathbb{P}(\tau_2 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_2(s)ds\right)$$

as in (4.1), we can define the default correlation $\rho^D \in [-1, 1]$ as

$$\begin{aligned}\rho^D &:= \frac{\mathrm{Cov}(\mathbb{1}_{\{\tau_1 \leqslant T\}}, \mathbb{1}_{\{\tau_2 \leqslant T\}})}{\sqrt{\mathrm{Var}[\mathbb{1}_{\{\tau_1 \leqslant T\}}]}\sqrt{\mathrm{Var}[\mathbb{1}_{\{\tau_2 \leqslant T\}}]}} \\ &= \frac{\mathbb{E}[\mathbb{1}_{\{\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T\}}] - \mathbb{E}[\mathbb{1}_{\{\tau_1 \leqslant T\}}]\mathbb{E}[\mathbb{1}_{\{\tau_2 \leqslant T\}}]}{\sqrt{\mathbb{E}[\mathbb{1}_{\{\tau_1 \leqslant T\}}] - (\mathbb{E}[\mathbb{1}_{\{\tau_1 \leqslant T\}}])^2}\sqrt{\mathbb{E}[\mathbb{1}_{\{\tau_2 \leqslant T\}}] - (\mathbb{E}[\mathbb{1}_{\{\tau_2 \leqslant T\}}])^2}} \\ &= \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}},\end{aligned}$$

and when the joint CDF $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$ is written using a copula C as in Sklar's Theorem 4.7, as

$$\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) = C(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)),$$

we find

$$\rho^D = \frac{C(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}}.$$

When the default probabilities are specified in the Merton model (8.2) of credit risk using two geometric Brownian motions

$$S_t^{(1)} := S_0^{(1)} \mathrm{e}^{\sigma_1 B_t + (\mu_1 - \sigma_1^2/2)t}, \quad S_t^{(2)} := S_0^{(2)} \mathrm{e}^{\sigma_2 B_t + (\mu_2 - \sigma_2^2/2)t}, \quad t \in [0, T],$$

with parameters $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$, we have

$$\mathbb{P}(\tau_i \leqslant T) = \mathbb{P}\big(S_T^{(i)} < K\big)$$

$$
\begin{aligned}
&= \mathbb{P}\left(\mathrm{e}^{\sigma_i B_T + (\mu_i - \sigma_i^2/2)T} < \frac{K}{S_0^{(i)}}\right) \\
&= \mathbb{P}\left(B_T \leqslant -\frac{(\mu_i - \sigma_i^2/2)T}{\sigma_i} + \frac{1}{\sigma_i}\log\frac{K}{S_0^{(i)}}\right) \\
&= \Phi\left(\frac{\log(K/S_0^{(i)}) - (\mu_i - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}\right), \qquad i = 1, 2.
\end{aligned}
$$

Using the Gaussian copula (4.6), we can model the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$ using the CDF Φ_Σ of the joint Gaussian distribution with covariance matrix Σ, as

$$
\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) = C_\Sigma(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)) \\
&\quad = \Phi_\Sigma\big(\Phi^{-1}(\mathbb{P}(\tau_1 \leqslant T)), \Phi^{-1}(\mathbb{P}(\tau_2 \leqslant T))\big) \\
&\quad = \Phi_\Sigma\left(\frac{\log(K/S_0^{(1)}) - (\mu_1 - \sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\log(K/S_0^{(2)}) - (\mu_2 - \sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right),
\end{aligned}
$$

hence the default correlation ρ^D becomes

$$
\begin{aligned}
\rho^D &= \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}} \\
&= \frac{\Phi_\Sigma\left(\frac{\log(K/S_0^{(1)}) - (\mu_1 - \sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\log(K/S_0^{(2)}) - (\mu_2 - \sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}}.
\end{aligned}
$$

When trying to build a dependence structure for the default times τ_1 and τ_2, the idea in Li (2000) is to use the Gaussian copula $C_\Sigma(x, y)$ with

$$
\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
$$

using a correlation coefficient $\rho \in [-1, 1]$, and to model the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$ as

$$
\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) := C_\Sigma\left(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)\right).
$$

In particular, it was suggested to use a single *average correlation* estimate, see (8.1) page 82 of the Credit Metrics™ Technical Document, see Gupton et al. (1997) and Appendix F therein.

Note that this approach should be applied with caution, as such simplifying assumptions on dependence and correlation can significantly underestimate joint default risk under extreme market conditions. The outcomes of this methodology have been discussed in a number of magazine articles in recent years, to name a few:

"Recipe for disaster: the formula that killed Wall Street", *Wired Magazine*, by F. Salmon (2009);

"The formula that felled Wall Street", *Financial Times Magazine*, by S. Jones (2009);

"Formula from hell", *Forbes.com*, by S. Lee (2009),

see also **here**.

On the other hand, a more proper definition of the default correlation ρ^D should be

$$\rho^D := \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}},$$

which requires the actual computation of the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$. An exact expression for this joint default probability in the first passage time Black-Cox model, and the associated correlation, have been obtained in Li and Krehbiel (2016).

Multiple default times

Consider now a sequence $(\tau_k)_{k=1,2,\ldots,n}$ of random default times and, for more flexibility, a standardized random variable M with probability density function $\phi(m)$ and variance $\mathrm{Var}[M] = 1$.

As in the Merton (1974) model, cf. § 8.1, a common practice, see Vašiček (1987), Gibson (2004), Hull and White (2004) is to parametrize the default probability associated to each τ_k by a conditioning of the form

$$\mathbb{P}(\tau_k \leqslant T \mid M = m) = \Phi\left(\frac{\Phi^{-1}(F_{\tau_k}(T)) - a_k m}{\sqrt{1 - a_k^2}}\right), \tag{8.8}$$

see (8.4), where the dependence between the τ_k's is controlled through the values of M, and

$$F_{\tau_k}(T) := \mathbb{P}(\tau_k \leqslant T)$$

is the cumulative distribution function of τ_k, $k = 1, 2, \ldots, n$, and $a_1, \ldots, a_n \in (-1, 1)$. Note that we have

$$\mathbb{P}(\tau_k \leqslant T) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m)\phi(m)dm$$
$$= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k m}{\sqrt{1-a_k^2}}\right)\phi(m)dm, \qquad (8.9)$$

and $\phi(m)$ can be typically chosen as a standard normal Gaussian probability density function.

Next, we present a dependence structure which implements of the conditional Gaussian copula correlation method in the case of multiple default times.

Definition 8.7. *Given $X_1, X_2, \ldots, X_n$ Gaussian samples defined as*

$$X_k := a_k M + \sqrt{1-a_k^2} Z_k, \quad k = 1, 2, \ldots, n, \qquad (8.10)$$

conditionally to M, where $Z_1, Z_2, \ldots, Z_n$ are normal random variables with same cumulative distribution function Φ, independent of M, we construct the correlated default times $(\tau_1, \ldots, \tau_n)$ as

$$\tau_k := F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)), \qquad (8.11)$$

where $F_{\tau_k}^{-1}$ is the inverse of the CDF F_{τ_k} of τ_k and Φ_{X_k} denotes the CDF of X_k, $k = 1, 2, \ldots, n$.

In Proposition 8.8 we compute the joint distribution of the default times $(\tau_1, \ldots, \tau_n)$ according to the above dependence structure.

Proposition 8.8. *The default times $(\tau_k)_{k=1,2,\ldots,n}$ have the joint distribution*

$$\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big), \qquad (8.12)$$

where

$$C(x_1, \ldots, x_n)$$
$$:= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_k}^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi_{X_k}^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm,$$

$x_1, x_2, \ldots, x_n \in [0, 1]$.

Proof. We start by showing that Definition 8.7 recovers the conditional distribution (8.8), as follows:

$$\mathbb{P}(\tau_k \leqslant T \mid M = m) = \mathbb{P}\big(F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)) \leqslant T \mid M = m\big)$$
$$= \mathbb{P}\big(\Phi_{X_k}(X_k) \leqslant F_{\tau_k}(T) \mid M = m\big)$$

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$$
\begin{aligned}
&= \mathbb{P}\big(X_k \leqslant \Phi^{-1}_{X_k}\big(F_{\tau_k}(T)\big) \mid M = m\big) \\
&= \mathbb{P}\left(a_k m + Z_k\sqrt{1-a_k^2} \leqslant \Phi^{-1}_{X_k}\big(F_{\tau_k}(T)\big)\right) \\
&= \mathbb{P}\left(Z_k\sqrt{1-a_k^2} \leqslant \Phi^{-1}_{X_k}\big(F_{\tau_k}(T)\big) - a_k m\right) \\
&= \mathbb{P}\left(Z_k \leqslant \frac{\Phi^{-1}_{X_k}\big(F_{\tau_k}(T)\big) - a_k m}{\sqrt{1-a_k^2}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}_{X_k}\big(\mathbb{P}(\tau_k \leqslant T)\big) - a_k m}{\sqrt{1-a_k^2}}\right), \qquad k = 1, 2, \ldots, n.
\end{aligned}
$$

Note that the above recovers the correct marginal distributions (8.9), *i.e.* we have

$$
\begin{aligned}
\mathbb{P}(\tau_k \leqslant y_k) &= \mathbb{P}(\tau_1 \leqslant \infty, \ldots, \tau_{k-1} \leqslant \infty, \tau_k \leqslant y_k, \tau_{k+1} \leqslant \infty, \ldots, \tau_n \leqslant \infty) \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}_{X_k}\big(\mathbb{P}(\tau_k \leqslant y_k)\big) - a_k m}{\sqrt{1-a_k^2}}\right)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m)\phi(m)dm, \quad k = 1, 2, \ldots, n.
\end{aligned}
$$

Knowing that, given the sample $M = m$, the default times τ_k, $k = 1, 2, \ldots, n$, are independent random variables, we can compute the joint distribution

$$
\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m) \\
&\qquad = \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \times \cdots \times \mathbb{P}(\tau_n \leqslant y_n \mid M = m),
\end{aligned}
$$

conditionally to $M = m$. This yields

$$
\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \cdots \mathbb{P}(\tau_n \leqslant y_n \mid M = m)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}_{X_1}\big(\mathbb{P}(\tau_1 \leqslant y_1)\big) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi^{-1}_{X_n}\big(\mathbb{P}(\tau_n \leqslant y_n)\big) - a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm,
\end{aligned}
$$

which is (8.12). □

The next corollary deals with the case where M is normally distributed.

Corollary 8.9. *Assume that M has the standard normal distribution with probability density function ϕ and is independent of $Z_1, \ldots, Z_n$. Then, the joint distribution of the default times $(\tau_k)_{k=1,2,\ldots,n}$ is given by*

$$
\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big),
$$

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where

$$C(x_1,\ldots,x_n)$$
$$:= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1)-a_1m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{\Phi^{-1}(x_n)-a_nm}{\sqrt{1-a_n^2}}\right)\phi(m)dm,$$

$x_1,x_2,\ldots,x_n \in [0,1]$, *is the Gaussian copula with covariance matrix*

$$\Sigma = \begin{bmatrix} 1 & a_1a_2 & \cdots & a_1a_{n-1} & a_1a_n \\ a_2a_1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & a_{n-1}a_n \\ a_na_1 & a_na_2 & \cdots & a_na_{n-1} & 1 \end{bmatrix}. \tag{8.13}$$

Proof. When the random variable M is normally distributed and independent of $Z_1,\ldots,Z_n$, the random vector $(X_1,\ldots,X_n)$ has the covariance matrix (8.13), and the function

$$C(x_1,\ldots,x_n)$$
$$:= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1)-a_1m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{\Phi^{-1}(x_n)-a_nm}{\sqrt{1-a_n^2}}\right)\phi(m)dm,$$

$x_1,x_2,\ldots,x_n \in [0,1]$, is a Gaussian copula on $[0,1]^n$, built as

$$C(x_1,\ldots,x_n) = F\big(\Phi^{-1}(x_1),\ldots,\Phi^{-1}(x_n)\big),$$

from the Gaussian cumulative distribution function

$$\begin{aligned} F(x_1,\ldots,x_n) &:= \int_{-\infty}^{\infty} \Phi\left(\frac{x_1-a_1m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{x_n-a_nm}{\sqrt{1-a_n^2}}\right)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left(Z_1 \leqslant \frac{x_1-a_1m}{\sqrt{1-a_1^2}}\right)\cdots\mathbb{P}\left(Z_n \leqslant \frac{x_n-a_nm}{\sqrt{1-a_n^2}}\right)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 \leqslant x_1,\ldots,X_n \leqslant x_n \mid M=m)\phi(m)dm \\ &= \mathbb{P}(X_1 \leqslant x_1,\ldots,X_n \leqslant x_n), \qquad 0 \leqslant x_1,x_2,\ldots,x_n \leqslant 1, \end{aligned}$$

of the vector $(X_1,\ldots,X_n)$, with covariance matrix given by (8.13). We conclude by Proposition 8.8. □

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Exercises

Exercise 8.1 Compute the conditional probability density function of the default time τ defined in (8.3).

Exercise 8.2 Credit Default Contract. The assets of a company are modeled using a geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ with drift $r > 0$ under the risk-neutral probability measure $\mathbb{P}^*$. A Credit Default Contract pays \$1 as soon as the asset S_t hits a level $K > 0$. Price this contract at time $t > 0$ assuming that $S_t > K$.

Exercise 8.3

a) Check that the vector $(X_1, X_2, \ldots, X_n)$ defined in (8.10) has the covariance matrix given by (8.13).
b) Show that the vector $(X_1, X_2, \ldots, X_n)$, with covariance matrix (8.13) has standard Gaussian marginals.
c) By computing explicitly the probability density function of $(X_1, \ldots, X_n)$, recover the fact that it is a jointly Gaussian random vector with covariance matrix (8.13).

Exercise 8.4 Compute the inverse Σ^{-1} of the covariance matrix (8.13) in case $n = 2$.