Risk-neutral hedging of interest rate derivatives

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Abstract

In this paper we review the hedging of interest rate derivatives priced under a risk-neutral measure, and we compute self-financing hedging strategies for various derivatives using the Clark-Ocone formula.

Key words: Bond markets, hedging, infinite-dimensional analysis, Clark-Ocone formula, swaptions, bond options, caplets.
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1 Introduction

While the pricing of interest rate derivatives is well understood, due notably to the use of the change of numeraire technique, the computation of hedging strategies for such derivatives presents several difficulties. In general, hedging strategies appear not to be unique and one is faced with the problem of choosing an appropriate tenor structure of bond maturities in order to correctly hedge maturity-related risks, see e.g. [3] in the jump case. In [6], self-financing hedging strategies have been computed for swaptions in a geometric Brownian model, using the associated forward measure. In [7] this approach has been extended to other interest rate derivatives using the Markov property and stochastic integral representation formulas under change of numeraire, which is a natural tool for the pricing of such derivatives.

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In this paper we focus on the hedging of interest rate derivatives under the risk-neutral probability measure $\mathbb{P}$ itself, using the general framework for the hedging of interest rate derivatives introduced in [1], [2], which is based on a cylindrical Wiener process $(W_t)_{t\in\mathbb{R}_+}$ with values in a Hilbert space H under $\mathbb{P}$. In particular, we compute hedging strategies for interest rate derivatives, using both Delta hedging and the Clark-Ocone formula. As in [7], we determine the relevant tenor structure from payoff structure of the claim, in such a way that the hedging strategy does not explicitly depend on bond volatilities.

We proceed as follows. The notation on bond markets and option pricing under the risk-neutral measure is introduced at the end of this Section. In Section 2 we derive self-financing hedging strategies for interest rate derivatives based on the Markov property. Finally in Section 3 we use the Clark-Ocone formula to compute self-financing hedging strategies for interest rate derivatives. We mainly consider three examples, namely swaptions, bond options, and caplets on the forward and LIBOR rates.

Notation

We work in the infinite dimensional framework of [1], [2]. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a cylindrical Wiener process $(W_t)_{t\in\mathbb{R}_+}$ with values in a Hilbert space H. The measure $\mathbb{P}$ is taken as a risk-neutral measure. Let $(r_t)_{t\in\mathbb{R}_+}$ denote a short term interest rate process adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(W_t)_{t\in\mathbb{R}_+}$, consider the bank account process

$$B_t = e^{\int_0^t r_s ds}, \qquad t \in \mathbb{R}_+.$$

By risk-neutral valuation under the measure $\mathbb{P}$, an $\mathcal{F}_T$-measurable claim with payoff ξ, maturity S and exercise date T, is priced at time t as

$$\mathbb{E}\left[e^{-\int_t^S r_s ds} B_S \tilde{\xi} \,\middle|\, \mathcal{F}_t \right] = B_t\, \mathbb{E}[\tilde{\xi} \mid \mathcal{F}_t], \qquad 0 \le t \le T < S, \tag{1.1}$$

where

$$\tilde{\xi} = B_S^{-1} \xi \in L^1(\mathbb{P}, \mathcal{F}_S)$$

denotes the discounted payoff of the claim.

We will work with a continuous $\mathcal{F}_t$-adapted asset price process $(X_t)_{t\in\mathbb{R}_+}$ taking values in a real separable Hilbert space F of real-valued functions on $\mathbb{R}_+$, usually a weighted Sobolev space F of real-valued functions on $\mathbb{R}_+$, cf. [5] and § 6.5.2 of [2]. In the sequel, $(X_t)_{t\in\mathbb{R}_+}$ will represent either a bond price curve taking values in the function space F, or a real-valued asset price when $F = \mathbb{R}$, cf. also [7]. We note that the discounted asset price

$$\tilde{X}_t := \frac{X_t}{B_t}, \qquad 0 \le t \le T,$$

is an F-valued martingale under the risk-neutral measure $\mathbb{P}$, provided it is integrable under $\mathbb{P}$. More precisely we will assume that $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ satisfies

$$d\tilde{X}_t = \sigma_t dW_t, \qquad t \in \mathbb{R}_+, \tag{1.2}$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ is an $\mathcal{L}_{HS}(H, F)$-valued adapted process of Hilbert-Schmidt operators from H to F.

2 Risk-neutral hedging in bond markets

Assume that the discounted claim $\tilde{\xi} \in L^2(\Omega)$ has the predictable representation

$$\tilde{\xi} = \mathbb{E}[\tilde{\xi}] + \int_0^T \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F}, \tag{2.1}$$

where $(\tilde{X}_t)_{t\in[0,T]}$ is given by (1.2) and $(\tilde{\phi}_t)_{t\in[0,T]}$ is a square-integrable F^* -valued adapted process of continuous linear mappings on F under $\mathbb{P}$, from which it follows that the discounted claim price

$$\tilde{V}_t := \mathbb{E}[\tilde{\xi} \mid \mathcal{F}_t], \qquad 0 \le t \le T,$$

is a martingale that can be decomposed as

$$\tilde{V}_t = \mathbb{E}[\tilde{\xi}] + \int_0^t \langle \tilde{\phi}_s, d\tilde{X}_s \rangle_{F^*,F}, \qquad 0 \le t \le T. \tag{2.2}$$

Consider a portfolio strategy $(\tilde{\phi}_t, \tilde{\eta}_t)$ with value

$$V_t := \langle \tilde{\phi}_t, X_t \rangle_{F^*,F} + \tilde{\eta}_t B_t = \int_T^\infty X_t(y)\tilde{\phi}_t(dy) + \tilde{\eta}_t B_t \tag{2.3}$$

where $\tilde{\phi}_t(dy)$ will denote the amount of bonds having maturity in $[y, y+dy]$ in the portfolio, and $\tilde{\eta}_t$ denotes the quantity invested in the money market account in the portfolio at time $t \in [0,T]$.

Definition 2.1 *The portfolio strategy $(\tilde{\phi}_t, \tilde{\eta}_t)$, $0 \le t \le T$, is said to be self-financing if*

$$dV_t = \langle \tilde{\phi}_t, dX_t \rangle_{F^*,F} + \tilde{\eta}_t dB_t, \qquad 0 \le t \le T. \tag{2.4}$$

We say that the portfolio strategy $(\tilde{\phi}_t, \tilde{\eta}_t)$ hedges the option with payoff ξ if for all $t \in [0,T]$, its value $\langle \tilde{\phi}_t, X_t \rangle_{F^*,F} + \tilde{\eta}_t B_t$ satisfies

$$\langle \tilde{\phi}_t, X_t \rangle_{F^*,F} + \tilde{\eta}_t B_t = \mathbb{E}\left[e^{-\int_t^S r_s ds} \xi \;\Big|\mathcal{F}_t \right],$$

i.e.

$$\tilde{V}_t = \langle \tilde{\phi}_t, \tilde{X}_t \rangle_{F^*,F} + \tilde{\eta}_t = \int_T^\infty \tilde{X}_t(y) \tilde{\phi}_t(dy) + \tilde{\eta}_t, \qquad 0 \le t \le T,$$

where $\tilde{X}_t = B_t^{-1} X_t$, $0 \le t \le T$, is the discounted asset price.

The next proposition is well known and is a particular case of a general change of numeraire argument, cf. e.g. [6], [7].

Proposition 2.2 *Letting $\tilde{\eta}_t = \tilde{V}_t - \langle \tilde{X}_t, \tilde{\phi}_t \rangle_{F^*,F}$, $0 \le t \le T$, where $(\tilde{\phi}_t)_{t\in[0,T]}$ satisfies* (2.1)*, the portfolio $(\tilde{\phi}_t, \tilde{\eta}_t)_{t\in[0,T]}$ with value*

$$V_t = \langle \tilde{\phi}_t, X_t \rangle_{F^*,F} + \tilde{\eta}_t B_t, \qquad 0 \le t \le T, \tag{2.5}$$

is self-financing and hedges the claim $\xi = B_S \tilde{\xi}$.

Proof. By (2.5) we have

$$V_t = B_t \tilde{V}_t = B_t \, \mathbb{E}[\tilde{\xi} \mid \mathcal{F}_t] = \mathbb{E}\left[e^{-\int_t^S r_s ds} \xi \Big| \mathcal{F}_t \right], \qquad 0 \le t \le T,$$

hence the portfolio $(\tilde{\phi}_t, \tilde{\eta}_t)_{t\in[0,T]}$ hedges the payoff $\xi = B_S \tilde{\xi}$. Next, we show that it is self-financing. We have

$$\begin{aligned} dV_t &= d(B_t \tilde{V}_t) \\ &= \tilde{V}_t dB_t + B_t d\tilde{V}_t + dB_t \cdot d\tilde{V}_t \end{aligned}$$

$$
\begin{aligned}
&= \tilde{V}_t dB_t + B_t \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F} + dB_t \cdot \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F} \\
&= \langle \tilde{\phi}_t, \tilde{X}_t \rangle_{F^*,F} dB_t + B_t \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F} + dB_t \cdot \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F} + (\tilde{V}_t - \langle \tilde{\phi}_t, \tilde{X}_t \rangle_{F^*,F}) dB_t \\
&= \langle \tilde{\phi}_t, d(B_t \tilde{X}_t) \rangle_{F*,F} + (\tilde{V}_t - \langle \tilde{\phi}_t, \tilde{X}_t \rangle_{F^*,F}) dB_t \\
&= \langle \tilde{\phi}_t, dX_t \rangle_{F^*,F} + \tilde{\eta}_t dB_t.
\end{aligned}
$$

where

$$
dB_t \cdot d\tilde{V}_t = dB_t \cdot \langle \tilde{\phi}_t, d\tilde{X}_t \rangle_{F^*,F} = 0.
$$

□

Next we recall how the process $(\tilde{\phi}_t)_{t \in \mathbb{R}_+}$ in the predictable representation (2.2) can be computed by the Clark-Ocone formula, cf. [1]. Let D denote the Malliavin gradient defined on smooth functionals of Brownian motion of the form

$$
\tilde{\xi} = f(W_{t_1}(h_1), \ldots, W_{t_n}(h_n)), \qquad 0 < t_1 < \cdots < t_n,
$$

$f \in \mathcal{C}_b^1(\mathbb{R}^n)$, $h_1, \ldots, h_n \in H$, as

$$
D_t \tilde{\xi} = \sum_{k=1}^{n} \mathbf{1}_{[0,t_k]}(t) \frac{\partial f}{\partial x_k}(W_{t_1}(h_1), \ldots, W_{t_n}(h_n)) \otimes h_k, \tag{2.6}
$$

cf. § 5.1.2 of [2], and extended by closability to its domain Dom (D).

To hedge a claim ξ in this setting, we decompose the discounted payoff $\tilde{\xi}$ as

$$
\tilde{\xi} = \mathbb{E}[\tilde{\xi}] + \int_0^T \langle \mathbb{E}[D_t \tilde{\xi} \mid \mathcal{F}_t], dW_t \rangle_H = \mathbb{E}[\tilde{\xi}] + \int_0^T \langle \alpha_t, dW_t \rangle_H \tag{2.7}
$$

where, by the Clark-Ocone formula,

$$
\alpha_t = \mathbb{E}[D_t \tilde{\xi} \mid \mathcal{F}_t], \tag{2.8}
$$

cf. Theorem 5.3 of [2].

From Relations (1.2) and (2.7) the process $(\tilde{\phi}_t)_{t \in \mathbb{R}_+}$ in (2.1) is given by

$$
\tilde{\phi}_t = (\sigma_t^*)^{-1} \alpha_t = (\sigma_t^*)^{-1} \, \mathbb{E}[D_t \tilde{\xi} \mid \mathcal{F}_t], \qquad 0 \le t \le T, \tag{2.9}
$$

provided $\sigma_t^* : H \to F$ is invertible, $0 \le t \le T$.

Markovian case

Next, assume in addition that $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ has the Markov property, and the dynamics

$$d\tilde{X}_t = \sigma_t(\tilde{X}_t)dW_t, \tag{2.10}$$

where $x \mapsto \sigma_t(x) \in \mathcal{L}_{HS}(H,F)$ is a Lipschitz function from F into the space of Hilbert-Schmidt operator from H to F, uniformly in $t \in \mathbb{R}_+$. In case $H = F = \mathbb{R}$ and $\sigma_t(\tilde{X}_t) = \sigma(t)\tilde{X}_t$, i.e. the martingale $(\tilde{X}_t)_{t\in[0,T]}$ is a geometric Brownian motion under $\mathbb{P}$ with deterministic variance $(\sigma(t))_{t\in[0,T]}$.

In the Markovian setting of (2.10), $D_t\tilde{X}_T$ can be computed as

$$D_t\tilde{X}_T = \tilde{Y}_{t,T}\sigma_t(\tilde{X}_t), \tag{2.11}$$

where $(\tilde{Y}_{t,T})_{t\in[0,T]}$ is solution of

$$\tilde{Y}_{s,t} = I_d + \int_s^t \nabla\sigma_u(\tilde{X}_u)\tilde{Y}_{s,u}dW_u, \qquad 0 \le s \le t, \tag{2.12}$$

cf. Proposition 6.7 of [2], hence in case $\tilde{\xi} = \tilde{g}(\tilde{X}_T)$ we get, assuming that $\tilde{g}$ is Lipschitz,

$$\begin{aligned}
\tilde{\phi}_t &= (\sigma_t^*(\tilde{X}_t))^{-1}\alpha_t \\
&= (\sigma_t^*(\tilde{X}_t))^{-1}\,\mathbb{E}\left[D_t\tilde{g}(\tilde{X}_T)\ \Big|\mathcal{F}_t\right] \\
&= (\sigma_t^*(\tilde{X}_t))^{-1}\,\mathbb{E}\left[(D_t\tilde{X}_T)^*\nabla\tilde{g}(\tilde{X}_T)\ \Big|\mathcal{F}_t\right] \\
&= (\sigma_t^*(\tilde{X}_t))^{-1}\,\mathbb{E}\left[(\tilde{Y}_{t,T}\sigma_t(\tilde{X}_t))^*\nabla\tilde{g}(\tilde{X}_T)\ \Big|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\tilde{Y}_{t,T}^*\nabla\tilde{g}(\tilde{X}_T)\ \Big|\mathcal{F}_t\right], \qquad 0 \le t \le T,
\end{aligned}$$

cf. [2] § 6.5.5.

The use of (2.8) can be somewhat limited since the application of D to $\tilde{\xi}$ can lead to technical difficulties due to the differentiation of $B_S^{-1} = \tilde{P}_S(S)$, and (2.12) can be difficult to solve.

In the remaining of this section we take $T = S$.

European options

We close this section with an application of the Delta hedging method to European type options with discounted payoff $\tilde{\xi} = \tilde{g}(\tilde{X}_T)$ where $\tilde{g} : F \to \mathbb{R}$ and $(\tilde{X}_t)_{t\in\mathbb{R}_+}$ has the Markov property as in (2.10), where $\sigma : \mathbb{R}_+ \times F \to \mathcal{L}_{HS}(H, F)$.

In this case the option with payoff $\xi = B_T\tilde{g}(\tilde{X}_T)$ is priced at time t as

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}\xi \middle| \mathcal{F}_t\right] = B_t \,\mathbb{E}\left[\tilde{g}(\tilde{X}_T)\middle|\mathcal{F}_t\right] = B_t\tilde{C}(t, \tilde{X}_t),$$

for some measurable function $\tilde{C}(t, x)$ on $\mathbb{R}_+ \times F$. However this formula allows one to deal with only a limited range of options, such as exchange options.

Assuming that the function $\tilde{C}(t, x)$ is $\mathcal{C}^2$ on $\mathbb{R}_+ \times F$, we have the following corollary of Proposition 2.2.

Corollary 2.3 *Letting* $\tilde{\eta}_t = \tilde{C}(t, \tilde{X}_t) - \langle\nabla\tilde{C}(t, \tilde{X}_t), \tilde{X}_t\rangle_{F^*,F}$, $0 \le t \le T$, *the portfolio* $(\nabla\tilde{C}(t, \tilde{X}_t), \tilde{\eta}_t)_{t\in[0,T]}$ *with value*

$$V_t = \tilde{\eta}_t B_t + \langle\nabla\tilde{C}(t, \tilde{X}_t), \tilde{X}_t\rangle_{F^*,F}, \qquad t \in \mathbb{R}_+,$$

is self-financing and hedges the claim $\xi = B_T\tilde{g}(\tilde{X}_T)$.

Proof. This result follows directly from Proposition 2.2 by noting that by Itô's formula, cf. Theorem 4.17 of [4], we have

$$d\tilde{C}(t, \tilde{X}_t) = \langle\nabla\tilde{C}(t, \tilde{X}_t), d\tilde{X}_t\rangle_{F^*,F}$$

By the martingale property of $\tilde{V}_t$ under $\mathbb{P}$ and the predictable representation (2.2) we have

$$d\tilde{V}_t = \langle\tilde{\phi}_t, d\tilde{X}_t\rangle_{F^*,F}$$

which ultimately gives us

$$\tilde{\phi}_t = \nabla\tilde{C}(t, \tilde{X}_t), \qquad 0 \le t \le T.$$

□

As a consequence the exchange call option with payoff $\xi = B_T \tilde{g}(\tilde{X}_T) = (X_T - \kappa B_T)^+$ on $(\tilde{X}_t)_{t\in[0,T]}$ a geometric Brownian motion under $\mathbb{P}$ with $(\sigma(t))_{t\in[0,T]}$ a deterministic function, the option price is given by the Margrabe formula

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} (X_T - \kappa B_T)^+ \Big| \mathcal{F}_t \right] = B_t \tilde{C}(t, \tilde{X}_t) = X_t \Phi_+(t, \kappa, \tilde{X}_t) - \kappa B_t \Phi_-(t, \kappa, \tilde{X}_t), \tag{2.13}$$

where the functions $\Phi_+(t, \kappa, x)$ and $\Phi_-(t, \kappa, x)$ are defined as

$$\Phi_+(t, \kappa, x) := \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2} \right) \quad \text{and} \quad \Phi_-(t, \kappa, x) := \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2} \right),$$

where

$$v(t,T) = \int_t^T |\sigma(s)|^2 ds, \qquad 0 \le t \le T.$$

By Corollary 2.3 and the relation $\frac{\partial \tilde{C}}{\partial x}(t,x) = \Phi_+(t, \kappa, x)$, $x \in \mathbb{R}$, applied to the function $\tilde{C}(t,x) = x\Phi_+(t,\kappa,x) - \kappa\Phi_-(t,\kappa,x)$, the portfolio

$$(\tilde{\phi}_t, \tilde{\eta}_t) = (\Phi_+(t, \kappa, \tilde{X}_t), -\kappa\Phi_-(t, \kappa, \tilde{X}_t)), \qquad 0 \le t \le T, \tag{2.14}$$

is self-financing and hedges the claim $(X_T - \kappa B_T)^+$.

In general, however, claim payoffs of the form $B_T \tilde{g}(\tilde{X}_T)$ are not frequent and in Section 3 we will use another method, i.e. the Clark-Ocone formula, to hedge interest rate derivatives. Note that the Delta hedging method requires the computation of the function $\tilde{C}(t,x)$ and that of the associated finite differences, and may not apply to path-dependent claims.

3 Hedging by the Clark-Ocone formula

In this section we compute hedging strategies for interest rate derivatives via the Clark-Ocone formula, and we refer to [8] for the pricing computations not included here. We consider a real-valued Wiener process $(W_t)_{t\in R_+}$ under a risk-neutral probability measure $\mathbb{P}$ and we take $(X_t)_{t\in\mathbb{R}_+} = (P_t)_{t\in\mathbb{R}_+}$, i.e. the bond price curve $(P_t)_{t\in\mathbb{R}_+}$ takes values in a Sobolev space F of real-valued functions on $\mathbb{R}_+$, cf. [5] and [1] for examples.

Let $\mu \in F^*$ denote a finite measure on $\mathbb{R}_+$ with support in $[T, \infty)$, and consider the asset price

$$P_t(\mu) = \langle \mu, P_t \rangle_{F^*,F} = \int_T^\infty P_t(y)\mu(dy).$$

In practice, $\mu(dy)$ and $\tilde{\phi}_t(dy)$ will be finite point measures, i.e. sums of the form

$$\tilde{\phi}_t(dy) = \sum_{k=i}^{j} \alpha_k \delta_{T_k}(dy)$$

of Dirac measures at the maturities $T_i, \dots, T_j$ of a given a tenor structure, in which $\alpha_k(t)$ represents the amount allocated to a bond with maturity T_k, $i \le k \le j$, in which case (2.5) reads

$$V_t = \sum_{k=i}^{j} \alpha_k(t) P_t(T_k) + \tilde{\eta}_t B_t, \qquad 0 \le t \le T,$$

We will assume that the dynamics of $(P_t)_{t \in R_+}$ is given by

$$dP_t = r_t P_t dt + P_t \zeta_t dW_t, \tag{3.1}$$

where $(\zeta_t)_{t\in[0,T]}$ is an $L_{HS}(H, F)$-valued deterministic mapping, with

$$d\tilde{P}_t(y) = \tilde{P}_t(y)\zeta_t(y)dW_t, \tag{3.2}$$

i.e. we take $\sigma_t(\tilde{P}_t) = \zeta_t(\cdot)\tilde{P}_t(\cdot)$ in (2.10). Consider a discounted payoff function of the form

$$\tilde{\xi} = \tilde{g}(\tilde{P}_T(\mu)), \tag{3.3}$$

with maturity T, where $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and

$$P_T(\mu) = \int_T^\infty P_T(x)\mu(dx).$$

The next result is stated for discounted payoffs.

Proposition 3.1 *Letting*

$$\tilde{\phi}_t(dx) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \tilde{g}'(\tilde{P}_T(\mu)) \frac{P_T(x)}{P_t(x)} \Big| \mathcal{F}_t \right] \mu(dx),$$

and $\tilde{\eta}_t = \tilde{V}_t - \langle \tilde{X}_t, \tilde{\phi}_t \rangle_{F^*,F}$, $0 \le t \le T$, *yields a self-financing hedging portfolio hedging the claim with payoff*

$$\xi = B_T \tilde{g}(\tilde{P}_T(\mu)).$$

Proof. We note that since $\tilde{\xi} = \tilde{g}(\tilde{P}_T(\mu))$, we have

$$D_t \tilde{\xi} = D_t \tilde{g}(\tilde{P}_T(\mu)) = \tilde{g}'(\tilde{P}_T(\mu)) D_t \tilde{P}_T(\mu),$$

where

$$D_t \tilde{P}_T(\mu) = \int_T^\infty \zeta_t(x) \tilde{P}_T(x) \mu(dx).$$

Therefore the process $(\alpha_t)_{t\in[0,T]}$ in (2.7) is given by

$$\begin{aligned} \alpha_t &= \mathbb{E}\left[D_t \tilde{\xi} \mid \mathcal{F}_t \right] \\ &= \mathbb{E}\left[\tilde{g}'(\tilde{P}_T(\mu)) D_t \tilde{P}_T(\mu) \mid \mathcal{F}_t \right] \\ &= \int_T^\infty \zeta_t(x)\, \mathbb{E}\left[\tilde{g}'(\tilde{P}_T(\mu)) \tilde{P}_T(x) \mid \mathcal{F}_t \right] \mu(dx), \end{aligned}$$

hence

$$\begin{aligned} \langle \alpha_t, dW_t \rangle_H &= \int_T^\infty \mathbb{E}\left[\tilde{g}'(\tilde{P}_T(\mu)) \tilde{P}_T(x) \mid \mathcal{F}_t \right] \mu(dx) \zeta_t(x) dW_t \\ &= \int_T^\infty \mathbb{E}\left[\tilde{g}'(\tilde{P}_T(\mu)) \frac{\tilde{P}_T(x)}{\tilde{P}_t(x)} \mid \mathcal{F}_t \right] \mu(dx) d\tilde{P}_t(x). \end{aligned}$$

From (2.7) the process $(\tilde{\phi}_t)_{t\in[0,T]}$ in (2.1) is given by

$$\tilde{\phi}_t(dx) = \mathbb{E}\left[\tilde{g}'(\tilde{P}_T(\mu)) \frac{\tilde{P}_T(x)}{\tilde{P}_t(x)} \Big| \mathcal{F}_t \right] \mu(dx),$$

and it remains to apply Proposition 2.2 with $(X_t)_{t\in\mathbb{R}_+} = (P_t)_{t\in\mathbb{R}_+}$. □

Next, we apply Proposition 3.1 to swaptions.

Swaptions on the LIBOR rate

Consider a tenor structure $\{T \le T_i, \ldots, T_j\}$ and the swaption on the LIBOR rate with payoff

$$\xi = (P_T(T_i) - P_T(T_j) - \kappa P(T, T_i, T_j))^+ \tag{3.4}$$

where

$$P(T, T_i, T_j) = \sum_{k=i}^{j-1} \tau_k P_T(T_{k+1})$$

is the annuity numeraire, with $\tau_k = T_{k+1} - T_k$, $k = i, \ldots, j-1$. The next corollary follows from Proposition 3.1.

Corollary 3.2 *Letting*

$$\begin{aligned}\tilde{\phi}_t(dx) &= \mathbb{E}\left[\mathbf{1}_{\{S(T,T_i,T_j)>\kappa\}} \frac{\tilde{P}_T(T_i)}{\tilde{P}_t(T_i)} \Big| \mathcal{F}_t\right] \delta_{T_i}(dx) \\ &\quad -(1+\kappa\tau_{j-1})\,\mathbb{E}\left[\mathbf{1}_{\{S(T,T_i,T_j)>\kappa\}} \frac{\tilde{P}_T(T_j)}{\tilde{P}_t(T_j)} \Big| \mathcal{F}_t\right] \delta_{T_j}(dx) \\ &\quad -\kappa \sum_{k=i+1}^{j-1} \tau_{k-1}\,\mathbb{E}\left[\mathbf{1}_{\{S(T,T_i,T_j)>\kappa\}} \frac{\tilde{P}_T(T_k)}{\tilde{P}_t(T_k)} \Big| \mathcal{F}_t\right] \delta_{T_k}(dx),\end{aligned}$$

and $\tilde{\eta}_t = 0$, $0 \le t \le T$, *yields a self-financing hedging portfolio hedging the claim with payoff* (3.4), *without any investment in the money market account, where*

$$S(T, T_i, T_j) = \frac{P_T(T_i) - P_T(T_j)}{P(T, T_i, T_j)}$$

is the swap rate.

Proof. We apply Proposition 3.1 with

$$\mu(dx) = \delta_{T_i}(dx) - \delta_{T_j}(dx) - \kappa \sum_{k=i}^{j-1} \tau_k \delta_{T_k}(dx),$$

after checking that $\tilde{\eta}_t = \tilde{V}_t - \langle \tilde{X}_t, \tilde{\phi}_t \rangle_{F^*,F} = 0$, $0 \le t \le T$. $\square$

The remaining of this paper is concerned with bond type options, which include caplets on the LIBOR and forward rates.

Bond type options

We consider a bond type option on $P_T(\mu)$ with (non-discounted) payoff $\xi = g(P_T(\mu))$, maturity S, and discount factor B_S^{-1}.

Proposition 3.3 *Letting*

$$\begin{aligned}\tilde{\phi}_t(dx) &= \mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(S)}g(P_T(\mu))\Big|\mathcal{F}_t\right]\delta_S(dx)-\mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(T)}P_T(\mu)g'(P_T(\mu))\Big|\mathcal{F}_t\right]\delta_T(dx)\\ &\quad+\mathbb{E}\left[\tilde{P}_S(S)g'(P_T(\mu))\frac{P_T(x)}{\tilde{P}_t(x)}\Big|\mathcal{F}_t\right]\mu(dx),\end{aligned}$$

and $\tilde{\eta}_t = \tilde{V}_t - \langle\tilde{\phi}_t, \tilde{P}_t\rangle_{F^*,F}$, $0 \le t \le T$, *yields a self-financing hedging portfolio hedging the claim with payoff* $\xi = g(P_T(\mu))$.

Proof. We have

$$D_t g(P_T(\mu)) = g'(P_T(\mu))\int_T^\infty P_T(x)(\zeta_t(x)-\zeta_t(T))\mu(dx).$$

and

$$\begin{aligned}\alpha_t &= \mathbb{E}[D_t\tilde{\xi}\mid\mathcal{F}_t]\\ &= \mathbb{E}[D_t(\tilde{P}_S(S)g(P_T(\mu)))|\mathcal{F}_t]\\ &= \mathbb{E}[g(P_T(\mu))D_t\tilde{P}_S(S)|\mathcal{F}_t]+\mathbb{E}\left[\tilde{P}_S(S)g'(P_T(\mu))\int_T^\infty P_T(x)(\zeta_t(x)-\zeta_t(T))\mu(dx)\Big|\mathcal{F}_t\right]\\ &= \zeta_t(S)\,\mathbb{E}[\tilde{P}_S(S)g(P_T(\mu))|\mathcal{F}_t]+\int_T^\infty(\zeta_t(x)-\zeta_t(T))\,\mathbb{E}\left[g'(P_T(\mu))\tilde{P}_S(S)P_T(x)\Big|\mathcal{F}_t\right]\mu(dx),\end{aligned}$$

and therefore

$$\begin{aligned}\langle\alpha_t, dW_t\rangle_H &= \mathbb{E}[\tilde{P}_S(S)g(P_T(\mu))|\mathcal{F}_t]\zeta_t(S)dW_t\\ &\quad+\int_T^\infty\mathbb{E}[g'(P_T(\mu))\tilde{P}_S(S)P_T(x)\mid\mathcal{F}_t]\mu(dx)(\zeta_t(x)-\zeta_t(T))dW_t\\ &= \mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(S)}g(P_T(\mu))\Big|\mathcal{F}_t\right]d\tilde{P}_t(S)\\ &\quad+\int_T^\infty\mathbb{E}\left[g'(P_T(\mu))\tilde{P}_S(S)\frac{P_T(x)}{\tilde{P}_t(x)}\Big|\mathcal{F}_t\right]d\tilde{P}_t(x)\mu(dx)\\ &\quad-\mathbb{E}\left[g'(P_T(\mu))\tilde{P}_S(S)\frac{P_T(\mu)}{\tilde{P}_t(T)}\Big|\mathcal{F}_t\right]d\tilde{P}_t(T).\end{aligned}$$

From (2.7) this implies that the process $(\tilde{\phi}_t)_{t\in[0,T]}$ in (2.1) is given by

$$\tilde{\phi}_t(dx) = \mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(S)}g(P_T(\mu))\Big|\mathcal{F}_t\right]\delta_S(dx)-\mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(T)}P_T(\mu)g'(P_T(\mu))\Big|\mathcal{F}_t\right]\delta_T(dx)$$

$$+\,\mathbb{E}\left[\tilde{P}_S(S)g'(P_T(\mu))\frac{P_T(x)}{\tilde{P}_t(x)}\Big|\mathcal{F}_t\right]\mu(dx),$$

which gives a self-financing hedging portfolio consisting of bonds with maturities S and T, after applying Proposition 2.2 with $(X_t)_{t\in\mathbb{R}_+} = (P_t)_{t\in\mathbb{R}_+}$. □

Bond call options

We consider a bond call option on $P_T(S)$, $S > T$, with payoff

$$\xi = (P_T(S) - \kappa)^+$$

and maturity T, and priced at time $t \in [0, T]$ as

$$\begin{aligned}
B_t\,\mathbb{E}[\tilde{\xi}|\mathcal{F}_t] &= B_t\,\mathbb{E}\left[e^{-\int_0^T r_s ds}(P_T(S)-\kappa)^+\Big|\mathcal{F}_t\right]\\
&= B_t\,\mathbb{E}\left[\tilde{P}_T(T)(P_T(S)-\kappa)^+ \mid \mathcal{F}_t\right]\\
&= P_t(S)\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}+\frac{\vartheta_{t,T}}{2}\right)-\kappa P_t(T)\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}-\frac{\vartheta_{t,T}}{2}\right),
\end{aligned}$$

where

$$\vartheta_{t,T}^2 = \int_t^T (\zeta_u(S)-\zeta_u(T))^2 du, \qquad 0 \le t \le T.$$

We have the following corollary of Proposition 3.3.

Corollary 3.4 *Letting*

$$\tilde{\phi}_t(dx) = \Phi\left(\frac{\vartheta_{t,T}}{2}+\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}\right)\delta_S(dx)-\kappa\Phi\left(-\frac{\vartheta_{t,T}}{2}+\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}\right)\delta_T(dx)$$

and $\tilde{\eta}_t = 0$, $0 \le t \le T$, *yields a self-financing hedging portfolio for the bond option on* $P_T(S)$, *consisting of bonds with maturities* S *and* T.

Proof. We apply Proposition 3.3 with $g(x) = (x-\kappa)^+$, $\mu(dx) = \delta_S(dx)$, and the discount factor B_T^{-1}. We find

$$\begin{aligned}
\tilde{\phi}_t(dx) &= -\kappa\,\mathbb{E}\left[\frac{\tilde{P}_T(T)}{\tilde{P}_t(T)}\mathbf{1}_{\{P_T(S)>\kappa\}}\Big|\mathcal{F}_t\right]\delta_T(dx)+\mathbb{E}\left[\tilde{P}_T(T)\mathbf{1}_{\{P_T(S)>\kappa\}}\frac{P_T(S)}{\tilde{P}_t(S)}\Big|\mathcal{F}_t\right]\delta_S(dx)\\
&= -\kappa\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}-\frac{\vartheta_{t,T}}{2}\right)\delta_T(dx)+\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(S)}{\kappa P_t(T)}+\frac{\vartheta_{t,T}}{2}\right)\delta_S(dx).
\end{aligned}$$

□

Caplets on the LIBOR rate

Next, we consider a caplet with payoff

$$\xi = (S-T)(L(T,T,S)-\kappa)^+ = (P_T(S)^{-1} - (1+\kappa(S-T)))^+, \qquad S > T, \quad (3.5)$$

and maturity S on the LIBOR rate

$$L(t,T,S) = \frac{P_t(T) - P_t(S)}{(S-T)P_t(S)}$$

Its price at time $t \in [0,T]$ is given by

$$\begin{aligned} B_t\,\mathbb{E}[\tilde{\xi} \mid \mathcal{F}_t] &= (S-T)B_t\,\mathbb{E}\left[e^{-\int_0^S r_s ds}(L(T,T,S)-\kappa)^+ \Big| \mathcal{F}_t\right] \\ &= B_t\,\mathbb{E}\left[\tilde{P}_S(S)(P_T(S)^{-1} - (1+\kappa(S-T)))^+ \Big| \mathcal{F}_t\right] \\ &= P_t(T)\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)} + \frac{\vartheta_{t,T}}{2}\right) \\ &\quad -(1+\kappa(S-T))P_t(T)\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)} - \frac{\vartheta_{t,T}}{2}\right). \end{aligned}$$

We have the following corollary of Proposition 3.3.

Corollary 3.5 *Letting*

$$\begin{aligned} \tilde{\phi}_t(dx) &= \Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)} + \frac{\vartheta_{t,T}}{2}\right)\delta_T(dx) \\ &\quad -(1+\kappa(S-T))\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)} - \frac{\vartheta_{t,T}}{2}\right)\delta_S(dx), \end{aligned}$$

and $\tilde{\eta}_t = 0$, $0 \le t \le T$, *yields a self-financing hedging portfolio consisting of bonds with maturities* S *and* T, *that hedges the claim with payoff* (3.5).

Proof. Applying Proposition 3.3 with $g(x) = (1/x - (1+\kappa(S-T)))^+$, $\mu(dx) = \delta_S(dx)$, and the discount factor B_S^{-1}, we get

$$\begin{aligned} \tilde{\phi}_t(dx) &= \mathbb{E}\left[(P_T(S))^{-1}\mathbf{1}_{\{P_T(S)^{-1} > 1+\kappa(S-T)\}}\frac{\tilde{P}_S(S)}{\tilde{P}_t(T)}\Big|\mathcal{F}_t\right]\delta_T(dx) \\ &\quad -(1+\kappa(S-T))\,\mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(S)}\mathbf{1}_{\{P_T(S)^{-1} > [1+\kappa(S-T)]\}}\Big|\mathcal{F}_t\right]\delta_S(dx) \end{aligned}$$

$$
\begin{aligned}
= \quad & \Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)}+\frac{\vartheta_{t,T}}{2}\right)\delta_T(dx)\\
& -(1+\kappa(S-T))\Phi\left(\frac{1}{\vartheta_{t,T}}\log\frac{P_t(T)}{(1+\kappa(S-T))P_t(S)}-\frac{\vartheta_{t,T}}{2}\right)\delta_S(dx).
\end{aligned}
$$

□

Caplets on the forward rate

Finally, we consider a caplet with payoff

$$
\xi = (S-T)(f(T,T,S)-\kappa)^+, \qquad S > T, \tag{3.6}
$$

and maturity S on the forward rate

$$
f(t,T,S) = -\frac{\log P_t(S) - \log P_t(T)}{S-T}
$$

Its price at time $t \in [0,T]$ is given by

$$
\begin{aligned}
& B_t\,\mathbb{E}[\tilde{\xi} \mid \mathcal{F}_t] = B_t\,\mathbb{E}\left[e^{-\int_0^S r_s ds}(S-T)(f(T,T,S)-\kappa)^+ \Big| \mathcal{F}_t\right]\\
&= \quad B_t\,\mathbb{E}\left[\tilde{P}_S(S)(-\log P_T(S)-\kappa(S-T))^+ \Big| \mathcal{F}_t\right]\\
&= \quad P_t(S)\frac{\vartheta_{t,T}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\vartheta_{t,T}^2}\left(\frac{\vartheta_{t,T}^2}{2}+\kappa(S-T)+\log\frac{P_t(S)}{P_t(T)}\right)^2\right)\\
&\quad -P_t(S)\left(\kappa(S-T)+\frac{\vartheta_{t,T}^2}{2}+\log\frac{P_t(S)}{P_t(T)}\right)\Phi\left(-\frac{1}{\vartheta_{t,T}}\left(\kappa(S-T)+\log\frac{P_t(S)}{P_t(T)}\right)-\frac{\vartheta_{t,T}}{2}\right).
\end{aligned}
$$

We have the following corollary of Proposition 3.3.

Corollary 3.6 *Letting*

$$
\begin{aligned}
\tilde{\phi}_t(dx) = {} & \frac{\vartheta_{t,T}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\vartheta_{t,T}^2}\left(\frac{\vartheta_{t,T}^2}{2}+\kappa(S-T)+\log\frac{P_t(S)}{P_t(T)}\right)^2\right)\delta_S(dx)\\
& -\left(\kappa(S-T)+\frac{\vartheta_{t,T}^2}{2}+1+\log\frac{P_t(S)}{P_t(T)}\right)\Phi\left(-\frac{1}{\vartheta_{t,T}}\left(\kappa(S-T)+\log\frac{P_t(S)}{P_t(T)}\right)-\frac{\vartheta_{t,T}}{2}\right)\delta_S(dx)\\
& +\frac{\tilde{P}_t(S)}{\tilde{P}_t(T)}\Phi\left(-\frac{1}{\vartheta_{t,T}}\left(\kappa(S-T)+\log\frac{P_t(S)}{P_t(T)}\right)-\frac{\vartheta_{t,T}}{2}\right)\delta_T(dx),
\end{aligned}
$$

and $\tilde{\eta}_t = 0$, $0 \le t \le T$, *yields a self-financing hedging portfolio consisting of bonds with maturities* S *and* T, *that hedges the claim with payoff* (3.6).

Proof. Applying Proposition 3.3 with $g(x) = (-\kappa(S-T) - \log x)^+$, $\mu(dx) = \delta_S(dx)$, and the discount factor B_S^{-1}, we have

$$\begin{aligned}
\tilde{\phi}_t(dx) = {} & \mathbb{E}\left[(-\log P_T(S) - \kappa(S-T))^+ \frac{\tilde{P}_S(S)}{\tilde{P}_t(S)} \Big| \mathcal{F}_t \right] \delta_S(dx) \\
& - \mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(S)} \mathbf{1}_{\{-\log P_T(S) > \kappa(S-T)\}} \Big| \mathcal{F}_t \right] \delta_S(dx) \\
& + \mathbb{E}\left[\frac{\tilde{P}_S(S)}{\tilde{P}_t(T)} \mathbf{1}_{\{-\log P_T(S) > \kappa(S-T)\}} \Big| \mathcal{F}_t \right] \delta_T(dx) \\
= {} & \frac{\vartheta_{t,T}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\vartheta_{t,T}^2} \left(\log \frac{P_t(S)}{P_t(T)} + \frac{\vartheta_{t,T}^2}{2} + \kappa(S-T) \right)^2 \right) \delta_S(dx) \\
& - \left(\log \frac{P_t(S)}{P_t(T)} + \kappa(S-T) + \frac{\vartheta_{t,T}^2}{2} + 1 \right) \Phi\left(-\frac{1}{\vartheta_{t,T}} \left(\log \frac{P_t(S)}{P_t(T)} + \kappa(S-T) \right) - \frac{\vartheta_{t,T}}{2} \right) \delta_S(dx) \\
& + \frac{\tilde{P}_t(S)}{\tilde{P}_t(T)} \Phi\left(-\frac{1}{\vartheta_{t,T}} \left(\log \frac{P_t(S)}{P_t(T)} + \kappa(S-T) \right) - \frac{\vartheta_{t,T}}{2} \right) \delta_T(dx).
\end{aligned}$$

□

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