# Convex concentration for some additive functionals of jump stochastic differential equations

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**Abstract**

Using forward-backward stochastic calculus we prove convex concentration inequalities for some additive functionals of the solution of stochastic differential equations with jumps admitting an invariant probability measure. As a consequence, transportation-information inequalities are obtained and bounds on option prices for interest rate derivatives are given as an application.

**Key words:** Convex concentration inequalities, transportation-information inequalities, stochastic differential equations with jumps, interest rate derivatives.
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## 1 Introduction

Two random variables $F$ and $G$ are said to satisfy a convex concentration inequality if

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)] \tag{1.1}$$

for all convex function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$. By classical arguments, the application of (1.1) to the convex function $\phi(x) = \exp(\lambda x)$, $\lambda > 0$, can be used to estimate the deviation probabilities of $F$ via the Laplace transform of $G$.

In this paper we establish a convex concentration inequality for additive functionals of the form

$$S_T = \int_0^T g(X_t)dt,$$

where $(X_t)_{t\in\mathbb{R}_+}$ is a $\mathbb{R}^d$-valued jump-diffusion process solution of (2.1) below and admitting an invariant distribution, and $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a sufficiently smooth function, cf. Theorem 2.1.

We apply the technique of [12], cf. also [17], [6], [11], which consists in rewriting $S_t$ as the half sum

$$S_t = \frac{\overleftarrow{M}_t^f + \overrightarrow{M}_t^f}{2},$$

of a forward martingale

$$\overrightarrow{M}_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds, \qquad t \in \mathbb{R}_+, \tag{1.2}$$

and a backward martingale $\overleftarrow{M}_t^f$ defined in (2.9) below, where $\mathcal{L}$ is the generator of $(X_t)_{t\in\mathbb{R}_+}$, cf. (2.5) below, and $f$ is in the space $\mathcal{C}_b^2(\mathbb{R}^d)$ of $\mathcal{C}^2$ functions on $\mathbb{R}^d$ which have bounded first and second derivatives.

In Proposition 2.1 below we derive a convex concentration inequality for $\overrightarrow{M}_T^f$ using forward-backward stochastic calculus, cf. [8], [3], [1], [2].

When $(X_t)_{t\in\mathbb{R}_+}$ is reversible with invariant probability measure $\mu$ on $\mathbb{R}^d$, both martingales have same distribution under $P_\mu$ and the convex concentration inequality for $S_T$ is obtained from that on $\overrightarrow{M}_T^f$ after solving the Poisson equation $-\mathcal{L}f = g$ for $f$, cf. Theorem 2.1 below.

The framework of mean-reverting and stationary processes is well adapted to the modeling of interest rates in finance, and we apply those inequalities to the derivation of bounds on the prices of interest rate derivatives for which the underlying interest rate processes are usually mean-reverting and stationary.

This paper is organized as follows. Section 2 focuses on convex concentration inequalities, first for the martingale with jumps $\overrightarrow{M}_t^f$, and then for $S_T = \int_0^T g(X_t)dt$. Some examples of processes satisfying the assumptions **(A)**, **(B)**, **(C)** and **(D)** below are also given at the end of Section 2 from [16]. Section 3 presents some consequences on related transportation-information inequalities. In Section 4 we apply those results to derive bounds on the prices of interest rate derivatives.

# 2 Convex concentration inequalities

Given $m(x, du)$ a non-negative Radon measure on $\mathbb{R}^n$, $x \in \mathbb{R}^d$, we consider three vector fields $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma_1 : \mathbb{R}^d \longrightarrow \mathcal{M}_{d\times n}$, and

$$\begin{aligned} \sigma_2 \ : \ \mathbb{R}^d &\longrightarrow L^2(\mathbb{R}^n, m(x, du); \mathbb{R}^d) \\ x &\longmapsto \sigma_2(x, \cdot), \end{aligned}$$

where $\mathcal{M}_{d\times n}$ denotes the space of $d \times n$ matrices.

We assume that

**(A)** $x \longmapsto b(x) \in \mathbb{R}^d$, $x \longmapsto \sigma_1(x) \in \mathcal{M}_{d\times n}$, and $x \longmapsto \sigma_2(x, \cdot) \in L^2(\mathbb{R}^n, m(x, du); \mathbb{R}^d)$ are continuously differentiable in $x \in \mathbb{R}^d$.

Consider the $\mathbb{R}^d$-valued Markov jump-diffusion process $(X_t)_{t\in\mathbb{R}_+}$, solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma_1(X_s)dW_s + \int_0^t \int_{\mathbb{R}^n} \sigma_2(X_{s^-}, u)\tilde{\omega}_X(ds, du), \qquad (2.1)$$

where $W_t$ is a $\mathbb{R}^n$-valued Brownian motion,

$$\tilde{\omega}_X(dt, du) = \omega_X(dt, du) - m(X_t, du)dt$$

is the compensated random measure with intensity $m(X_t, du)dt$ on $\mathbb{R}^n \times \mathbb{R}_+$, defined on a well-filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and $X_0$ is a (random) initial condition independent of $\omega_X(dt, du)$.

Since the coefficients $b$, $\sigma_1$ and $\sigma_2$ are bounded on compacts of $\mathbb{R}_+ \times \mathbb{R}^d$, it can be shown that SDE in (2.1) admits a unique solution $X_t(x)$, cf. Theorem 13.58, Theorem 14.80 of [7], pages 434, 438 and 481, using the results on martingale problems for discontinuous processes of [9], [10], [15]. In addition, $X_t$ is a right-continuous process admitting left limits $X_{t^-}$, $t \in \mathbb{R}_+$.

We also assume that

**(B)** the process $(X_t)_{t\in\mathbb{R}_+}$ admits a unique invariant probability distribution $\mu$ on $\mathbb{R}^d$, and

**(C)** we have $\|\sigma_1(x)\|_{\mathrm{HS}} \leq \sigma_{1,\infty} < \infty$, $x \in \mathbb{R}^d$, and

$$|\sigma_2(x,u)| \leq \sigma_{2,\infty}(u), \qquad m(x,du)dx - a.e. \tag{2.2}$$

for some measurable function $u \longmapsto \sigma_{2,\infty}(u)$ on $\mathbb{R}^n$, and

$$m(x,du) \leq n(du), \qquad x \in \mathbb{R}^d. \tag{2.3}$$

where $n(du)$ is a measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} (e^{\lambda\sigma_{2,\infty}(u)} - \lambda\sigma_{2,\infty}(u) - 1)n(du) < \infty, \qquad \lambda > 0.$$

Given $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, let $\overrightarrow{M}_t^f$ be the $\mu$-local martingale defined by

$$\overrightarrow{M}_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds, \qquad t \in \mathbb{R}_+, \tag{2.4}$$

where

$$\begin{aligned} \mathcal{L}f(x) \;&=\; \langle b(x), \nabla f(x)\rangle_{\mathbb{R}^d} + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\partial_{i,j}f(x) \\ &\quad + \int_{\mathbb{R}^n} (f(x+\sigma_2(x,u)) - f(x) - \langle \nabla f(x), \sigma_2(x,u)\rangle_{\mathbb{R}^d}) m(x,du), \end{aligned} \tag{2.5}$$

$f \in \mathcal{C}_b^2(\mathbb{R}^d)$, is the generator of $(X_t)_{t\in\mathbb{R}_+}$, with $(a_{i,j}) = (\sigma_1\sigma_1^\dagger)_{i,j}$, and $\nabla$ is the gradient on $\mathbb{R}^d$. The next proposition states a convex concentration inequality for $\overrightarrow{M}_T^f$.

**Proposition 2.1** *Let $T > 0$. Under Conditions* **(A)**, **(B)**, *and* **(C)**, *for all $\phi$ in the set $\mathscr{C}_c$ of $\mathcal{C}^2$-convex functions with convex derivative on $\mathbb{R}$, we have*

$$\mathbb{E}\left[\phi(\overrightarrow{M}_T^f)\right] \leq \mathbb{E}\left[\phi\left(\bar{W}\left(T\|f\|_{\mathrm{Lip}}^2\sigma_{1,\infty}^2\right) + \|f\|_{\mathrm{Lip}}\int_0^T\int_{\mathbb{R}^n}\sigma_{2,\infty}(u)\tilde{\omega}(ds,du)\right)\right], \quad (2.6)$$

*where $\overrightarrow{M}_t^f$ is defined by (2.4), $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ is Lipschitz, $\tilde{\omega}(ds,du)$ is a compensated Poisson random measure with intensity $n(du)ds$, and $\bar{W}(\sigma^2)$ is an independent centered Gaussian random variable with variance $\sigma^2 > 0$.*

*Proof.* By Itô's formula we have

$$\langle\overrightarrow{M}^f\rangle_t = \int_0^t \Gamma(f)(X_s)ds, \qquad t \in \mathbb{R}_+,$$

where

$$\Gamma(f)(x) = \sum_{i,j=1}^d a_{i,j}(x)\partial_i f(x)\partial_j f(x) + \int_{\mathbb{R}^n}|f(x+\sigma_2(x,u)) - f(x)|^2 m(x,du),$$

$x \in \mathbb{R}^d$, is the carré du champ operator of $\mathcal{L}$. Therefore we have

$$\begin{aligned}
\langle(\overrightarrow{M}^f)^c\rangle_t &= \sum_{i,j=1}^d \int_0^t a_{i,j}(X_s)\partial_i f(X_s)\partial_j f(X_s)ds \\
&= \sum_{k=1}^n\sum_{i,j=1}^d \int_0^t \sigma_{i,k}^{(1)}(X_s)\sigma_{j,k}^{(1)}(X_s)\partial_i f(X_s)\partial_j f(X_s)ds \\
&\leq \sum_{k=1}^n \int_0^t \left|\sum_{i=1}^d \sigma_{i,k}^{(1)}(X_s)\partial_i f(X_s)\right|^2 ds \\
&\leq \sum_{k=1}^n\sum_{i=1}^d \int_0^t |\sigma_{i,k}^{(1)}(X_s)|^2 \sum_{i=1}^d |\partial_i f(X_s)|^2 ds \\
&\leq \|f\|_{\mathrm{Lip}}^2 \int_0^t \|\sigma_1(X_s)\|_{\mathrm{HS}}^2 ds \\
&\leq t\|f\|_{\mathrm{Lip}}^2\sigma_{1,\infty}^2, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

while we have

$$|\Delta\overrightarrow{M}_t^f| := |\overrightarrow{M}_t^f - \overrightarrow{M}_{t^-}^f| \leq \|f\|_{\mathrm{Lip}}|\Delta X_t|.$$

Next we define the backward martingale $(M_t^*)_{t\in[0,T]}$ with respect to its own backward filtration $(\mathcal{F}_t^*)_{t\in[0,T]}$, as

$$M_t^* = \bar{W}\left(\|f\|_{\rm Lip}^2\sigma_{1,\infty}^2 T\right) - \bar{W}\left(\|f\|_{\rm Lip}^2\sigma_{1,\infty}^2 t\right) + \|f\|_{\rm Lip}\int_t^T\int_{\mathbb{R}^n}\sigma_{2,\infty}(u)\tilde{\omega}(ds,du),$$

$t\in[0,T]$, where $\tilde{\omega}$ is a compensated Poisson random measure with intensity $n(du)ds$, independent of $\omega_X$.

By the forward-backward Itô formula, cf. Theorem 8.1 of [8], for any $\phi\in\mathscr{C}_c$ we have

$$\begin{aligned}
\mathbb{E}[\phi(\overrightarrow{M}_t^f + M_t^*)] - \mathbb{E}[\phi(\overrightarrow{M}_0^f + M_0^*)] &= \frac{1}{2}\mathbb{E}\left[\int_0^t \phi''(\overrightarrow{M}_s^f + M_s^*)(d\langle(\overrightarrow{M}^f)^c\rangle_s - d\langle(M^*)^c\rangle_s)\right] \\
&+\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi(\overrightarrow{M}_s^f + M_s^*, \Delta\overrightarrow{M}_s^f)\omega_X(ds,du)\right] \\
&-\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi(\overrightarrow{M}_s^f + M_s^*, \|f\|_{\rm Lip}\sigma_{2,\infty}(u))\tilde{\omega}(ds,du)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\int_0^t \phi''(\overrightarrow{M}_s^f + M_s^*)(d\langle(\overrightarrow{M}^f)^c\rangle_s - d\langle(M^*)^c\rangle_s)\right] \\
&+\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi(\overrightarrow{M}_s^f + M_s^*, \|f\|_{\rm Lip}\sigma_2(X_{s^-},u))\omega_X(ds,du)\right] \\
&-\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi(\overrightarrow{M}_s^f + M_s^*, \|f\|_{\rm Lip}\sigma_{2,\infty}(u))\tilde{\omega}(ds,du)\right] \\
&\leq \mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi\left(\overrightarrow{M}_s^f + M_s^*, \|f\|_{\rm Lip}\sigma_2(X_s,u)\right)m(X_s,du)ds\right] \\
&-\mathbb{E}\left[\int_0^t\int_{\mathbb{R}^n}\psi\left(\overrightarrow{M}_s^f + M_s^*, \|f\|_{\rm Lip}\sigma_{2,\infty}(u)\right)n(du)ds\right] \\
&\leq 0,
\end{aligned}$$

from (2.3), where

$$\psi(x,y) := \phi(x+y) - \phi(x) - y\phi'(x) = y^2\int_0^1(1-\tau)\phi''(x+\tau y)d\tau \geq 0,$$

is a non-negative and non-decreasing function of $y$. This shows that

$$\mathbb{E}[\phi(\overrightarrow{M}_t^f + M_t^*)] \leq \mathbb{E}[\phi(\overrightarrow{M}_0^f + M_0^*)], \qquad t\in[0,T],$$

which yields (2.6) by letting $t=T$. □

Next, we derive a convex concentration inequality for the additive functional

$$S_T = \int_0^T g(X_t)dt,$$

where $g$ is a function on $\mathbb{R}^d$ such that $\mu(g) = 0$.

We make the following assumption.

**(D)** The process $(X_t)_{t\in\mathbb{R}_+}$ is reversible with respect to the invariant probability measure $\mu$, i.e.

$$\langle \mathcal{L}f, g\rangle_{L^2(\mathbb{R}^d,\mu)} = \langle f, \mathcal{L}g\rangle_{L^2(\mathbb{R}^d,\mu)}, \qquad f, g \in \mathcal{C}_b^2(\mathbb{R}^d).$$

Note that the Poisson equation (2.7) below admits the trivial Lipschitz solution $f(x) = \sum_{i=1}^d x_i$ when $g(x) = -\sum_{i=1}^d b_i(x)$, $x \in \mathbb{R}^d$.

**Theorem 2.1** *Assume that Conditions* **(A)**, **(B)**, **(C)** *and* **(D)** *hold. Consider $g$ a Lipschitz function on $\mathbb{R}^d$ such that $\mu(g) = 0$ and such that the Poisson equation*

$$\mathcal{L}f = -g, \tag{2.7}$$

*admits a Lipschitz solution $f \in \mathcal{C}_b^2(\mathbb{R}^d)$. Then we have*

$$\mathbb{E}_\mu\left[\phi\left(\int_0^T g(X_t)dt\right)\right] \leq \mathbb{E}\left[\phi\left(\bar{W}\left(T\sigma_{1,\infty}^2\|f\|_{\mathrm{Lip}}^2\right) + \|f\|_{\mathrm{Lip}}\int_0^T\int_{\mathbb{R}^n}\sigma_{2,\infty}(u)\tilde{\omega}(ds,du)\right)\right], \tag{2.8}$$

*where $\tilde{\omega}(ds, du)$ is a compensated Poisson random measure with intensity $n(du)ds$ and $\bar{W}(\sigma^2)$ is an independent centered Gaussian random variable with variance $\sigma^2 > 0$.*

*Proof.* Inspired by Lyons-Zheng's forward-backward martingale decomposition, cf. [12] and also [17], [6], [11], we define

$$\overleftarrow{M}_t^f = f(X_0) - f(X_t) - \int_0^t \mathcal{L}f(X_s)ds, \tag{2.9}$$

$t \in \mathbb{R}_+$, where $f$ solves the Poisson equation (2.7). Obviously we have

$$S_T = \int_0^T g(X_t)dt = -\int_0^T \mathcal{L}f(X_t)dt = \frac{\overleftarrow{M}_T^f + \overrightarrow{M}_T^f}{2},$$

and by the reversibility assumption **(D)**, $\overrightarrow{M}_T^f$ and $\overleftarrow{M}_T^f$ have the same distribution under $\mathbb{P}_\mu$. Therefore, for any convex function $\phi$ we have

$$\mathbb{E}_\mu[\phi(S_T)] = \mathbb{E}_\mu\left[\phi\left(\frac{\overleftarrow{M}_T^f + \overrightarrow{M}_T^f}{2}\right)\right] \le \frac{1}{2}\mathbb{E}_\mu\left[\phi(\overleftarrow{M}_T^f) + \phi(\overrightarrow{M}_T^f)\right] = \mathbb{E}_\mu\left[\phi(\overrightarrow{M}_T^f)\right],$$

and we conclude by Proposition 2.1. □

Taking $\phi(x) = e^{\lambda x}$, $\lambda > 0$, in Theorem 2.1 we obtain the following corollary on the Laplace transform of $S_T$, which can be used in deviation bounds by Chebyshev-type arguments.

**Corollary 2.2** *Assume that Conditions* **(A)**, **(B)**, **(C)** *and* **(D)** *hold and that* (2.7) *admits a Lipschitz solution $f$ for $g$ a Lipschitz function on $\mathbb{R}^d$ such that $\mu(g) = 0$. Then we have*

$$\mathbb{E}[e^{\lambda(S_T - \mathbb{E}[S_T])}] \le \exp\left(\frac{\lambda^2}{2} T\|f\|_{\mathrm{Lip}}^2 \sigma_{1,\infty}^2 + T\beta(\lambda\|f\|_{\mathrm{Lip}})\right), \tag{2.10}$$

*where*

$$\beta(\lambda) := \int_{\mathbb{R}^n} (e^{\lambda\sigma_{2,\infty}(u)} - \lambda\sigma_{2,\infty}(u) - 1) n(du) < \infty,$$

*and $n(du)$ is defined in* (2.3).

*Proof.* Although this result is a direct consequence of Theorem 2.1, we present an independent derivation. By the definition (2.5) of $\mathcal{L}$ we have

$$\begin{aligned} & e^{-f(x)}(\mathcal{L}e^f)(x) \\ = \;& \mathcal{L}f(x) + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\partial_i f(x)\partial_j f(x) + \int_{\mathbb{R}^n} h\Big(f(x+\sigma_2(x,u)) - f(x)\Big) m(x,du), \end{aligned}$$

$f \in \mathcal{C}_b^2(\mathbb{R}^d)$, where

$$h(x) = e^x - x - 1, \qquad x \in \mathbb{R}.$$

Let

$$\begin{aligned} Z_t^f \;&:=\; \exp\left(f(X_t) - f(X_0) - \int_0^t (e^{-f}\mathcal{L}e^f)(X_s)ds\right) \\ &=\; \exp\left(\overrightarrow{M}_t^f + \int_0^t \mathcal{L}f(X_s) - \int_0^t (e^{-f}\mathcal{L}e^f)(X_s)ds\right) \end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigg(\overrightarrow{M}_t^f - \frac{1}{2}\int_0^t \sum_{i,j=1}^d a_{i,j}(X_s)\partial_i f(X_s)\partial_j f(X_s)ds \\
&\quad - \int_0^t \int_{\mathbb{R}^n} h\Big(f(X_s+\sigma_2(X_s,u)) - f(X_s)\Big)m(X_s,du)ds\Bigg), \qquad t \in \mathbb{R}_+,
\end{aligned}
$$

which is both a local martingale and a supermartingale by Lemma 3.3 in [5], for all initial distribution $\rho \in M_1(\mathbb{R}^d)$ absolutely continuous with respect to $\mu$.

Define now the sub-probability measure $\mathbb{Q}$ by

$$
d\mathbb{Q} = Z^{\lambda f} d\mathbb{P}
$$

on the path space. We have

$$
\begin{aligned}
\mathbb{E}_\mu\left[\exp\left(\lambda \overrightarrow{M}_t^f\right)\right] &= \mathbb{E}_{\mathbb{P}_\mu}\Bigg[Z_t^{\lambda f}\exp\Bigg(\frac{\lambda^2}{2}\int_0^t \sum_{i,j=1}^d a_{i,j}(X_s)\partial_i f(X_s)\partial_j f(X_s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}^n} h\Big(\lambda f(X_s+\sigma_2(X_{s^-},u)) - \lambda f(X_s)\Big)m(X_s,du)ds\Bigg)\Bigg] \\
&= \mathbb{E}_{\mathbb{Q}_\mu}\Bigg[\exp\Bigg(\frac{\lambda^2}{2}\int_0^t \sum_{i,j=1}^d a_{i,j}(X_s)\partial_i f(X_s)\partial_j f(X_s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}^n} h\Big(\lambda f(X_s+\sigma_2(X_s,u)) - \lambda f(X_s)\Big)m(X_s,du)ds\Bigg)\Bigg] \\
&\le \exp\left(\frac{\lambda^2}{2}t\|f\|_{\mathrm{Lip}}^2\sigma_{1,\infty}^2 + t\beta\big(\|f\|_{\mathrm{Lip}}\lambda\big)\right),
\end{aligned}
$$

which recovers (2.10). □

**Examples**

We now produce some examples of processes $(X_t)_{t\in\mathbb{R}_+}$ that satisfies Assumptions **(A)**, **(B)**, **(C)** and **(D)**, based on Example 1.2 in [16].

Taking $d = n = 1$ and assuming that the invariant measure $\mu(dx) = \rho(x)dx$ has a differentiable density $\rho(x)$ on $\mathbb{R}$, we let $a(x) = \sigma_1^2(x)$, $\sigma_2(x,u) = u - x$,

$$
b(x) = a(x)\frac{\rho'(x)}{\rho(x)} + a'(x),
$$

and

$$m(x, du) = j(|u - x|)\rho(u)du,$$

where $j(u)$ is a non-negative Borel measurable function on $\mathbb{R}$.

Then by Theorem 1.1 of [16], the process $(X_t)_{t\in\mathbb{R}_+}$ with generator

$$\begin{aligned}\mathcal{L}f(x) &= b(x)f'(x) + a(x)f''(x) \\ &\quad + \int_{\mathbb{R}} (f(x+z) - f(x) - zf'(x))j(|z|)\rho(x+z)dz \\ &= b(x)f'(x) + a(x)f''(x) \\ &\quad + \int_{\mathbb{R}} (f(x+\sigma_2(x,u)) - f(x) - \sigma_2(x,u)f'(x))m(x,du),\end{aligned}$$

$f \in \mathcal{C}_b^2(\mathbb{R})$, is symmetric with respect to $\mu(dx) = \rho(x)dx$, provided

$$\int_{\mathbb{R}^d} |z|m(x, dz) < \infty, \qquad x \in \mathbb{R}^d.$$

Assuming in addition that the function $z \longmapsto j(z)$ is bounded by 1 on $\mathbb{R}$ and supported in $B(0, K)$, we have

$$|\sigma_2(x,u)| = |x - u| \le K, \qquad m(x, du) \le \rho(u)du - a.e., \quad x \in \mathbb{R},$$

i.e. (2.2) and (2.3) are satisfied with $\sigma_{2,\infty}(u) = K > 0$ and $n(du) = \mu(du) = \rho(u)du$.

Again by Example 1.3 in [16], when $d = n = 1$ and $\rho(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard Gaussian density with $\sigma_1(x) = a$ constant, then $b(x) = -x$ and we find

$$\begin{aligned}\mathcal{L}f(x) &= -xf'(x) + \frac{1}{2}\sigma_1^2 f''(x) \\ &\quad + \int_{\mathbb{R}} (f(x+\sigma_2(x,u)) - f(x) - \sigma_2(x,u)f'(x))j(|x-u|)m(du),\end{aligned} \tag{2.11}$$

$f \in \mathcal{C}_b^2(\mathbb{R})$, and the process $(X_t)_{t\in\mathbb{R}_+}$ with generator $\mathcal{L}$ is symmetric with respect to the Gaussian measure $\mu(dx) = e^{-x^2/2}dx/\sqrt{2\pi}$.

# 3 Transportation-information inequality

In this section 3 we present some consequences of Theorem 2.1 on related transportation-information inequalities. Given $p \ge 1$, the $L^p$-Wasserstein distance between two probability measures $\mu$ and $\nu$ on $\mathbb{R}^d$ is defined by

$$W_{p,d}(\mu,\nu) = \inf \left( \int \int |d(x,y)|^p d\pi(x,y) \right)^{1/p},$$

where $d(x,y)$ is the distance on $\mathbb{R}^d$ and the infimum is taken over all probability measures $\pi$ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal distribution $\mu$ and $\nu$. Given $\alpha$ a non-decreasing, left-continuous function on $\mathbb{R}_+$ which vanishes at 0, we say that $\mu$ satisfies a transportation-information inequality $W_1 I$ with deviation function $\alpha$ if

$$\alpha(W_{1,d}(\nu,\mu)) \leq I(\nu|\mu), \qquad \nu \in M_1(\mathbb{R}^d), \tag{3.1}$$

for some given probability measure $\mu$, cf. Guillin *et al.* [6]. Here, $I(\nu|\mu)$ is the Fisher-Donsker-Varadhan information of $\nu$ with respect to $\mu$

$$I(\nu|\mu) = \begin{cases} \mathcal{E}(\sqrt{f},\sqrt{f}) := -\langle \mathcal{L}\sqrt{f},\sqrt{f}\rangle_{L^2(\mathbb{R},\mu)}, & \text{if } d\nu = f d\mu, \quad \sqrt{f} \in \mathbb{D}(\mathcal{E}), \\ +\infty & \text{otherwise} \end{cases} \tag{3.2}$$

associated with the Dirichlet form $\mathcal{E}$ on $L^2(\mu)$ with domain $\mathbb{D}(\mathcal{E})$. From the characterization result Corollary 2.4 of [6] and the convex concentration inequality (2.8), we obtain the following result under the hypotheses of Theorem 2.1.

**Theorem 3.1** *Assume that Conditions* **(A)**, **(B)**, **(C)** *and* **(D)** *hold. The invariant probability $\mu$ satisfies the $W_1 I$ transportation-information inequality* (3.1) *with deviation function*

$$\alpha(r) := \sup_{\lambda>0} \left( \lambda r - \frac{1}{2}\beta(2\|f\|_{\mathrm{Lip}}\lambda) - \lambda^2 \|f\|_{\mathrm{Lip}}^2 \sigma_{1,\infty}^2 \right) \tag{3.3}$$

*for $d(x,y) = |x-y|$ the Euclidean distance on $\mathbb{R}^d$.*

*Proof.* For any initial probability measure $\nu$ absolutely continuous with respect to $\mu$ with $d\nu/d\mu \in L^2(\mu)$ and $r, T > 0$, by Theorem 2.1, applied to $\phi(x) = e^{\lambda x}$ with $\lambda > 0$, we have

$$\begin{aligned} \mathbb{P}_\nu\left(\frac{1}{T}S_T - \mu(g) > r\right) &= \mathbb{P}_\nu\left(S_T - \mathbb{E}_\mu[S_T] > rT\right) \\ &\leq \inf_{\lambda>0} e^{-\lambda r T}\mathbb{E}_\nu\left[\exp(\lambda(S_T - \mathbb{E}_\mu[S_T]))\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \frac{d\nu}{d\mu} \right\|_2 \inf_{\lambda>0} e^{-\lambda r T} \left| \mathbb{E}_\mu \left[ \exp\left( 2\lambda (S_T - \mathbb{E}_\mu[S_T]) \right) \right] \right|^{1/2} \\
&\leq \left\| \frac{d\nu}{d\mu} \right\|_2 \inf_{\lambda>0} \exp\left( -\lambda r T + \lambda^2 T \sigma_{1,\infty}^2 \|f\|_{\mathrm{Lip}}^2 + \frac{T}{2} \beta\big(2\lambda \|f\|_{\mathrm{Lip}}\big) \right),
\end{aligned}
$$

which, by Corollary 2.4 of [6], implies the $\alpha$-$W_1 I$ inequality (3.1) with deviation function $\alpha(r)$ given by (3.3). □

# 4 Application to interest rate derivatives

Theorem 2.1 applies typically to the derivation of bounds on options such as Asian options on the average $S_T = \int_0^T g(X_t) dt$ of a asset price $g(X_t)$. However, the processes we consider are mostly mean-reverting stationary processes and as such they are more frequently used for the modeling of instantaneous interest rates than for stock prices, cf. e.g. [4]. This is the case in particular for the Vasicek model which relies on the Gaussian Ornstein-Uhlenbeck process, cf. e.g. [14] and references therein for a review.

Consequently we present an application to the pricing of interest rate derivatives.

**Bond pricing**

Assuming that the short term interest rate $r_t$ is modeled as $r_t = b(X_t)$, the bond price

$$
E\left[ \exp\left( - \int_0^T r_s ds \right) \right]
$$

can be bounded by Theorem 2.1 as

$$
\begin{aligned}
E\left[ \exp\left( - \int_0^T r_s ds \right) \right] &\leq E\left[ \exp\left( -\bar{W}\left( T\sigma_{1,\infty}^2 \right) + \int_0^T \int_{\mathbb{R}^n} \sigma_{2,\infty}(u) \tilde{\omega}(ds, du) \right) \right] \\
&\leq \exp\left( T\sigma_{1,\infty}^2/2 + T \int_{\mathbb{R}^n} (e^{\sigma_{2,\infty}(u)} - \sigma_{2,\infty}(u) - 1) n(du) \right),
\end{aligned}
$$

where $\bar{W}(\sigma^2)$ is a centered Gaussian random variable with variance $\sigma^2 > 0$.

**Caplet pricing**

In this case, the process $b(X_t)$ is used to model an instantaneous forward rate, and the integral

$$f(t,t,T) = \int_t^T b(X_s)ds$$

represents a spot forward interest rate process under the forward measure.

The price of an interest rate cap with strike $\kappa = \mathbb{E}_\mu[f(t,t,T)]$ on $f(t,t,T)$ can be bounded as follows:

$$\mathbb{E}_\mu[(f(t,t,T)-\kappa)^+] \leq \mathbb{E}\left[\left(\bar{W}\left((T-t)\sigma_{1,\infty}^2\right) + \int_t^T \int_{\mathbb{R}^n} \sigma_{2,\infty}(u)\tilde{\omega}(ds,du) - \kappa\right)^+\right]$$

$$= \mathbb{E}\left[\left(\bar{W}\left((T-t)\sigma_{1,\infty}^2\right) + K(\tilde{\omega}([0,T-t]\times\mathbb{R}^n) - (T-t)n(\mathbb{R}^n)) - \kappa\right)^+\right]$$

$$= e^{-(T-t)n(\mathbb{R}^n)} \sum_{k=0}^\infty \frac{((T-t)n(\mathbb{R}^n))^k}{k!} \mathbb{E}\left[\left(\bar{W}\left((T-t)\sigma_{1,\infty}^2\right) + Kk - K(T-t)n(\mathbb{R}^n) - \kappa\right)^+\right]$$

$$= e^{-(T-t)n(\mathbb{R}^n)} \frac{1}{\sigma_{1,\infty}\sqrt{2(T-t)\pi}} \sum_{k=0}^\infty \frac{((T-t)n(\mathbb{R}^n))^k}{k!} \int_{-Kk+K(T-t)n(\mathbb{R}^n)+\kappa}^\infty (x + Kk - K(T-t)n(\mathbb{R}^n) - \kappa)\, e^{-x^2/(2(T-t)\sigma_{1,\infty}^2)}dx$$

$$= e^{-(T-t)n(\mathbb{R}^n)} \sigma_{1,\infty}\sqrt{\frac{(T-t)}{2\pi}} \sum_{k=0}^\infty \frac{((T-t)n(\mathbb{R}^n))^k}{k!} \left(e^{-\frac{(-Kk+K(T-t)n(\mathbb{R}^n)+\kappa)^2}{(2(T-t)\sigma_{1,\infty}^2}} + (Kk - K(T-t)n(\mathbb{R}^n) - \kappa)\,\Phi\left(\frac{-Kk+K(T-t)n(\mathbb{R}^n)+\kappa}{(T-t)\sigma_{1,\infty}^2}\right)\right),$$

where $\Phi$ is the standard Gaussian distribution function.

In the case of a Gaussian stationary distribution with mean reversion coefficient $b(x) = -x$ the covariance of the process can be difficult to compute, so that the above bounds can remain useful even in this case.

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