Chapter 6
Value at Risk

Value at Risk (VaR) is one of the most basic and widely used measures of risk and estimates the potential loss on a given investment over a specified time horizon. This chapter begins with a review of risk measures in general, including quantile-based risk measures, before providing a mathematical treatment of Value at Risk, along with experiments using actual financial data sets.

6.1 Risk Measures

Risk measures have two objectives:

i) to provide a measure for risk, and

ii) to determine an adequate level of capital reserves that matches the current level of risk.

In what follows, the potential losses associated to a given risk will be modeled by the values of a random variable X.

Definition 6.1. *A* risk measure *is a mapping that assigns a value V_X to a given loss random variable X.*

For insurance companies, which need to hold a capital in order to meet future liabilities, the capital C_X required to face the risk induced by a potential loss $X \geqslant 0$ can be defined as

$$C_X := V_X - L_X, \tag{6.1}$$

where

a) V_X stands for an upper "reasonable" estimate of the potential loss associated to X.

b) L_X represents the *liabilities* of the company.

In other words, managing risk means here determining a level V_X of provision or capital requirement that will not be "too much" exceeded by X. When $L_X < 0$ the amount $-L_x > 0$ corresponds to a debt owed by the company, while $L_X > 0$ corresponds to positive liabilities such as deferred revenue or to a debt owed to the company.

Some examples of risk measures (Hardy (2006))

a) The *expected value premium principle* is the risk measure defined by

$$\mathrm{E}_X := \mathbb{E}[X] + \alpha\mathbb{E}[X] \tag{6.2}$$

for some $\alpha \geqslant 0$. For $\alpha = 0$, $\mathrm{E}_X := \mathbb{E}[X]$ it is called the *pure premium* risk measure.

b) The *standard deviation premium principle* is the risk measure defined by

$$\mathrm{SD}_X := \mathbb{E}[X] + \alpha\sqrt{\mathrm{Var}[X]} \tag{6.3}$$

for some $\alpha \geqslant 0$, where $\mathrm{Var}[X]$ denotes the variance of X.

Conditional expectations

In order to proceed with more examples of risk measures, we will need to use conditional expectations, see *e.g.* Lemma A.19 for the following proposition. In what follows, we let $\mathbb{1}_A$ denote the *indicator function* of any event A subset of Ω, defined as

$$\mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ 0 \text{ if } \omega \notin A. \end{cases}$$

Lemma 6.2 generalizes the classical construction of conditional probabilities as

$$\mathbb{P}(B \mid A) = \frac{1}{\mathbb{P}(A)}\mathbb{P}(B \cap A).$$

Lemma 6.2. *Let A be an event such that $\mathbb{P}(A) > 0$. The* conditional expectation *of $X : \Omega \longrightarrow \mathbb{N}$ given the event A satisfies the relation*

$$\mathbb{E}[X \mid A] := \frac{1}{\mathbb{P}(A)} \mathbb{E}\left[X \mathbb{1}_A\right].$$

For example, consider the sample space $\Omega = \{1, 3, -1, -2, 5, 7\}$ with the (non-uniform) probability measure given by

$$\mathbb{P}(\{-1\}) = \mathbb{P}(\{-2\}) = \mathbb{P}(\{1\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{7\}) = \frac{1}{7}, \ \mathbb{P}(\{5\}) = \frac{2}{7},$$

and the random variable

$$X : \Omega \longrightarrow \mathbb{Z}$$

given by

$$X(k) = k, \qquad k = 1, 3, -1, -2, 5, 7.$$

Here, $\mathbb{E}[X \mid X > 1]$ denotes the expected value of X given the event

$$A := \{X > 1\} = \{3, 5, 7\} \subset \Omega,$$

i.e. the mean value of X given that X is strictly greater than one, with $\mathbb{P}(X > 1) = 4/7$. This conditional expectation can be computed using conditional probabilities as

$$\begin{aligned}
&\mathbb{E}[X \mid X > 1] \\
&= 3 \times \mathbb{P}(X = 3 \mid X > 1) + 5 \times \mathbb{P}(X = 5 \mid X > 1) + 7 \times \mathbb{P}(X = 7 \mid X > 1) \\
&= 3 \times \frac{1}{4} + 5 \times \frac{2}{4} + 7 \times \frac{1}{4} \\
&= \frac{3 + 2 \times 5 + 7}{4}.
\end{aligned}$$

On the other hand, using Lemma 6.2, we have

$$\begin{aligned}
&\frac{1}{\mathbb{P}(X > 1)} \mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right] \\
&= \frac{1}{\mathbb{P}(X > 1)} \left(3 \times \mathbb{P}(X = 3) + 5 \times \mathbb{P}(X = 5) + 7 \times \mathbb{P}(X = 7)\right) \\
&= \frac{1}{4/7} \left(3 \times \frac{1}{7} + 5 \times \frac{2}{7} + 7 \times \frac{1}{7}\right) \\
&= \frac{3 + 5 \times 2 + 7}{4},
\end{aligned}$$

where the truncated expectation $\mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right]$ is given by

$$\mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right] = \frac{3 + 2 \times 5 + 7}{7}.$$

c) The *Conditional Tail Expectation* (CTE) of X given that $X < 0$ is the risk measure defined as the conditional mean

$$\mathrm{CTE}^X := \mathbb{E}[X \mid X < 0] = \frac{\mathbb{E}[X \mathbb{1}_{\{X<0\}}]}{\mathbb{P}(X < 0)}. \tag{6.4}$$

Next, we consider the following market returns data.

```r
library(quantmod); getSymbols("^HSI",from="2013-06-01",to="2014-10-01",src="yahoo")
stock<-Ad(`HSI`);returns <- as.vector((stock-lag(stock))/lag(stock));
times=index(stock);m=mean(returns[returns<0],na.rm=TRUE)
dev.new(width=16,height=7);par(oma=c(0,1,0,0))
plot(times,returns,pch=19,cex=.4,col="blue",ylab="",xlab="",main='',las=1,cex.lab=1.8,
  cex.axis=1.8, lwd=3); segments(x0=times,x1=times,y0=0,y1=returns,col="blue")
abline(h=m,col="red",lwd=3); length(returns)
```

Listing 6.1: R code - Market returns and CTE.

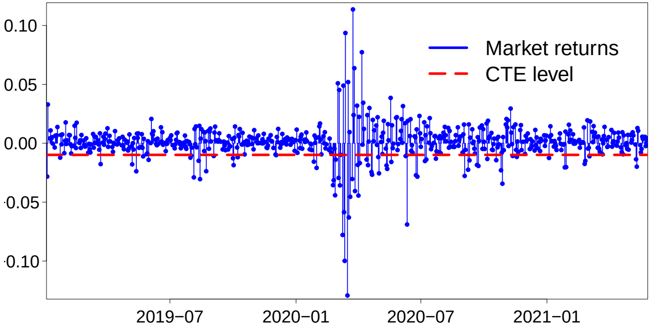

Fig. 6.1: Estimating liabilities by the conditional mean $\mathbb{E}[X \mid X < 0]$.

```python
import yfinance as yf, pandas as pd, matplotlib.pyplot as plt
hsi= yf.download("^HSI", start="2013-06-01", end="2014-10-01")
stock= hsi['Close']; returns= (stock - stock.shift(1))/ stock.shift(1)
returns= returns.dropna(); times= returns.index; m= returns[returns < 0].mean();
plt.figure(figsize=(16, 7)); plt.plot(times, returns, 'o', ms=4, c='b')
plt.vlines(times, 0, returns, color='b', lw=1.5); plt.axhline(y=float(m), color='r', lw=3)
plt.xlim(times[0], times[-1]); plt.tight_layout(pad=0); plt.show()
print(f"Mean of negative returns: {float(m):.6f}"); print(f"Number of returns: {len(returns)}")
```

Listing 6.2: Python code - Market returns and CTE.

The conditional tail expectation (CTE) (6.4) estimated in Figure 6.1 can also be computed using the next code, which also implements the statement of Lemma 6.2.

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```r
returns=returns[!is.na(returns)];condmean=mean(returns[returns<0]);n=length(returns)
sum=sum(returns[returns<0]);prop=length(returns[returns<0])/length(returns)
condmean; sum/prop/n
condmean=mean(returns[returns<(-.025)]);n=length(returns);sum=sum(returns[returns<(-.025)])
prop=length(returns[returns<(-.025)])/length(returns); condmean; sum/prop/n
```

Listing 6.3: R code - CTE Computation.

```python
cndmean=returns[returns<0].dropna().mean(); n=len(returns)
sum_neg=returns[returns<0].dropna().sum();prp=len(returns[returns < 0].dropna())/n
print(f"cndmean: {float(cndmean):.6f}");print(f"sum/prp/n: {float(sum_neg/(prp*n)):.6f}")
cndmean=returns[returns<(-.025)].dropna().mean();n=len(returns)
sum_neg=returns[returns<(-.025)].dropna().sum();prp=len(returns[returns<(-.025)].dropna())/n
print(f"cndmean: {float(cndmean):.6f}");print(f"sum/prp/n: {float(sum_neg/(prp*n)):.6f}")
```

Listing 6.4: Python code - CTE Computation.

Coherent risk measures

Definition 6.3. *A risk measure V is said to be* coherent *if it satisfies the following four properties, for any two random variables X, Y:*

i) Monotonicity:

$$X \leqslant Y \Longrightarrow V_X \leqslant V_Y,$$

ii) (Positive) homogeneity:

$$V_{\lambda X} = \lambda V_X, \qquad \text{for constant } \lambda > 0,$$

iii) Translation invariance:

$$V_{\mu + X} = \mu + V_X, \qquad \text{for constant } \mu > 0,$$

iv) Subadditivity:

$$V_{X+Y} \leqslant V_X + V_Y.$$

Subadditivity means that the combined risk of several portfolios is lower than the sum of risks of those portfolios, as happens usually through *portfolio diversification*. For example, one person traveling might insure the unlikely loss of her phone for $V_X = \$100$. However, two people traveling together might want to insure the phone loss event at a level V_{X+Y} lower than $V_X + V_Y = \$100 + \$100$ as the simultaneous loss of both phones during a same trip seems even more unlikely.

The concept of subadditivity is common in most pricing engines, as shown in the following example:

Price(  ) $\leqslant$ Price() + Price() + Price().

The *expectation* of random variables

$$\mathrm{E}_X := \mathbb{E}[X],$$

or *pure premium* risk measure, is an example of a coherent (and additive) risk measure satisfying the above conditions (i)-(iv).

Definition 6.4. *A* distortion risk measure *is a risk measure of the form*

$$\mathrm{D}_X = \mathbb{E}[X f_X(X)],$$

where f_X is a distortion function, *i.e. a nonnegative, non-decreasing function such that*

i) $f_{\mu+X}(\mu + x) = f_X(x)$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$,

ii) $f_{\lambda X}(\lambda x) = f_X(x)$, $x \in \mathbb{R}$, $\lambda > 0$,

iii) $\mathbb{E}[f_X(X)] = 1$.

The *Conditional Tail Expectation* (6.4) is an example of distortion risk measure with distortion function

$$f_X(x) = \frac{1}{\mathbb{P}(X < 0)} \mathbb{1}_{\{x<0\}},$$

see Proposition 7.7. We note that distortion risk measures are positive homogeneous and translation invariant.

i) Positive homogeneity. For any $\lambda > 0$, we have

$$\begin{aligned}
\mathrm{D}_{\lambda X} = \mathbb{E}[\lambda X f_{\lambda X}(\lambda X)] &= \mathbb{E}[\lambda X f_X(X)] \\
&= \lambda \mathbb{E}[X f_X(X)] \\
&= \lambda \mathrm{D}_X.
\end{aligned}$$

ii) Translation invariance. For any $\mu \in \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{D}_{\mu+X} &= \mathbb{E}[(\mu + X) f_{\mu+X}(\mu + X)] \\
&= \mathbb{E}[(\mu + X) f_X(X)] \\
&= \mathbb{E}[X f_X(X)] + \mu \mathbb{E}[f_X(X)] \\
&= \mu + \mathbb{E}[X f_X(X)] \\
&= \mu + \mathrm{D}_X.
\end{aligned}$$

See (7.5) and (7.10) below for examples of distortion risk measures.

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6.2 Quantile Risk Measures

Definition 6.5. *The* Cumulative Distribution Function *(CDF) of a random variable X is the function*

$$F_X : \mathbb{R} \longrightarrow [0,1]$$

defined by

$$F_X(x) := \mathbb{P}(X \leqslant x), \qquad x \geqslant 0.$$

Any cumulative distribution function F_X satisfies the following properties:

i) $x \mapsto F_X(x)$ is non-decreasing,

ii) $x \mapsto F_X(x)$ is right-continuous,

iii) $\lim_{x\to\infty} F_X(x) = 1$,

iv) $\lim_{x\to-\infty} F_X(x) = 0$.

Cumulative distribution functions can be discontinuous functions, as illustrated in Figure 6.2 with

$$\mathbb{P}(X=0) = \mathbb{P}(X \leqslant 0) - \mathbb{P}(X < 0) = 0.25 > 0.$$

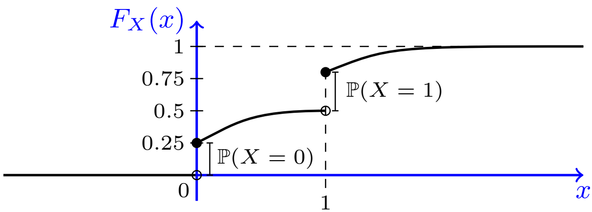


Fig. 6.2: Cumulative distribution function with discontinuities.

Proposition 6.6 shows in particular that cumulative distribution functions admit left limits.

Proposition 6.6. *For any non-decreasing sequence $(x_n)_{n\geqslant 1}$ converging to $x \in \mathbb{R}$, we have*

$$\lim_{y\nearrow x} F_X(y) = \lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n) = \mathbb{P}(X < x). \qquad (6.5)$$

Proof. By (A.7), we have

$$\mathbb{P}(X < x) = \mathbb{P}(X \in (-\infty, x))$$

$$
\begin{aligned}
&= \mathbb{P}\left(X \in \bigcup_{n\geqslant 1} (-\infty, x_n] \right) \\
&= \lim_{n\to\infty} \mathbb{P}(X \in (-\infty, x_n]) \\
&= \lim_{n\to\infty} F_X(x_n).
\end{aligned}
$$

□

As a consequence of Proposition 6.6, the gap generated by a discontinuity of the CDF F_X at a point $x \in \mathbb{R}$ is given by

$$
F_X(x) - \lim_{y \nearrow x} F_X(y) = \mathbb{P}(X \leqslant x) - \mathbb{P}(X < x) = \mathbb{P}(X = x).
$$

The following codes are plotting the graphs of the normal, exponential and Poisson CDFs.

```r
x <- seq(-4, 4, length=1000)
plot(x,pnorm(x,mean=0,sd=1), type="l", lwd=3, xlab= 'x', ylab= '', main= '', col='blue',
  ylim= c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='i', yaxs='i'); grid(4,10,lwd=2)
plot(x,pexp(x,1), type="l", lwd=3, xlab= 'x', ylab= '', main= '', col='blue', ylim=
  c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='i', yaxs='i'); grid(4,10,lwd=2)
plot(x,ppois(x,1), type="l", lwd=3, xlab= 'x', ylab= '', main= '', col='blue', ylim=
  c(-0.001,1.002), las=1, cex.lab=1, cex.axis=1, xaxs='i', yaxs='i'); grid(8,10,lwd=2)
```

Listing 6.5: R code - Cumulative distribution functions.

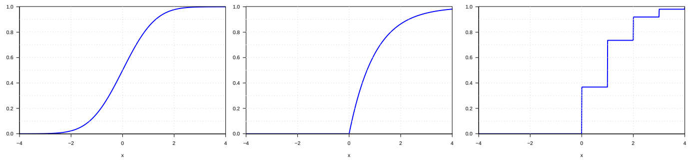
```python
import numpy as np, matplotlib.pyplot as plt
from scipy.stats import norm, expon, poisson; x= np.linspace(-4, 4, 1000)
list(map(lambda y: (plt.figure(), plt.plot(x, y, 'b-', lw=3), plt.xlabel('x'), plt.ylim(-0.001, 1.002),
  plt.grid(True, lw=2), plt.gca().set(xlim=(x.min(), x.max()), xmargin=0, ymargin=0),
  plt.show()), [norm.cdf(x), expon.cdf(x), poisson.cdf(x, mu=1)]))
```

Listing 6.6: Python code - Cumulative distribution functions.

Figure 6.3-(a) shows the continuous Cumulative Distribution Function

$$
F_X(x) := \mathbb{P}(X \leqslant x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} dy, \qquad x \geqslant 0,
$$

of a Gaussian random variable $X \simeq \mathcal{N}(0,1)$.



(a) Normal CDF. (b) Exponential CDF. (c) Poisson CDF.

Fig. 6.3: Cumulative distribution functions.

On the other hand, if $F_X(x)$ is differentiable in $x \in \mathbb{R}$ then the distribution of the random variable X is said to admit a *probability density function* (PDF) $f_X(x)$ given as the derivative

$$\varphi_X(x) = F_X'(x), \qquad x \geqslant 0.$$

The following R code is plotting the graphs of the normal and uniform PDFs.

```
x <- seq(-5, 5, length=1000)
plot(x, dnorm(x, mean=0, sd=1), type="l", lwd=3, xlab= 'x', ylab= '',main= '', col='blue',
  ylim=c(-0.001,1.002), las=1, cex.lab=1.5, cex.axis=1.5,xaxs='i',yaxs='i'); grid(4,10,lwd=2)
plot(x, dunif(x), type="l", lwd=3, xlab= 'x', ylab= '',main= '', col='blue', xlim=c(-0.5,1.5),
  ylim=c(-0.001,1.002), las=1,cex.lab=1.5, cex.axis=1.5,xaxs='i', yaxs='i'); grid(4,10,lwd=2)
```

Listing 6.7: R code - Probability density functions.

The following Python code is plotting the graph of the normal PDF.

```
import scipy.stats as stats, numpy as np, matplotlib.pyplot as plt
x= np.linspace(-5, 5, num=1000); y= stats.norm.pdf(x, loc=0, scale=1)
plt.plot(x, y, 'b-', lw=3); plt.xlabel('x'); plt.ylabel(''); plt.title(''); plt.xlim(-5, 5)
plt.xticks(fontsize=14); plt.yticks(fontsize=14)
plt.grid(True, which='both', ls='-', lw=2, alpha=.7); plt.show()
```

Listing 6.8: Python code - Gaussian probability density function.

Definition 6.7. *(Embrechts and Hofert 2013, Definition 1). The* generalized inverse $\widetilde{F}_X$ *of the* Cumulative Distribution Function

$$x \mapsto F_X(x) = \mathbb{P}(X \leqslant x), \qquad x \geqslant 0,$$

of X is defined as

$$\widetilde{F}_X(x) := \inf\big\{y \in \mathbb{R} \ : \ F_X(y) \geqslant x\big\}, \quad x \in \mathbb{R}.$$

The generalized inverse $\widetilde{F}_X$ is:

- *left-continuous*,
- it admits *right limits*, and satisfies:
- $F_X\big(\widetilde{F}_X(u)\big) = u$ for $u \in \{0,1\} \cup \operatorname{range}(F_X)$, and

- $u \leqslant F_X(x) \iff \widetilde{F}_X(u) \leqslant x$ for $u \in [0,1]$ and $x \in \mathbb{R}$, (6.2.6)

see Proposition 1-(2)-(4)-(5) in Embrechts and Hofert (2013).

```
lambda=2; Fm<- function(z) {1-exp(-lambda * z)}; max= 3; z <- seq(0, max, length.out= 10000)
plot(z,z, col= "red", lwd=3, xlim=c(0,max), type='l', ylim= c(0,max), ylab="", main="",
  xaxs="i", yaxs="i", cex.axis=1.6, cex.lab=1.6); lines(z,Fm(z),col= "blue", lwd=3.8)
inverse <- function(z) {-log(1-z)/lambda}; lines(z,inverse(z), col= "purple", lwd=3.8)
abline(h=1,lwd=2); abline(v=1,lwd=2);grid(col="black",lwd=2)
```

Listing 6.9: R code - Exponential CDF and its inverse.

```
import numpy as np, matplotlib.pyplot as plt; lam= 2; Fm= lambda z: 1 - np.exp(-lam * z)
inverse=lambda z: -np.log(1 - z)/ lam; max_val= 3; z= np.linspace(0, max_val, 1000)
plt.plot(z,Fm(z),'b-',lw=2.8);plt.plot(z,inverse(z),c='m',lw=2.8);plt.xlim(0,max_val);plt.xlabel('z')
plt.plot(z,z,'r-',lw=2);plt.axhline(1,lw=1);plt.axvline(1,lw=1);plt.ylim(0,max_val);plt.ylabel('')
plt.grid(True,c='k');plt.gca().set(xmargin=0,ymargin=0);plt.tick_params(labelsize=12);plt.show()
```

Listing 6.10: Python code - Exponential CDF and its inverse.

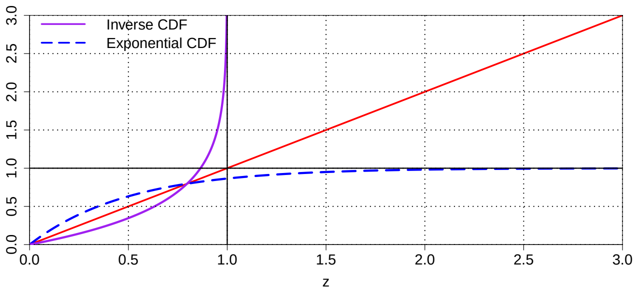


Fig. 6.4: Exponential CDF (blue) and its inverse (purple).

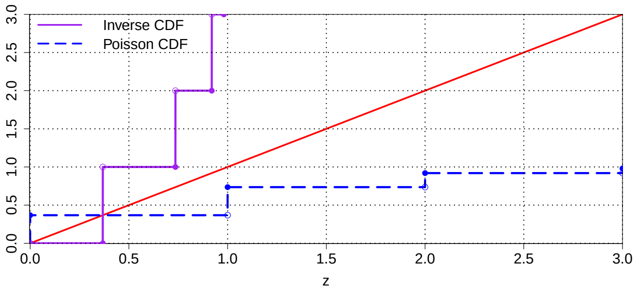


Fig. 6.5: Poisson CDF (blue) and its generalized inverse (purple).

```r
lambda=1; max=4; x<-seq(0,max,1); y<-cumsum(dpois(x,lambda))
Fm <- function(z) {stepfun(c(0,x),c(0,0,y))(z)}; z <- seq(-0.003, max-1, length.out= 10000)
plot(z,z,col="red",lwd=3,xlim= c(-0.003,max-1),type='l', ylim= c(-0.004,max-1), ylab="",
  main="", xaxs="i", yaxs="i", cex.axis=1.6, cex.lab=1.6); lines(z,Fm(z), col= "blue", lwd=3.7)
gen_inverse <- function(z) {stepfun(c(y[-max-1]),c(x))(z)}
lines(z,gen_inverse(z),col="purple",lwd=3.7);grid(col="black",lwd=2)
points(x[1:max],y[1:max],col="blue",pch=16,cex=1.6)
points(x,c(0,y[-max]),col="blue",pch=21,cex=1.6)
points(y[1:max],x[1:max],col="purple",pch=16,cex=1.6)
points(c(0,y[-max]),x,col="purple",pch=21,cex=1.6)
```

Listing 6.11: R code - Poisson CDF and its generalized inverse.

```python
import numpy as np, matplotlib.pyplot as plt; from scipy.stats import poisson;
lambda_ =1;max_ =4;x=np.arange(0,max_ +1,1);y=np.cumsum(poisson.pmf(x,lambda_));
Fm= lambda z: np.piecewise(z, [z < 0] + [(z >= xi) for xi in [0] + list(x)], [0, 0] + list(y));
z=np.linspace(-0.003,max_ -1,10000);plt.figure(figsize=(10,8));plt.plot(z,Fm(z),c='b',lw=2.7);
plt.xlim(-0.003, max_ - 1); plt.ylim(-0.004, max_ - 1);
plt.gca().set_xmargin(0); plt.gca().set_ymargin(0); plt.tick_params(axis='both', labelsize=16);
gen_inverse=lambda z: np.piecewise(z,[z < y[0]]+[(z >= yi) for yi in y[:-1]], [x[0]]+list(x[1:]));
plt.plot(z, gen_inverse(z), c='m', lw=2.7); plt.plot(z, z, c='r', lw=2.7);
plt.scatter(x[1:max_], y[1:max_], c='b', marker='o', s=60);
plt.scatter(x, np.concatenate(([0], y[:-1])), c='b', marker='o', fc='none', s=60, lw=2.7);
plt.scatter(y[1:max_], x[1:max_], c='m', marker='o', s=60); plt.grid(c='k')
plt.scatter(np.concatenate(([0],y[:-1])),x,c='m',marker='o',fc='none',s=60,lw=2.7);plt.show()
```

Listing 6.12: Python code - Poisson CDF and its generalized inverse.

Definition 6.8. *Given X a random variable with cumulative distribution function $F_X : \mathbb{R} \longrightarrow [0,1]$ and a level $p \in (0,1)$, the p-quantile q_X^p of X is defined as the* generalized inverse

$$q_X^p := \widetilde{F}_X(p) = \inf\{x \in \mathbb{R} \ : \ F_X(x) \geqslant p\}. \tag{6.7}$$

$$x \mapsto F_X(x) := \mathbb{P}(X \leqslant x), \qquad x \geqslant 0.$$

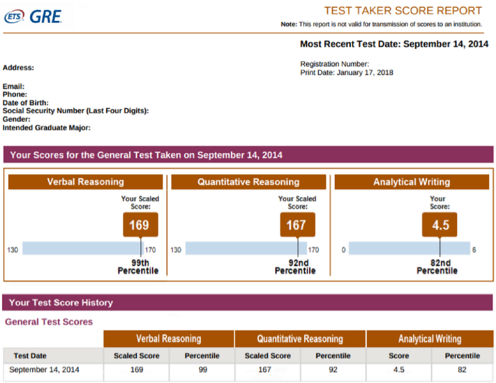
of X, see Definition 6.7. As a consequence, we have the following proposition.

Proposition 6.9.

i) The function $p \mapsto q_X^p$ is a non-decreasing, left-continuous function of $p \in [0,1]$, and it admits limits on the right.
ii) For all $p \in [0,1]$ and $x \in \mathbb{R}$, we have

$$p \leqslant F_X(x) \Longleftrightarrow q_X^p \leqslant x.$$

Proof. (i) follows from Proposition 1-(2) in Embrechts and Hofert (2013), since $F_X(x)$ is non-decreasing in $x \in \mathbb{R}$, and (ii) follows from Proposition 1-(5) therein, since $F_X(x)$ is right-continuous in $x \in \mathbb{R}$. □


GRE

TEST TAKER SCORE REPORT

Note: This report is not valid for transmission of scores to an institution.

Most Recent Test Date: September 14, 2014

Address:

Registration Number:
Print Date: January 17, 2018

Email:
Phone:
Date of Birth:
Social Security Number (Last Four Digits):
Gender:
Intended Graduate Major:

Your Scores for the General Test Taken on September 14, 2014

Verbal Reasoning	Quantitative Reasoning	Analytical Writing
Your Scaled Score: 169	Your Scaled Score: 167	Your Score: 4.5
130 – 170	130 – 170	0 – 6
99th Percentile	92nd Percentile	82nd Percentile

Your Test Score History

General Test Scores

	Verbal Reasoning		Quantitative Reasoning		Analytical Writing	
Test Date	Scaled Score	Percentile	Scaled Score	Percentile	Score	Percentile
September 14, 2014	169	99	167	92	4.5	82

Fig. 6.6: Example of quantiles given as percentiles.

Quantiles of common distributions

The quantiles of various distributions can be obtained in R.

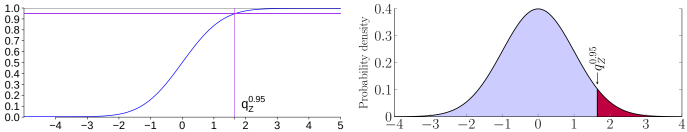
- *Gaussian distribution.* The following command shows that the 95%-quantile of a $\mathcal{N}(0,1)$ Gaussian random variable is 1.644854.

```
qnorm(.95, mean=0, sd=1)
```

Listing 6.13: R code - Normal quantiles.

```
from scipy.stats import norm
quantile_95= norm.ppf(0.95, loc=0, scale=1); print(quantile_95)
```

Listing 6.14: Python code - Normal quantiles.



(a) Gaussian quantile $q_Z^{0.95}$ and CDF. (b) Gaussian quantile $q_Z^{0.95}$ and PDF.

Fig. 6.7: Gaussian quantile $q_Z^p = 1.644854$ at $p = 0.95$.

- *Exponential distribution.* The following command displays the 95%-quantile of an exponentially distributed random variable with CDF

$$\mathbb{P}(X \leqslant x) = 1 - \mathrm{e}^{-\lambda x}, \qquad x \geqslant 0.$$

```r
qexp(.95, 1)
```

Listing 6.15: R code - Exponential quantiles.

```python
from scipy.stats import expon
expon.ppf(0.95, scale=1/1)  # scale= 1/ rate
```

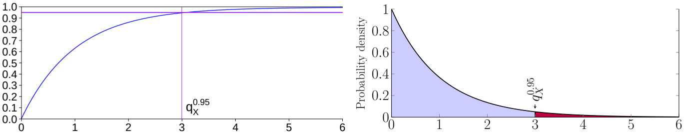
Listing 6.16: Python code - Exponential quantiles.

By equating $\mathbb{P}(X \leqslant q_X^p) = p$, we find

$$\begin{aligned} q_X^p &= \inf\left\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\right\} \\ &= -\frac{1}{\lambda}\log(1-p) \\ &= \mathbb{E}[X]\log\frac{1}{1-p}, \end{aligned}$$

and when $p = 95\%$ and $\lambda = 1$ this yields

$$q_X^p = 2.995732 \simeq 2.996\mathbb{E}[X].$$



(a) Exponential quantile $q_Z^{0.95}$ and CDF. (b) Exponential quantile $q_Z^{0.95}$ and PDF.

Fig. 6.8: Exponential quantile $q_X^p = 2.995732$ at $p = 0.95$.

- *Student distribution*. The following command displays the 90%-quantile of a Student t-distributed random variable with 5 degrees of freedom, which is 1.475884.

```r
qt(.90, df=5)
```

Listing 6.17: R code - Student quantiles.

```python
from scipy.stats import t
t.ppf(0.90, df=5)
```

Listing 6.18: Python code - Student quantiles.

- *Bernoulli distribution*. Consider the Bernoulli random variable $X \in \{0, 1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \qquad \mathbb{P}(X = 0) = 98\%.$$

In this case, we check from Figure 6.9 that $q_X^{0.99} = 1$.

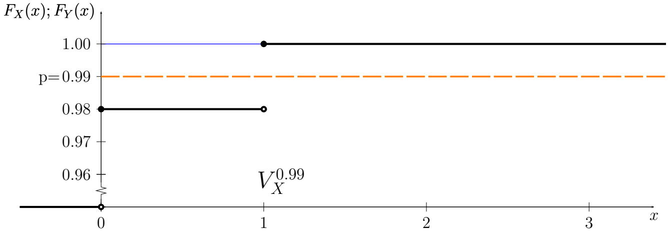


Fig. 6.9: Cumulative distribution function of X.

Empirical cumulative distribution function

The empirical Cumulative Distribution Function counts the proportion of data points which are less than a given value $x \in \mathbb{R}$.

Definition 6.10. *The* empirical Cumulative Distribution Function *(CDF) of an N-point data set $\{x_1, x_2, x_3, \ldots, x_N\}$ is estimated as*

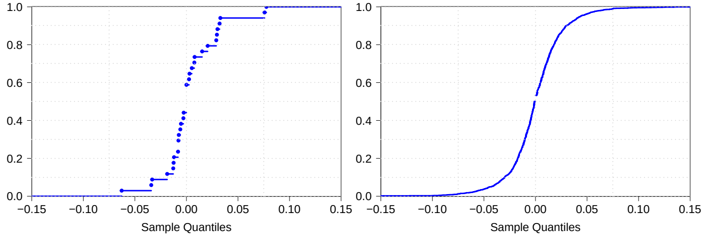
$$F_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_i \leqslant x\}}, \qquad x \in \mathbb{R}.$$

```
getSymbols("^STI",from="1990-01-03",to="2015-01-03",src="yahoo")
getSymbols("1800.HK",from=Sys.Date()-50,to=Sys.Date(),src="yahoo");stock=Ad(`1800.HK`)
stk.rtn=(stock-lag(stock))/lag(stock);stk.rtn=stk.rtn[!is.na(stk.rtn)];
  stk.ecdf=ecdf(as.vector(stk.rtn))
plot(stk.ecdf, xlab= 'Sample Quantiles', ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab= '',
  lwd= 3, main= '',col='blue', las=1, cex.lab=1.5, cex.axis=1.5, xaxs='i', yaxs='i')
grid(4,10,lwd=2)
```

Listing 6.19: R code - Empirical CDF with small data set.

```python
import yfinance as yf,numpy as np,matplotlib.pyplot as plt
from datetime import datetime,timedelta
data=yf.download("1800.HK", start=(datetime.now()-timedelta(days=50)).date(),
 end=datetime.now().date())
returns= data['Close']['1800.HK'].pct_change().dropna();sr,n=np.sort(returns), len(returns);
x=np.concatenate([[sr[0]-0.03],sr]);y=np.concatenate([[0],np.arange(1,n+1)/n]);
plt.figure(figsize=(10,6));plt.plot(np.repeat(x,2)[1:-1],np.repeat(y,2)[:-2],'b-',lw=2)
plt.yticks(np.linspace(0,1,5),fontsize=13);plt.scatter(sr,np.arange(1,n+1)/n,c='b',s=30)
plt.ylim(0,1);plt.xlim(-0.15,0.15);plt.xticks(fontsize=13);plt.xlabel('Quantiles',fontsize=15)
plt.grid(True,ls='--',alpha=.7);plt.tight_layout();plt.show()
```

Listing 6.20: Python code - Empirical CDF with small data set.



(a) Empirical CDF on 50 samples. (b) Empirical CDF on 2463 samples.

Fig. 6.10: Empirical cumulative distribution functions.

Note that the empirical distribution function in Figure 6.10-a) has a visible discontinuity (or gap) at $x = 0$, whose height 0.05483347 is given by

```r
getSymbols("1800.HK",from=Sys.Date()-3650,to=Sys.Date(),src="yahoo")
stock=Ad(`1800.HK`);stk.rtn=(stock-lag(stock))/lag(stock);stk.ecdf=ecdf(as.vector(stk.rtn))
plot(stk.ecdf, xlab= 'Sample Quantiles', ylim=c(-0.001,1.002), xlim=c(-0.15,0.15), ylab= '',
 lwd= 2, main= '',col='blue', cex=1, las=1, cex.lab=1.5, cex.axis=1.5, xaxs='i', yaxs='i')
grid(4,10,lwd=2)
```

Listing 6.21: R code - Empirical CDF with large data set.

```r
length(stk.rtn[stk.rtn==0])/length(stk.rtn)
```

Listing 6.22: R code - Estimation of gap at zero.

```python
import yfinance as yf, numpy as np,matplotlib.pyplot as plt
from datetime import datetime, timedelta
dat=yf.download("1800.HK",start=(datetime.now()-timedelta(days=3650)).date(),
 end=datetime.now().date())
returns= dat['Close']['1800.HK'].pct_change().dropna(); sr, n= np.sort(returns), len(returns)
x= np.concatenate([[sr[0]-0.03], sr]); y= np.concatenate([[0], np.arange(1,n+1)/n])
plt.figure(figsize=(10,6)); plt.plot(np.repeat(x,2)[1:-1], np.repeat(y,2)[:-2], 'b-', lw=.3)
plt.scatter(sr, np.arange(1,n+1)/n, c='b', s=1); plt.ylim(0,1); plt.xlim(-0.15,0.15);
plt.xlabel('Sample Quantiles',fontsize=15);plt.grid(True,ls='--',alpha=.7);plt.tight_layout()
plt.xticks(fontsize=13); plt.yticks(np.linspace(0,1,5), fontsize=13); plt.show()
```

Listing 6.23: Python code- Empirical CDF with large data set.

```python
np.sum(returns == 0)/ len(returns)
```

Listing 6.24: Python code - Estimation of gap at zero.

6.3 Value at Risk (VaR)

Consider a random variable X used to model the potential losses associated to a given risk. The probability $\mathbb{P}(X > V)$ that X exceeds the level V is of a capital importance. Choosing the value of V such that for example

$$\mathbb{P}(X \leqslant V) \geqslant 0.95, \quad \textit{i.e.} \quad \mathbb{P}(X > V) \leqslant 0.05,$$

means that insolvency will occur with probability less that 5%. In this setting, the 95%-quantile risk measure is the smallest value of $x \in \mathbb{R}$ such that

$$\mathbb{P}(X \leqslant x) \geqslant 0.95, \quad \textit{i.e.} \quad \mathbb{P}(X > x) \leqslant 0.05.$$

More precisely, we have the following definition.

Definition 6.11. *The Value at Risk V_X^p of a random variable X at the level $p \in (0, 1)$ is the p-quantile of X defined by*

$$V_X^p := \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p\}. \tag{6.8}$$

In other words, for some decreasing sequence $(x_n)_{n \geqslant 1}$ such that

$$\mathbb{P}(X \leqslant x_n) \geqslant p \quad \text{for all} \quad n \geqslant 1,$$

we have

$$V_X^p := \lim_{n \to \infty} x_n. \tag{6.9}$$

Similarly to the above, the function $p \mapsto V_X^p$ coincides with the *generalized inverse* $\widetilde{F}_X$ of the *Cumulative Distribution Function* $\mapsto F_X$ of X, and from Proposition 6.9-(i) we have the following result.

Proposition 6.12. *The function $p \mapsto V_X^p$ is a* non-decreasing, left-continuous *function of $p \in [0, 1]$, and it admits limits on the right.*

In particular, if F_X is continuous and strictly increasing it admits an inverse $F_X^{-1} = \widetilde{F}_X$, and in this case V_X^p is given by

$$V_X^p = F_X^{-1}(p), \qquad p \in (0, 1).$$

Proposition 6.13. *The Value at Risk V_X^p of X at the level $p \in (0, 1)$ satisfies the properties*

$$\mathbb{P}\big(X < V_X^p\big) \leqslant p \leqslant \mathbb{P}\big(X \leqslant V_X^p\big), \tag{6.10}$$

and

$$\mathbb{P}\big(X > V_X^p\big) \leqslant 1 - p \leqslant \mathbb{P}\big(X \geqslant V_X^p\big). \tag{6.11}$$

In particular, if $\mathbb{P}(X = V_X^p) = 0$*, then we have*

$$p = \mathbb{P}\big(X < V_X^p\big) = \mathbb{P}(X \leqslant V_X^p). \tag{6.12}$$

Proof. Using the decreasing sequence $(x_n)_{n\geqslant 1}$ in (6.9) and the right continuity of the cumulative distribution function F_X, we have

$$\begin{aligned}\mathbb{P}\big(X \leqslant V_X^p\big) &= \mathbb{P}(X \leqslant \lim_{n\to\infty} x_n)\\ &= F_X\big(\lim_{n\to\infty} x_n\big)\\ &= \lim_{n\to\infty} F_X(x_n)\\ &= \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n)\\ &\geqslant p.\end{aligned}$$

On the other hand, if $\mathbb{P}\big(X < V_X^p\big) > p$ then there is a strictly increasing sequence $(y_n)_{n\geqslant 1}$ such that

$$\lim_{n\to\infty} y_n = V_X^p$$

and by (6.5) we have

$$\mathbb{P}\big(X < V_X^p\big) = \lim_{n\to\infty} \mathbb{P}(X \leqslant y_n) > p,$$

in which case there would exist $n \geqslant 1$ such that $y_n < V_X^p$ and $\mathbb{P}(X \leqslant y_n) > p$, which contradicts (6.8). Relations (6.11)-(6.12) are direct consequences of (6.10). □

When $\mathbb{P}\big(X = V_X^p\big) > 0$ we may have $\mathbb{P}\big(X > V_X^p\big) = 0$, for example in the case of a Bernoulli random variable $X \in \{0,1\}$ with the distribution

$$\mathbb{P}(X = 1) = 2\%, \qquad \mathbb{P}(X = 0) = 98\%,$$

see Figure 6.9. Proposition 6.14 also follows from the Definition 6.11 of V_X^p and Proposition 6.9-(ii).

Proposition 6.14. *For all* $x \in \mathbb{R}$ *we have*

$$V_X^p \leqslant x \quad \Longleftrightarrow \quad \mathbb{P}(X \leqslant x) \geqslant p. \tag{6.13}$$

Proof. $\Leftarrow$) If $\mathbb{P}(X \leqslant x) \geqslant p$ then we have

$$V_X^p = \inf\{y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \leqslant x.$$

$\Rightarrow$) On the other hand, choosing a strictly decreasing sequence $(x_n)_{n\geqslant 1}$ such that

$$\lim_{n\to\infty} x_n = V_X^p \quad \text{and} \quad \mathbb{P}(X \leqslant x_n) \geqslant p, \qquad n \geqslant 1,$$

if $V_X^p \leqslant x$ we have

$$\mathbb{P}(X \leqslant x) \geqslant \mathbb{P}(X \leqslant V_X^p) = \lim_{n\to\infty} \mathbb{P}(X \leqslant x_n) \geqslant p$$

by the right continuity of the cumulative distribution function F_X of X. □

On the other hand, the Value at Risk V_X^p does not reveal any information on *how large* losses can be beyond V_X^p, see Chapter 7 for details. Proposition 6.15 shows how to estimate Value at Risk when switching the sign of the data.

Proposition 6.15. *Assume that the cumulative distribution function F_X is continuous and strictly increasing. Then, we have*

$$V_{-X}^p = -V_X^{1-p}, \qquad p \in (0,1). \tag{6.14}$$

Proof. Since F_X is continuous, we have

$$\begin{aligned} F_{-X}(x) &= \mathbb{P}(-X \leqslant x) \\ &= \mathbb{P}(X \geqslant -x) \\ &= 1 - \mathbb{P}(X < -x) \\ &= 1 - \mathbb{P}(X \leqslant -x) \\ &= 1 - F_X(-x), \end{aligned}$$

hence, taking $x := F_{-X}^{-1}(p)$, we have

$$p = F_{-X}\big(F_{-X}^{-1}(p)\big) = 1 - F_X\big(-F_{-X}^{-1}(p)\big),$$

or

$$F_X\big(-F_{-X}^{-1}(p)\big) = 1 - p$$

i.e.

$$F_{-X}^{-1}(p) = -F_X^{-1}(1-p),$$

which yields

$$V_{-X}^p = F_{-X}^{-1}(p) = -F_X^{-1}(1-p) = -V_X^{1-p}, \qquad p \in (0,1).$$

□

In Figure 6.11 we choose a continuous CDF F_X with

$$F_{-X}(x) = 1 - F_X(-x), \qquad x \in \mathbb{R}.$$

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In this case, the continuity of F_X ensures the symmetry property around the origin

$$V^p_{-X} = -V_X^{1-p},$$

see Proposition 6.15.

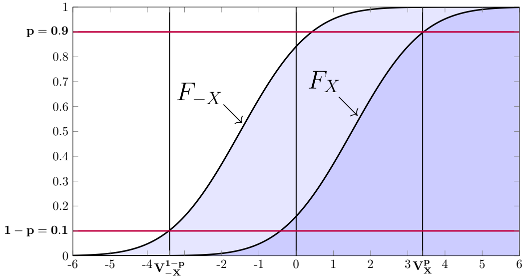

Fig. 6.11: Continuous CDF and symmetric VaR.

On the other hand, in Figure 6.12 we consider X with distribution

$$\mathbb{P}(X = 1) = 0.5, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3.4) = 0.2,$$

hence

$$\mathbb{P}(-X = -3.4) = 0.2, \quad \mathbb{P}(-X = -2) = 0.3, \quad \mathbb{P}(-X = -1) = 0.5,$$

and we check that in this discontinuous case the relation $V^q_{-X} = -V_X^{1-q}$ fails for $p = 0.8$, although it still holds for $p' = 0.9$.

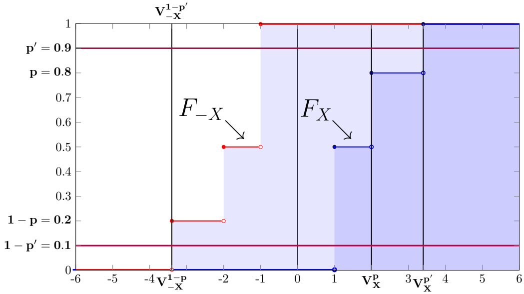

Fig. 6.12: Discontinuous CDF and nonsymmetric VaR.

Next, we check the properties of Value at Risk. Although Value at Risk satisfies the monotonicity, positive homogeneity and translation invariance properties, it is *not* subadditive in general. Namely, the Value at Risk V_{X+Y}^p of $X+Y$ may be larger than the sum $V_X^p + V_Y^p$.

Proposition 6.16. *Value at Risk V_X^p is*

- *monotone,*
- *positive homogeneous,*
- *translation invariant.*

However, it is not subadditive, and therefore it is not a coherent risk measure.

Proof.

a) *Monotonicity.*

Value at Risk is a monotone risk measure. If $X \leqslant Y$ then

$$\mathbb{P}(Y \leqslant x) = \mathbb{P}(X \leqslant Y \leqslant x) \leqslant \mathbb{P}(X \leqslant x), \qquad x \geqslant 0,$$

hence

$$\mathbb{P}(Y \leqslant x) \geqslant p \quad \Longrightarrow \quad \mathbb{P}(X \leqslant x) \geqslant p, \qquad x \geqslant 0,$$

which shows that

$$V_X^p \leqslant V_Y^p$$

by (6.8).

b) *Positive homogeneity and translation invariance.*

Value at Risk satisfies the positive homogeneity and translation invariance properties. For any $\mu \in \mathbb{R}$ and $\lambda > 0$, we have

$$\begin{aligned}
V_{\mu+\lambda X}^p &= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(\mu + \lambda X \leqslant x) \geqslant p\} \\
&= \inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant (x-\mu)/\lambda) \geqslant p\} \\
&= \inf\{\mu + \lambda y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \\
&= \mu + \lambda \inf\{y \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant y) \geqslant p\} \\
&= \mu + \lambda V_X^p.
\end{aligned}$$

c) *Subadditivity and coherence.*

We show that Value at Risk is *not subadditive* by considering two *independent* Bernoulli random variables $X, Y \in \{0,1\}$ having the same distribution

$$\begin{cases} \mathbb{P}(X=1) = \mathbb{P}(Y=1) = 2\%, \\ \mathbb{P}(X=0) = \mathbb{P}(Y=0) = 98\%, \end{cases}$$

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hence $V_X^{0.975} = V_Y^{0.975} = 0$.

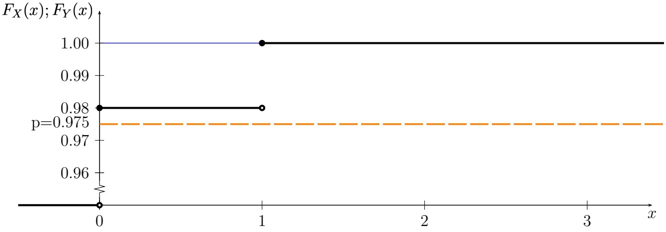

Fig. 6.13: Cumulative distribution function of X and Y.

On the other hand, we have

$$\begin{cases} \mathbb{P}(X+Y=2) = \mathbb{P}(X=1 \text{ and } Y=1) = (0.02)^2 = 0.04\%, \\ \mathbb{P}(X+Y=1) = 2 \times 0.02 \times 0.98 = 3.92\%, \\ \mathbb{P}(X+Y=0) = \mathbb{P}(X=0 \text{ and } Y=0) = (0.98)^2 = 96.04\%, \end{cases}$$

hence

$$V_{X+Y}^{0.975} = 1 > V_X^{0.975} + V_Y^{0.975} = 0.$$

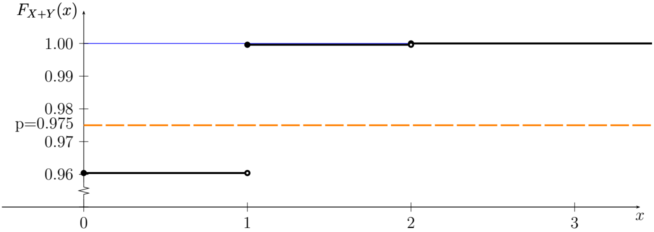

Fig. 6.14: Cumulative distribution function of $X+Y$.

□

In Proposition 6.17 we use the standard Gaussian Cumulative Distribution Function (CDF)

$$\Phi(x) := \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R},$$

of a standard normal random variable $Z \simeq \mathcal{N}(0,1)$.

Proposition 6.17. Gaussian Value at Risk. *Given* $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$, *we have*

$$V_X^p = \mu_X + \sigma_X q_Z^p \tag{6.15}$$

where the normal quantile $q_Z^p = V_Z^p$ *at the level* p *satisfies*

$$\Phi(q_Z^p) = \mathbb{P}\big(Z \leqslant q_Z^p\big) = p \quad \textit{for} \quad Z \simeq \mathcal{N}(0,1),$$

i.e.

$$q_Z^p = \Phi^{-1}(p) \quad \textit{and} \quad V_X^p = \mu_X + \sigma_X \Phi^{-1}(p).$$

Proof. We represent the random variable $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ as

$$X = \mu_X + \sigma_X Z,$$

where $Z \simeq \mathcal{N}(0,1)$ is a standard normal random variable, and use the relation

$$\begin{aligned}
p &= \mathbb{P}(X \leqslant V_X^p) \\
&= \mathbb{P}(\mu_X + \sigma_X Z \leqslant V_X^p) \\
&= \mathbb{P}(Z \leqslant (V_X^p - \mu_X)/\sigma_X) \\
&= \mathbb{P}(Z \leqslant q_Z^p),
\end{aligned}$$

which holds provided that $V_X^p = \mu_X + \sigma_X q_Z^p$. □

We also note that if $X \simeq \mathcal{N}(\mu_X, \sigma_X^2)$ then $-X \simeq \mathcal{N}(-\mu_X, \sigma_X^2)$, hence

$$\begin{aligned}
V_{-X}^p &= -\mu_X + \sigma_X q_Z^p \\
&= -\mu_X - \sigma_X q_Z^{1-p} \\
&= -V_X^{1-p},
\end{aligned}$$

which is consistent with (6.14).

Lemma 6.18. *For any two random variables* X *and* Y, *we have*

$$\sqrt{\mathrm{Var}[X+Y]} \leqslant \sqrt{\mathrm{Var}[X]} + \sqrt{\mathrm{Var}[Y]}.$$

Proof. By (6.15), we have

$$\begin{aligned}
&\mathrm{Var}[X+Y] \\
&= \mathbb{E}\big[(X+Y)^2\big] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}\big[X^2\big] + \mathbb{E}\big[Y^2\big] + 2\mathbb{E}[XY] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]
\end{aligned}$$

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$$
\begin{aligned}
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y) \qquad (6.16)\\
&\leqslant \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\sqrt{\mathbb{E}\big[(X-\mathbb{E}[X])^2\big]}\sqrt{\mathbb{E}\big[(Y-\mathbb{E}[Y])^2\big]}\\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\sqrt{\mathrm{Var}[X]}\sqrt{\mathrm{Var}[Y]} \qquad (6.17)\\
&= \big(\sqrt{\mathrm{Var}[X]} + \sqrt{\mathrm{Var}[Y]}\big)^2,
\end{aligned}
$$

where, from (6.16) to (6.17) we applied the *Cauchy-Schwarz* inequality. □

Remark 6.19 shows that, although Value at Risk is *not sub-additive* in general, it is sub-additive (and therefore coherent) on (not necessarily independent) Gaussian random variables.

Remark 6.19. *If X and Y are two Gaussian random variables, we have*

$$V_{X+Y}^p \leqslant V_X^p + V_Y^p.$$

Proof. By Lemma 6.18 the standard deviations of $X+Y$ and X, Y satisfy

$$\sigma_{X+Y} \leqslant \sigma_X + \sigma_Y,$$

hence if X and Y are Gaussian random variables, by (6.15) we have

$$
\begin{aligned}
V_{X+Y}^p &= \mu_{X+Y} + \sigma_{X+Y} q_Z^p\\
&= \mu_X + \mu_Y + \sigma_{X+Y} q_Z^p\\
&\leqslant \mu_X + \mu_Y + (\sigma_X + \sigma_Y) q_Z^p\\
&= V_X^p + V_Y^p.
\end{aligned}
$$

□

6.4 Numerical Estimates

In this section we implement the numerical computation of Value at Risk, see also § 6.1.1 of Mina and Xiao (2001). In the case where we care about negative return values, Definition 6.11 is replaced with

$$\overline{V}_X^p := \mathrm{Sup}\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \leqslant 1-p\}. \qquad (6.18)$$

In case the CDF of X is continuous, we note the relation

$$
\begin{aligned}
\overline{V}_X^p &= \mathrm{Sup}\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \leqslant 1-p\}\\
&= -\inf\{-x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \leqslant 1-p\}\\
&= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant -x) \leqslant 1-p\}\\
&= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(-X \geqslant x) \leqslant 1-p\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\inf\{x \in \mathbb{R} \ : \ 1 - \mathbb{P}(-X \geqslant x) \geqslant p\} \\
&= -\inf\{x \in \mathbb{R} \ : \ \mathbb{P}(-X \leqslant x) \geqslant p\} \\
&= -V^p_{-X},
\end{aligned}
$$

hence the relation

$$
\overline{V}^p_X = -V^p_{-X} = V^{1-p}_X \tag{6.19}
$$

which is obtained from Proposition 6.15 when the cumulative distribution function F_X is continuous and strictly increasing.

```r
library(quantmod); getSymbols("0700.HK",from="2010-01-03",to="2018-02-01",src="yahoo")
stock=Ad(`0700.HK`);chartSeries(stock,up.col="blue",theme="white")
stk.rtn=(stock-lag(stock))/lag(stock)[-1]; returns <- as.numeric(stk.rtn[!is.na(stk.rtn)])
VaR <- function(p){r=-returns; ordered<-r[order(r)]; v=tail(ordered,1); i=length(r)
while(length(r[r<=v])/length(r)>=p) {i=i-1;VaR=v;v=ordered[i]}; return (-VaR)}
var1=VaR(0.95); var2=-quantile(-returns, probs = 0.95,type=1);times=index(stock)
length(returns[returns<var1])/length(returns); length(returns[returns<var2])/length(returns)
chartSeries(stk.rtn,up.col="blue",theme="white");abline(h=var1,col="red",lwd=3)
```

Listing 6.25: R code - Estimation of the Value at Risk (6.19).

The historical 95%-Value at Risk over N samples $(x_i)_{i=1,2,\ldots,N}$ can be estimated by inverting the *empirical cumulative distribution function* $F_N(x)$, and is found to be $\overline{V}^{95\%}_X = -0.03117879$. It can be recovered using the type=1 option in the R quantile command.

```python
import yfinance as yf,pandas as pd,numpy as np,matplotlib.pyplot as plt;
stock_data=yf.download("0700.HK","2010-01-03","2018-02-01"); stock=stock_data['Close'];
plt.figure(figsize=(12,6)); plt.plot(stock.index,stock,c='b')
plt.title("Stock Price"); plt.grid(True); plt.show();
stock_rtn=(stock-stock.shift(1))/stock.shift(1); stock_rtn=stock_rtn.dropna();
returns=np.array(stock_rtn); returns=-returns[~np.isnan(returns)];
VaR= lambda p: ((rr := -returns), (ordered := np.sort(rr)), (n := len(rr)), (taken :=
 list(__import__('itertools').takewhile( lambda v: (rr[rr <= v].size/ n) >= p, ordered[::-1]))),
 -taken[-1] if taken else -ordered[-1])[4]; var1=VaR(0.95);print(f"VaR(0.95): {var1}")
var2=pd.Series(returns).quantile(0.05,interpolation='nearest');
len(returns[returns<var1])/len(returns); len(returns[returns<var2])/len(returns);
plt.figure(figsize=(12,6)); plt.plot(stock_rtn.index,stock_rtn,c='b',lw=1);
plt.axhline(y=var1,c='r',lw=3); plt.title("Returns with VaR Line"); plt.grid(True); plt.show()
```

Listing 6.26: Python **3.8** code - Estimation of the Value at Risk (6.19).

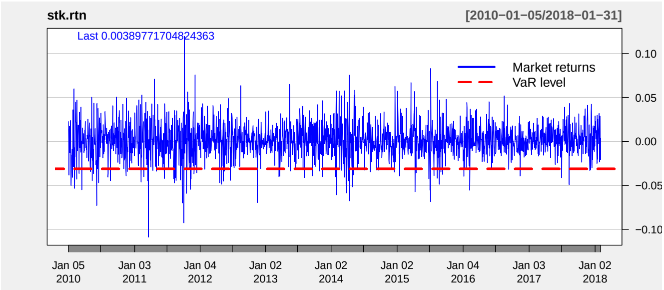

Fig. 6.15: Market returns *vs.* Value at Risk level in red.

The Gaussian 95%-Value at Risk is estimated from (6.15) with $p = 0.95$ as

$$\overline{V}_X^p = V_X^{1-p} = \mu + \sigma q_Z^{1-p} = \mu - \sigma q_Z^p,$$

where $-\mu = \mathbb{E}[-X]$ and $\sigma^2 = \mathrm{Var}[-X]$, and is found equal to

$$\overline{V}_X^{95\%} = -0.0311592.$$

according to Proposition 6.17, using following code.

```
m=mean(stk.rtn,na.rm=TRUE);s=sd(stk.rtn,na.rm=TRUE);q=qnorm(.95,mean=0,sd=1);m-s*q
```

Listing 6.27: R code - Computation of Gaussian Value at Risk.

```
import numpy as np, pandas as pd, yfinance as yf, matplotlib.pyplot as plt
from scipy.stats import norm
st=yf.download("0700.HK",start="2010-01-03",end="2018-02-01");adj_close=st['Close']
st.rtn=adj_close.pct_change().dropna();m=st.rtn.mean();
s=st.rtn.std();q=norm.ppf(0.95);(m-s*q).item()
```

Listing 6.28: Python code - Computation of Gaussian Value at Risk.

Note that here, we are concerned about large negative returns, which explains the negative sign in $m - s * q$.

Lemma 6.20 extends Lemma 4.5, and is useful for random simulation purposes. It will also be used in the proof of Propositions 7.6 and 7.13 below.

Lemma 6.20. *Any random variable X with CDF F_X can be represented in distribution as*

$$X := V_X^U = \widetilde{F}_X(U), \tag{6.20}$$

where U is a uniformly distributed random variable on the interval $[0, 1]$.

Proof. It suffices to note that by (6.13) and (6.20) we have

$$\begin{aligned}F_X(x) &= \mathbb{P}\big(V_X^U \leqslant x\big)\\ &= \mathbb{P}(U \leqslant \mathbb{P}(X \leqslant x))\\ &= \mathbb{P}(X \leqslant x), \quad x \geqslant 0.\end{aligned}$$

□

Exercises

Exercise 6.1

a) Is the expected value premium principle (6.2) a coherent risk measure?
b) Is the standard deviation premium principle (6.3) a coherent risk measures?

Exercise 6.2 Consider a random variable X having the Pareto distribution with probability density function

$$f_X(x) = \frac{\gamma\theta^\gamma}{(\theta + x)^{\gamma+1}}, \qquad x \geqslant 0.$$

a) Compute the cumulative distribution function

$$F_X(x) := \int_0^x f_X(y)dy, \qquad x \geqslant 0.$$

b) Compute the value at risk V_X^p at the level p for any θ and γ, and then for $p = 99\%$, $\theta = 40$ and $\gamma = 2$.

Exercise 6.3 Consider a random variable X with the following cumulative distribution function:

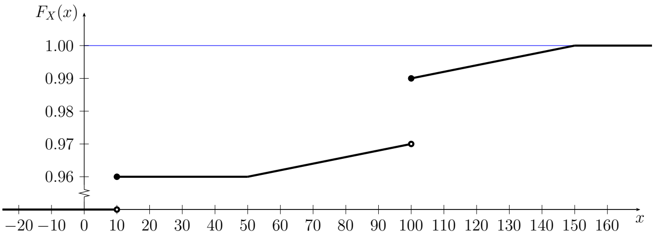

Fig. 6.16: Cumulative distribution function of X.

a) Give the value of $\mathbb{P}(X = 100)$.

b) Give the value of V_X^q for all q in the interval $[0.97, 0.99]$.
c) Compute the value of V_X^q for all q in the interval $[0.99, 1]$.

Hint: We have

$$F_X(x) = \mathbb{P}(X \leqslant x) = 0.99 + 0.01 \times \frac{x - 100}{50}, \qquad x \in [100, 150].$$

Exercise 6.4 Discrete distribution. Consider $X \in \{10, 100, 110\}$ with the distribution

$$\mathbb{P}(X = 10) = 90\%, \quad \mathbb{P}(X = 100) = 9.5\%, \quad \mathbb{P}(X = 110) = 0.5\%.$$

Compute the value at risk $V_X^{99\%}$.

Exercise 6.5 Exponential distribution. Assume that X has an exponential distribution with parameter $\lambda > 0$ and mean $1/\lambda$, *i.e.*

$$\mathbb{P}(X \leqslant x) = 1 - \mathrm{e}^{-\lambda x}, \qquad x \geqslant 0.$$

a) Compute
$$V_X^p := \inf \big\{ x \in \mathbb{R} \ : \ \mathbb{P}(X \leqslant x) \geqslant p \big\}$$
and $V_X^{95\%}$.
b) Assuming that the liabilities of a company are estimated by $\mathbb{E}[X]$, compute the amount of required capital C_X from (6.1).

Exercise 6.6 Given X a random variable having the geometric distribution with
$$\mathbb{P}(X = k) = (1 - p)^k p, \qquad k \geqslant 0,$$
compute the conditional expectation $\mathbb{E}[X \mid X \geqslant a]$ for $a > 0$.

Exercise 6.7 Estimating risk probabilities from moments.

a) Show that for every $r > 0$

$$V_X^p \leqslant \left(\frac{\mathbb{E}[|X|^r]}{1 - p} \right)^{1/r} = \frac{\|X\|_{L^r(\Omega)}}{(1 - p)^{1/r}},$$

where $\|X\|_{L^r(\Omega)} := (\mathbb{E}[|X|^r])^{1/r}$.

Hint: Use the argument of the Markov inequality.
b) Give an upper bound for $V_X^{95\%}$ when $p = 95\%$ and $r = 1$.

Exercise 6.8 We consider a discrete random variable X having the following distribution.

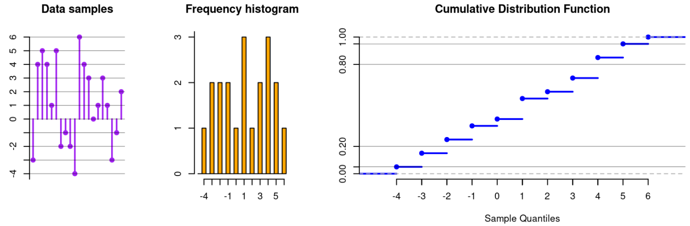

Fig. 6.17: Graphs of data samples, histogram, and empirical CDF.

a) Find the following quantities for the above data set, and mark their values on the graph.

  i) Historical "Academic" Value at Risk at $p = 0.95$. $\mathrm{VaR}^{95}_{\mathrm{A}c\text{-H}} =$ ________

  ii) Historical "Academic" Value at Risk at $p = 0.80$. $\mathrm{VaR}^{80}_{\mathrm{A}c\text{-H}} =$ ________

  iii) Historical "Practitioner" Value at Risk at $p = 0.95$. $\overline{\mathrm{VaR}}^{95}_{\mathrm{P}r\text{-H}} =$ ________

  iv) Historical "Practitioner" Value at Risk at $p = 0.80$. $\overline{\mathrm{VaR}}^{80}_{\mathrm{P}r\text{-H}} =$ ________

b) Knowing that mean=1.15, sd=3.048, qnorm(0.95)=1.645 and qnorm(0.80)=0.842, compute (from Proposition 6.17):

  i) Gaussian "Academic" Value at Risk at $p = 0.95$. $\mathrm{VaR}^{95}_{\mathrm{A}c\text{-G}} =$ ________

  ii) Gaussian "Academic" Value at Risk at $p = 0.80$. $\mathrm{VaR}^{80}_{\mathrm{A}c\text{-G}} =$ ________

  iii) Gaussian "Practitioner" Value at Risk at $p = 0.95$. $\overline{\mathrm{VaR}}^{95}_{\mathrm{P}r\text{-G}} =$ ________

  iv) Gaussian "Practitioner" Value at Risk at $p = 0.80$. $\overline{\mathrm{VaR}}^{80}_{\mathrm{P}r\text{-G}} =$ ________