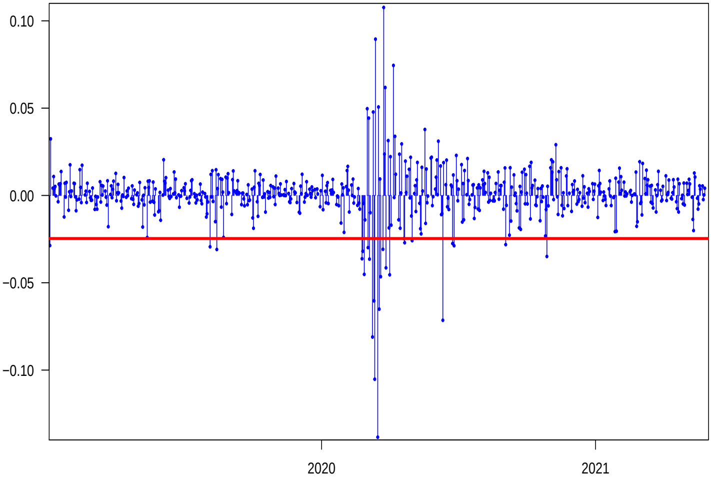
Nicolas Privault

Notes on
Financial Risk and Analytics

With and Python



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Preface

Risk measurement and the modeling of extreme events are present in many aspects of our daily lives, although their significance is often underestimated. The types of risks to which we are typically exposed can be roughly classified into: market risk, liquidity risk, credit risk, counterparty risk, model risk, and estimation risk. Climate risk and green investment analytics have also become important areas of interest.

This book introduces mathematical tools for financial risk modeling and related analytics with R and Python. Topics include stochastic modeling using random processes with jumps and time series, Value at Risk and Expected Shortfall, and structures of random dependence for multivariate modelling, all of which are relevant to risk analytics and financial risk management.

Prerequisites and audience

This book is largely self-contained but assumes a basic knowledge of undergraduate probability and statistics, including concepts such as random variables and probability distributions. Complements are provided where necessary - for example, on correlation and dependence, or cumulative distribution functions and their generalized inverses.

As such, the book is suitable for advanced undergraduate or beginning graduate students interested in the mathematical foundations of risk management. It can be used for modules such as *Foundations of Risk Analytics*, *Statistical Learning for Risk Modeling*, *Mathematical Tools for Risk Management*, *Actuarial and Insurance Statistics*, *Credit Risk*, *Quantitative Methods for Risk Analysis*, *Risk Management and Derivatives*, *Financial Mathematics*, and *Computational Finance*, in BSc and MSc programs in risk analytics, financial mathematics, actuarial science, and data analytics.

The book can also serve as a resource for multiple modules within programs in finance and economics, business, mathematics and statistics, and as preparation for more advanced courses. In comparison with related references, it also focuses on proof derivations, especially in Part II on risk measures. Other relevant references include Gourieroux and Jasiak (2010) for a time-series-based framework, Bluhm et al. (2024) for credit risk and correlated default, McNeil et al. (2015), Baesens et al. (2016) and Pfaff (2016) for hands-on analytics, and Chen et al. (2024) which also covers machine learning tools.

Exercises

This text includes 83 practice exercises of varying difficulty levels, covering both computational and applied aspects. Complete and detailed solutions to the exercises are provided in the accompanying solutions manual. Clicking on an exercise number inside the solution section will take the reader to the corresponding problem inside the main text. Conversely, clicking on the problem number sends the reader to the associated solution; however, this feature should not be misused.

Computer codes

The document contains 120 Python compact code blocks also available as notebooks at

https://github.com/nprivaul/Python_Notebooks/

and 121 R scripts and instructions of minimal length, also available as notebooks at

https://github.com/nprivaul/R_Notebooks/

which are designed to be run locally and in sequence within each chapter.

Organization

The book is organized into three parts. Part I provides the foundations of random processes, the stochastic modeling of market returns, and their correlation and dependence properties. Part II focuses on classical risk measures, while Part III presents more advanced concepts in credit risk.

The diagram below illustrates the dependencies among the different chapters of the book.

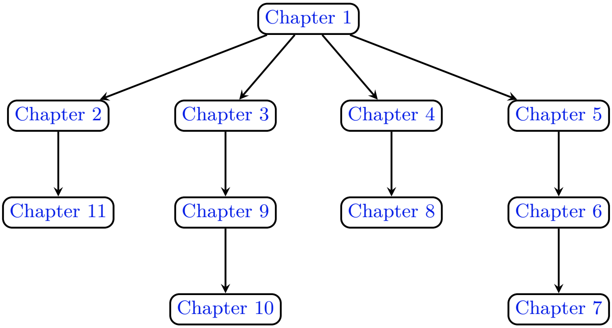

Fig. 1: Plan of the book.

Part I: Stochastic Modeling: Part I introduces tools for stochastic modeling with applications in option pricing, portfolio allocation, and insurance. Chapter 1 begins with random walks and geometric Brownian motion for financial modeling, and also covers more advanced tools such as Gram–Charlier expansions for modeling market returns. Chapter 2 presents discrete- and continuous-time modeling of time-dependent events using time series. Chapter 3 focuses on jump processes and risk distributions, such as compound Poisson distributions, with applications to insurance risk. This is also relevant to liability, catastrophe and operational risks such as business or event risk. Correlation and dependence are addressed in Chapter 4 using copulas.

Part II: Risk Measures: Part II focuses on risk measures relevant to financial, investment, market and non-catastrophe risks, beginning with the superhedging risk measure in Chapter 5. Chapters 6 and 7 provide a detailed treatment of Value at Risk, Tail Value at Risk, and Expected Shortfall, including a proof that Expected Shortfall and Tail Value at Risk are coherent risk measures.

Part III: Credit Risk: Part III presents structural and reduced-form approaches to credit risk and valuation in Chapters 8 and 9. Credit default is analyzed through defaultable bonds, Credit Default Swaps (CDS), and collateralized debt obligations (CDOs) in Chapter 10 on credit derivatives. Chapter 11 is devoted to credit scoring, covering the main theoretical tools used by credit rating practitioners, such as discriminant analysis and logistic regression.

The material in this book has been used for teaching in the Masters of Science programs in Financial Engineering (MFE), Analytics (MSA), and Business Analytics (MSBA) at the Nanyang Technological University, and for a course at the Nanjing University of Science and Technology at the invitation of Prof Wang Liang. The pdf file contains external links and 169 figures, including 11 animated figures and an embedded video in Figure 1.9, which may require using Acrobat Reader for viewing on the complete pdf file.

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