

# Chapter 7

## Expected Shortfall

This chapter presents the construction of Tail Value at Risk (TVaR) and Expected Shortfall (ES), which, unlike Value at Risk, are coherent risk measures. Tail Value at Risk at a confidence level  $p \in (0, 1)$  is defined as the average of losses suffered in the worst  $(1 - p)\%$  outcomes. Expected Shortfall provides an alternative computation of Tail Value at Risk by averaging losses that exceed the VaR level. Experiments using financial data sets are also included.

---

<b>7.1 Tail Value at Risk (TVaR)</b> .....	<b>201</b>
<b>7.2 Conditional Tail Expectation (CTE)</b> .....	<b>202</b>
<b>7.3 Expected Shortfall (ES)</b> .....	<b>206</b>
<b>7.4 Numerical Estimates</b> .....	<b>214</b>
<b>Exercises</b> .....	<b>219</b>

---

### 7.1 Tail Value at Risk (TVaR)

A basic shortcoming of Value at Risk is failing to provide information on the behavior of probability distribution tails beyond  $V_X^p$ . The next figure illustrates the limitations of Value at Risk, namely its inability to capture the properties of a probability distribution beyond  $V_X^p$ .<sup>†</sup>

---

<sup>†</sup> “Value at Risk is like an airbag that works all the time, except when you have a car accident”. - D. Einhorn, hedge fund manager.



Fig. 7.1: Two distributions having the same Value at Risk  $V_X^{95\%} = 2.145$ .

The Tail Value at Risk (or Conditional Value at Risk) aims at providing a solution to the tail distribution problem observed with Value at Risk at the level  $p \in (0, 1)$  by averaging over confidence levels ranging from  $p$  to 1.

**Definition 7.1.** *The Tail Value at Risk (TVaR) of a random variable  $X$  at the level  $p \in (0, 1)$  is defined by the average*

$$\text{TVaR}_X^p := \frac{1}{1-p} \int_p^1 V_X^q dq. \tag{7.1}$$

We note the following property.

**Proposition 7.2.** *The Tail Value at Risk (TVaR) and Value at Risk (VaR) satisfy the following inequality:*

$$\text{TVaR}_X^p \geq V_X^p, \quad p \in (0, 1).$$

*Proof.* Since the function  $p \mapsto V_X^p$  is non-decreasing by Proposition 6.12, we have

$$\text{TVaR}_X^p = \frac{1}{1-p} \int_p^1 V_X^q dq \geq \frac{1}{1-p} \int_p^1 V_X^p dq = V_X^p.$$

□

## 7.2 Conditional Tail Expectation (CTE)

Recall that by Lemma A.18, given an event  $A$  such that  $\mathbb{P}(A) > 0$ , the conditional expectation of  $X : \Omega \rightarrow \mathbb{N}$  given the event  $A$  satisfies

$$\mathbb{E}[X | A] = \frac{1}{\mathbb{P}(A)} \mathbb{E}[X \mathbb{1}_A],$$

see Section 6.1 for an example.

**Definition 7.3.** Consider a random variable  $X$  such that  $\mathbb{P}(X > V_X^p) > 0$ . The Conditional Tail Expectation of  $X$  at the level  $p \in (0, 1)$  is the quantity

$$\text{CTE}_X^p := \mathbb{E}[X \mid X > V_X^p] = \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}].$$

The use of the strict inequality “>” in the definition of the Conditional Tail Expectation allows us to avoid any dependence on  $\mathbb{P}(X = V_X^p)$ , and to consider risky values strictly beyond  $V_X^p$ . The Conditional Tail Expectation is also called Conditional Value at Risk (CVaR).

Examples of Conditional Tail Expectations can be computed using the following code in which computation is done on sign-changed data, *i.e.* according to the “practitioner” point of view.

```
1 library(quantmod); getSymbols("^HSI",from="2013-06-01",to="2014-10-01",src="yahoo")
2 data=Ad("HSI");returns=(data-lag(data))/lag(data);returns=as.numeric(returns[is.na(returns)])
3 VaR(.95);var=-quantile(-returns,probs=.95,type=1);cte=mean(returns[returns<as.numeric(var)])
```

Listing 7.1: R code - Computation of conditional tail expectation.

```
1 import yfinance as yf, numpy as np, pandas as pd;
2 data= yf.download("^HSI", start="2013-06-01", end="2014-10-01")
3 close_prices= data["Close"]["HSI"]; returns= close_prices.pct_change().dropna(); VaR(0.95)
4 var=pd.Series(returns).quantile(0.05,interpolation='nearest');cte=np.mean(returns[returns<var])
5 print(f'95% Historical VaR: {var:.6f}'); print(f'Conditional Tail Expectation: {cte:.6f}')
```

Listing 7.2: Python code - Computation of conditional tail expectation.

Proposition 7.4 shows more precisely by which amount the Conditional Tail Expectation exceeds the Value at Risk.

**Proposition 7.4.** Let  $X$  be a random variable  $X$  such that  $\mathbb{P}(X > V_X^p) > 0$ . For any  $p \in (0, 1]$  we have  $\text{CTE}_X^p > \mathbb{E}[X]$  and  $\text{CTE}_X^p > V_X^p$  with, more precisely,

$$\text{CTE}_X^p = \mathbb{E}[X \mid X > V_X^p] = V_X^p + \mathbb{E}[(X - V_X^p)^+ \mid X > V_X^p].$$

*Proof.* We have

$$\begin{aligned} \mathbb{E}[X \mid X > V_X^p] &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}] \\ &= \frac{1}{\mathbb{P}(X > V_X^p)} (\mathbb{E}[(X - V_X^p) \mathbb{1}_{\{X > V_X^p\}}] + V_X^p \mathbb{E}[\mathbb{1}_{\{X > V_X^p\}}]) \\ &= \frac{1}{\mathbb{P}(X > V_X^p)} (\mathbb{E}[(X - V_X^p)^+] + V_X^p \mathbb{P}(X > V_X^p)) \end{aligned}$$

$$\begin{aligned} &= V_X^p + \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[(X - V_X^p)^+] \\ &= V_X^p + \mathbb{E}[(X - V_X^p)^+ | X > V_X^p]. \end{aligned}$$

See Exercise 7.2-(h) for a proof of  $\text{CTE}_X^p > \mathbb{E}[X]$ . □

Next, we check that when  $\mathbb{P}(X = V_X^p) = 0$ , the Conditional Tail Expectation coincides with the Tail Value at Risk. Note that in this case, we have

$$\mathbb{P}(X > V_X^p) = 1 - p > 0$$

by (6.11) in Proposition 6.13.

**Proposition 7.5.** *Assume that  $\mathbb{P}(X = V_X^p) = 0$ . Then, we have*

$$\text{CTE}_X^p = \text{TV}_X^p,$$

i.e.

$$\text{CTE}_X^p = \mathbb{E}[X | X > V_X^p] = \mathbb{E}[X | X \geq V_X^p] = \frac{1}{1-p} \int_p^1 V_X^q dq = \text{TV}_X^p. \quad (7.2)$$

*Proof.* By Lemma 6.20 we construct  $X$  as  $X = V_X^U$  where  $U$  is uniformly distributed on  $[0, 1]$ , with

$$U \geq p \implies V_X^U \geq V_X^p \iff X \geq V_X^p,$$

and

$$U \leq p \implies V_X^U \leq V_X^p \iff X \leq V_X^p,$$

hence

$$X > V_X^p \iff V_X^U > V_X^p \implies U > p.$$

Since  $\mathbb{P}(X = V_X^p) = 0$  we find that, with probability 1,

$$U \geq p \iff U > p \iff V_X^U \geq V_X^p \iff X \geq V_X^p \iff X > V_X^p,$$

hence

$$\begin{aligned} \text{CTE}_X^p &= \mathbb{E}[X | X > V_X^p] \\ &= \mathbb{E}[V_X^U | V_X^U > V_X^p] \\ &= \mathbb{E}[V_X^U | U \geq p] \\ &= \frac{1}{\mathbb{P}(U \geq p)} \mathbb{E}[V_X^U \mathbb{1}_{\{U \geq p\}}] \end{aligned}$$

$$= \frac{1}{1-p} \int_p^1 V_X^q dq.$$

□

Figure 7.2 shows the locations of Value at Risk and Conditional Tail Expectation on a given data set. Note that here, the computation is done on sign-changed data according to Proposition 6.15, *i.e.* the output is computed according to the “practitioner” point of view.

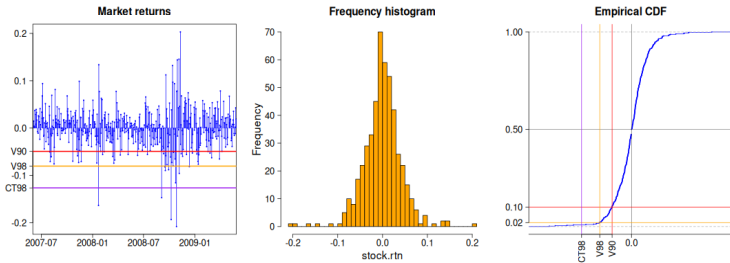


Fig. 7.2: Value at Risk and Conditional Tail Expectation.

The Conditional Tail Expectation of a Gaussian  $\mathcal{N}(\mu, \sigma^2)$  random variable is computed in the next proposition, see also Proposition 6.17.

**Proposition 7.6.** Gaussian CTE and TVAR. *Given  $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$  we have*

$$\text{CTE}_X^p = \text{TVAR}_X^p = \mu_X + \frac{\sigma_X}{1-p} \phi(q_Z^p) = \mu_X + \frac{\sigma_X}{(1-p)\sqrt{2\pi}} e^{-(q_Z^p)^2/2}, \quad (7.3)$$

where  $q_Z^p = \Phi^{-1}(p)$  is the Gaussian quantile of  $Z \simeq \mathcal{N}(0, 1)$  at the level  $p \in (0, 1)$  and

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R},$$

is the standard normal probability density function.

*Proof.* Using the relation  $\mathbb{P}(X \geq V_X^p) = \mathbb{P}(X > V_X^p) = 1 - p$ , cf. Proposition 6.14, by Proposition 7.5 we have

$$\begin{aligned} \text{CTE}_X^p &= \text{TVAR}_X^p \\ &= \mathbb{E}[X \mid X > V_X^p] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}] \\
 &= \frac{1}{1-p} \int_{V_X^p}^{\infty} x e^{-(x-\mu_X)^2/(2\sigma_X^2)} \frac{dx}{\sqrt{2\pi\sigma_X^2}} \\
 &= \frac{\mu_X}{1-p} \int_{V_X^p}^{\infty} e^{-(x-\mu_X)^2/(2\sigma_X^2)} \frac{dx}{\sqrt{2\pi\sigma_X^2}} \\
 &\quad + \frac{1}{1-p} \int_{V_X^p}^{\infty} (x-\mu_X) e^{-(x-\mu_X)^2/(2\sigma_X^2)} \frac{dx}{\sqrt{2\pi\sigma_X^2}} \\
 &= \frac{\mu_X}{1-p} \mathbb{P}(X \geq V_X^p) + \frac{\sigma_X^2}{(1-p)\sqrt{2\pi\sigma_X^2}} \left[ -e^{-(x-\mu_X)^2/(2\sigma_X^2)} \right]_{V_X^p}^{\infty} \\
 &= \mu_X + \frac{\sigma_X^2}{(1-p)\sqrt{2\pi\sigma_X^2}} e^{-((V_X^p-\mu_X)/\sigma_X)^2/2} \\
 &= \mu_X + \frac{\sigma_X}{(1-p)\sqrt{2\pi}} e^{-(q_Z^p)^2/2} \\
 &= \mu_X + \frac{\sigma_X}{1-p} \phi(q_Z^p),
 \end{aligned}$$

due to the rescaling relation  $V_X^p = \mu_X + \sigma_X q_Z^p$ , cf. (6.15). □

### 7.3 Expected Shortfall (ES)

There are several variants for the definition of the Expected Shortfall  $ES_X^p$ . Next is a frequently used definition.

**Definition 7.7.** *The Expected Shortfall  $ES_X^p$  of a random variable  $X$  at the level  $p \in (0, 1)$  is defined by*

$$ES_X^p := V_X^p + \frac{1}{1-p} \mathbb{E}[(X - V_X^p)^+]. \tag{7.4}$$

Note that the quantity  $\mathbb{E}[(X - V_X^p)^+]$  in (7.4) can be interpreted as a European call option price. Proposition 7.8 provides an alternative expression for the expected shortfall  $ES_X^p$ .

**Proposition 7.8.** *The Expected Shortfall of  $X$  at the level  $p \in (0, 1)$  can be written as*

$$ES_X^p = \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)).$$

*Proof.* By Lemma 6.2, we have

$$\begin{aligned}
 \text{ES}_X^p &= V_X^p + \frac{1}{1-p} \mathbb{E}[(X - V_X^p)^+] \\
 &= V_X^p + \frac{1}{1-p} \mathbb{E}[(X - V_X^p) \mathbb{1}_{\{X \geq V_X^p\}}] \\
 &= V_X^p + \frac{\mathbb{P}(X \geq V_X^p)}{1-p} \mathbb{E}[X - V_X^p \mid X \geq V_X^p] \\
 &= V_X^p + \frac{1}{1-p} (\mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] - V_X^p \mathbb{E}[\mathbb{1}_{\{X \geq V_X^p\}}]) \\
 &= V_X^p + \frac{1}{1-p} (\mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] - V_X^p \mathbb{P}(X \geq V_X^p)) \\
 &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)).
 \end{aligned}$$

□

When  $\mathbb{P}(X = V_X^p) = 0$ , Proposition 7.8 also yields

$$\text{ES}_X^p = \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X > V_X^p)).$$

**Proposition 7.9.** *The Conditional Tail Expectation  $\text{CTE}_X^p$  at the level  $p \in (0, 1)$  can be written as the distortion risk measure*

$$\text{CTE}_X^p := \mathbb{E}[X f_X(X)], \tag{7.5}$$

where  $f_X$  is the function defined by

$$f_X(x) := \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{1}_{\{x > V_X^p\}}, \quad x \in \mathbb{R}. \tag{7.6}$$

*Proof.* We check that the function  $f_X$  defined in (7.6) is a *distortion function* according to Definition 6.4. Indeed, by Proposition 6.16 we have

$$\begin{aligned}
 f_{\mu+X}(\mu+x) &= \frac{1}{\mathbb{P}(\mu+X > V_{\mu+X}^p)} \mathbb{1}_{\{\mu+x > V_{\mu+X}^p\}} \\
 &= \frac{1}{\mathbb{P}(\mu+X > \mu + V_X^p)} \mathbb{1}_{\{\mu+x > \mu + V_X^p\}} \\
 &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{1}_{\{x > V_X^p\}} \\
 &= f_X(x), \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R},
 \end{aligned}$$

and

$$\begin{aligned}
 f_{\lambda X}(\lambda x) &= \frac{1}{\mathbb{P}(\lambda X > V_{\lambda X}^p)} \mathbb{1}_{\{\lambda x > V_{\lambda X}^p\}} \\
 &= \frac{1}{\mathbb{P}(\lambda X > \lambda V_X^p)} \mathbb{1}_{\{\lambda x > \lambda V_X^p\}} \\
 &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{1}_{\{x > V_X^p\}} \\
 &= f_X(x), \quad x \in \mathbb{R}, \quad \lambda > 0.
 \end{aligned}$$

Finally, we note that

$$\begin{aligned}
 \mathbb{E}[f_X(X)] &= \mathbb{E}\left[\frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{1}_{\{x > V_X^p\}}\right] \\
 &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}\left[\mathbb{1}_{\{x > V_X^p\}}\right] \\
 &= \frac{\mathbb{P}(X > V_X^p)}{\mathbb{P}(X > V_X^p)} \\
 &= 1.
 \end{aligned}$$

□

**Proposition 7.10.** *When  $\mathbb{P}(X = V_X^p) = 0$ , the Expected Shortfall  $\text{ES}_X^p$  coincides with the Conditional Tail Expectation  $\text{CTE}_X^p$  and with the Tail Value at Risk  $\text{TV}_X^p$ , i.e., we have*

$$\text{ES}_X^p = \mathbb{E}[X \mid X > V_X^p] = \mathbb{E}[X \mid X \geq V_X^p] = \text{TV}_X^p.$$

*Proof.* By Relation (6.12) in Proposition 6.13, when  $\mathbb{P}(X = V_X^p) = 0$  we have

$$p = \mathbb{P}(X \leq V_X^p) \quad \text{and} \quad 1 - p = \mathbb{P}(X > V_X^p) = \mathbb{P}(X \geq V_X^p),$$

hence

$$\begin{aligned}
 \text{ES}_X^p &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] \\
 &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}] \\
 &= \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}] \\
 &= \mathbb{E}[X \mid X > V_X^p]
 \end{aligned}$$

$$= \text{TV}_X^p,$$

by Proposition 7.5. □

From Propositions 7.8 and 7.10, we deduce that

$$\text{ES}_X^p = \begin{cases} \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] = \mathbb{E}[X | X > V_X^p] = \text{TV}_X^p & \text{if } \mathbb{P}(X = V_X^p) = 0, \\ \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)) & \text{if } \mathbb{P}(X = V_X^p) \geq 0. \end{cases}$$

In particular, by Propositions 7.6 and 7.10, the Gaussian Expected Shortfall of  $X \simeq \mathcal{N}(\mu, \sigma^2)$  at the level  $p \in (0, 1)$  is also given by

$$\text{ES}_X^p = \text{CTE}_X^p = \mu + \frac{\sigma}{1-p} \phi(\Phi^{-1}(p)) = \mu + \frac{\sigma}{(1-p)\sqrt{2\pi}} e^{-(\Phi^{-1}(p))^2/2}.$$

**Proposition 7.11.** *The Expected Shortfall  $\text{ES}_X^p$  at the level  $p \in (0, 1)$  can be written as the distortion risk measure*

$$\text{ES}_X^p = \mathbb{E}[X f_X(X)], \tag{7.7}$$

where the function  $f_X$  defined by

$$f_X(x) := \frac{1}{1-p} \mathbb{1}_{\{x > V_X^p\}} + \mathbb{1}_{\{\mathbb{P}(X = V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{(1-p)\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{x = V_X^p\}},$$

$x \in \mathbb{R}$ , is a distortion function.

*Proof.* By Proposition 7.8, we have

$$\begin{aligned} \text{ES}_X^p &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)) \\ &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \mathbb{1}_{\{\mathbb{P}(X = V_X^p) > 0\}} \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)) \\ &= \frac{1}{1-p} \mathbb{E} \left[ \left( \mathbb{1}_{\{X \geq V_X^p\}} + \mathbb{1}_{\{\mathbb{P}(X = V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X \geq V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{X = V_X^p\}} \right) X \right] \\ &= \frac{1}{1-p} \mathbb{E} \left[ \left( \mathbb{1}_{\{X \geq V_X^p\}} - \mathbb{1}_{\{\mathbb{P}(X = V_X^p) > 0\}} \mathbb{1}_{\{X = V_X^p\}} \right. \right. \\ &\quad \left. \left. + \mathbb{1}_{\{\mathbb{P}(X = V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{X = V_X^p\}} \right) X \right] \end{aligned}$$

$$= \frac{1}{1-p} \mathbb{E} \left[ \left( \mathbb{1}_{\{X > V_X^p\}} + \mathbb{1}_{\{\mathbb{P}(X=V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{X=V_X^p\}} \right) X \right].$$

In order to show that  $f_X$  is a distortion function according to Definition 6.4 we can proceed as in the proof of Proposition 7.9, and in particular we can check that

$$\begin{aligned} \mathbb{E}[f_X(X)] &= \frac{1}{1-p} \mathbb{E} \left[ \mathbb{1}_{\{X > V_X^p\}} + \mathbb{1}_{\{\mathbb{P}(X=V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{X=V_X^p\}} \right] \\ &= \frac{1}{1-p} \left( \mathbb{E}[\mathbb{1}_{\{X > V_X^p\}}] + 1-p - \mathbb{P}(X > V_X^p) \right) \\ &= \frac{1}{1-p} \left( \mathbb{P}(X > V_X^p) + 1-p - \mathbb{P}(X > V_X^p) \right) \\ &= 1. \end{aligned} \tag{7.8}$$

□

By Lemma 6.20 and Proposition 7.11, we also have

$$ES_X^p = \int_0^1 V_X^q f_X(V_X^q) dq,$$

and we check that the distortion function  $f_X$  of Proposition 7.11 is a non-decreasing function that satisfies

$$f_X(x) \leq \frac{1}{1-p}, \quad x \in \mathbb{R},$$

by (6.11). The following proposition, see Acerbi and Tasche (2001), shows that in general, the Expected Shortfall at the level  $p \in (0, 1)$  coincides with the Tail Value at Risk  $TV_X^p$ .

**Theorem 7.12.** *The Expected Shortfall  $ES_X^p$  coincides with the Tail Value at Risk  $TV_X^p$  for any  $p \in (0, 1)$ , i.e. we have*

$$ES_X^p = TV_X^p = \frac{1}{1-p} \int_p^1 V_X^q dq.$$

*Proof.* Constructing  $X$  as  $X = V_X^U$  where  $U$  is uniformly distributed on  $[0, 1]$  as in Lemma 6.20, by Proposition 6.12 we have

$$U \geq p \implies V_X^U \geq V_X^p \implies X \geq V_X^p$$

and

$$(U < p \text{ and } X \geq V_X^p) \implies (V_X^U \leq V_X^p \text{ and } X \geq V_X^p)$$

$$\begin{aligned} &\implies (X \leq V_X^p \text{ and } X \geq V_X^p) \\ &\implies X = V_X^p. \end{aligned}$$

Hence by (7.4) and the relations

$$1 - p = \mathbb{E}[\mathbb{1}_{\{U \geq p\}}] \quad \text{and} \quad \mathbb{P}(X \geq V_X^p) = \mathbb{E}[\mathbb{1}_{\{X \geq V_X^p\}}],$$

we have

$$\begin{aligned} V_X^p(1 - p - \mathbb{P}(X \geq V_X^p)) &= -V_X^p \mathbb{E}[\mathbb{1}_{\{X \geq V_X^p\}} - \mathbb{1}_{\{U \geq p\}}] \\ &= -V_X^p \mathbb{E}[\mathbb{1}_{\{X \geq V_X^p\} \setminus \{U \geq p\}}] \\ &= -V_X^p \mathbb{E}[\mathbb{1}_{\{X \geq V_X^p\} \cap \{U < p\}}] \\ &= -\mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\} \cap \{U < p\}}], \end{aligned}$$

hence

$$\begin{aligned} \text{ES}_X^p &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] + \frac{V_X^p}{1-p} (1-p - \mathbb{P}(X \geq V_X^p)) \\ &= \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\}}] - \frac{1}{1-p} \mathbb{E}[X \mathbb{1}_{\{X \geq V_X^p\} \cap \{U < p\}}] \\ &= \frac{1}{1-p} \mathbb{E}[V_X^U \mathbb{1}_{\{V_X^U \geq V_X^p\}}] - \frac{1}{1-p} \mathbb{E}[V_X^U \mathbb{1}_{\{V_X^U \geq V_X^p\}} \mathbb{1}_{\{U < p\}}] \\ &= \frac{1}{1-p} \mathbb{E}[V_X^U \mathbb{1}_{\{V_X^U \geq V_X^p\}} \mathbb{1}_{\{U \geq p\}}] \\ &= \frac{1}{1-p} \mathbb{E}[V_X^U \mathbb{1}_{\{U \geq p\}}] \\ &= \frac{1}{1-p} \int_p^1 V_X^q dq, \end{aligned}$$

which is the Tail Value at Risk  $\text{TV}_X^p$ . □

**Theorem 7.13.** *Expected Shortfall  $\text{ES}_X^p$  and Tail Value at Risk  $\text{TV}_X^p$  are coherent risk measures.*

*Proof.* Since the Expected Shortfall  $\text{ES}_X^p$  is a *distortion risk measure* by Proposition 7.11, we can conclude to positive homogeneity and translation invariance as done after Definition 6.4.

Alternatively, as  $\text{ES}_X^p$  coincides with  $\text{TV}_X^p$  for all  $p \in (0, 1)$  from Theorem 7.12, we can use Relation (7.1) in Definition 7.1 or Relation (7.4) in Definition 7.7 to proceed as follows.

(i) Monotonicity. If  $X \leq Y$ , since Value at Risk is monotone by Proposition 6.16, we have

$$\begin{aligned} \text{ES}_X^p &= \text{TV}_X^p \\ &= \frac{1}{1-p} \int_p^1 V_X^q dq \\ &\leq \frac{1}{1-p} \int_p^1 V_Y^q dq \\ &= \text{TV}_Y^p \\ &= \text{ES}_Y^p \end{aligned}$$

for all  $p \in (0, 1)$ .

(ii) Homogeneity and translation invariance. Similarly, since Value at Risk satisfies the homogeneity and translation invariance properties, for all  $\mu \in \mathbb{R}$  and  $\lambda > 0$  we have

$$\begin{aligned} \text{ES}_{\mu+\lambda X}^p &= \text{TV}_{\mu+\lambda X}^p \\ &= \frac{1}{1-p} \int_p^1 V_{\mu+\lambda X}^q dq \\ &= \frac{1}{1-p} \int_p^1 (\mu + \lambda V_X^q) dq \\ &= \mu + \lambda \frac{1}{1-p} \int_p^1 V_X^q dq \\ &= \mu + \lambda \text{TV}_X^p \\ &= \mu + \lambda \text{ES}_X^p \end{aligned}$$

for all  $p \in (0, 1)$ .

(iii) Sub-additivity. By Proposition 7.11, we have

$$\begin{aligned} &(1-p) (\text{ES}_{X+Y}^p - \text{ES}_X^p - \text{ES}_Y^p) \\ &= \mathbb{E}[(X+Y)f_{X+Y}(X+Y)] - \mathbb{E}[Xf_X(X)] - \mathbb{E}[Yf_Y(Y)] \\ &= \mathbb{E}[X(f_{X+Y}(X+Y) - f_X(X))] + \mathbb{E}[Y(f_{X+Y}(X+Y) - f_Y(Y))] \\ &= V_X^p \mathbb{E}[f_{X+Y}(X+Y) - f_X(X)] + \mathbb{E}[(X - V_X^p)(f_{X+Y}(X+Y) - f_X(X))] \\ &\quad + V_Y^p \mathbb{E}[f_{X+Y}(X+Y) - f_Y(Y)] + \mathbb{E}[(Y - V_Y^p)(f_{X+Y}(X+Y) - f_Y(Y))] \\ &= (1-1)V_X^p + \mathbb{E}[(X - V_X^p)(f_{X+Y}(X+Y) - f_X(X))] \\ &\quad + (1-1)V_Y^p + \mathbb{E}[(Y - V_Y^p)(f_{X+Y}(X+Y) - f_Y(Y))] \\ &\leq 0, \end{aligned}$$

where we have used (7.8) and the following facts.

- When  $x - V_X^p < 0$ , we have

$$\begin{aligned}
 (1-p)(f_{X+Y}(x+y) - f_X(x)) &= \mathbb{1}_{\{x+y > V_{X+Y}^p\}} - \mathbb{1}_{\{x > V_X^p\}} \\
 &+ \mathbb{1}_{\{\mathbb{P}(X+Y=V_{X+Y}^p) > 0\}} \frac{1-p - \mathbb{P}(X+Y > V_{X+Y}^p)}{\mathbb{P}(X+Y = V_{X+Y}^p)} \mathbb{1}_{\{x+y=V_{X+Y}^p\}} \\
 &- \mathbb{1}_{\{\mathbb{P}(X=V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{x=V_X^p\}} \\
 &= \mathbb{1}_{\{x+y > V_{X+Y}^p\}} + \mathbb{1}_{\{\mathbb{P}(X+Y=V_{X+Y}^p) > 0\}} \frac{1-p - \mathbb{P}(X+Y > V_{X+Y}^p)}{\mathbb{P}(X+Y = V_{X+Y}^p)} \mathbb{1}_{\{x+y=V_{X+Y}^p\}} \\
 &\geq 0, \quad x < V_X^p,
 \end{aligned}$$

where we applied (6.11).

- When  $x - V_X^p > 0$ , we have

$$\begin{aligned}
 (1-p)(f_{X+Y}(x+y) - f_X(x)) &= \mathbb{1}_{\{x+y > V_{X+Y}^p\}} - \mathbb{1}_{\{x > V_X^p\}} \\
 &+ \mathbb{1}_{\{\mathbb{P}(X+Y=V_{X+Y}^p) > 0\}} \frac{1-p - \mathbb{P}(X+Y > V_{X+Y}^p)}{\mathbb{P}(X+Y = V_{X+Y}^p)} \mathbb{1}_{\{x+y=V_{X+Y}^p\}} \\
 &- \mathbb{1}_{\{\mathbb{P}(X=V_X^p) > 0\}} \frac{1-p - \mathbb{P}(X > V_X^p)}{\mathbb{P}(X = V_X^p)} \mathbb{1}_{\{x=V_X^p\}} \\
 &= \mathbb{1}_{\{x+y > V_{X+Y}^p\}} - \mathbb{1}_{\{x > V_X^p\}} \\
 &+ \mathbb{1}_{\{\mathbb{P}(X+Y=V_{X+Y}^p) > 0\}} \frac{1-p - \mathbb{P}(X+Y > V_{X+Y}^p)}{\mathbb{P}(X+Y = V_{X+Y}^p)} \mathbb{1}_{\{x+y=V_{X+Y}^p\}} \\
 &\leq \mathbb{1}_{\{x+y > V_{X+Y}^p\}} - \mathbb{1}_{\{x > V_X^p\}} + \mathbb{1}_{\{x+y=V_{X+Y}^p\}} \\
 &= \mathbb{1}_{\{x+y \geq V_{X+Y}^p\}} - \mathbb{1}_{\{x > V_X^p\}} \\
 &\leq 0, \quad x > V_X^p,
 \end{aligned}$$

where we applied (6.11).

□

Note that in general, the Conditional Tail Expectation is not a coherent risk measure when  $\mathbb{P}(X = V_X^p) > 0$ .

## 7.4 Numerical Estimates

In case we care about negative return values, Definitions 7.3 and 7.7 are replaced with

$$\overline{\text{CTE}}_X^p := \mathbb{E}[X \mid X < \overline{V}_X^p] = \frac{1}{\mathbb{P}(X < \overline{V}_X^p)} \mathbb{E}[X \mathbb{1}_{\{X < \overline{V}_X^p\}}]$$

and


$$\overline{\text{ES}}_X^p := \overline{V}_X^p - \frac{1}{1-p} \mathbb{E}[(\overline{V}_X^p - X)^+].$$

From Proposition 6.15, when the cumulative distribution function  $F_X$  is continuous and strictly increasing, we have

$$\begin{aligned} \overline{\text{CTE}}_X^p &= \overline{\text{ES}}_X^p \\ &= \mathbb{E}[X \mid X < \overline{V}_X^p] \\ &= \mathbb{E}[X \mid X < V_X^{1-p}] \\ &= \frac{1}{\mathbb{P}(X < V_X^{1-p})} \mathbb{E}[X \mathbb{1}_{\{X < V_X^{1-p}\}}] \\ &= -\frac{1}{\mathbb{P}(-X > -V_X^{1-p})} \mathbb{E}[-X \mathbb{1}_{\{-X > -V_X^{1-p}\}}] \\ &= -\mathbb{E}[-X \mid -X > -V_X^{1-p}] \\ &= -\mathbb{E}[-X \mid -X > V_{-X}^p] \\ &= -\text{CTE}_{-X}^p \\ &= -\text{ES}_{-X}^p. \end{aligned}$$

The 95% historical Conditional Tail Expectation is  $\text{CTE}_X^{95\%} = -0.02117477$ , and can be computed from the empirical Conditional Tail Expectation (CTE) as follows.

```
1 mean(returns[returns<(Var(0.95))],na.rm=TRUE)
```

Listing 7.3:  code - Computation of conditional tail expectation.

```
1 var2=pd.Series(returns).quantile(0.05,interpolation='nearest');returns[returns < var2].mean()
```

Listing 7.4: Python code - Computation of conditional tail expectation.

The Gaussian Expected Shortfall can be computed from (7.3) (after sign inversion) as


$$\begin{aligned}
 \text{ES}_X^p &= -\text{ES}_{-X}^{1-p} \\
 &= -\left(-\mu + \frac{\sigma}{1-p}\phi\left(V_Z^{1-p}\right)\right) \\
 &= \mu - \frac{\sigma}{1-p}\phi\left(V_Z^p\right) \\
 &= \mu - \frac{\sigma}{(1-p)\sqrt{2\pi}}e^{-(V_Z^p)^2/2},
 \end{aligned}$$

which yields the output  $-0.01916536$  in the following code.

```

1 q=qnorm(.95, mean=0, sd=1); mu=mean(returns,na.rm=TRUE)
  sigma=sd(returns,na.rm=TRUE); mu-sigma*dnorm(q)/0.05

```

Listing 7.5:  code - Gaussian expected shortfall.

```

1 import numpy as np; from scipy.stats import norm
2 q= norm.ppf(0.95); mu= returns.mean(); sigma= returns.std()
  result= mu - sigma * norm.pdf(q) / 0.05; print(f"Result: {float(result):.6f}")

```


Listing 7.6: Python code - Gaussian expected shortfall.

The expected shortfall can be computed using the following code, with output  $-0.02022518$ .

```

1 ES <- function(p){r=-returns; return (-(-VaR(p)+
  (mean(r[r]>=-VaR(p)))-(-VaR(p)))/length(r)/(1-p)))
  ES(0.95)

```

Listing 7.7:  code - Computation of expected shortfall.

```

1 ES=lambda p: ((v:=-VaR(p)), (r:=-returns), (b:=r[r>=v]),-(v+(b.mean()-v)*len(b)
  /len(r)/(1-p)))[3]
2 ES(0.95)

```


Listing 7.8: Python code - Computation of expected shortfall.

The TVaR can be recovered using the following code, with output  $-0.02022518$ .

```

1 TVaR <- function(p){r=-returns; ordered=r[order(r)]; l=length(ordered); i=1; TVaR=0;
  v=ordered[i];
2 while(length(r[r<=v])/length(r)>=p) {v=ordered[i]; j=1; while (v==ordered[i-j]) {j=j+1}; i=i-j;
  TVaR=TVaR+v*min(j/l,(i+j)/l-p); v=ordered[i]}; return (-TVaR/(1-p))}
  TVaR(0.95)

```

Listing 7.9:  code - Computation of Tail Value at Risk.

```

1 TVaR = lambda p: -(sum(v * min(cnt/l, (end_idx + 1)/l - p)
  for ordered in [np.sort(-returns)]
3 for l in [len(ordered)]
  for v, start_idx, cnt in zip(*np.unique(ordered, return_index=True, return_counts=True))
5 for end_idx in [start_idx + cnt - 1]
  if (end_idx + 1)/l >= p) / (1-p))
7 TVaR(0.95)

```

Listing 7.10: Python code - Computation of Tail Value at Risk.

The attached [R code 1](#) and [R code 2](#) compute the Expected Shortfall from the practitioner and academic points of views, and compare their outputs to the that of the PerformanceAnalytics package, as illustrated in Figure 7.3.

```
> source("comparison.R")
Number of samples= 265
VaR 95 = -0.03420879 , Threshold= 0.9433962
CTE 95 = -0.04646176
ES 95 = -0.04623058
Historical VaR 95 0= -0.03316604
Gaussian VaR 95 = -0.03209374
Historical ES 95 = -0.04552403
Gaussian ES 95 = -0.04043227
```

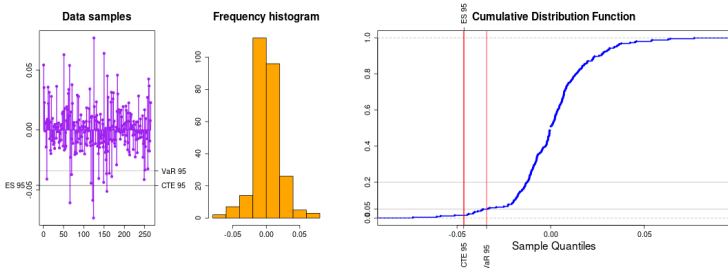


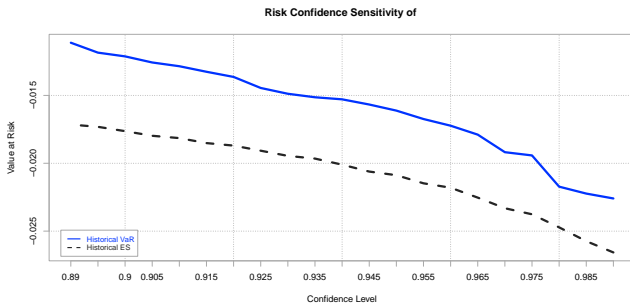
Fig. 7.3: Value at Risk and Expected Shortfall.

## Value at Risk *vs.* Expected Shortfall

The next code uses the PerformanceAnalytics [R](#) package.

```
1 library(PerformanceAnalytics); dev.new(width=16,height=8)
  chart.VaRSensitivity(ts(returns),methods=c("HistoricalVaR","HistoricalES"),
    colorset=bluefocus, lwd=4)
```

Listing 7.11: [R](#) code - Comparison of VaR and expected shortfall.

Fig. 7.4: Value at Risk *vs.* Expected Shortfall.

```

1 import matplotlib.pyplot as plt; import numpy as np; from scipy.stats import norm
2 conf_l= np.arange(0.90, 0.995, 0.005); hist_var= [returns.quantile(1 - cl) for cl in conf_l]
3 hist_es= [returns[returns <= returns.quantile(1 - cl)].mean() for cl in conf_l]
4 plt.figure(figsize=(16, 8)); plt.plot(conf_l, hist_var, 'b-', lw=4, label='Historical VaR')
5 plt.plot(conf_l, hist_es, 'k--', lw=4, label='Historical ES'); plt.grid(True, alpha=.3)
6 plt.tight_layout(); plt.xlabel('Confidence Level'); plt.ylabel('Value at Risk')
7 plt.title('VaR Sensitivity'); plt.legend(); plt.show()

```

Listing 7.12: Python code - Comparison of VaR and expected shortfall.

## Historical *vs.* Gaussian risk measures

```

1 dev.new(width=16,height=8)
2 chart.VaRSensitivity(ts(returns),methods=c("HistoricalVaR","GaussianVaR"),
3 colorset=bluefocus, lwd=4)

```

Listing 7.13: code - Comparison of Gaussian and historical VaR.

```

1 import matplotlib.pyplot as plt, numpy as np; from scipy.stats import norm
2 conf_l= np.arange(0.90, 0.995, 0.005); hist_var= [returns.quantile(1 - cl) for cl in conf_l]
3 gaussian_var=[returns.mean()+returns.std()*norm.ppf(1-cl) for cl in conf_l]
4 plt.figure(figsize=(16,8)); plt.plot(conf_l, hist_var, 'b-', lw=4, label='Hist. VaR'); plt.ylabel('Value')
5 plt.plot(conf_l, gaussian_var, 'k--', lw=4, label='Gaussian VaR'); plt.xlabel('Confidence Level')
6 plt.title('VaR Sensitivity'); plt.legend(); plt.grid(True, alpha=.3); plt.tight_layout(); plt.show()

```

Listing 7.14: Python code - Comparison of Gaussian and historical VaR.

Figure 7.5 uses the above code to compare the historical and Gaussian values at risk.

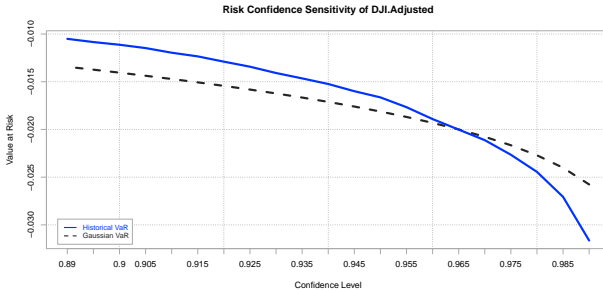



Fig. 7.5: Historical *vs.* Gaussian estimates of Value at Risk.

```

1 dev.new(width=16,height=8)
2 chart.VaRSensitivity(ts(returns),methods=c("HistoricalES","GaussianES"),colorset=bluefocus,
  lwd=4)

```

Listing 7.15:  code - Gaussian *vs.* historical expected shortfall.

```

1 import matplotlib.pyplot as plt; import numpy as np; from scipy.stats import norm
2 conf_l= np.arange(0.90, 0.995, 0.005)
3 hist_es= [returns[returns <= returns.quantile(1 - cl)].mean() for cl in conf_l]
4 gauss_es=[returns.mean()-returns.std()*norm.pdf(norm.ppf(cl))/(1-cl) for cl in conf_l]
5 plt.figure(figsize=(16,8));plt.plot(conf_l,hist_es,'b-',lw=4,label='Historical ES')
6 plt.plot(conf_l,gauss_es,'k-',lw=4,label='Gaussian ES');plt.xlabel('Confidence Level')
7 plt.ylabel('Value');plt.title('ES Sensitivity');plt.legend()
8 plt.grid(True,alpha=.3);plt.tight_layout();plt.show()

```

Listing 7.16: Python code - Gaussian *vs.* historical expected shortfall.

In Figure 7.6 we compare the Gaussian and historical estimates of Expected Shortfall.

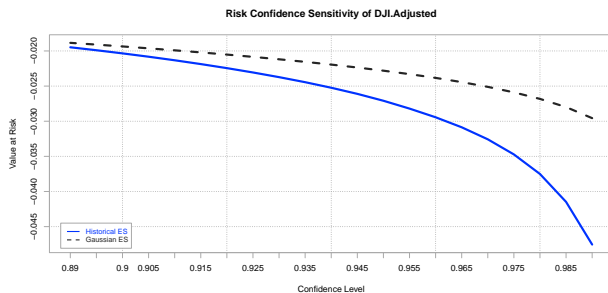


Fig. 7.6: Quantile function.

In Table 7.1 we summarize some properties of risk measures.

Risk Measure	Additivity	Homogeneity	Subadditivity	Coherence
$V_X$	✓	✓	✗	✗
$\text{CTE}_X$	✓	✓	✗	✗
$\text{TV}_X$	✓	✓	✓	✓
$\text{ES}_X$	✓	✓	✓	✓

Table 7.1: Summary of Risk Measures.

Note that Value at Risk  $V_X^p$  is *coherent* on Gaussian random variables according to Remark 6.19. Similarly, the Conditional Tail Expectation  $\text{CTE}_X^p$  is *coherent* on random variables having a continuous CDF by Proposition 7.5 and Theorem 7.13.

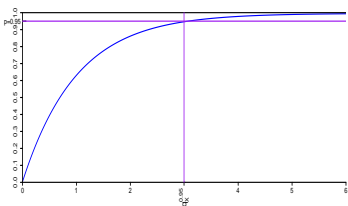
## Exercises

**Exercise 7.1** Let  $X$  denote an exponentially distributed random variable with parameter  $\lambda > 0$ , *i.e.* the distribution of  $X$  has the cumulative distribution function (CDF)

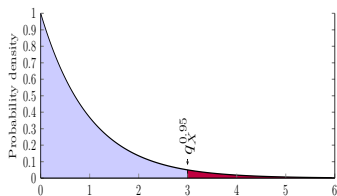
$$F_X(x) = \mathbf{P}(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0,$$

and the probability density function (PDF)

$$f_X(x) = F'_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$



(a) Exponential quantile and CDF.



(b) Exponential PDF.

a) Compute the conditional tail expectation

$$\mathbb{E}[X \mid X > \text{VaR}_X^p] = \frac{1}{\mathbb{P}(X > \text{VaR}_X^p)} \int_{\text{VaR}_X^p}^{\infty} x f_X(x) dx.$$

b) Compute the tail value at risk

$$\text{TV}_X^p = \frac{1}{1-p} \int_p^1 V_X^q dq.$$

**Exercise 7.2** Let  $X$  denote an (integrable) random variable, and let  $z \in \mathbb{R}$ . Show that the following statements hold.

- $\mathbb{E}[X \mid X \geq z] \geq z$ , provided that  $\mathbb{P}(X \geq z) > 0$ .
- $\mathbb{E}[X \mid X \geq z] \geq \mathbb{E}[X]$ , provided that  $\mathbb{P}(X \geq z) > 0$ .
- $\mathbb{E}[X \mid X > z] > z$ , provided that  $\mathbb{P}(X > z) > 0$ .
- $\mathbb{E}[X \mid X > z] \geq \mathbb{E}[X]$ , provided that  $\mathbb{P}(X > z) > 0$ .
- $\mathbb{E}[X \mid X > z] = \mathbb{E}[X \mid X \geq z]$ , provided that  $\mathbb{P}(X > z) > 0$  and  $\mathbb{P}(X = z) = 0$ .
- $\mathbb{E}[X \mid X > z] > \mathbb{E}[X \mid X \geq z]$ , provided that  $\mathbb{P}(X = z) > 0$ .
- $\mathbb{E}[X \mid X > z] > \mathbb{E}[X]$ , provided that  $\mathbb{P}(X > z) > 0$  and  $\mathbb{P}(X \leq z) > 0$ .
- $\text{CTE}_X^p > \mathbb{E}[X]$ , provided that  $\mathbb{P}(X > z) > 0$ .

**Exercise 7.3** Consider the following data set.

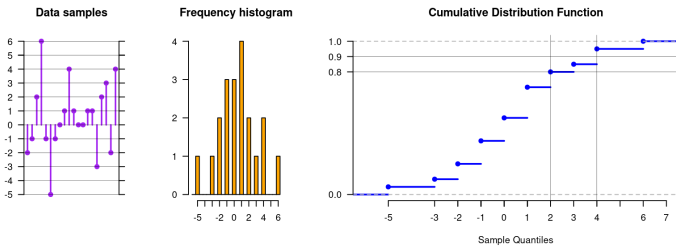


Fig. 7.8: Data set and cumulative distribution function.

Find the Value at Risk  $\text{VaR}_X^p$  and the Conditional Tail Expectation  $\text{CTE}_X^p = \mathbb{E}[X \mid X > \text{VaR}_X^p]$  and mark their values on the graph in the following cases.

- $p = 0.9$ .
- $p = 0.8$ .

#### Exercise 7.4

Consider the following data set.

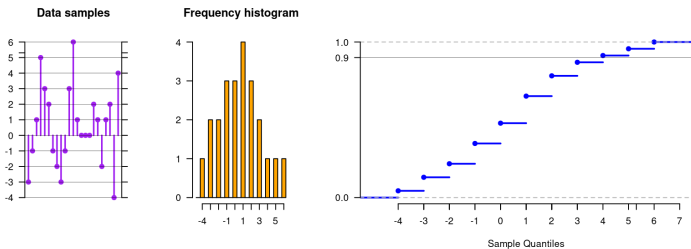


Fig. 7.9: Data set and cumulative distribution function.

For the above data set represented by the random variable  $X$ , compute the numerical values of the following quantities, with  $p = 0.9$ .

- $\text{VaR}_X^{90}$ ,
- $\mathbb{E}[X \mathbb{1}_{\{X > V_X^{90}\}}]$ ,
- $\mathbb{P}(X > V_X^{90})$ ,

- d)  $\text{CTE}_X^{90} = \mathbb{E}[X \mid X > V_X^{90}] = \mathbb{E}[X \mathbb{1}_{\{X > V_X^{90}\}}] / \mathbb{P}(X > V_X^{90})$ ,
- e)  $\mathbb{E}[X \mathbb{1}_{\{X \geq V_X^{90}\}}]$ ,
- f)  $\mathbb{P}(X \geq V_X^{90})$ ,
- g)  $\text{ES}_X^{90} = \frac{1}{1-p} (\mathbb{E}[X \mathbb{1}_{\{X \geq V_X^{90}\}}] + V_X^{90} (1-p - \mathbb{P}(X \geq V_X^{90})))$ ,
- h)  $\text{TV}_X^{90} = \frac{1}{1-p} \int_p^1 V_X^q dq$ ,

and mark the values of  $\text{VaR}_X^{90\%}$ ,  $\text{CTE}_X^{90\%}$ ,  $\text{ES}_X^{90\%}$ ,  $\text{TV}_X^{90\%}$  on the above graph.

**Exercise 7.5** Consider a random variable  $X \in \{10, 100, 150\}$  with the distribution

$$\mathbb{P}(X = 10) = 96\%, \quad \mathbb{P}(X = 100) = 3\%, \quad \mathbb{P}(X = 150) = 1\%.$$

Compute

- a) the Value at Risk  $V_X^{98\%}$ ,
- b) the Tail Value at Risk  $\text{TV}_X^{98\%}$ ,
- c) the Conditional Tail Expectation  $\mathbb{E}[X \mid X > V_X^{98\%}]$ , and
- d) the Expected Shortfall  $E_X^{98\%}$ .

**Exercise 7.6** Consider two independent random variables  $X$  and  $Y$  with same distribution given by

$$\mathbb{P}(X = 0) = \mathbb{P}(Y = 0) = 90\% \quad \text{and} \quad \mathbb{P}(X = 100) = \mathbb{P}(Y = 100) = 10\%.$$

- a) Plot the cumulative distribution function of  $X$  on the following graph:

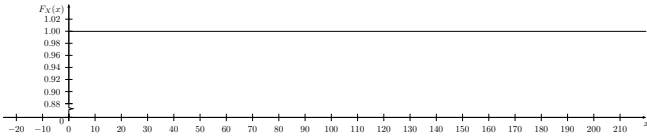
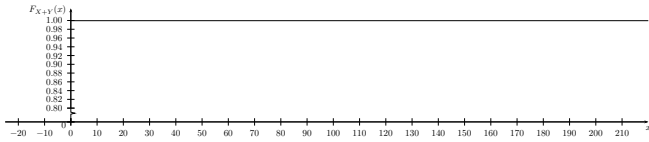


Fig. 7.10: Cumulative distribution function of  $X$ .

- b) Plot the cumulative distribution function of  $X + Y$  on the following graph:

Fig. 7.11: Cumulative distribution function of  $X + Y$ .

- c) Give the values at risk  $V_{X+Y}^{99\%}$ ,  $V_{X+Y}^{95\%}$ ,  $V_{X+Y}^{90\%}$ .  
 d) Compute the *Tail Value at Risk*

$$\text{TV}_X^{90\%} := \frac{1}{1-p} \int_p^1 V_X^q dq$$

at the level  $p = 90\%$ .

- e) Compute the *Tail Value at Risk*

$$\text{TV}_{X+Y}^p := \frac{1}{1-p} \int_p^1 V_{X+Y}^q dq$$

at the levels  $p = 90\%$  and  $p = 80\%$ .

Exercise 7.7 (Exercise 6.3 continued).

- a) Compute the Tail Value at Risk

$$\text{TV}_X^p := \frac{1}{1-p} \int_p^1 V_X^q dq$$

for all  $p$  in the interval  $[0.99, 1]$ , and give the value of  $\text{TV}_X^{99\%}$ .

- b) Taking  $p = 0.98$ , compute the Conditional Tail Expectation

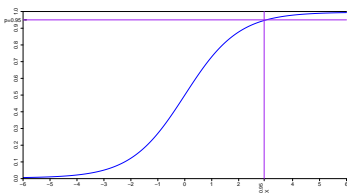
$$\text{CTE}_X^{98\%} = \mathbb{E}[X \mid X > V_X^{98\%}] = \frac{1}{\mathbb{P}(X > V_X^p)} \mathbb{E}[X \mathbb{1}_{\{X > V_X^p\}}].$$

Exercise 7.8 We assume that the payoff  $X$  of a portfolio follows the standard logistic distribution with cumulative distribution function (CDF)

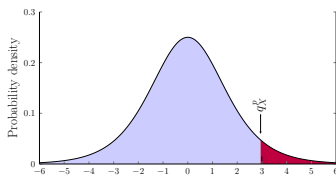
$$F_X(x) = \mathbb{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R},$$

and the probability density function (PDF)

$$f_X(x) = F'_X(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad x \in \mathbb{R}.$$



(a) Logistic quantile and CDF.



(b) Logistic PDF.

- a) Compute the quantile  $q_X^p = \text{VaR}_X^p$  of  $X$ , defined by the relation

$$F_X(q_X^p) = \mathbb{P}(X \leq \text{VaR}_X^p) = p,$$

at any level  $p \in [0, 1]$ .

- b) Compute the conditional tail expectation

$$\mathbb{E}[X \mid X > \text{VaR}_X^p] = \frac{1}{\mathbb{P}(X > \text{VaR}_X^p)} \int_{\text{VaR}_X^p}^{\infty} x f_X(x) dx.$$

Hint. We have

$$\int_a^{\infty} \frac{x e^{-x}}{(1 + e^{-x})^2} dx = \log(1 + e^a) - \frac{a e^a}{1 + e^a}, \quad a \in \mathbb{R}.$$

- c) Compute the tail value at risk

$$\text{TV}_X^p = \frac{1}{1-p} \int_p^1 V_X^q dq.$$

Hint. We have  $\int_p^1 \log q dq = p - 1 - p \log p$ ,  $p \in (0, 1)$ .

### Exercise 7.9

- a) Show that for any random variable  $Z$  with probability density function  $f_Z : \mathbb{R} \rightarrow \mathbb{R}_+$  we have

$$q \mathbb{P}(Z \geq q) \leq \mathbb{E}[Z \mathbb{1}_{\{Z \geq q\}}] = \int_q^{\infty} x f_Z(x) dx, \quad q \geq 0. \quad (7.9)$$

- b) Compute the left hand side and right hand side of the inequality (7.9) when  $Z \simeq \mathcal{N}(0, 1)$  has the standard normal distribution and  $q$  is the quantile  $q_Z^p$  of  $Z$  at the level  $p \in [0, 1]$ .

- c) Given  $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$  a Gaussian random variable with mean  $\mu_X$  and variance  $\sigma_X^2$ , show that the Gaussian Value at Risk

$$V_X^p = \mu_X + \sigma_X q_Z^p$$

is upper bounded by the Gaussian conditional tail expectation

$$\text{CTE}_X^p = \mu_X + \frac{\sigma_X}{1-p} \phi(q_Z^p)$$

where

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R},$$

is the standard normal probability density function.