Chapter 6
Value at Risk

Value at risk (VaR) is probably the most basic and widely used measure of risk. It relies on estimating the amount that can potentially be lost on a given investment within a certain time range. This chapter starts with a review the concept of risk measure in general, including quantile risk measures, before providing a mathematical treatment of Value at Risk, together with experiments based on actual financial data sets.

6.1 Risk Measures

Risk measures have two objectives:

i) to provide a measure for risk, and

ii) to determine an adequate level of capital reserves that matches the current level of risk.

In what follows, the potential losses associated to a given risk will be modeled by the values of a random variable X.

Definition 6.1. *A* risk measure *is a mapping that assigns a value V_X to a given loss random variable X.*

For insurance companies, which need to hold a capital in order to meet future liabilities, the capital C_X required to face the risk induced by a potential loss X can be defined as

$$C_X := V_X - L_X, \tag{6.1}$$

where

a) V_X stands for an upper "reasonable" estimate of the potential loss associated to X.

b) L_X represents the *liabilities* of the company.

   In other words, managing risk means here determining a level V_X of provision or capital requirement that will not be "too much" exceeded by X. When $L_X < 0$ the amount $-L_x > 0$ corresponds to a debt owed by the company, while $L_X > 0$ corresponds to positive liabilities such as deferred revenue or to a debt owed to the company.

Some examples of risk measures (Hardy (2006))

a) The *expected value premium principle* is the risk measure defined by

$$V_X := \mathbb{E}[X] + \alpha \mathbb{E}[X]$$

   for some $\alpha \geqslant 0$. For $\alpha = 0$, $V_X := \mathbb{E}[X]$ it is called the *pure premium* risk measure.

b) The *standard deviation premium principle* is the risk measure defined by

$$V_X := \mathbb{E}[X] + \alpha \sqrt{\mathrm{Var}[X]}$$

   for some $\alpha \geqslant 0$, where $\mathrm{Var}[X]$ denotes the variance of X.

In order to proceed with more examples of risk measures, we will need to use conditional expectations, see *e.g.* Lemma A.15 for the following proposition. The what follows, we let $\mathbb{1}_A$ denote the *indicator function* of any event A subset of Ω, defined as

$$\mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ \\ 0 \text{ if } \omega \notin A. \end{cases}$$

Proposition 6.2. *Let A be an event such that $\mathbb{P}(A) > 0$. The* conditional expectation *of $X : \Omega \longrightarrow \mathbb{N}$ given the event A satisfies the relation*

$$\mathbb{E}[X \mid A] := \frac{1}{\mathbb{P}(A)} \mathbb{E}\left[X \mathbb{1}_A\right].$$

For example, consider the sample space $\Omega = \{1, 3, -1, -2, 5, 7\}$ with the non-uniform probability measure given by

$$\mathbb{P}(\{-1\}) = \mathbb{P}(\{-2\}) = \mathbb{P}(\{1\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{7\}) = \frac{1}{7}, \ \mathbb{P}(\{5\}) = \frac{2}{7},$$

and the random variable

$$X : \Omega \longrightarrow \mathbb{Z}$$

given by

$$X(k) = k, \qquad k = 1, 3, -1, -2, 5, 7.$$

Here, $\mathbb{E}[X \mid X > 1]$ denotes the expected value of X given the event

$$A := \{X > 1\} = \{3, 5, 7\} \subset \Omega,$$

i.e. the mean value of X given that X is strictly greater than one. This conditional expectation can be computed as

$$\begin{aligned}
&\mathbb{E}[X \mid X > 1] \\
&= 3 \times \mathbb{P}(X = 3 \mid X > 1) + 5 \times \mathbb{P}(X = 5 \mid X > 1) + 7 \times \mathbb{P}(X = 7 \mid X > 1) \\
&= 3 \times \frac{1}{4} + 5 \times \frac{2}{4} + 7 \times \frac{1}{4} \\
&= \frac{3 + 2 \times 5 + 7}{4} \\
&= \frac{1}{4/7}\left(3 \times \frac{1}{7} + 5 \times \frac{2}{7} + 7 \times \frac{1}{7}\right) \\
&= \frac{1}{\mathbb{P}(X > 1)}\left(3 \times \mathbb{P}(X = 3) + 5 \times \mathbb{P}(X = 5) + 7 \times \mathbb{P}(X = 7)\right) \\
&= \frac{1}{\mathbb{P}(X > 1)} \mathbb{E}\big[X \mathbb{1}_{\{X > 1\}}\big],
\end{aligned}$$

where $\mathbb{P}(X > 1) = 4/7$ and the truncated expectation $\mathbb{E}\big[X \mathbb{1}_{\{X > 1\}}\big]$ is given by

$$\mathbb{E}\big[X \mathbb{1}_{\{X > 1\}}\big] = \frac{3 + 2 \times 5 + 7}{7}.$$

c) The *Conditional Tail Expectation* (CTE) of X given that $X > 0$ is the risk measure defined as the conditional mean

$$V^X := \mathbb{E}[X \mid X > 0] = \frac{\mathbb{E}[X \mathbb{1}_{\{X > 0\}}]}{\mathbb{P}(X > 0)}. \tag{6.2}$$

Next, we consider the following market returns data.

```r
library(quantmod)
getSymbols("^HSI",from="2013-06-01",to="2014-10-01",src="yahoo")
stock<-Ad(`HSI`);returns <- as.vector((stock-lag(stock))/lag(stock));
times=index(stock);m=mean(returns[returns<0],na.rm=TRUE)
dev.new(width=16,height=7);par(oma=c(0,1,0,0))
plot(times,returns,pch=19,cex=0.4,col="blue", ylab="", xlab="", main = '', las=1, cex.lab=1.8,
      cex.axis=1.8, lwd=3)
segments(x0 = times, x1 = times, y0 = 0, y1 = returns,col="blue")
abline(h=m,col="red",lwd=3); length(returns)
```

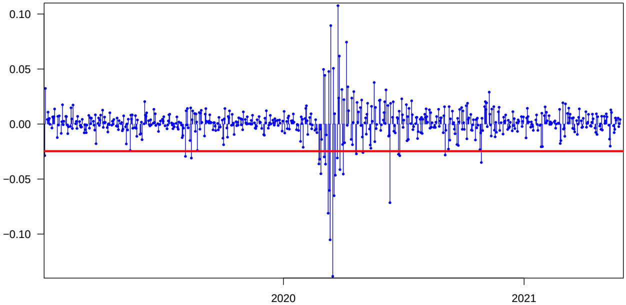

Fig. 6.1: Estimating liabilities by the conditional mean $\mathbb{E}[X \mid X < 0]$ over 346 market returns.

The conditional tail expectation (CTE) (6.2) estimated in Figure 6.1 can also be computed using the next R code, which also implements the statement of Proposition 6.2.

```r
returns <- returns[!is.na(returns)]
condmean<-mean(returns[returns<0])
n <- length(returns); sum<-sum(returns[returns<0])
proportion<-length(returns[returns<0])/length(returns)
condmean; sum/proportion/n
condmean<-mean(returns[returns<(-0.025)])
n <- length(returns); sum<-sum(returns[returns<(-0.025)])
proportion<-length(returns[returns<(-0.025)])/length(returns)
condmean; sum/proportion/n
```

Coherent risk measures

Definition 6.3. *A risk measure V is said to be* coherent *if it satisfies the following four properties, for any two random variables X, Y:*

i) Monotonicity:

$$X \leqslant Y \Longrightarrow V_X \leqslant V_Y,$$

ii) (Positive) homogeneity:

$$V_{\lambda X} = \lambda V_X, \qquad \textit{for constant } \lambda > 0,$$

iii) Translation invariance:

$$V_{\mu + X} = \mu + V_X, \qquad \textit{for constant } \mu > 0,$$

iv) Subadditivity:

$$V_{X+Y} \leqslant V_X + V_Y.$$

Subadditivity means that the combined risk of several portfolios is lower than the sum of risks of those portfolios, as happens usually through *portfolio diversification*. For example, one person traveling might insure the unlikely loss of her phone for $V_X = \$100$. However, two people traveling together might want to insure the phone loss event at a level V_{X+Y} lower than $V_X + V_Y = \$100 + \$100$ as the simultaneous loss of both phones during a same trip seems even more unlikely.

The concept of subadditivity is common in most pricing engines, as shown in the following example:

Price(  ) $\leqslant$ Price() $+$ Price() $+$ Price().

The *expectation* of random variables

$$V_X := \mathbb{E}[X],$$

or *pure premium* risk measure, is an example of a coherent (and additive) risk measure satisfying the above conditions (i)-(iv).

Definition 6.4. *A distortion risk measure is a risk measure of the form*

$$M_X = \mathbb{E}[X f_X(X)],$$

where f_X is a distortion function, *i.e. a non-negative, non-decreasing function such that*

i) $f_{\mu+X}(\mu + x) = f_X(x)$, $x \geqslant 0$, $\lambda > 0$, $\mu \geqslant 0$,

ii) $f_{\lambda X}(\lambda x) = f_X(x)$, $x \geqslant 0$, $\lambda > 0$, $\mu \geqslant 0$,

iii) $\mathbb{E}[f_X(X)] = 1$.

We note that distortion risk measures are positive homogeneous and translation invariant. Indeed,

i) for any $\lambda > 0$, we have

$$\begin{aligned} M_{\lambda X} = \mathbb{E}[\lambda X f_{\lambda X}(\lambda X)] &= \mathbb{E}[\lambda X f_X(X)] \\ &= \lambda \mathbb{E}[X f_X(X)] \\ &= \lambda M_X, \end{aligned}$$

which shows the (positive) homogeneity.

ii) For any $\mu \geqslant 0$, we have

$$\begin{aligned} M_{\mu+X} &= \mathbb{E}[(\mu + X) f_{\mu+X}(\mu + X)] \\ &= \mathbb{E}[(\mu + X) f_X(X)] \\ &= \mathbb{E}[X f_X(X)] + \mu \mathbb{E}[f_X(X)] \\ &= \mu + \mathbb{E}[X f_X(X)] \\ &= \mu + M_X, \end{aligned}$$

which shows the translation invariance.

See (7.2) and (7.6) below for examples of distortion risk measures.

6.2 Quantile Risk Measures

Definition 6.5. *The* Cumulative Distribution Function *(CDF) of a random variable X is the function*

$$F_X : \mathbb{R} \longrightarrow [0, 1]$$

defined by

$$F_X(x) := \mathbb{P}(X \leqslant x), \qquad x \geqslant 0.$$

Any cumulative distribution function F_X satisfies the following properties:

i) $x \mapsto F_X(x)$ is non-decreasing,

ii) $x \mapsto F_X(x)$ is right-continuous,

iii) $\lim_{x \to \infty} F_X(x) = 1$,

iv) $\lim_{x \to -\infty} F_X(x) = 0$.

Cumulative distribution functions can be discontinuous functions, as illustrated in Figure 6.2 with

$$\mathbb{P}(X = 0) = \mathbb{P}(X \leqslant 0) - \mathbb{P}(X < 0) = 0.25 > 0.$$

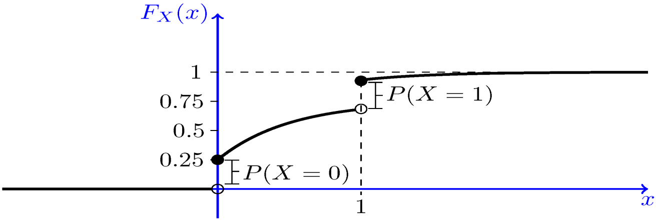

Fig. 6.2: Cumulative distribution function with discontinuities.*

Proposition 6.6 shows in particular that cumulative distribution functions admit left limits.

Proposition 6.6. *For any non-decreasing sequence* $(x_n)_{n\geqslant 1}$ *converging to* $x \in \mathbb{R}$*, we have*

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n) = \mathbb{P}(X < x). \tag{6.3}$$

Proof. By (A.7), we have

$$\begin{aligned}\mathbb{P}(X < x) &= \mathbb{P}(X \in (-\infty, x)) \\ &= \mathbb{P}\left(X \in \bigcup_{n\geqslant 1} (-\infty, x_n]\right) \\ &= \lim_{n\to\infty} \mathbb{P}(X \in (-\infty, x_n]) \\ &= \lim_{n\to\infty} F_X(x_n).\end{aligned}$$

□

As a consequence of Proposition 6.6 below, the gap generated by any discontinuity of a CDF at the point $x \in \mathbb{R}$, is given by

$$\mathbb{P}(X = x) = \mathbb{P}(X \leqslant x) - \mathbb{P}(X < x) = F_X(x) - \lim_{y \nearrow x} F_X(y).$$

* Picture taken from https://www.probabilitycourse.com/.

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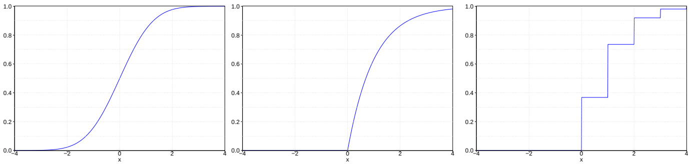
```
x <- seq(-4, 4, length=1000)
plot(x, pnorm(x, mean=0, sd=1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
     ylim=c(-0.001,1.002), las=1, cex.lab=2.5, cex.axis=2.5, xaxs='i', yaxs='i')
grid(4, 10, lwd = 2)
plot(x, pexp(x, 1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
     ylim=c(-0.001,1.002), las=1, cex.lab=2.5, cex.axis=2.5, xaxs='i', yaxs='i')
grid(4, 10, lwd = 2)
plot(x, ppois(x, 1), type="l", lwd=3, xlab = 'x', ylab = '', main = '', col='blue',
     ylim=c(-0.001,1.002), las=1, cex.lab=2.5, cex.axis=2.5, xaxs='i', yaxs='i')
grid(4, 10, lwd = 2)
```

Figure 6.3-(a) shows the continuous Cumulative Distribution Function

$$F_X(x) := \mathbb{P}(X \leqslant x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} dy, \qquad x \geqslant 0,$$

of a Gaussian random variable $X \simeq \mathcal{N}(0,1)$.



(a) Normal CDF. (b) Exponential CDF. (c) Poisson CDF.

Fig. 6.3: Cumulative distribution functions.

On the other hand, if $F_X(x)$ is differentiable in $x \in \mathbb{R}$ then the distribution of the random variable X is said to admit a *probability density function* (PDF) $f_X(x)$ given as the derivative

$$f_X(x) = F_X'(x), \qquad x \geqslant 0.$$

Definition 6.7. *Given X a random variable with cumulative distribution function $F_X : \mathbb{R} \longrightarrow [0,1]$ and a level $p \in (0,1)$, the p-quantile q_X^p of X is defined by*

$$q_X^p := \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\}. \tag{6.4}$$

We note that by (6.4), the function $p \mapsto q_X^p$ is the *generalized inverse* $F_X^{-1}(x)$ of the *Cumulative Distribution Function*

$$x \mapsto F_X(x) := \mathbb{P}(X \leqslant x), \qquad x \geqslant 0.$$

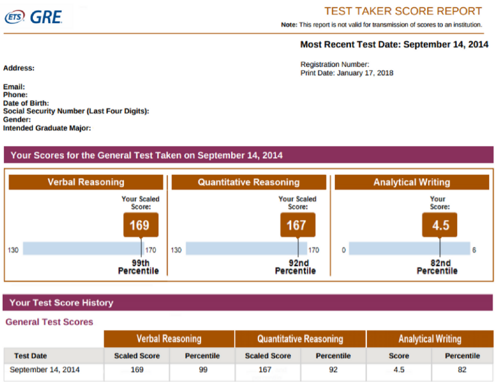
of X, see Definition 1 in Embrechts and Hofert (2013). As a consequence, we have the following.

Proposition 6.8.

i) The function $p \mapsto q_X^p$ *is a non-decreasing, left-continuous function of* $p \in [0, 1]$*, and it admits limits on the right.*
ii) For all $p \in [0, 1]$ *and* $x \in \mathbb{R}$ *we have*

$$p \leqslant F_X(x) \Longleftrightarrow q_X^p \leqslant x.$$

Proof. (i) follows from Proposition 1-(2) in Embrechts and Hofert (2013), since $F_X(x)$ is non-decreasing in $x \in \mathbb{R}$, and (ii) follows from Proposition 1-(5) in Embrechts and Hofert (2013), since $F_X(x)$ is right-continuous in $x \in \mathbb{R}$. □


ETS GRE

TEST TAKER SCORE REPORT

Note: This report is not valid for transmission of scores to an institution.

Most Recent Test Date: September 14, 2014

Address:

Registration Number:
Print Date: January 17, 2018

Email:
Phone:
Date of Birth:
Social Security Number (Last Four Digits):
Gender:
Intended Graduate Major:

Your Scores for the General Test Taken on September 14, 2014

Verbal Reasoning	Quantitative Reasoning	Analytical Writing
Your Scaled Score: 169	Your Scaled Score: 167	Your Score: 4.5
130 – 170	130 – 170	0 – 6
99th Percentile	92nd Percentile	82nd Percentile

Your Test Score History

General Test Scores

	Verbal Reasoning		Quantitative Reasoning		Analytical Writing	
Test Date	Scaled Score	Percentile	Scaled Score	Percentile	Score	Percentile
September 14, 2014	169	99	167	92	4.5	82

Fig. 6.4: Example of quantiles given as percentiles.

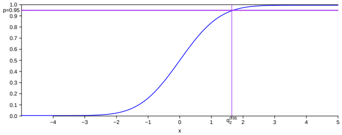
Quantiles of common distributions

The quantiles of various distributions can be obtained in R.

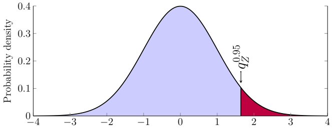
- *Gaussian distribution.* The command

```
qnorm(.95, mean=0, sd=1)
```

shows that the 95%-quantile of a $\mathcal{N}(0,1)$ Gaussian random variable is 1.644854.



(a) Gaussian quantile and CDF.



(b) Gaussian quantile and CDF.

Fig. 6.5: Gaussian quantile $q_Z^p = 1.644854$ at $p = 0.95$.

- *Exponential distribution.* The command

```
qexp(.95, 1)
```

displays the 95%-quantile of an exponentially distributed random variable with CDF

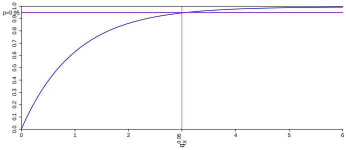
$$\mathbb{P}(X \leqslant x) = 1 - \mathrm{e}^{-\lambda x}, \qquad x \geqslant 0.$$

By equating $\mathbb{P}(X \leqslant q_X^p) = p$, we find

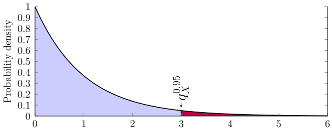
$$\begin{aligned} q_X^p &= \inf\left\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\right\} \\ &= -\frac{1}{\lambda}\log(1-p) \\ &= \mathbb{E}[X]\log\frac{1}{1-p}, \end{aligned}$$

and when $p = 95\%$ and $\lambda = 1$ this yields

$$q_X^p = 2.995732 \simeq 2.996\mathbb{E}[X].$$



(a) Exponential quantile and CDF.



(b) Exponential quantile and CDF.

Fig. 6.6: Exponential quantile $q_X^p = 2.995732$ at $p = 0.95$.

- *Student distribution.* The command

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```
qt(.90, df=5)
```

displays the 90%-quantile of a Student t-distributed random variable with 5 degrees of freedom, which is 1.475884.

- *Bernoulli distribution.* Consider the Bernoulli random variable $X \in \{0, 1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \qquad \mathbb{P}(X = 0) = 98\%.$$

In this case, we check from Figure 6.7 that $q_X^{0.99} = 1$.

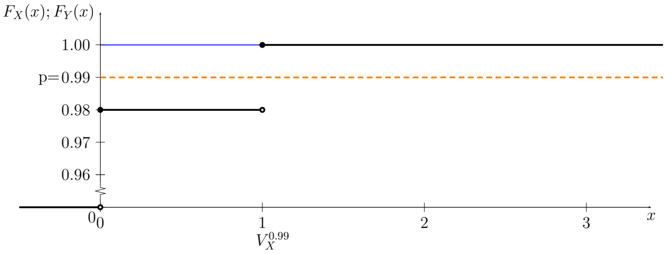


Fig. 6.7: Cumulative distribution function of X.

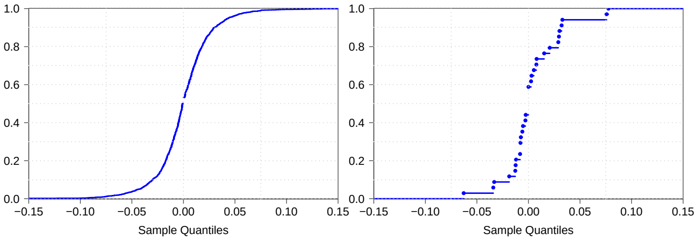
Empirical Cumulative Distribution Function

Definition 6.9. *The* empirical Cumulative Distribution Function *(CDF) of an N-point data set $\{x_1, x_2, x_3, \ldots, x_N\}$ is estimated as*

$$F_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_i \leqslant x\}}, \qquad x \geqslant 0.$$

```
getSymbols("^STI",from="1990-01-03",to="2015-01-03",src="yahoo")
getSymbols("1800.HK",from=Sys.Date()-50,to=Sys.Date(),src="yahoo")
stock=Ad(`1800.HK`);stock.rtn=(stock-lag(stock))/lag(stock);
stock.rtn <- stock.rtn[!is.na(stock.rtn)]
stock.ecdf=ecdf(as.vector(stock.rtn))
plot(stock.ecdf, xlab = 'Sample Quantiles', ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab = '',
       lwd = 3, main = '',col='blue', las=1, cex.lab=1.5, cex.axis=1.5, xaxs='i', yaxs='i')
```

```
getSymbols("1800.HK",from=Sys.Date()-3650,to=Sys.Date(),src="yahoo")
stock=Ad(`1800.HK`);stock.rtn=(stock-lag(stock))/lag(stock);
stock.ecdf=ecdf(as.vector(stock.rtn))
plot(stock.ecdf, xlab = 'Sample Quantiles', ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab = '',
       lwd = 2, main = '',col='blue', cex=1, las=1, cex.lab=1.5, cex.axis=1.5, xaxs='i', yaxs='i')
grid(4, 10, lwd = 2)
```



(a) Empirical CDF on 2463 samples. (b) Empirical CDF on 50 samples.

Fig. 6.8: Empirical cumulative distribution functions.

Note that the empirical distribution function in Figure 6.8-a) has a visible discontinuity (or gap) at $x = 0$, whose height 0.05483347 is given by

```
length(stock.rtn[stock.rtn==0])/length(stock.rtn)
```

6.3 Value at Risk (VaR)

Consider a random variable X used to model the potential losses associated to a given risk. The probability $\mathbb{P}(X > V)$ that X exceeds the level V is of a capital importance. Choosing the value of V such that for example

$$\mathbb{P}(X \leqslant V) \geqslant 0.95, \quad i.e. \quad \mathbb{P}(X > V) \leqslant 0.05,$$

means that insolvency will occur with probability less that 5%. In this setting, the 95%-quantile risk measure is the smallest value of V such that

$$\mathbb{P}(X \leqslant V) \geqslant 0.95, \quad i.e. \quad \mathbb{P}(X > V) \leqslant 0.05.$$

More precisely, we have the following definition.

Definition 6.10. *The Value at Risk* V_X^p *of a random variable* X *at the level* $p \in (0, 1)$ *is the* p*-quantile of* X *defined by*

$$V_X^p := \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\}. \tag{6.5}$$

In other words, for some decreasing sequence $(x_n)_{n\geqslant 1}$ such that

$$\mathbb{P}(X \leqslant x_n) \geqslant p \quad \text{for all} \quad n \geqslant 1,$$

we have

$$V_X^p := \lim_{n\to\infty} x_n. \tag{6.6}$$

Similarly to the above, the function $p \mapsto V_X^p$ is the *generalized inverse* $F_X^{-1}(x)$ of the *Cumulative Distribution Function* $\mapsto F_X$ of X, and from Proposition 6.8-(i) we have the following result.

Proposition 6.11. *The function $p \mapsto V_X^p$ is a* non-decreasing, left-continuous *function of $p \in [0,1]$, and it admits limits on the right.*

In particular, if F_X is continuous and strictly increasing it admits an inverse F_X^{-1}, and in this case we have

$$V_X^p = F_X^{-1}(p), \qquad p \in (0,1).$$

Proposition 6.12. *The Value at Risk V_X^p of X at the level $p \in (0,1)$ satisfies*

$$\mathbb{P}\big(X < V_X^p\big) \leqslant p \leqslant \mathbb{P}\big(X \leqslant V_X^p\big), \tag{6.7}$$

and

$$0 \leqslant 1 - p - \mathbb{P}\big(X > V_X^p\big) \leqslant \mathbb{P}\big(X = V_X^p\big). \tag{6.8}$$

In particular, if $\mathbb{P}(X = V_X^p) = 0$, then we have

$$p = \mathbb{P}\big(X < V_X^p\big) = \mathbb{P}\big(X \leqslant V_X^p\big). \tag{6.9}$$

Proof. Using the decreasing sequence $(x_n)_{n\geqslant 1}$ in (6.6) and the right continuity of the cumulative distribution function F_X, we have

$$\begin{aligned}
\mathbb{P}\big(X \leqslant V_X^p\big) &= \mathbb{P}\big(X \leqslant \lim_{n\to\infty} x_n\big) \\
&= F_X\big(\lim_{n\to\infty} x_n\big) \\
&= \lim_{n\to\infty} F_X(x_n) \\
&= \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n) \\
&\geqslant p.
\end{aligned}$$

On the other hand, if $\mathbb{P}\big(X < V_X^p\big) > p$ then there is a strictly increasing sequence $(y_n)_{n\geqslant 1}$ such that

$$\lim_{n\to\infty} y_n = V_X^p$$

and by (6.3) we have

$$\mathbb{P}\big(X < V_X^p\big) = \lim_{n\to\infty} \mathbb{P}(X \leqslant y_n) > p,$$

in which case there would exist $n \geqslant 1$ such that $y_n < V_X^p$ and $\mathbb{P}(X \leqslant y_n) > p$, which contradicts (6.5). Regarding the inequality (6.8), from (6.7) we have

$$\begin{aligned}\mathbb{P}\big(X = V_X^p\big) &= \mathbb{P}\big(X \leqslant V_X^p\big) - \mathbb{P}\big(X < V_X^p\big)\\ &= 1 - \mathbb{P}\big(X > V_X^p\big) - \mathbb{P}\big(X < V_X^p\big)\\ &\geqslant 1 - p - \mathbb{P}\big(X < V_X^p\big)\\ &= \mathbb{P}\big(X \geqslant V_X^p\big) - p\\ &\geqslant 0.\end{aligned}$$

The inequality (6.9) is similarly a consequence of (6.8). □

When $\mathbb{P}\big(X = V_X^p\big) > 0$ we may have $\mathbb{P}\big(X > V_X^p\big) = 0$, for example in the case of a Bernoulli random variable $X \in \{0,1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \qquad \mathbb{P}(X = 0) = 98\%,$$

see Figure 6.7. The next proposition also follows from the Definition 6.10 of V_X^p and Proposition 6.8-(ii).

Proposition 6.13. *For all $x \in \mathbb{R}$ we have*

$$V_X^p \leqslant x \quad \Longleftrightarrow \quad \mathbb{P}(X \leqslant x) \geqslant p. \tag{6.10}$$

Proof. $\Leftarrow$) If $\mathbb{P}(X \leqslant x) \geqslant p$ then we have

$$V_X^p = \inf\{y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \leqslant x.$$

$\Rightarrow$) On the other hand, choosing a strictly decreasing sequence $(x_n)_{n\geqslant 1}$ such that

$$\lim_{n\to\infty} x_n = V_X^p \quad \text{and} \quad \mathbb{P}(X \leqslant x_n) \geqslant p, \qquad n \geqslant 1,$$

if $V_X^p \leqslant x$ we have

$$\mathbb{P}(X \leqslant x) \geqslant \mathbb{P}(X \leqslant V_X^p) = \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n) \geqslant p$$

by the right continuity of the cumulative distribution function F_X of X. □

On the other hand, the Value at Risk V_X^p does not reveal any information on *how large* losses can be beyond V_X^p, see Chapter 7 for details. The next

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proposition shows how to estimate Value at Risk when switching the sign of the data.

Proposition 6.14. *Assume that the cumulative distribution function F_X is continuous and strictly increasing. Then, we have*

$$V_{-X}^{p} = -V_X^{1-p}, \qquad p \in (0,1). \tag{6.11}$$

Proof. Since F_X is continuous, we have

$$\begin{aligned} F_{-X}(x) &= \mathbb{P}(-X \leqslant x) \\ &= \mathbb{P}(X \geqslant -x) \\ &= 1 - \mathbb{P}(X < -x) \\ &= 1 - \mathbb{P}(X \leqslant -x) \\ &= 1 - F_X(-x), \end{aligned}$$

hence, taking

$$x := F_{-X}^{-1}(p),$$

we have

$$p = F_{-X}\big(F_{-X}^{-1}(p)\big) = 1 - F_X\big(-F_{-X}^{-1}(p)\big),$$

or

$$F_X\big(-F_{-X}^{-1}(p)\big) = 1 - p$$

i.e.

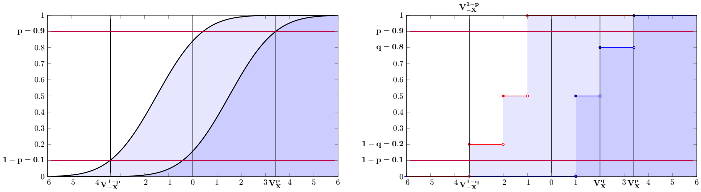
$$F_{-X}^{-1}(p) = -F_X^{-1}(1-p),$$

which yields

$$V_{-X}^{p} = F_{-X}^{-1}(p) = -F_X^{-1}(1-p) = -V_X^{1-p}, \qquad p \in (0,1).$$

□

Figure 6.9-(a) shows an example where the continuity of F_X ensures the symmetry property $V_{-X}^{p} = -V_X^{1-p}$ of Proposition 6.14. On the other hand, Figure 6.9-(b) shows that in the discontinuous case the relation $V_{-X}^{q} = -V_X^{1-q}$ fails for $q = 0.8$, although it holds for $p = 0.9$.



(a) Continuous CDF.

(b) Discontinuous CDF.

Fig. 6.9: Symmetric and nonsymmetric VaR.

Next, we check the properties of Value at Risk.

a) *Monotonicity.* Value at Risk is a monotone risk measure.

Proof. If $X \leqslant Y$ then

$$\mathbb{P}(Y \leqslant x) = \mathbb{P}(X \leqslant Y \leqslant x) \leqslant \mathbb{P}(X \leqslant x), \qquad x \geqslant 0,$$

hence

$$\mathbb{P}(Y \leqslant x) \geqslant p \quad \Longrightarrow \quad \mathbb{P}(X \leqslant x) \geqslant p, \qquad x \geqslant 0,$$

which shows that

$$V_X^p \leqslant V_Y^p$$

by (6.5). □

b) *Positive homogeneity and translation invariance.* Value at Risk satisfies the positive homogeneity and translation invariance properties.

Proof. For any $\mu \in \mathbb{R}$ and $\lambda > 0$, we have

$$\begin{aligned} V_{\mu+\lambda X}^p &= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(\mu + \lambda X \leqslant x) \geqslant p\} \\ &= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant (x-\mu)/\lambda) \geqslant p\} \\ &= \inf\{\mu + \lambda y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \\ &= \mu + \lambda \inf\{y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \\ &= \mu + \lambda V_X^p. \end{aligned}$$

□

c) *Subadditivity and coherence.* Although Value at Risk satisfies the monotonicity, positive homogeneity and translation invariance properties, it is *not* subadditive in general. Namely, the Value at Risk V_{X+Y}^p of $X + Y$ may be larger than the sum $V_X^p + V_Y^p$. Therefore, Value at Risk is *not* a coherent risk measure.

Proof. We show that Value at Risk is *not subadditive* by considering two *independent* Bernoulli random variables $X, Y \in \{0,1\}$ having the same distribution

$$\begin{cases} \mathbb{P}(X=1) = \mathbb{P}(Y=1) = 2\%, \\ \mathbb{P}(X=0) = \mathbb{P}(Y=0) = 98\%, \end{cases}$$

hence $V_X^{0.975} = V_Y^{0.975} = 0$.

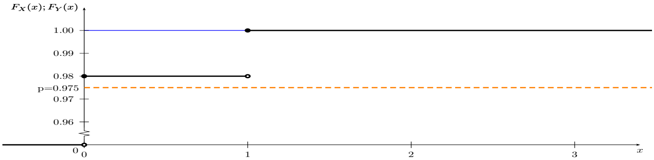


Fig. 6.10: Cumulative distribution function of X and Y.

On the other hand, we have

$$\begin{cases} \mathbb{P}(X+Y=2) = \mathbb{P}(X=1 \text{ and } Y=1) = (0.02)^2 = 0.04\%, \\ \mathbb{P}(X+Y=1) = 2 \times 0.02 \times 0.98 = 3.92\%, \\ \mathbb{P}(X+Y=0) = \mathbb{P}(X=0 \text{ and } Y=0) = (0.98)^2 = 96.04\%, \end{cases}$$

hence

$$V_{X+Y}^{0.975} = 1 > V_X^{0.975} + V_Y^{0.975} = 0.$$

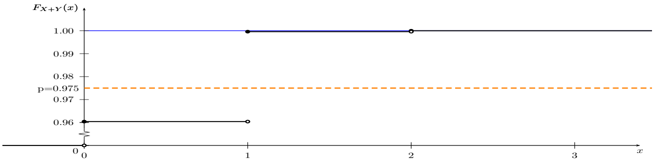


Fig. 6.11: Cumulative distribution function of $X+Y$.

□

In the next proposition, we use the standard Gaussian Cumulative Distribution Function (CDF)

$$\Phi(x) := \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi T}}, \qquad x \in \mathbb{R},$$

of a standard normal random variable $Z \simeq \mathcal{N}(0,1)$.

Proposition 6.15. Gaussian Value at Risk. *Given* $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$, *we have*

$$V_X^p = \mu_X + \sigma_X q_Z^p \tag{6.12}$$

where the normal quantile $q_Z^p = V_Z^p$ *at the level* p *satisfies*

$$\Phi(q_Z^p) = \mathbb{P}(Z \leqslant q_Z^p) = p \quad \textit{for} \quad Z \simeq \mathcal{N}(0,1),$$

i.e.

$$q_Z^p = \Phi^{-1}(p) \quad \textit{and} \quad V_X^p = \mu_X + \sigma_X \Phi^{-1}(p).$$

Proof. We represent the random variable $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ as

$$X = \mu_X + \sigma_X Z,$$

where $Z \simeq \mathcal{N}(0,1)$ is a standard normal random variable, and use the relation

$$\begin{aligned} p &= \mathbb{P}(X \leqslant V_X^p) \\ &= \mathbb{P}(\mu_X + \sigma_X Z \leqslant V_X^p) \\ &= \mathbb{P}(Z \leqslant (V_X^p - \mu_X)/\sigma_X) \\ &= \mathbb{P}(Z \leqslant q_Z^p), \end{aligned}$$

which holds provided that $V_X^p = \mu_X + \sigma_X q_Z^p$. □

We also note that if $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ then $-X \simeq \mathcal{N}(-\mu_X, \sigma_X^2)$, hence

$$\begin{aligned} V_{-X}^p &= -\mu_X + \sigma_X q_Z^p \\ &= -\mu_X - \sigma_X q_Z^{1-p} \\ &= -V_X^{1-p}, \end{aligned}$$

which is consistent with (6.11).

The next remark shows that, although Value at Risk is *not sub-additive* in general, it is sub-additive (and therefore coherent) on (not necessarily independent) Gaussian random variables.

Remark 6.16. *If* X *and* Y *are two Gaussian random variables, we have*

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$$V^p_{X+Y} \leqslant V^p_X + V^p_Y.$$

Proof. By (6.12), for any two random variables X and Y, we have

$$\begin{aligned}
&\sigma^2_{X+Y} = \mathrm{Var}[X+Y] \\
&= \mathbb{E}\big[(X+Y)^2\big] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}\big[X^2\big] + \mathbb{E}\big[Y^2\big] + 2\mathbb{E}[XY] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y) \qquad (6.13) \\
&\leqslant \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\sqrt{\mathbb{E}\big[(X-\mathbb{E}[X])^2\big]}\sqrt{\mathbb{E}\big[\big(Y-\mathbb{E}[Y]\big)^2\big]} \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\sqrt{\mathrm{Var}[X]}\sqrt{\mathrm{Var}[Y]} \qquad (6.14) \\
&= \Big(\sqrt{\mathrm{Var}[X]} + \sqrt{\mathrm{Var}[Y]}\Big)^2,
\end{aligned}$$

where, from (6.13) to (6.14) we applied the *Cauchy-Schwarz* inequality, hence $\sigma_{X+Y} \leqslant \sigma_X + \sigma_Y$. Assuming that X and Y are Gaussian, by (6.12) we find

$$\begin{aligned}
V^p_{X+Y} &= \mu_{X+Y} + \sigma_{X+Y} q^p_Z \\
&= \mu_X + \mu_Y + \sigma_{X+Y} q^p_Z \\
&\leqslant \mu_X + \mu_Y + (\sigma_X + \sigma_Y) q^p_Z \\
&= V^p_X + V^p_Y.
\end{aligned}$$

□

6.4 Numerical estimates

In this section we are using the PerformanceAnalytics R package, see also § 6.1.1 of Mina and Xiao (2001). In case we care about negative return values, Definition 6.10 is replaced with

$$\overline{V}^p_X := \mathrm{Sup}\{x \in \mathbb{R} \ : \ \mathbb{P}(X \geqslant x) \leqslant 1-p\}. \qquad (6.15)$$

In case the CDF of X is continuous, we note the relation

$$\begin{aligned}
\overline{V}^p_X &= \mathrm{Sup}\{x \in \mathbb{R} \ : \ \mathbb{P}(X \geqslant x) \leqslant 1-p\} \\
&= -\inf\{-x \in \mathbb{R} \ : \ \mathbb{P}(X \geqslant x) \leqslant 1-p\} \\
&= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \geqslant -x) \leqslant 1-p\}
\end{aligned}$$

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$$= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(-X \geqslant x) \leqslant 1 - p\}$$
$$= -\inf\{x \in \mathbb{R} \ : \ 1 - \mathbb{P}(-X \geqslant x) \geqslant p\}$$
$$= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(-X \leqslant x) \geqslant p\}$$
$$= -V^p_{-X},$$

hence the relation

$$\overline{V}^p_X = -V^p_{-X} = V^{1-p}_X$$

which is obtained from Proposition 6.14 when the cumulative distribution function F_X is continuous and strictly increasing.

```
install.packages("PerformanceAnalytics")
library(PerformanceAnalytics)
getSymbols("0700.HK",from="2010-01-03",to="2018-02-01",src="yahoo")
stock=Ad(`0700.HK`);chartSeries(stock,up.col="blue",theme="white")
stock.rtn=(stock-lag(stock))/lag(stock)[-1];stock.rtn <- stock.rtn[!is.na(stock.rtn)]
dev.new(width=16,height=7); chart.CumReturns(stock.rtn, main="Cumulative Returns")
var=VaR(stock.rtn, p=.95, method="historical");var
length(stock.rtn[stock.rtn<var[1]])/length(stock.rtn)
times=index(stock);chartSeries(stock.rtn,up.col="blue",theme="white")
abline(h=var,col="red",lwd=3)
```

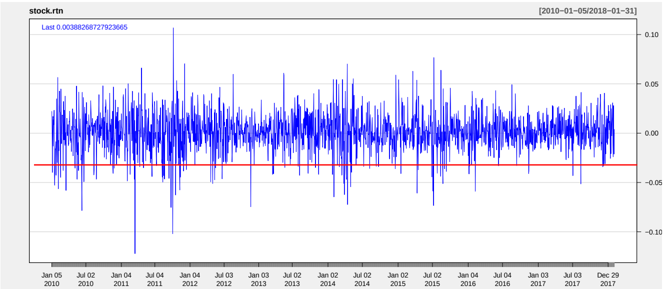

Fig. 6.12: Market returns *vs.* Value at Risk.

The historical 95%-Value at Risk over N samples $(x_i)_{i=1,2,\ldots,N}$ can be estimated by inverting the *empirical cumulative distribution function* $F_N(x)$, and is found to be $\overline{V}^{95\%}_X = -0.03165963$.

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```
VaR(stock.rtn, p=.95, method="gaussian",invert="FALSE")
VaR(stock.rtn, p=.95, method="gaussian",invert="TRUE")
```

The Gaussian 95%-Value at Risk is estimated from (6.12) with $p = 0.95$ as

$$\overline{V}_X^p = V_X^{1-p} = \mu + \sigma q_Z^{1-p} = \mu - \sigma q_Z^p,$$

where $-\mu = \mathbb{E}[-X]$ and $\sigma^2 = \mathrm{Var}[-X]$, and is found equal to $\overline{V}_X^{95\%} = -0.03115425$. It can be recovered up to approximation according to Proposition 6.15 from the following R code, which yields -0.0311592.

```
m=mean(stock.rtn,na.rm=TRUE); s=sd(stock.rtn,na.rm=TRUE)
q=qnorm(.95, mean=0, sd=1); m-s*q
```

Note that here we are concerned about large negative returns, which explains the negative sign in $m - s * q$.

The next lemma is useful for random simulation purposes, and it will also be used in the proof of Propositions 7.4 and 7.10 below.

Lemma 6.17. *Any random variable X can be represented as*

$$X = V_X^U = F_X^{-1}(U),$$

where U a uniformly distributed random variable on $[0,1]$.

Proof. It suffices to note that by (6.10) we have

$$\mathbb{P}\big(V_X^U \leqslant x\big) = \mathbb{P}(U \leqslant \mathbb{P}(X \leqslant x)) = \mathbb{P}(X \leqslant x) = F_X(x), \quad x \geqslant 0.$$

□

Exercises

Exercise 6.1 Consider a random variable X having the Pareto distribution with probability density function

$$f_X(x) = \frac{\gamma\theta^\gamma}{(\theta + x)^{\gamma+1}}, \qquad x \geqslant 0.$$

a) Compute the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \qquad x \geqslant 0.$$

b) Compute the value at risk V_X^p at the level p for any θ and γ, and then for $p = 99\%$, $\theta = 40$ and $\gamma = 2$.

Exercise 6.2 Consider a random variable X with the following cumulative distribution function:

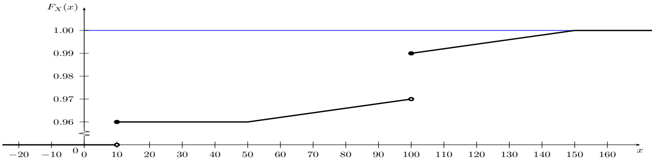


Fig. 6.13: Cumulative distribution function of X.

a) Give the value of $\mathbb{P}(X = 100)$.
b) Give the value of V_X^q for all q in the interval $[0.97, 0.99]$.
c) Compute the value of V_X^q for all q in the interval $[0.99, 1]$.

Hint: We have

$$F_X(x) = \mathbb{P}(X \leqslant x) = 0.99 + 0.01 \times \frac{x - 100}{50}, \qquad x \in [100, 150].$$

Exercise 6.3 Discrete distribution. Consider $X \in \{10, 100, 110\}$ with the distribution

$$\mathbb{P}(X = 10) = 90\%, \quad \mathbb{P}(X = 100) = 9.5\%, \quad \mathbb{P}(X = 110) = 0.5\%.$$

Compute the value at risk $V_X^{99\%}$.

Exercise 6.4 Exponential distribution. Assume that X has an exponential distribution with parameter $\lambda > 0$ and mean $1/\lambda$, *i.e.*

$$\mathbb{P}(X \leqslant x) = 1 - \mathrm{e}^{-\lambda x}, \qquad x \geqslant 0.$$

a) Compute

$$V_X^p := \inf \{ x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p \}$$

and $V_X^{95\%}$.

b) Assuming that the liabilities of a company are estimated by $\mathbb{E}[X]$, compute the amount of required capital C_X from (6.1).

Exercise 6.5 Given X a random variable having the geometric distribution with

$$\mathbb{P}(X = k) = (1-p)^k p, \qquad k \geqslant 0,$$

compute the conditional expectation $\mathbb{E}[X \mid X \geqslant a]$ for $a > 0$.

Exercise 6.6 Estimating risk probabilities from moments.

a) Show that for every $r > 0$

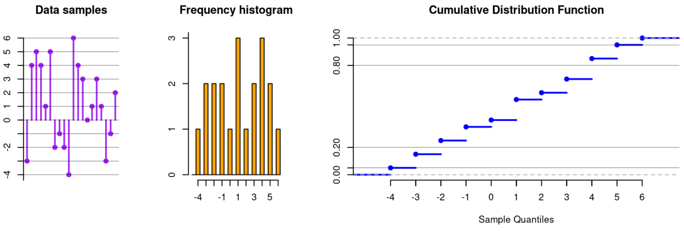
$$V_X^p \leqslant \left(\frac{\mathbb{E}[|X|^r]}{1-p} \right)^{1/r} = \frac{\|X\|_{L^r(\Omega)}}{(1-p)^{1/r}},$$

where $\|X\|_{L^r(\Omega)} := (\mathbb{E}[|X|^r])^{1/r}$.

Hint: Use the argument of the Markov inequality.

b) Give an upper bound for $V_X^{95\%}$ when $p = 95\%$ and $r = 1$.

Exercise 6.7 We consider a discrete random variable X having the following distribution.



a) Find the following quantities for the above data set, and mark their values on the graph.

i) Historical "Academic" Value at Risk at $p = 0.95$. $\mathrm{VaR}^{95}_{\mathrm{Ac\text{-}H}} =$ ________

ii) Historical "Academic" Value at Risk at $p = 0.80$. $\mathrm{VaR}^{80}_{\mathrm{Ac\text{-}H}} =$ ________

iii) Historical "Practitioner" Value at Risk at $p = 0.95$. $\overline{\mathrm{VaR}}^{95}_{\mathrm{Pr\text{-}H}} =$ ________

iv) Historical "Practitioner" Value at Risk at $p = 0.80$. $\overline{\mathrm{VaR}}^{80}_{\mathrm{Pr\text{-}H}} =$ ________

b) Knowing that mean=1.15, sd=3.048, qnorm(0.95)=1.645 and qnorm(0.80)=0.842, compute (from Proposition 6.15):

i) Gaussian "Academic" Value at Risk at $p = 0.95$. $\mathrm{VaR}^{95}_{\mathrm{A}c\text{-}\mathrm{G}} =$ ________

ii) Gaussian "Academic" Value at Risk at $p = 0.80$. $\mathrm{VaR}^{80}_{\mathrm{A}c\text{-}\mathrm{G}} =$ ________

iii) Gaussian "Practitioner" Value at Risk at $p = 0.95$. $\overline{\mathrm{VaR}}^{95}_{\mathrm{P}r\text{-}\mathrm{G}} =$ ________

iv) Gaussian "Practitioner" Value at Risk at $p = 0.80$. $\overline{\mathrm{VaR}}^{80}_{\mathrm{P}r\text{-}\mathrm{G}} =$ ________

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