Infinite divisibility of interpolated gamma powers

Nicolas Privault*
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

Dichuan Yang†
Department of Mathematics
City University of Hong Kong
Tat Chee Avenue
Kowloon Tong
Hong Kong

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Abstract

This paper is concerned with the distribution properties of the binomial $aX + bX^\alpha$, where X is a gamma random variable. We show in particular that $aX + bX^\alpha$ is infinitely divisible for all $\alpha \in [1, 2]$ and $a, b \in \mathbb{R}_+$, and that for $\alpha = 2$ the second order polynomial $aX + bX^2$ is a generalized gamma convolution whose Thorin density and Wiener-gamma integral representation are computed explicitly. As a byproduct we deduce that fourth order multiple Wiener integrals are in general not infinitely divisible.

Key words: Infinite divisibility; complete monotonicity; gamma distribution; generalized gamma convolutions; powers of random variables.
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1 Introduction

The power X_β^α of order $\alpha \in (-\infty, -1] \cup [1, \infty)$ of a gamma random variable X_β with shape parameter $\beta > 0$ is known to be infinitely divisible, and in addition it belongs to the class of generalized gamma convolutions (GGCs), cf. [10], which is made of random variables Z whose Laplace transform can be expressed as

$$E[e^{-sZ}] = \exp\left(-cs - \int_0^\infty \log\left(1 + \frac{s}{t}\right)\mu(dt)\right), \qquad s \geq 0, \tag{1.1}$$

*nprivault@ntu.edu.sg
†dyang5@student.cityu.edu.hk

where $c \geq 0$ and the Thorin measure $\mu(dt)$ of Z satisfies

$$\int_{0^+}^{1} |\log t| \mu(dt) < \infty \quad \text{and} \quad \int_{1}^{\infty} \frac{1}{t} \mu(dt) < \infty, \tag{1.2}$$

cf. page 29 of [1]. We refer to [5] for a complete survey with recent results on generalized gamma convolutions.

In particular, the random variable X_β^α is a GGC for all $\alpha \geq 1$, cf. Example 4.3.4 page 60 of [1], and its Thorin measure $\mu_{0,\alpha}(dx)$ has total mass

$$\mu_{0,\alpha}([0,\infty)) = \sup \left\{ \nu > 0 \ : \ \lim_{x \searrow 0} \frac{f_{X_\beta^\alpha}(x)}{x^{\nu-1}} = 0 \right\} = \frac{\beta}{\alpha}, \tag{1.3}$$

cf. Theorem 4.4 of [1], and cumulative function

$$F_{0,\alpha} : \mathbb{R}_+ \to [0, \beta/\alpha).$$

In addition, X_β^α is also a GGC with $\mu_{0,\alpha}([0,\infty)) = \infty$ if $\alpha < -1$.

In this paper we deal with the binomial $aX_\beta + bX_\beta^\alpha$, $\alpha \geq 0$, and show in Proposition 2.1 that $aX_\beta + bX_\beta^\alpha$ is infinitely divisible for all $\alpha \in [1,2]$ and $a, b \in \mathbb{R}_+$. For $\alpha = 2$ we consider the problem of addition and interpolation of two *dependent* GGCs by proving that the (non-central) gamma square $aX_\beta + bX_\beta^2$ is a generalized gamma convolution for all $\beta > 0$, whose Thorin density $\varphi_{a,b}$ is computed explicitly in Proposition 3.1 as

$$\varphi_{a,b}(y) = \frac{\beta(1+ay)e^{(1-ay)^2/(4by)}}{2^\beta b^{1/2} \pi^{3/2} y^{3/2} \left(\Gamma(1/2+\beta/2) + 2^{2-2\beta} F^2 \left(\frac{|1-ay|}{2\sqrt{by}} \right) / \Gamma(1/2+\beta/2) \right)},$$

$y \in \mathbb{R}_+$, where F is the complex error function. In Proposition 3.4 we also compute the Wiener-gamma integral representation

$$aX_\beta + bX_\beta^2 \simeq \int_0^\beta h_{a,b}(s) d\gamma_s,$$

where $(\gamma_s)_{s \in \mathbb{R}_+}$ is a standard gamma process, and $h_{a,b}$ is given by (3.9) below.

The case $\beta = 1/2$ allows us to consider powers of the square $Z^2 \simeq X_{1/2}$ of a centered Gaussian random variable Z. As an application of Proposition 3.1 for $\beta = 1/2$ we find that $|Z|^\alpha + aZ^2$ is infinitely divisible for all $\alpha \in [2, 4]$ and $a \geq 0$, and, in particular, that $Z^4 + aZ^2$ is infinitely divisible for $a \geq 0$. In Section 4 we show that $Z^4 + aZ^2$ is not infinitely divisible for all $a < 0$, showing in particular that fourth order multiple stochastic integrals with respect to Brownian motion are not infinitely divisible random variables, although first and second order Brownian stochastic integrals are known to be infinitely divisible.

We proceed as follows. After recalling some basic results on infinite divisibility at the end of this introduction, we consider interpolated gamma powers of the form $aX_\beta + bX_\beta^\alpha$ in Section 2. The case of second order polynomials of the form $aX_\beta + bX_\beta^2$ as generalized gamma convolutions is discussed in Section 3 with explicit calculations of Thorin measures and Wiener-gamma representations, based in part on Theorem 2.3 of [5]. In Section 4 we consider the infinite divisibility of multiple Wiener integrals. Section 5 contains the complete monotonicity results needed in this paper, and in the Appendix Section 6 we extend to the gamma case the results of [5] on the computation of Thorin densities for exponential random variables.

Infinite divisibility and complete monotonicity

We close this introduction with a review of the links between infinite divisibility and complete monotonicity. Recall that a nonnegative random variable $Z \geq 0$ is infinitely divisible if and only if its Laplace transform takes the form

$$E[e^{-sZ}] = \exp\left(-cs - \int_0^\infty (1 - e^{-sx})\nu(dx)\right), \qquad s \in \mathbb{R}_+, \tag{1.4}$$

where $c \geq 0$ and $\nu(dx)$ is a measure on $\mathbb{R}_+$ such that

$$\int_0^\infty (1 \wedge x)\nu(dx) < \infty.$$

In this paper we are mainly concerned with the infinite divisibility of lower (or upper) bounded random variables, for which we will use the following criterion, cf. Theorem XIII.7.1 of [4], Theorem III.4.1 of [9], or Theorem 5.9 of [8]. Recall that a $\mathcal{C}^\infty$ function $f : (0, \infty) \to \mathbb{R}$ is completely monotone if

$$(-1)^n f^{(n)}(x) \geq 0, \qquad x \in \mathbb{R}_+,$$

for all integer $n \geq 0$.

Theorem 1.1 *Let Z be a nonnegative random variable with Laplace transform*

$$\Psi_Z(s) = E[e^{-sZ}], \qquad s \in \mathbb{R}_+.$$

Then Z is infinitely divisible if and only if

$$s \mapsto -\frac{\partial}{\partial s} \log \Psi_Z(s)$$

is completely monotone on $\mathbb{R}_+$.

Proof. By the Bernstein theorem, cf. e.g. Theorem 3.2 of [8], the $\mathcal{C}^\infty$ function

$$\varphi(s) = -\log \Psi_Z(s), \qquad s \in \mathbb{R}_+,$$

has the representation

$$\varphi(s) = cs + \int_0^\infty (1 - e^{-sx})\nu(dx), \qquad s \in \mathbb{R}_+, \tag{1.5}$$

where $c \in \mathbb{R}_+$ and $\nu(dx)$ is a measure on $\mathbb{R}_+$ such that $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$, if and only if φ' is completely monotone. $\square$

In addition it follows from (1.1), (1.4), Frullani's identity

$$\log\left(1 + \frac{s}{t}\right) = \int_0^\infty (1 - e^{-sx})e^{-xt}\frac{dx}{x}, \qquad s, t \in \mathbb{R}_+,$$

that the Lévy measure $\nu(dx)$ is linked to the Thorin measure $\mu(dx)$ by the relation

$$\nu(dx) = \frac{1}{x}\int_0^\infty e^{-xz}\mu(dz)dx, \qquad x > 0.$$

2 Interpolated gamma powers

Let X_β denote a gamma random variable with density

$$f_{X_\beta}(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1}e^{-x}, \qquad x > 0,$$

and shape parameter $\beta > 0$. Recall that for any $\alpha \in \mathbb{R} \setminus \{0\}$ the random variable X_β^α has density

$$f_{X_\beta^\alpha}(x) = \frac{1}{|\alpha|\Gamma(\beta)} x^{-1+\beta/\alpha}e^{-x^{1/\alpha}}, \qquad x > 0,$$

which is a Weibull probability density when $\alpha > 0$ and $\beta = 1$. On the other hand, X_β^α is not infinitely divisible for $\alpha \in (0,1)$ and it is unknown whether X_β^α is infinitely divisible when $\alpha \in (-1,0)$, cf. Example 4.3.4 page 60 and § III page 67 of [1].

In this section we prove the following result.

Proposition 2.1 *Let $\beta > 0$ and $a, b \in \mathbb{R}_+$. The random variable $aX_\beta + bX_\beta^\alpha$ is infinitely divisible for all $\alpha \in [1,2]$.*

Proof. This result is a consequence of Theorem 1.1, Lemma 2.2 below, and Corollary 5.2 which states that the function

$$s \mapsto \frac{\eta + a\alpha(1+as)^{\alpha-1}}{\eta s + (1+as)^\alpha} = \frac{\partial}{\partial s}\log(\eta s + (1+as)^\alpha) \qquad (2.1)$$

is completely monotone for all $\alpha \in [0,2]$ and $a, \eta \in \mathbb{R}_+$. □

The next Lemma 2.2 has been used in the proof of Proposition 2.1.

Lemma 2.2 *Let $a \in \mathbb{R}_+$ and $\alpha \geq 1$. We have*

$$\Psi_{aX_\beta + X_\beta^\alpha}(s) = \exp\left(-\int_0^\infty \log\left((1+as)^\alpha + \frac{s}{t}\right)\mu_{0,\alpha}(dt)\right), \qquad s \in \mathbb{R}_+. \qquad (2.2)$$

Proof. We have

$$\begin{aligned}
\Psi_{aX_\beta + X_\beta^\alpha}(s) &= E[e^{-s(X_\beta^\alpha + aX_\beta)}] \\
&= \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-s(x^\alpha + ax)} x^{\beta-1} e^{-x} dx \\
&= \frac{1}{\Gamma(\beta)}\int_0^\infty x^{\beta-1} e^{-sx^\alpha - x(1+as)} dx \\
&= (1+as)^{-\beta}\frac{1}{\Gamma(\beta)}\int_0^\infty x^{\beta-1} e^{-s(1+as)^{-\alpha}x^\alpha - x} dx \\
&= (1+as)^{-\beta} E[e^{-s(1+as)^{-\alpha}X_\beta^\alpha}] \\
&= (1+as)^{-\beta}\Psi_{X_\beta^\alpha}(s(1+as)^{-\alpha}), \qquad s \in \mathbb{R}_+.
\end{aligned}$$

Hence from (1.1) and (1.3) we get

$$\Psi_{aX_\beta + X_\beta^\alpha}(s) = (1+as)^{-\beta}\Psi_{X_\beta^\alpha}(s(1+as)^{-\alpha})$$

$$\begin{aligned}
&= (1+as)^{-\beta} \exp\left(-\int_0^\infty \log\left(1+\frac{s}{t}(1+as)^{-\alpha}\right)\mu_{0,\alpha}(dt)\right) \\
&= (1+as)^{-\beta} \exp\left(-\int_0^\infty \log\left((1+as)^{\alpha}+\frac{s}{t}\right)\mu_{0,\alpha}(dt) + \alpha\log(1+as)\mu_{0,\alpha}([0,\infty))\right) \\
&= \exp\left(-\int_0^\infty \log\left((1+as)^{\alpha}+\frac{s}{t}\right)\mu_{0,\alpha}(dt)\right),
\end{aligned}$$

since $\mu_{0,\alpha}([0,\infty)) = \beta/\alpha$ by (1.3). □

Note that the transformations of Lemma 2.2 are not applicable to the Lévy-Khintchine formula (1.1), as a consequence the Lévy measure of $aX_\beta + X_\beta^\alpha$ does not seem to be computable from the Lévy measure of X_β^α.

We close this section with some remarks on the case $\alpha \notin [1,2]$. When $\alpha \in (0,1)$ the random variable X_β^α is not infinitely divisible, and from Remark 5.3 below we conjecture that $X_\beta^\alpha + aX_\beta$ is not infinitely divisible for $\alpha > 2$.

When $\alpha \geq 1$ is an integer we may decompose the polynomial $s \mapsto (1+as)^\alpha + s/t$ in (2.2), $t > 0$, as

$$(1+as)^{\alpha} + \frac{s}{t} = \prod_{k=1}^{\alpha}\left(1+\frac{s}{g_k^a(t)}\right) \geq 0, \tag{2.3}$$

where $g_1^a(t), \ldots, g_\alpha^a(t)$ are the (complex) roots of $s \mapsto (1-as)^\alpha - s/t$, counted with their multiplicities. Then Frullani's identity yields

$$\log\left((1+as)^{\alpha}+\frac{s}{t}\right) = \sum_{k=1}^{\alpha}\log\left(1+\frac{s}{g_k^a(t)}\right) = \int_0^\infty (1-e^{-sx})\sum_{k=1}^{\alpha} e^{-xg_k^a(t)}\frac{dx}{x},$$

and the Lévy measure $\nu(dx)$ of $X_\beta^\alpha + aX_\beta$, if it exists, is given from Lemma 2.2 by

$$\nu(dx) = \frac{1}{x}\sum_{k=1}^{\alpha}\int_0^\infty e^{-xg_k^a(t)}\mu_{0,\alpha}(dt)dx,$$

which is real since the roots $\{g_1^a(t), \ldots, g_n^a(t)\}$ are either real or complex conjugate.

In Section 3 we will consider the case $\alpha = 2$ where both roots are real, $\sum_{k=1}^2 e^{-xg_k^a(t)}$ is positive, and $aX_\beta + X_\beta^2$ is infinitely divisible. Numerical computations not presented here have shown that the sum $\sum_{k=1}^k e^{-xg_k^a(t)}$ is not always positive when $\alpha \geq 3$.

3 Second order polynomials

In this section we consider the case $\alpha = 2$ and $a \in \mathbb{R}_+$. The probability density function of $aX_\beta + X_\beta^2$ is given for all $\beta > 0$ by

$$f_a(x) = \frac{2^{1-\beta}}{\sqrt{a^2+4x}} \frac{1}{\Gamma(\beta)} \left(-a + \sqrt{a^2+4x}\right)^{\beta-1} e^{-\frac{-a+\sqrt{a^2+4x}}{2}}, \qquad x > 0,$$

and it is log-convex only for $\beta \in (0,1]$, which shows that $aX_\beta + X_\beta^2$ is infinitely divisible for $\beta \in (0,1]$ and $a > 0$, by e.g. Theorem 51.4 of [7].

In case $\beta = 1$ the density function f_a is hyperbolically completely monotone (HCM, cf. § 5.1 of [1]), hence $aX_1 + X_1^2$ is a GGC by Theorem 5.1.2 page 71 of [1], or Theorem 5.18 of [9].

That f_a is HCM follows from the facts that the function $x \mapsto (a^2+4x)^{-1/2}$ is HCM by page 68 of [1], and the function $x \mapsto e^{-c\sqrt{x}}$ is HCM and decreasing for $c > 0$, cf. Property (iv) page 68 of [1]. Then, by Property (xi) of [1], $x \mapsto e^{-c\sqrt{a^2+4x}}$ is HCM, and f_a is HCM since the product of two HCM functions is HCM. Note also that $x \mapsto e^{-c(\sqrt{a^2+4x}-a)}$ is HCM because it is the Laplace transform of a tempered stable distribution with parameter 1/2 which is a GGC, cf. Theorem 6.1.1 page 90 of [1].

In Proposition 3.1 below we prove that $aX_\beta + bX_\beta^2$ is a GGC for all $\beta > 0$ and $a, b \geq 0$ by resorting directly to the definition (1.1) of the GGC class.

Proposition 3.1 *For all $a \in \mathbb{R}_+$, $b > 0$ and $\beta > 0$, the random variable $aX_\beta + bX_\beta^2$ is a GGC whose Thorin measure $\mu_{a,2}(dy)$ has the density*

$$\varphi_{a,b}(y) = \frac{\beta(1+ay)e^{(1-ay)^2/(4by)}}{2^\beta b^{1/2}\pi^{3/2}y^{3/2}\left(\Gamma(1/2+\beta/2) + 2^{2-2\beta}F^2\left(\frac{|1-ay|}{2\sqrt{by}}\right)/\Gamma(1/2+\beta/2)\right)}, \quad (3.1)$$

$y > 0$, where

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{z^2} dz, \qquad x \in \mathbb{R}_+, \quad (3.2)$$

is the complex error function.

Proof. We use the expression (3.3) of the Thorin density $\varphi_{0,1}(y)$ of X_β^2 given in Lemma 3.2, and apply Lemma 3.3. □

Proposition 3.1 follows from the next Lemmas 3.2 and 3.3 below. Lemma 3.2 is an application for $\alpha = 2$ of Proposition 6.1 which relies on Theorem 2.3 of [5], cf. the Appendix Section 6.

Lemma 3.2 *The distribution function $F_{0,2}$ of the Thorin measure $\mu_{0,2}(dy)$ of X_β^2 is given by*

$$F_{0,2}(y) = \frac{\beta}{\pi} \arctan\left(\frac{\Gamma(\beta/2+1/2)}{2^{1-\beta}F(1/(2\sqrt{y}))}\right), \qquad y > 0,$$

for all $\beta > 0$. In particular, $F_{0,2}(y)$ is absolutely continuous and $\mu_{0,2}(dy)$ admits a density $\varphi_{0,1}(y)$ with respect to the Lebesgue measure, given by

$$\varphi_{0,1}(y) = \frac{\beta e^{1/(4y)}}{2^\beta \pi^{3/2} y^{3/2}\left(\Gamma(1/2+\beta/2) + 2^{2-2\beta}F^2(1/(2\sqrt{y}))/\Gamma(1/2+\beta/2)\right)}, \qquad (3.3)$$

$y > 0$.

The next Lemma 3.3 shows that the Thorin density $\varphi_{a,b}(x)$ of $aX_\beta + bX_\beta^2$ can be computed from the Thorin density $\varphi_{0,1}(x)$ of X_β^2, and relies on Proposition 6.1 in the appendix Section 6 for the absolute continuity of $F_{0,2}$ and the existence of $\varphi_{0,1}$.

Lemma 3.3 *For all $a \in \mathbb{R}_+$, $b > 0$ and $\beta > 0$, the random variable $aX_\beta + bX_\beta^2$ is a GGC whose Thorin density $\varphi_{a,b}$ satisfies*

$$\varphi_{a,b}(x) = b\frac{1+ax}{|1-ax|^3}\varphi_{0,1}\left(\frac{bx}{(1-ax)^2}\right), \qquad x > 0,$$

with total mass

$$\mu_{a,2}((0,\infty)) = \beta$$

if $a > 0$, while $\mu_{0,2}((0,\infty)) = \beta/2$.

Proof. Letting

$$g_a^{\pm}(t) = \frac{1}{2a^2t} + \frac{1}{a} \pm \sqrt{\left(\frac{1}{2a^2t} + \frac{1}{a}\right)^2 - \frac{1}{a^2}} > 0, \qquad t > 0, \qquad (3.4)$$

with

$$g_a^+ : (0,\infty) \longrightarrow (1/a,\infty) \qquad \text{and} \qquad g_a^- : (0,\infty) \longrightarrow (0,1/a), \qquad (3.5)$$

$g_a^-(t)g_a^+(t) = 1/a^2$, $t > 0$, and

$$\lim_{a\to 0} g_a^-(t) = t, \qquad \lim_{a\to 0} g_a^+(t) = \infty, \qquad t \in \mathbb{R}_+.$$

Letting $\mu_{a,2}^+(dt)$, resp. $\mu_{a,2}^-(dt)$ denote the image measures of $\mu_{0,2}(dt)$ by g_a^+, resp. g_a^-, by (2.3) and Lemma 2.2 we have

$$\begin{aligned}
\Psi_{aX_\beta+X_\beta^2}(s) &= E[e^{-s(aX_\beta+X_\beta^2)}] \\
&= \exp\left(-\int_0^\infty \log\left((1+as)^2+\frac{s}{t}\right)\mu_{0,2}(dt)\right) \\
&= \exp\left(-\int_0^\infty \log\left(1+\left(\frac{1}{t}+2a\right)s+a^2s^2\right)\mu_{0,2}(dt)\right) \\
&= \exp\left(-\int_0^\infty \log\left(\left(1+\frac{s}{g_a^+(t)}\right)\left(1+\frac{s}{g_a^-(t)}\right)\right)\mu_{0,2}(dt)\right) \\
&= \exp\left(-\int_0^\infty \log\left(1+\frac{s}{g_a^-(t)}\right)\mu_{0,2}(dt)-\int_0^\infty \log\left(1+\frac{s}{g_a^+(t)}\right)\mu_{0,2}(dt)\right) \\
&= \exp\left(-\int_0^{1/a} \log\left(1+\frac{s}{t}\right)\mu_{a,2}^-(dt)-\int_{1/a}^\infty \log\left(1+\frac{s}{t}\right)\mu_{a,2}^+(dt)\right),
\end{aligned}$$

$s \in \mathbb{R}_+$, which shows that the Thorin measure $\mu_{a,2}(dx)$ of $aX_\beta + X_\beta^2$ satisfies

$$\mu_{a,2}(dx) = \mu_{a,2}^+(dx) + \mu_{a,2}^-(dx).$$

Next, denoting by $F_{a,2}(x) = \mu_{a,2}([0,x])$ the cumulative distribution function of the Thorin measure $\mu_{a,2}(dx)$, we have

$$\begin{aligned}
F_{a,2}(x) &= \int_0^\infty \mathbf{1}_{[0,x]}(g_a^-(t))\mu_{0,2}(dt) + \int_0^\infty \mathbf{1}_{[0,x]}(g_a^+(t))\mu_{0,2}(dt) \\
&= \mathbf{1}_{\{x\le 1/a\}}\int_0^\infty \mathbf{1}_{[0,x]}(g_a^-(t))\mu_{0,2}(dt) + \mathbf{1}_{\{x>1/a\}}\frac{\beta}{2} + \mathbf{1}_{\{x>1/a\}}\int_0^\infty \mathbf{1}_{[0,x]}(g_a^+(t))\mu_{0,2}(dt) \\
&= \mathbf{1}_{\{x\le 1/a\}}\int_0^{(g_a^-)^{-1}(x)} \mu_{0,2}(dt) + \mathbf{1}_{\{x>1/a\}}\left(\frac{\beta}{2}+\int_{(g_a^+)^{-1}(x)}^\infty \mu_{0,2}(dt)\right) \\
&= \mathbf{1}_{\{x\le 1/a\}}\int_0^{(g_a^-)^{-1}(x)} \mu_{0,2}(dt) + \mathbf{1}_{\{x>1/a\}}\left(\beta-\int_0^{(g_a^+)^{-1}(x)} \mu_{0,2}(dt)\right) \\
&= F_{0,2}\left(\frac{x}{(1-ax)^2}\right)\mathbf{1}_{\{x\le 1/a\}} + \left(\beta - F_{0,2}\left(\frac{x}{(1-ax)^2}\right)\right)\mathbf{1}_{\{x>1/a\}}, \qquad (3.6)
\end{aligned}$$

$x \in \mathbb{R}_+$, $a > 0$. This shows in particular that

$$F_{a,2}(1/a) = F_{0,2}(+\infty) = \mu_{0,2}((0,\infty)) = \beta/2,$$

by (1.3), and

$$F_{a,2}(\infty) = \mu_{a,2}((0,\infty)) = \beta,$$

$a > 0$.

By Proposition 6.1 below the function $F_{0,2}$ is absolutely continuous and by differentiation with respect to x we obtain

$$\varphi_{a,1}(x) = \frac{1+ax}{|1-ax|^3}\varphi_{0,1}\left(\frac{x}{(1-ax)^2}\right), \qquad x > 0,$$

which gives $\varphi_{a,b}(x)$ in (3.1) by (3.3) and the rescaling relation

$$\varphi_{a,b}(x) = b\varphi_{a/b,1}(bx), \qquad x > 0,$$

for $a \in \mathbb{R}_+$ and $b > 0$.

In order to conclude that $aX_\beta + X_\beta^2$ is a GGC with Thorin measure $\mu_{a,2}(dt) = \mu^+_{a,2}(dt) + \mu^-_{a,2}(dt)$ it suffices to check that Condition (1.2) holds. We have

$$\begin{aligned}
&\int_{0^+}^{1/a} |\log t| \mu^-_{a,2}(dt) + \int_{1/a}^\infty \frac{1}{t}\mu^+_{a,2}(dt) \\
&= \int_{0^+}^\infty \left|\log\frac{1}{g_a^-(t)}\right| \mu_{0,2}(dt) + \int_{0^+}^\infty \frac{1}{g_a^+(t)}\mu_{0,2}(dt) \\
&\leq c_a\frac{\beta}{2} + c_a\int_{0^+}^\infty |\log t|\mu_{0,2}(dt) + a\mu_{0,2}((0,\infty)) \\
&< \infty,
\end{aligned}$$

by (3.5) for some $c_a > 0$ since

$$\begin{aligned}
g_a^-(t) &= \frac{1}{2a^2t} + \frac{1}{a} - \sqrt{\left(\frac{1}{2a^2t} + \frac{1}{a}\right)^2 - \frac{1}{a^2}} \\
&= \frac{1}{2a^2t} + \frac{1}{a} - \frac{1}{2a^2t}\sqrt{1+4at} \\
&= \frac{1}{2a^2t} + \frac{1}{a} - \frac{1}{2a^2t}\left(1 + 2at - \frac{1}{8}4^2a^2t^2\right) + o(t) \\
&= t + o(t), \qquad t \searrow 0,
\end{aligned}$$

which implies

$$\begin{aligned}
\log \frac{1}{g_a^-(t)} &= -\log g_a^-(t) \\
&= -\log(t + o(t)) \\
&= -\log t - \log(1 + o(t)/t) \\
&\leq c + |\log t|, \qquad 0 < t < 1,
\end{aligned}$$

for some $c > 0$. □

Note that from (3.6) we have

$$F_{a,2}(x/a) = F_{0,2}\left(\frac{x}{a(1-x)^2}\right)\mathbf{1}_{\{x\leq 1\}} + \left(\beta - F_{0,2}\left(\frac{x}{a(1-x)^2}\right)\right)\mathbf{1}_{\{x>1\}},$$

which converges to

$$\lim_{a\to\infty} F_{a,2}(x/a) = \beta\mathbf{1}_{[1,\infty)}(x)$$

as a goes to infinity, and we recover

$$\lim_{b\to 0} \varphi_{a,b}(x)dx = \beta\delta_{1/a}(dx),$$

which is the Thorin measure of a $\Gamma(a, \beta)$ random variable.

By (3.1) we also find

$$\varphi_{a,b}(1/a) = \frac{2^{1-\beta}\beta}{\Gamma(1/2+\beta/2)}\frac{a^{3/2}}{\pi^{3/2}b^{1/2}}, \qquad a, b > 0,$$

and when $\beta = 1$ we have

$$\varphi_{a,b}(x) = \frac{1+ax}{2\pi^{3/2}b^{1/2}x^{3/2}}\frac{e^{|1-ax|^2/(4bx)}}{1+F^2\left(|1-ax|/(2\sqrt{bx})\right)}, \qquad x > 0.$$

Figure 1 below shows a graph of the Thorin density

$$\begin{aligned}
\varphi_{p,1-p}(x) &= (1-p)\frac{1+px}{|1-px|^3}\varphi_{0,1}\left(\frac{(1-p)x}{(1-px)^2}\right) \\
&= \frac{1+px}{2\pi^{3/2}(1-p)^{1/2}x^{3/2}}\frac{e^{(1-px)^2/(4(1-p)x)}}{1+F^2\left(|1-px|/(2\sqrt{x(1-p)})\right)}, \quad x > 0,
\end{aligned}$$

of $pX_1 + (1-p)X_1^2$ for $\beta = 1$ and different values of $p \in [0,1]$, which interpolate between the Thorin measure $\delta_1(dx)$ of X_1 and the Thorin density $\varphi_{0,1}(x)$ of X_1^2.

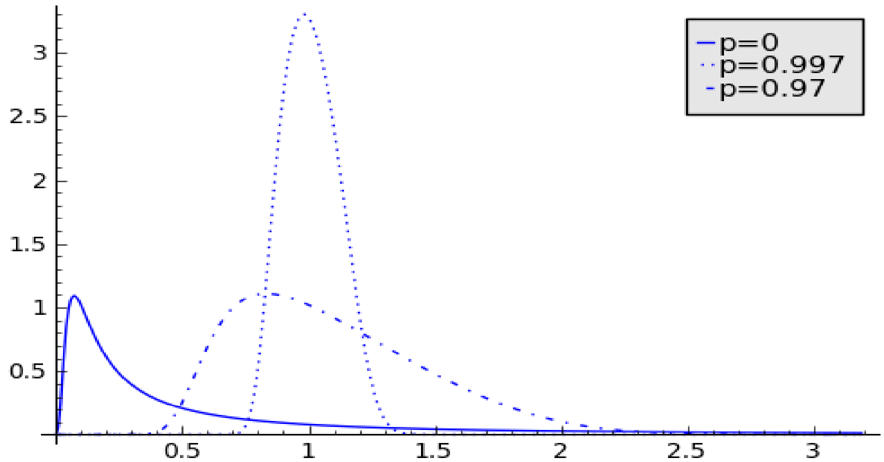

Figure 1: Graphs of $x \mapsto \varphi_{p,1-p}(x)$ for $p = 0$, $p = 0.97$, and $p = 0.997$.

Here the shape parameter is plotted as a function of the scale parameter. Note that the total mass of the Thorin density $\varphi_{p,1-p}$ on $(0,\infty)$ is 1 for $p > 0$ and 1/2 for $p = 0$.

Wiener-gamma representation

By Proposition 1.1 of [5], the gamma square X_β^2 can be written as the Wiener-gamma integral

$$X_\beta^2 \simeq \int_0^{\beta/2} h_{0,1}(s) d\gamma_s = \int_0^\infty \frac{1}{t} d\gamma_{F_{0,2}(t)},$$

with

$$h_{0,1}(s) = \frac{1}{(F_{0,2})^{-1}(s)}, \qquad 0 \leq s < \beta/2,$$

where $(\gamma_s)_{s\in\mathbb{R}_+}$ is a standard gamma process, i.e. the Lévy process whose increments $\gamma_t - \gamma_s$ are gamma distributed with shape parameter $t - s$, $0 \leq s \leq t$, where

$$F_{0,2}(x) = \mu_{0,2}([0,x]), \qquad x > 0,$$

is the cumulative distribution function of the Thorin measure $\mu_{0,2}$ of X_β^2, with total mass $\beta/2$. By inverting $F_{0,2}$ in (6.3), i.e.

$$F_{0,2}(y) = \frac{\beta}{\pi} \arctan\left(\frac{\Gamma(\beta/2 + 1/2)}{2^{1-\beta} F(1/(2\sqrt{y}))}\right),$$

we find

$$h_{0,1}(y) = \frac{1}{(F_{0,2})^{-1}(y)} = 4\left|F^{-1}\left(2^{\beta-1}\Gamma(1/2+\beta/2)\cot(\pi y/\beta)\right)\right|^2, \qquad y \in (0, \beta/2), \tag{3.7}$$

where $F(x)$ is the complex error function (3.2) and $\cot x = 1/\tan x$, $x \in \mathbb{R} \setminus \{0\}$.

In particular when $\beta = 1$ we have

$$h_{0,1}(y) = 4 \left| F^{-1}(\cot(\pi y)) \right|^2, \qquad y \in [0, 1/2].$$

The Wiener-gamma integral representation provides a stochastic integral expression of the Lévy process associated to $aX_\beta + bX_\beta^2$ and indexed by the shape parameter.

Proposition 3.4 *For all $a, b \in \mathbb{R}_+$ and $\beta > 0$ the random variable $aX_\beta + bX_\beta^2$ admits the representation*

$$aX_\beta + bX_\beta^2 \simeq \int_0^\beta h_{a,b}(s) d\gamma_s, \tag{3.8}$$

in law, with

$$h_{a,b}(s) = \begin{cases} a + \dfrac{b}{2}\left(1 + \sqrt{1 + \dfrac{4a}{bh_{0,1}(s)}}\right) h_{0,1}(s), & 0 < s < \beta/2, \\ \\ a + \dfrac{b}{2}\left(1 - \sqrt{1 + \dfrac{4a}{bh_{0,1}(\beta - s)}}\right) h_{0,1}(\beta - s), & \beta/2 < s \leq \beta, \end{cases} \tag{3.9}$$

where $h_{0,1}(s)$ is given by (3.7).

Proof. By (3.6) we have

$$\begin{aligned} F_{a,2}(x) &= F_{0,2}\left(\frac{x}{(1-ax)^2}\right) \mathbf{1}_{\{x \leq 1/a\}} + \left(\beta - F_{0,2}\left(\frac{x}{(1-ax)^2}\right)\right) \mathbf{1}_{\{x > 1/a\}} \\ &= \left(\frac{1}{h_{0,1}}\right)^{-1}\left(\frac{x}{(1-ax)^2}\right) \mathbf{1}_{\{x \leq 1/a\}} + \left(\beta - \left(\frac{1}{h_{0,1}}\right)^{-1}\left(\frac{x}{(1-ax)^2}\right)\right) \mathbf{1}_{\{x > 1/a\}} \\ &= \left(\frac{1}{h_{0,1}}\right)^{-1}\left((g_a^-)^{-1}(x)\right) \mathbf{1}_{\{x \leq 1/a\}} + \left(\beta - \left(\frac{1}{h_{0,1}}\right)^{-1}\left((g_a^+)^{-1}(x)\right)\right) \mathbf{1}_{\{x > 1/a\}}, \end{aligned}$$

$x > 0$, where

$$\begin{aligned} (g_a^+)^{-1} : \quad (1/a, \infty) &\longrightarrow (0, \infty) \\ x &\longmapsto \frac{x}{(1-ax)^2}, \end{aligned}$$

and

$$(g_a^-)^{-1} : \quad (0, 1/a) \longrightarrow (0, \infty)$$

$$x \quad \longmapsto \frac{x}{(1-ax)^2},$$

hence the Wiener-gamma representation of $aX_\beta + X_\beta^2$ is given by

$$\begin{aligned}
h_{a,1}(s) &= \frac{1}{(F_{a,2})^{-1}(s)} \\
&= \begin{cases} \dfrac{1}{g_a^-\left(1/h_{0,1}(s)\right)}, & 0 < s < F_{a,2}(1/a) = \beta/2, \\[2ex] \dfrac{1}{g_a^+\left(1/h_{0,1}(\beta - s)\right)}, & \beta/2 = F_{a,2}(1/a) < s \leq \beta, \end{cases} \\
&= \begin{cases} a^2 g_a^+\left(\dfrac{1}{h_{0,1}(s)}\right), & 0 < s < \beta/2, \\[2ex] a^2 g_a^-\left(\dfrac{1}{h_{0,1}(\beta - s)}\right), & \beta/2 < s \leq \beta, \end{cases} \qquad (3.10) \\
&= \begin{cases} \dfrac{1}{2}h_{0,1}(s) + a + \sqrt{\dfrac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)}, & 0 < s < \beta/2, \\[2ex] \dfrac{1}{2}h_{0,1}(\beta - s) + a - \sqrt{\dfrac{h_{0,1}^2(\beta - s)}{4} + a h_{0,1}(\beta - s)}, & \beta/2 < s \leq \beta. \end{cases} \\
&= \begin{cases} a + \dfrac{1}{2}h_{0,1}(s)\left(1 + \sqrt{1 + \dfrac{4a}{h_{0,1}(s)}}\right), & 0 < s < \beta/2, \\[2ex] a + \dfrac{1}{2}h_{0,1}(\beta - s)\left(1 - \sqrt{1 + \dfrac{4a}{h_{0,1}(\beta - s)}}\right), & \beta/2 < s \leq \beta. \end{cases} \qquad (3.11)
\end{aligned}$$

To conclude the proof we use the rescaling relation

$$h_{a,b}(s) = b h_{a/b,1}(s), \qquad s > 0,$$

for $b > 0$. $\square$

We note that $h_{a,b}(\beta/2) = a$, and $h_{a,0}(s) = a$, $s \in (0, \beta]$. When $b > 0$ we also have $h_{0,b}(0) = +\infty$, and $h_{a,b}(\beta) = 0$ since

$$\lim_{x \to \infty} \frac{x}{2} + a - \sqrt{\frac{x^2}{4} + ax} = \lim_{x \to \infty} \frac{x}{2}\left(1 + \frac{2a}{x} - \sqrt{1 + \frac{4a}{x}}\right) = 0.$$

Figure 2 below shows a graph of the Wiener-gamma integrand $x \mapsto h_{p,1-p}(x)$ of $pX_1 + (1-p)X_1^2$ for $\beta = 1$ and different values of $p \in [0,1]$. Here the scale parameter is plotted as a function of the shape parameter.

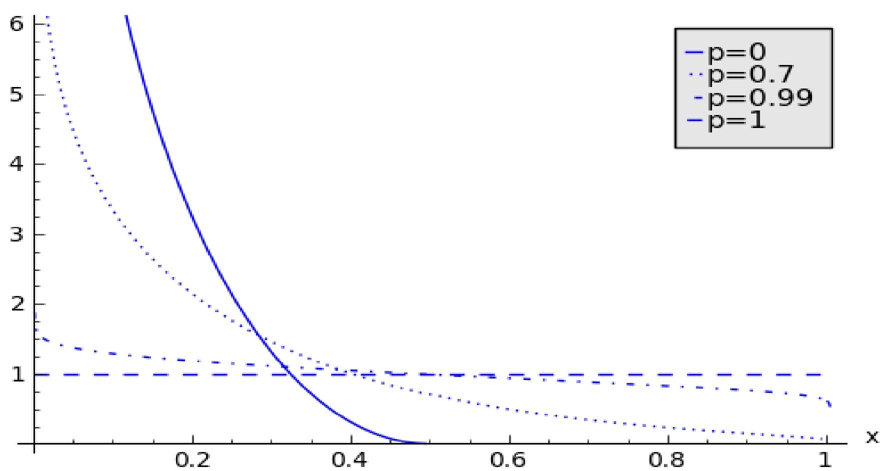

Figure 2: Graphs of $x \mapsto h_{p,1-p}(x)$ for $p = 0$, $p = 0.7$, $p = 0.99$, and $p = 1$.

As can be expected, by (3.9) we also have

$$\lim_{b \to 0} h_{a,b}(y) = h_{a,0}(y) = a, \qquad y \in (0, \beta).$$

Finally, denoting by $(\tilde{\gamma}_s)_{s \in \mathbb{R}_+}$ another standard gamma process independent of $(\gamma_s)_{s \in \mathbb{R}_+}$, by (3.10) and (3.11) we have

$$\begin{aligned}
aX_\beta + X_\beta^2 &= \int_0^\beta h_{a,1}(s) d\gamma_s \\
&= a^2 \int_0^{\beta/2} g_a^+ \left(\frac{1}{h_{0,1}(s)} \right) d\gamma_s + a^2 \int_{\beta/2}^{\beta} g_a^- \left(\frac{1}{h_{0,1}(\beta - s)} \right) d\gamma_s \\
&= a^2 \int_0^{\beta/2} g_a^+ \left(\frac{1}{h_{0,1}(s)} \right) d\gamma_s + a^2 \int_0^{\beta/2} g_a^- \left(\frac{1}{h_{0,1}(s)} \right) d\tilde{\gamma}_s \\
&= \frac{1}{2} \int_0^{\beta/2} h_{0,1}(s) d\gamma_s + \int_0^{\beta/2} a d\gamma_s + \int_0^{\beta/2} \sqrt{\frac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)} d\gamma_s \\
&\quad + \frac{1}{2} \int_0^{\beta/2} h_{0,1}(s) d\tilde{\gamma}_s + \int_0^{\beta/2} a d\tilde{\gamma}_s - \int_0^{\beta/2} \sqrt{\frac{h_{0,1}^2(s)}{4} + a h_{0,1}(s)} d\tilde{\gamma}_s \\
&= a(\gamma(\beta/2) + \tilde{\gamma}(\beta/2)) + \int_0^{\beta/2} h_{0,1}(s) d\gamma_s
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^{\beta/2} \frac{h_{0,1}(s)}{2} \left(-1 + \sqrt{1 + a\frac{4}{h_{0,1}(s)}} \right) (d\gamma_s - d\tilde{\gamma}_s) \\
= \; & a(\gamma(\beta/2) + \tilde{\gamma}(\beta/2)) + \int_0^{\beta/2} h_{0,1}(s) d\gamma_s \\
& +2a \int_0^{\beta/2} \left(1 + \sqrt{1 + \frac{4a}{h_{0,1}(s)}} \right)^{-1} (d\gamma_s - d\tilde{\gamma}_s),
\end{aligned}
$$

where we also used (3.8).

This provides a linearization of $aX_\beta + bX_\beta^2$ in γ_s, into the sum of a $\Gamma(a, \beta)$ random variable, a squared gamma variable, and a remainder which is an extended GGC in the sense of Chapter 7 of [1], and goes to 0 as a tends to 0.

4 Multiple Wiener integrals

We now consider some examples of non-infinite divisibility of second order polynomials in the case $a < 0$, with application to the fourth order multiple Wiener integral of $f^{\otimes 4}$, with $\|f\|_{L^2(\mathbb{R}_+)} = 1$, which can be written as

$$
\begin{aligned}
I_4(f^{\otimes 4}) &= 4! \int_0^\infty f(t_4) \int_0^{t_4} f(t_3) \int_0^{t_3} f(t_2) \int_0^{t_2} f(t_1) dB_{t_1} dB_{t_2} dB_{t_3} dB_{t_4} \\
&= H_4(I_1(f)) \\
&= (I_1(f))^4 - 6(I_1(f))^2 + 3,
\end{aligned}
$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and $H_4(x) = x^4 - 6x^2 + 3$ is the Hermite polynomial of degree 4 and $I_1(f) = \int_0^\infty f(t) dB_t$ is the first order Wiener integral, cf. e.g. Proposition 5.1.3 of [6], and $X_{1/2} = I_1(f)^2$ is a gamma random variable with shape parameter $\beta = 1/2$.

Proposition 4.1 *The fourth order multiple Wiener integral $I_4(f^{\otimes 4})$, with $\|f\|_{L^2(\mathbb{R}_+)} = 1$, is not infinitely divisible.*

Proof. It follows from Lemma 4.2 and Figure 3 below* that the function $-\frac{\partial \log \Psi_{-6,1/2}}{\partial s}(s)$ is not to be completely monotone since its third derivative does not have constant sign.

*Figures 3 and 4 have been checked independently using Mathematica and Sage.

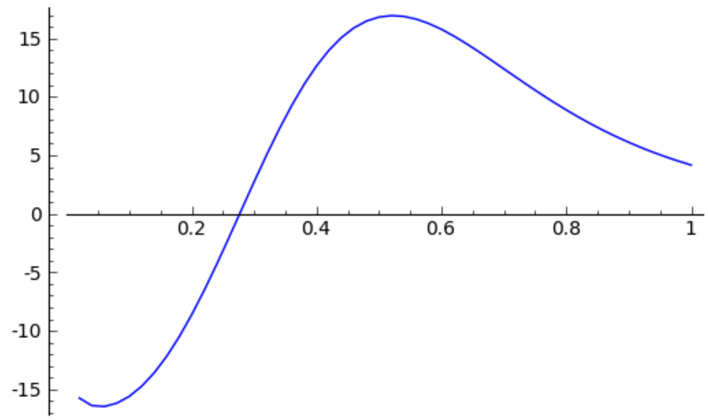

Figure 3: Graph of $s \mapsto -\partial^3/\partial s^3 \log \Psi_{-6,0.5}(s)$.

Consequently the fourth order multiple Wiener integral

$$I_4(f^{\otimes 4}) \simeq X_{1/2}^2 - 6X_{1/2} + 3$$

of $f^{\otimes 4}$, with $\|f\|_{L^2(\mathbb{R}_+)} = 1$, is not infinitely divisible by Theorem 1.1. $\square$

Lemma 4.2 *Letting*

$$\Psi_{a,\beta}(s) := E[e^{-s(X_\beta + a/2)^2}], \quad \textit{and} \quad \Gamma_{a,\beta}(s) = \int_0^\infty x^{\beta-1} e^{-s(x^2+ax)-x} dx,$$

$s \in \mathbb{R}_+$, we have

$$-\frac{\partial}{\partial s} \log \Psi_{a,\beta}(s) = \frac{a^2}{4} + \frac{\int_0^\infty x^\beta (x+a) e^{-s(x^2+ax)-x} dx}{\int_0^\infty x^{\beta-1} e^{-s(x^2+ax)-x} dx} = \frac{a^2}{4} + \frac{\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s)}{\Gamma_{a,\beta}(s)},$$

$$-\frac{\partial^2}{\partial s^2} \log \Psi_{a,\beta}(s) = -\frac{\Gamma_{a,\beta+4}(s) + 2a\Gamma_{a,\beta+3}(s) + a^2\Gamma_{a,\beta+2}(s)}{\Gamma_{a,\beta}(s)} + \frac{(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))^2}{(\Gamma_{a,\beta}(s))^2},$$

and

$$\begin{aligned}-\frac{\partial^3}{\partial s^3} \log \Psi_{a,\beta}(s) = {} & \frac{\Gamma_{a,\beta+6}(s) + 3a\Gamma_{a,\beta+5}(s) + 3a^2\Gamma_{a,\beta+4}(s) + a^3\Gamma_{a,\beta+3}(s)}{\Gamma_{a,\beta}(s)} \\ & -3\frac{(\Gamma_{a,\beta+4}(s) + 2a\Gamma_{a,\beta+3}(s) + a^2\Gamma_{a,\beta+2}(s))(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))}{(\Gamma_{a,\beta}(s))^2} \\ & +2\frac{(\Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s))^3}{(\Gamma_{a,\beta}(s))^3}, \qquad s \in \mathbb{R}_+.\end{aligned}$$

Proof. It suffices to note that we have

$$\Psi_{a,\beta}(s) = e^{-sa^2/4}\frac{\Gamma_{a,\beta}(s)}{\Gamma(\beta)},$$

and the relation

$$-\frac{\partial}{\partial s}\Gamma_{a,\beta}(s) = \Gamma_{a,\beta+2}(s) + a\Gamma_{a,\beta+1}(s), \qquad s \in \mathbb{R}_+.$$

$\square$

When $\beta = 1$ we have

$$\Psi_{a,1}(s) = \sqrt{\frac{\pi}{s}} e^{a/2+1/(4s)}\Phi\left(-a\sqrt{\frac{s}{2}} - \frac{1}{\sqrt{2s}}\right),$$

and

$$-\frac{\partial}{\partial s}\log\Psi_{a,1}(s) = \frac{1}{2s} + \frac{1}{4s^2} + \frac{(a-1/s)e^{-s(\frac{1}{2s}+a/2)^2}}{4\sqrt{\pi s}\Phi\left(-(a+1/s)\sqrt{s/2}\right)}, \qquad s > 0.$$

5 Complete monotonicity

In this section we prove the complete monotonicity results used in Proposition 2.1.

Lemma 5.1 *Let $a, \eta \in \mathbb{R}_+$.*

i) Let $\alpha \in [0,1]$. The function

$$s \mapsto \frac{1}{\xi + \eta s + (1+as)^\alpha}, \qquad s \in \mathbb{R}_+,$$

is completely monotone for all $\xi \in \mathbb{R}_+$.

ii) Let $\alpha \in [1,2]$. The function

$$s \mapsto \frac{1+as}{(1+as)^\alpha + \eta s}, \qquad s \in \mathbb{R}_+,$$

is completely monotone.

Proof. If $\alpha \in [0,1]$, consider the nonnegative function

$$h(s) = (1+as)^{\alpha} + \eta s + \xi, \qquad s \in \mathbb{R}_+,$$

whose derivative

$$h'(s) = \alpha a(1+as)^{\alpha-1} + \eta$$

is completely monotone on $\mathbb{R}_+$. By Criterion XIII.4.2 of [4], the function

$$s \mapsto \frac{1}{h(s)} = \frac{1}{\xi + \eta s + (1+as)^{\alpha}}$$

is completely monotone on $\mathbb{R}_+$. Next if $\alpha \in [1,2]$, consider the nonnegative function

$$h(s) = (1+as)^{\alpha-1} + \eta \frac{s}{1+as},$$

whose derivative

$$h'(s) = (\alpha-1)a(1+as)^{\alpha-2} + \frac{\eta}{(1+as)^2} \geq 0$$

is completely monotone on $\mathbb{R}_+$. Again by Criterion XIII.4.2 of [4] the function

$$s \mapsto \frac{1}{h(s)} = \frac{1}{(1+as)^{\alpha-1} + \eta s/(1+as)} = \frac{1+as}{(1+as)^{\alpha} + \eta s}$$

is completely monotone on $\mathbb{R}_+$. □

Remark: Alternatively we may note that the function

$$s \mapsto (1+as)^{\alpha} + \eta s$$

is a complete Bernstein function for all $\alpha \in [0,1]$ from § 15.2.2 of [8], hence

$$s \mapsto \frac{1}{\xi + \eta s + (1+as)^{\alpha}}$$

is a Stieltjes function for all $\xi \geq 0$ by Theorem 7.5 of [8], hence it is completely monotone. When $\alpha \in [1,2]$, the function

$$s \mapsto (1+as)^{\alpha-1} + \eta \frac{s}{1+as}$$

is also a complete Bernstein function from § 15.2.4 of [8], hence

$$s \mapsto \frac{1}{(1+as)^{\alpha-1} + \eta s(1+as)^{-1}} = \frac{1+as}{(1+as)^{\alpha} + \eta s}$$

is also a Stieltjes function which is completely monotone.

Corollary 5.2 *For all $a, b, c, \eta \in \mathbb{R}_+$ and all $\alpha \in [0, 2]$, the function*

$$s \mapsto \frac{c + b(1 + as)^{\alpha-1}}{\eta s + (1 + as)^\alpha} \tag{5.1}$$

is completely monotone.

Proof. If $\alpha \in [0, 1]$, we multiply

$$s \mapsto \frac{1}{\xi + \eta s + (1 + as)^\alpha},$$

of Lemma 5.1-*i*) by the completely monotone function

$$s \mapsto a\eta + \alpha(1 + as)^{\alpha-1}.$$

If $\alpha \in [1, 2]$ we multiply

$$s \mapsto \frac{1 + as}{(1 + as)^\alpha + \eta s}$$

of Lemma 5.1-*ii*) by the completely monotone function

$$s \mapsto a\alpha(1 + as)^{\alpha-2},$$

to get that

$$s \mapsto \frac{a\alpha(1 + as)^{\alpha-1}}{(1 + as)^\alpha + \eta s}, \qquad s \in \mathbb{R}_+,$$

is completely monotone on $\mathbb{R}_+$.

On the other hand by Lemma 5.1, the function

$$s \mapsto \frac{1 + as}{(1 + as)^\alpha + \eta s} \times \frac{1}{1 + as} = \frac{1}{(1 + as)^\alpha + \eta s} \qquad s \in \mathbb{R}_+,$$

is also completely monotone on $\mathbb{R}_+$ for $\alpha \in [1, 2]$. □

Remark 5.3 *There exist values of $\alpha > 2$ for which the function* (5.1) *is not completely monotone.*

Remark 5.3 is illustrated for $a = 1$ and $\eta = 100$ in Figure 4 below, in which differentiation up to the 18th order is required for $\alpha = 3$, and the values taken by the derivative are of order 10^{-20}.

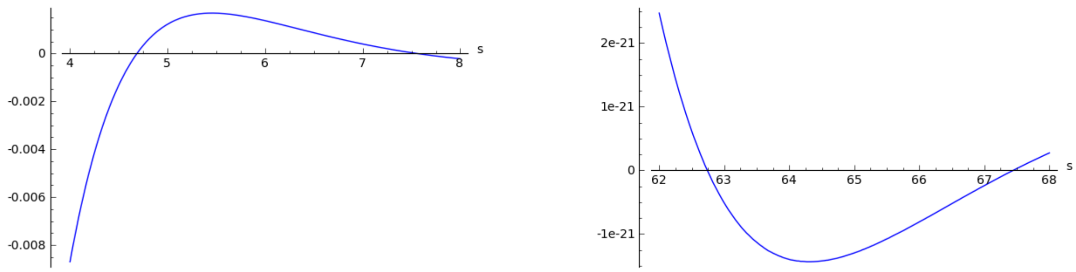

Figure 4: Graphs of $s \mapsto \partial^2/\partial s^2 \log(100s + (1+s)^4)$ and $s \mapsto \partial^{17}/\partial s^{17} \log(100s + (1+s)^3)$.

6 Appendix

In this appendix we prove Lemma 3.2 by extending to all values of $\beta > 0$ the results stated for $\beta = 1$ in § 2.6a of [5] on the computation of Thorin measures for exponential random variables. The function Λ_t, $t > 0$, is defined by

$$\Lambda_t(y) := 1 - \frac{1}{\pi t} \arctan\left(\frac{\sin(\pi t)}{y + \cos(\pi t)}\right), \qquad y > 0.$$

We let $f_{1/\alpha}$ denote the probability density function of a stable random variable with parameter $1/\alpha$.

Proposition 6.1 *Let X_β be a gamma random variable with shape parameter $\beta > 0$. Let $\alpha > 1$. Then the distribution function $F_{0,\alpha}$ of the Thorin measure $\mu_{0,\alpha}$ of X_β^α satisfies*

$$F_{0,\alpha}(y) = \frac{\beta}{\alpha} - \frac{\beta}{\alpha}\Lambda_{1/\alpha}\left(\frac{\Gamma(1+\beta/\alpha)}{\Gamma(1+\beta)}\frac{\sin(\pi/\alpha)}{\pi y}\int_y^\infty \frac{1}{(x-y)^{1/\alpha}}\frac{f_{1/\alpha}(x)}{f_{1/\alpha}(y)}dx\right), \qquad y \in \mathbb{R}_+. \tag{6.1}$$

In particular, $F_{0,\alpha}(y)$ is absolutely continuous and $\mu_{0,\alpha}$ admits a density with respect to the Lebesgue measure.

Proof. In this proof we use the notation of [5]. The random variable X_β^α is a $(\beta/\alpha, G)$-GGC, where G is a random variable with distribution function $F_G(x) = \alpha F_{0,\alpha}(x)/\beta$ and such that $E[\log^+(1/G)] < \infty$. By Theorem 4.1.1 page 49 of [1], X_β^α can be written as the gamma mixture

$$X_\beta^\alpha \simeq Y_{\beta/\alpha} D_{1/\alpha}(G),$$

where $Y_{\beta/\alpha}$ is a gamma random variable with parameter β/α, and $D_{1/\alpha}(G)$ is a positive independent random variable which is an example of a Dirichlet mean, cf. [3]

and Relation (38) in [5].

By the duality Theorem 2.1.3-(*ii*) in [5], the density

$$f_{X_\beta^\alpha}(x) = \frac{1}{|\alpha|\Gamma(\beta)} x^{\beta/\alpha-1} e^{-x^{1/\alpha}}, \qquad x > 0,$$

rewrites as

$$f_{X_\beta^\alpha}(x) = \frac{1}{\Gamma(\beta/\alpha)} x^{\beta/\alpha-1} e^{\beta E[\log(G)]/\alpha} E[e^{-xD_{1/\alpha}(1/G)}], \qquad x > 0,$$

which, by identification, yields

$$e^{\beta E[\log(G)]/\alpha} = \frac{\Gamma(\beta/\alpha)}{|\alpha|\Gamma(\beta)} = \frac{\Gamma(1+\beta/\alpha)}{\Gamma(1+\beta)}, \tag{6.2}$$

which extends Relation (158) of [5] to $\beta > 1$, and

$$E[e^{-xD_{1/\alpha}(1/G)}] = e^{-x^{1/\alpha}}, \qquad x \in \mathbb{R}_+,$$

i.e. $D_{1/\alpha}(1/G) = S_{1/\alpha}$ is a stable random variable with exponent $1/\alpha$. For completeness we need to check that the argument of page 385-386 of [5], which is stated therein for $\beta = 1$, also extends to all $\beta > 0$. By Relation (42) in § 1.4.d-(*iii*) in [5] we have

$$D_1\left(\frac{1}{GZ_{1/\alpha}}\right) \simeq \beta_{1,1-1/\alpha} D_{1/\alpha}(1/G),$$

where $Z_{1/\alpha}$ is a Bernoulli random variable with parameter $1/\alpha$ and $\beta_{1,1-1/\alpha}$ is a beta random variable, which shows by convolution that the density of $D_1\left(\frac{1}{GZ_{1/\alpha}}\right)$ is

$$f_{D_1\left(\frac{1}{GZ_{1/\alpha}}\right)}(y) = \frac{\sin(\pi/\alpha)}{\pi y} \int_y^\infty \frac{1}{(x/y-1)^{1/\alpha}} f_{1/\alpha}(x) dx.$$

Now, by Relation (4.19.2) on page 112 of [2] we have

$$X_\beta^\alpha \simeq \frac{X_\beta}{S_{1/\alpha}}$$

where $S_{1/\alpha}$ is a stable random variable with parameter $1/\alpha$. Hence, since

$$\frac{X_\beta}{S_{1/\alpha}} \simeq X_\beta^\alpha \simeq Y_{\beta/\alpha} D_{1/\alpha}(G) \simeq Y_\beta \beta_{1,1-1/\alpha} D_{1/\alpha}(G) \simeq Y_\beta D_1\left(\frac{G}{Z_{1/\alpha}}\right),$$

we get

$$D_1\left(\frac{G}{Z_{1/\alpha}}\right) \simeq \frac{1}{S_{1/\alpha}},$$

and by Relation (164) page 386 of [5] we get

$$f_{D_1\left(\frac{G}{Z_{1/\alpha}}\right)}(x) = \frac{1}{x^2} f_{1/\alpha}(1/x).$$

Next, Theorem 2.3-4) of [5] shows that the distribution function F_G of G satisfies

$$\begin{aligned}
1-\frac{\alpha}{\beta}F_{0,\alpha}(y) &= F_{1/G}(1/y) \\
&= \Lambda_{1/\alpha}\left(\frac{f_{D_1\left(\frac{1}{GZ_{1/\alpha}}\right)}(y)e^{\beta E[\log(G)]/\alpha}}{y^{1/\alpha-2}f_{D_1\left(\frac{G}{Z_{1/\alpha}}\right)}(1/y)}\right) \\
&= \Lambda_{1/\alpha}\left(\frac{\Gamma(1+\beta/\alpha)}{\Gamma(1+\beta)}\frac{\sin(\pi/\alpha)}{\pi y}\int_y^\infty \frac{1}{(x-y)^{1/\alpha}}\frac{f_{1/\alpha}(x)}{f_{1/\alpha}(y)}dx\right),
\end{aligned}$$

since $e^{\beta E[\log G]/\alpha} = \Gamma(1+\beta/\alpha)/\Gamma(1+\beta)$ by (6.2). □

Finally we prove Lemma 3.2 as a corollary of Proposition 6.1.

Proof of Lemma 3.2. When $\alpha = 2$ we have

$$\Lambda_{1/2}(y) = 1 - \frac{2}{\pi}\arctan\frac{1}{y} = \frac{2}{\pi}\arctan y, \qquad y > 0,$$

hence Relation (6.1) of Proposition 6.1 and Legendre's duplication formula (cf. [4], p. 64):

$$\frac{\Gamma(1+\beta/2)}{\sqrt{\pi}\Gamma(1+\beta)} = \frac{1}{2^\beta\Gamma(1/2+\beta/2)}$$

give

$$\begin{aligned}
F_{0,2}(y) &= \frac{2\beta}{\alpha\pi}\arctan\left(\frac{2^{-\beta}}{y\sqrt{\pi}\Gamma(1/2+\beta/2)}\int_y^\infty \frac{1}{(x-y)^{1/2}}\frac{f_{1/2}(x)}{f_{1/2}(y)}dx\right)^{-1} \\
&= \frac{\beta}{\pi}\arctan\left(\frac{2^{-\beta}\sqrt{y}e^{1/(4y)}}{\sqrt{\pi}\Gamma(1/2+\beta/2)}\int_y^\infty \frac{1}{\sqrt{x-y}}e^{-1/(4x)}\frac{dx}{x^{3/2}}\right)^{-1} \\
&= \frac{\beta}{\pi}\arctan\left(\frac{2^{2-\beta}}{\sqrt{\pi}\Gamma(1/2+\beta/2)}\int_0^{1/(2\sqrt{y})} e^{z^2}dz\right)^{-1} \\
&= \frac{\beta}{\pi}\arctan\left(\frac{\Gamma(\beta/2+1/2)}{2^{1-\beta}F(1/(2\sqrt{y}))}\right), \qquad (6.3)
\end{aligned}$$

where we applied the change of variable $z^2 = 1/(4y) - 1/(4x)$, $x, y > 0$. □

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