(Probabilités)

An Analytic Approach to Stochastic Calculus

Nicolas Privault

Une approche analytique du calcul stochastique

Abstract - *An Itô type stochastic differential calculus is constructed in an analytic way, without use of the notion of filtration, for processes that satisfy certain homogeneity and smoothness conditions. This calculus is developed in the framework of Lie groups.*

Résumé - Nous construisons de façon analytique, sans la notion de filtration, un calcul différentiel stochastique du type Itô pour des processus qui satisfont certaines conditions d'homogénéité et de régularité. Ce calcul est développé dans le cadre des groupes de Lie.

Version française abrégée

Soit G un groupe de Lie connexe de dimension d dont l'algèbre de Lie $\mathcal{G}$ a pour base $(X_1, \ldots, X_d)$. Soient $(B_t)_{t\in\mathbb{R}_+}$ un mouvement brownien à valeurs dans $\mathcal{G}$, et N une mesure aléatoire de Poisson sur $\mathsf{G} \times \mathbb{R}_+$, d'intensité $\mu(d\sigma)dt$. Soit $\mathcal{H}$ l'ensemble des fonctions de $\mathcal{C}_c(\mathsf{G} \times \mathbb{R}_+)$ qui s'annulent sur $\{e\} \times \mathbb{R}_+$ et telles que $\sigma \mapsto f(\sigma, t)$ est différentiable en e, $t \in \mathbb{R}_+$. Nous définissons l'intégrale stochastique compensée des processus adaptés dans $L^2(\Omega) \otimes \mathcal{H}$ par

$$\int_{\mathsf{G}}\int_0^\infty u_{\gamma,t} \diamond d\tilde{M}_{\gamma,t} = \int_{\mathsf{G}}\int_0^\infty u_{\gamma,t}(N(d\gamma, dt) - \mu(d\gamma)dt) + \int_0^\infty X_i u_{e,t} dB^i(t), \qquad (1)$$

L'intégrale stochastique multiple $I_n(h_n)$ de $h_n \in \mathcal{H}^{\circ n}$ est définie par récurrence à partir de (1). Soit $\Gamma(\mathcal{H})$ l'espace de Fock algébrique sur $\mathcal{H}$ avec son opérateur d'annihilation ∇^-, identifié à un sous-espace de $L^2(\Omega)$ en associant $h_n \in \mathcal{H}^{\circ n}$ à $I_n(h_n)$, $n \in \mathbb{N}$. Le gradient D du calcul de Malliavin est donnée par $(DF, h) = \int_0^\infty h_t \nabla^-_{e,t} F dt$, $h \in L^2(\mathbb{R}_+, \mathcal{G})$. On définit, par dérivation des noyaux de $\mathcal{H}$ et seconde quantification, des opérateurs mutuellement adjoints $a^\ominus$ et $\nabla^\oplus$. Pour $u \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+)$, soient $\mathcal{G}_u$ et $\mathcal{A}_u$ définis sur $\mathcal{S}$ par

$$\mathcal{G}_u F = \frac{1}{2}\int_0^\infty u_{e,t} \sum_{i=1}^{i=n} (X_i \nabla^-_{e,t})^2 F dt, \quad \mathcal{A}_u F = \int_0^\infty \int_{\mathsf{G}} u_{\gamma,t} \nabla^-_{\gamma,t} F \mu(d\gamma)dt.$$

Soit $\mathcal{V}$ l'ensemble des processus $(X_t, u^t)_{t\in\mathbb{R}_+}$ où $(X_t)_{t\in\mathbb{R}_+}$ est une famille de variables aléatoires suffisament régulières et $(u^t)_{t\in\mathbb{R}_+} \subset \mathcal{C}^1_c(\mathbb{R}_+)$ est une famille de fonctions, telles que la condition suivante d'homogéneité soit satisfaite pour tout $t > 0$:

$$\mathcal{U}_{\varepsilon u^t} X_t = X_{t-\varepsilon}, \ a.s.,$$

ε dans un voisinage de zéro, où $\mathcal{U}_{\varepsilon u^t}$ est défini par (11).

Théorème *La formule d'Itô s'écrit comme suit pour $(X_t, u^t)_{t\in\mathbb{R}_+} \in \mathcal{V}$:*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t a^\ominus_{u^s}[f(X_s)]ds + \int_0^t f'(X_s)\mathcal{G}_{u^s} X_s ds \\ &\quad + \frac{1}{2}\int_0^t (u^s DX_s, DX_s) f''(X_s) ds + \int_0^t \mathcal{A}_{u^s} f(X_s) ds, \end{aligned}$$

$t \in \mathbb{R}_+$, *pour tout polynôme* f.

La deuxième partie de cette Note est consacrée à la construction d'un opérateur gradient par différences finies à gauche pour les fonctionnelles d'un processus de Lévy sur G. On montre que si μ est la mesure de Haar sur G unimodulaire alors le gradient à gauche sur G est obtenu à partir de l'opérateur d'annihilation ∇^- par une transformation unitaire de $\mathcal{H}$, et que par différentiation on retrouve les opérateurs de l'analyse stochastique sur l'espace de Lie-Wiener.

1 Notation and preliminaries

Let G be a connected Lie group of dimension d whose Lie algebra $\mathcal{G}$ of left-invariant vector fields has basis $(X_1, \ldots, X_d)$. Let $(B_t)_{t \in \mathbb{R}_+}$ be a standard $\mathcal{G}$-valued Brownian motion, and let N be a Poisson random measure on $\mathsf{G} \times \mathbb{R}_+$ with intensity $\mu(d\sigma)dt$, μ finite and diffuse, on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{H}$ denote the vector space of functions in $\{f \in \mathcal{C}_c^1(\mathsf{G} \times \mathbb{R}_+) \ : \ f(e,t) = 0, \ t \in \mathbb{R}_+\}$ such that $\sigma \mapsto f(\sigma, t)$ is differentiable at e, $t \in \mathbb{R}_+$, equipped with the norm

$$\| f \|_{\mathcal{H}}^2 = \| f \|_{L^2(\mathsf{G} \times \mathbb{R}_+)}^2 + \| (X_1 f(e, \cdot), \ldots, X_d f(e, \cdot)) \|_{L^2(\mathbb{R}_+, \mathcal{G})}^2,$$

and let $(\cdot, \cdot)$ denote the scalar product in $L^2(\mathbb{R}_+, \mathcal{G})$. In this Note, "$\otimes$" and "$\circ$" denote the algebraic ordinary and symmetric tensor products. We define the compensated stochastic integral of an adapted process $u \in L^2(\Omega) \otimes \mathcal{H}$ as

$$\int_{\mathsf{G}} \int_0^\infty u_{\gamma,t} \diamond d\tilde{M}_{\gamma,t} = \int_{\mathsf{G}} \int_0^\infty u_{\gamma,t} (N(d\gamma, dt) - \mu(d\gamma)dt) + \int_0^\infty X_i u_{e,t} dB^i(t), \qquad (2)$$

with the isometry property $E\left[\left(\int_{\mathsf{G}} \int_0^\infty u_{\gamma,t} \diamond d\tilde{M}_{\gamma,t} \right)^2 \right] = E\left[\| u \|_{\mathcal{H}}^2 \right]$. The multiple stochastic integral $I_n(h_n)$ of $h_n \in \mathcal{H}^{\circ n}$ is defined by induction. Let $\Gamma(\mathcal{H})$ denote the algebraic Fock space on the normed vector space $\mathcal{H}$, with annihilation and creation operators ∇^-, ∇^+. Elements of $\Gamma(\mathcal{H})$ are identified to random variables in $L^2(\Omega)$ by associating $h_n \in \mathcal{H}^{\circ n}$ to its multiple stochastic integral $I_n(h_n)$. Let $\mathcal{S}$ denote the vector space generated by elements of the form

$$I_n(f_1 \circ \cdots \circ f_n), \quad f_1, \ldots, f_n \in \mathcal{H}, \quad n \in \mathbb{N}.$$

We will need the multiplication formula between n-th and first order integrals, written as

$$I_n(f^{\circ n}) I_1(g) = I_{n+1}(g \circ f^{\circ n}) + n I_n((fg) \circ f^{\circ(n-1)}) + n(f,g)_{\mathcal{H}} I_{n-1}(f^{\circ(n-1)}), \qquad (3)$$

$f, g \in \mathcal{H}$, or equivalently as

$$F \nabla^+(u) = \nabla^+(uF) + \nabla^+(u \nabla F) + (u, \nabla^- F)_{\mathcal{H}}, \quad F \in \mathcal{S}, \ u \in \mathcal{H}. \qquad (4)$$

As a consequence of (3), $\mathcal{S}$ is an algebra contained in $L^p(\Omega)$, $p \geq 2$. The annihilation operator ∇^- on $\Gamma(\mathcal{H})$ is interpreted as a finite difference operator:

$$\nabla_{\gamma,t}^- F = F\left(\dot{B}, \dot{N}(\cdot) + \delta_{\gamma,t}(\cdot) \right) - F, \quad (\gamma, t) \in \mathsf{G} \times \mathbb{R}_+, \quad F \in \mathcal{S}. \qquad (5)$$

The derivative $D : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbb{R}_+, \mathcal{G})$ of the Malliavin calculus is obtained by differentiating $\nabla_{\gamma,t}^-$:

$$(DF, h) = \int_0^\infty h_t \nabla_{e,t}^- F dt, \quad F \in \mathcal{S}, \ h \in L^2(\mathbb{R}_+, \mathcal{G}).$$

2 Analytic construction of stochastic calculus

In this section we construct a stochastic differential calculus in a purely analytic way. The annihilation operator ∇^- is interpreted as a directional derivative, constructed by shifts of trajectories. On the other hand, a stochastic process can be also perturbed in the time direction, and we start by identifying the operators on Fock space that are associated to such time perturbations.

2.1 First order differential operators and stochastic differentials

We denote by $\mathcal{U}$ the set of processes that can be written as $\sum_{i=1}^{i=n} F_i u_i$, where $u_1, \ldots, u_n \in \mathcal{C}_c^1(\mathsf{G} \times \mathbb{R}_+)$, and $F_1, \ldots, F_n \in \mathcal{S}$, $n \geq 1$. Let ∂ and ∂^* denote the operator of partial differentiation and integration with respect to $t \in \mathbb{R}^+$ on $\mathcal{C}_1(\mathsf{G} \times \mathbb{R}_+)$.

Definition 1 *We define in three steps the first order unbounded differential operators which will represent the Itô differentials.*

1. *For $g \in L^2(\mathsf{G} \times \mathbb{R}_+)$ let $a_g, a_g^* : L^2(\mathsf{G} \times \mathbb{R}_+) \to L^2(\mathsf{G} \times \mathbb{R}_+)$ be defined by*

$$a_g f = -\partial^* g \partial f, \quad \text{and} \quad a_g^* f = \partial(f \partial^* g), \quad f \in \mathcal{C}_c^1(\mathsf{G} \times \mathbb{R}_+).$$

2. *By second quantization we define $a_g^\ominus, a_g^\oplus$ on $\mathcal{S}$ as $a_g^\ominus = d\Gamma(a_g)$, and $a_g^\oplus = d\Gamma(a_g^*)$.*

3. *Finally, for $u = \sum_{i=1}^{i=n} F_i u_i \in \mathcal{U}$ we define $a_u^\ominus$ on $\mathcal{S}$ and $\nabla^\oplus$ on $\mathcal{U}$ by*

$$a_u^\ominus F = \sum_{i=1}^{i=n} F_i a_{u_i}^\ominus F, \quad \text{and} \quad \nabla^\oplus(u) = \sum_{i=1}^{i=n} a_{u_i}^\oplus F_i.$$

The duality relation between $a^\ominus$ and $\nabla^\oplus$ is $E[a_u^\ominus F] = E[F \nabla^\oplus(u)]$, $F \in \mathcal{S}$, $u \in \mathcal{U}$. Moreover, $\nabla^\oplus(u) = 0$ and $E[a_u^\ominus F] = 0$, $F \in \mathcal{S}$, if $u \in \mathcal{U}$ is adapted. The definition of $a_u^\ominus$ can be extended to $u \in L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ because there is a closable gradient operator $\nabla^\ominus : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbb{R}_+)$ (denoted by $\tilde{\nabla}$ in [5]) that satisfies $a_u^\ominus F = (\nabla^\ominus F, u)_{L^2(\mathbb{R}_+)}$, $u \in L^2(\Omega) \otimes L^2(\mathbb{R}_+)$, $F \in \mathcal{S}$. Hence $\nabla^\oplus : L^2(\Omega) \otimes L^2(\mathbb{R}_+) \to L^2(\Omega)$ is closable, of domain $Dom(\nabla^\oplus)$. The following is the product rule for $a_u^\ominus$, which can be proved from (3):

$$a_u^\ominus(FG) = F a_u^\ominus G + G a_u^\ominus F - (u \nabla^- F, \nabla^- G)_{\mathcal{H}}, \quad F, G \in \mathcal{S}, \; u \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+). \tag{6}$$

In the Wiener case, i.e. for $\mu = 0$, (6) implies for f polynomial:

$$a_u^\ominus f(F) = f'(F) a_u^\ominus F - \frac{1}{2} f''(F)(uDF, DF), \quad F \in \mathcal{S}, \; u \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+). \tag{7}$$

By duality, (6) is equivalent to the analog of (4) for $a^\ominus$ and $\nabla^\oplus$:

$$F \nabla^\oplus(u) = \nabla^\oplus(uF) + a_u^\ominus F - \nabla^+(u \nabla^- F), \quad F \in \mathcal{S}, \; u \in \mathcal{U}.$$

2.2 Second order differential operators and generators

For $u \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+)$ we define the operators $\mathcal{G}_u$, $\mathcal{A}_u$ on $\mathcal{S}$ by

$$\mathcal{G}_u F = \frac{1}{2} \int_0^\infty u_{e,t} \sum_{i=1}^{i=n} (X_i \nabla^-_{e,t})^2 F dt \quad \textit{and} \quad \mathcal{A}_u F = \int_0^\infty \int_{\mathsf{G}} u_{\gamma,t} \nabla^-_{\gamma,t} F \mu(d\gamma) dt, \quad F \in \mathcal{S}.$$

For constant u, the operator $\mathcal{G}_u$ is the Gross Laplacian, cf. [2]. We have

$$\mathcal{G}_u f(F) = f'(F) \mathcal{G}_u F + \frac{1}{2} (uDF, DF) f''(F), \quad F \in \mathcal{S}, \tag{8}$$

for f polynomial, and

$$(\mathcal{G}_u + \mathcal{A}_u)(FG) = F(\mathcal{G}_u + \mathcal{A}_u)G + G(\mathcal{G}_u + \mathcal{A}_u)F + (u\nabla^- F, \nabla^- G)_{\mathcal{H}}, \tag{9}$$

$F, G \in \mathcal{S}$, $u \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+)$.

2.3 Analytic stochastic differentials and calculus

Relations (6) and (9) show that $a_u^\ominus + \mathcal{G}_u + \mathcal{A}_u$ is a derivation operator on $\mathcal{S}$:

$$(a_u^\ominus + \mathcal{G}_u + \mathcal{A}_u) f(F) = f'(F)(a_u^\ominus + \mathcal{G}_u + \mathcal{A}_u)F, \quad F \in \mathcal{S}, \quad u \in \mathcal{U}, \tag{10}$$

for f polynomial. Let $\mathcal{D} \supset \mathcal{S}$ denote the vector space dense in $L^2(\Omega)$ generated by

$$\{I_n(h_1 \circ \cdots \circ h_n) \ : \ h_1, \ldots, h_n \in \cap_{p \geq 2} L^p(\mathsf{G} \times \mathbb{R}_+), \ n \in \mathbb{N}\}.$$

For $h \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+)$ with $\| h \|_\infty < 1$, let $\nu_h(\sigma, t) = t + \partial^* h(\sigma, t)$, $(\sigma, t) \in \mathsf{G} \times \mathbb{R}_+$. For $F \in \mathcal{D}$ we define

$$\mathcal{U}_h F = f(J_1(g_1 \circ \nu_h), \ldots, J_1(g_m \circ \nu_h)), \tag{11}$$

where $f(J_1(g_1), \ldots, J_1(g_m))$ denotes the expression of F as a polynomial in *non-compensated* single stochastic integrals obtained from (3).

Proposition 1 *From (10), we obtain*

$$-\frac{d}{d\varepsilon} \mathcal{U}_{\varepsilon h} F_{|\varepsilon=0} = a_h^\ominus F + \mathcal{G}_h F + \mathcal{A}_h F, \ \textit{a.s.}, \quad F \in \mathcal{S}, \ h \in \mathcal{C}_b(\mathsf{G} \times \mathbb{R}_+). \tag{12}$$

The transformation $\mathcal{U}_h$ is closely related to time changes as it can be shown that for $F \in \mathcal{S}$ there is a version $\hat{F}$ of F such that $\mathcal{U}_{\varepsilon h} F$ is a.s. equal to the functional $\mathcal{T}_h \hat{F}$, defined by evaluating $\hat{F}$ at time-changed trajectories whose jumps are obtained from the jumps of $N(dx, ds)$ via the mapping $(\sigma, t) \mapsto (\sigma, \nu_h(\sigma, t))$, and whose continuous part is given by the time-changed Brownian motion $B^h_{\nu_h(0,t)} = B_t$, $t \in \mathbb{R}_+$.

Definition 2 *We denote by $\mathcal{V}$ the class of processes $(X_t, u^t)_{t \in \mathbb{R}_+}$ where $(X_t)_{t \in \mathbb{R}_+} \subset \mathcal{S}$ is a family of random variables and $(u^t)_{t \in \mathbb{R}_+} \subset \mathcal{C}^1_c(\mathbb{R}_+)$ is a family of functions, such that the following homogeneity condition is satisfied:*

$$\mathcal{U}_{\varepsilon u^t} X_t = X_{t-\varepsilon}, \ \textit{a.s.}, \ \textit{for } \varepsilon \textit{ in a neighborhood of zero}, \ \forall t > 0.$$

Compared to other extensions of the Itô formula, cf. [3] and the references therein, the following result does not contain additional terms due to non-adaptedness, and it does not make use of Skorohod integral processes.

Theorem 1 *The Itô formula for $(X_t, u^t)_{t\in\mathbb{R}_+} \in \mathcal{V}$ is written as*

$$f(X_t) - f(X_0) = \int_0^t a^{\ominus}_{u^s}[f(X_s)]ds + \int_0^t f'(X_s)\mathcal{G}_{u^s}X_s ds + \frac{1}{2}\int_0^t (u^s DX_s, DX_s)f''(X_s)ds + \int_0^t \mathcal{A}_{u^s} f(X_s)ds, \tag{13}$$

$t \in \mathbb{R}_+$, for f polynomial.

In the Wiener case, $\mu = 0$, $\mathcal{A}_{u^s}X_s = 0$, $s \in \mathbb{R}_+$, and the "martingale" and "generator" parts of the process $(f(X_t))_{t\in\mathbb{R}_+}$ can be separated as follows:

$$f(X_t) \;=\; f(X_0) + \int_0^t \left(f'(X_s)a^{\ominus}_{u^s}X_s - \frac{1}{2}(u^s DX_s, DX_s)f''(X_s)\right) ds + \int_0^t \mathcal{G}_{u^s} f(X_s)ds,$$

$t \in \mathbb{R}_+$. In order to link (13) to the classical Itô formula we consider the approximation $B^n_t = \int_0^\infty e^t_n(s)dB_s$, $n \in \mathbb{N}$, of $(B_t)_{t\in\mathbb{R}_+}$, where $(e^t_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\infty_c([0,t+1],[0,1])$ is a sequence that converges pointwise to $1_{[0,t]}$ with $e^t_n = 1$ on $[0,t]$, $t > 0$. For every $t > 0$ let $u^t \in \mathcal{C}^1_c([t/2, 3t/4])$ with $\int_0^t u^t(s)ds = 1$. Then $\mathcal{U}_{\varepsilon u^t}B^n_t = B^n_{t-\varepsilon}$, $0 < \varepsilon < t/4$, $t \in \mathbb{R}_+$, and $(u^t DB^n_t, DB^n_t) = 1$, $\mathcal{G}_{u^t}f(B^n_t)$ converges to $\frac{1}{2}f''(B_t)$, hence $\int_0^t a^{\ominus}_{u^s}f(B^n_s)ds$ converges to $\int_0^t f'(B_s)dB_s$ in $L^2(\Omega)$ as $n \to \infty$, $t \in \mathbb{R}_+$.

3 Left and right difference operators

In this section we construct a stochastic calculus of variations for functionals of a Lévy process on G. From results in [1], N and B define a process $(\phi_t)_{t\in\mathbb{R}_+}$ with values in G via the stochastic differential equation written with the uncompensated differential $\diamond dM$ as

$$f(\phi_t) = f(e) + \int_0^t (f(\phi_{s^-}\sigma) - f(\phi_{s^-}))\diamond dM_{\sigma,s} + \int_0^t \mathcal{A}f(\phi_{s^-})ds, \tag{14}$$

$t \in \mathbb{R}_+$, $f \in \mathcal{C}^2(\mathsf{G})$, where $\mathcal{A}$ is the generator of $(\phi_t)_{t\in\mathbb{R}_+}$:

$$\mathcal{A}f(\gamma) = \frac{1}{2}\sum_{i=1}^{i=d} X_i^2 f(\gamma) + \int_{\mathsf{G}} (f(\gamma\sigma) - f(\gamma))\mu(d\sigma). \tag{15}$$

The mapping $\phi : \Omega \to \mathcal{D}_{\mathsf{G}}$ defines an image measure ν on the set $\mathcal{D}_{\mathsf{G}}$ of cadlag functions from $\mathbb{R}_+$ to G. We denote by $\mathcal{P}$ the set of functionals of the form $f_1(\phi_{t_1})\cdots f_n(\phi_{t_n})$, $f_1,\ldots,f_n \in \mathcal{C}^2_b(\mathsf{G})$, $t_1,\ldots,t_n$, $n \in \mathbb{N}$, and by $\mathcal{W}$ the set of processes of the form $\sum_{i=1} u_i F_i$, $F_1,\ldots,F_n \in \mathcal{P}$, $u_1,\ldots,u_n \in \mathcal{H}$, $n \in \mathbb{N}$.

Definition 3 *The left finite difference operator $L : L^2(\mathcal{D}_{\mathsf{G}}) \to L^2(\mathcal{D}_{\mathsf{G}}) \otimes L^2(\mathsf{G}\times\mathbb{R}_+)$ is defined on $\mathcal{P}$ from*

$$L_{\sigma,s}f(\phi_t) = 1_{[0,t]}(s)(f(\sigma\phi_t) - f(\phi_t)), \quad (\sigma,s) \in \mathsf{G}\times\mathbb{R}_+,\ t \in \mathbb{R}_+,$$

and by use of the finite difference product identity

$$L_{\sigma,s}(FG) = FL_{\sigma,s}G + GL_{\sigma,s}F + (L_{\sigma,s}F)(L_{\sigma,s}G), \quad (\sigma,s) \in \mathsf{G}\times\mathbb{R}_+,\ F,G \in \mathcal{P}.$$

We assume that μ is the left and right invariant Haar measure on G unimodular, that the inner product of $\mathcal{G}$ is invariant under inner automorphisms, and define the unitary operator $\theta : \mathcal{H} \rightarrow \mathcal{H}$ as $h(\sigma, s) \mapsto h(\phi_s^{-1}\sigma\phi_s, s)$. Then the operator L is closable, due to the relation

$$(LF) \circ \phi = \theta \circ \nabla^-(F \circ \phi), \quad F \in \mathcal{P}. \tag{16}$$

The left divergence operator L^* is the adjoint of L:

$$E[(LF, u)_{\mathcal{H}}] = E[FL^*(u)], \quad F \in \mathcal{P}, \quad u \in \mathcal{W}.$$

Since θ is unitary, we obtain $(L^*u) \circ \phi = (\nabla^+ \circ \theta^{-1})(u \circ \phi)$, $u \in \mathcal{W}$. The Lie-Wiener (left) derivative $\mathcal{L} : L^2(d\nu) \rightarrow L^2(d\nu) \otimes L^2(\mathbb{R}_+, \mathcal{G})$, cf. [4], [6], is obtained from the finite difference operator by differentiation at e, i.e. $(\mathcal{L}F, h) = \int_0^\infty h_s L_{e,s} F ds$, $F \in \mathcal{P}$. The relationship between $\mathcal{G}_u + \mathcal{A}_u$ and the generator $\mathcal{A}$ of $(\phi_t)_{t\in\mathbb{R}_+}$ is given by the relation

$$[\mathcal{A}f](\phi_t) = \lim_{s\uparrow t} \sum_{i=1}^{i=d} (X_i \nabla^-_{e,s})^2 f(\phi_t) + \int_{\mathsf{G}} \nabla^-_{\sigma,s} f(\phi_t)\mu(d\sigma), \quad f \in \mathcal{C}^2(\mathbb{R}), \ t \in \mathbb{R}_+.$$

References

[1] D. Applebaum and H. Kunita. Lévy flows on manifolds and Lévy processes on Lie groups. *Journal of Mathematics of Kyoto University*, 33:1103–1123, 1993.

[2] T. Hida, H. H. Kuo, J. Potthoff, and L. Streit. *White Noise, An Infinite Dimensional Calculus.* Kluwer Acad. Publ., Dordrecht, 1993.

[3] D. Nualart. *The Malliavin Calculus and Related Topics.* Probability and its Applications. Springer-Verlag, 1995.

[4] M. Pontier and A.S. Üstünel. Analyse stochastique sur l'espace de Lie-Wiener. *C. R. Acad. Sci. Paris Sér. I Math.*, 313:313–316, 1991.

[5] N. Privault. Calculus on Fock space and a quantum non-adapted Itô formula. *C. R. Acad. Sci. Paris Sér. I Math.*, 323:927–932, 1996.

[6] A.S. Üstünel. Stochastic analysis on Lie groups. In L. Decreusefond, J. Gjerde, B. Øksendal, and A.S. Üstünel, editors, *Stochastic Analysis and Related Topics VI: The Geilo Workshop*, Progress in Probability, pages 129–158, Geilo, 1996. Birkäuser.

Equipe d'Analyse et Probabilités
Université d'Evry
Boulevard des Coquibus
F-91025 Evry Cedex
e-mail: privault@lami.univ-evry.fr

and

Centre de Recerca Matemàtica
Universitat Autònoma de Barcelona
Apartat 50
E-08193 Bellaterra (Barcelona)