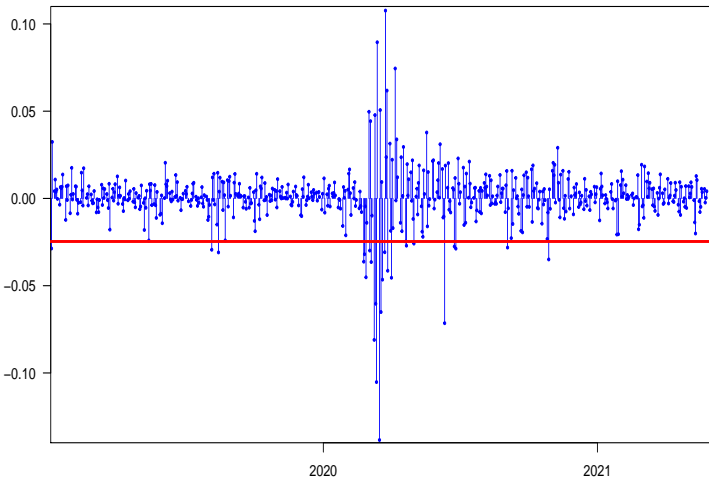


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Notes on

Financial Risk and Analytics



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<https://personal.ntu.edu.sg/nprivault/index.html>

Preface

The topics covered in these notes include an introduction to stochastic modeling with discrete-valued stochastic processes, a basic coverage of Value at Risk and Expected Shortfall, as well as structures of random dependence. Various types of risk, see *e.g.* [Gourieroux and Jasiak \(2010\)](#), can be classified into: market risk, liquidity risk, credit risk, counterparty risk, model risk, estimation risk. For insurance businesses, a more detailed classification can be set as follows.

a) Financial risk

Investment risk

- Credit risk,
- Market risk (*e.g.* depreciation),
- Counterparty risk.


Liability risk

- Catastrophe risk,
- Non-catastrophe risk (*e.g.* claim volatility).


b) Operational risk

- Business risk (*e.g.* lower production),
- Event risk (*e.g.* system failure),
- Policy risk ...

Part I introduces tools for stochastic modeling, with applications in option pricing, portfolio allocation, and insurance, starting with the use of random walks and geometric Brownian motion for financial modeling in Chapter 1. This is followed by the discrete and continuous-time modeling of time-dependent events using time series and processes with jumps, respectively in Chapters 2 and 3. In particular, the risk theory considered in Chapter 3 is relevant to liability, catastrophe and operational risks such as business or event risk. Correlation and dependence are treated via the used of copulas in Chapter 4.

Financial, investment, market and non-catastrophe risks are dealt with in Part II which focuses on risk measures. This includes the superhedging risk measure in Chapter 5, and Value at Risk and Expected Shortfall in Chapter 6 and 7. Chapter 8 is devoted to credit scoring, using discriminant analysis and logistic regression. Risk theory and credit scoring are presented with illustrative examples in .

Credit risk is considered Part III, including the structural and reduced-form approaches to credit risk and valuation in Chapters 9 and 10. Credit default is treated via defaultable bonds, Credit Default Swaps (CDS) and collateralized debt obligations (CDOs) in Chapter 11 on credit derivatives.

Parts of this material have been used for teaching in the Masters of Science in Financial Engineering (MFE), Analytics (MSA), and Business Analytics (MSBA) at the Nanyang Technological University in Singapore. The pdf file contains external links and 157 figures, including 10 animated figures and an embedded video in Figure 1.8, that may require using Acrobat Reader for viewing on the complete pdf file. It also contains 14 Python codes and 80  codes.

This text also includes 69 exercises with solutions. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on the problem number sends the reader to the corresponding solution, however this feature should not be misused.

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