Combinatorics of Poisson stochastic integrals with random integrands

Nicolas Privault

November 24, 2016

Abstract We present a self-contained account of recent results on moment identities for Poisson stochastic integrals with random integrands, based on the use of functional transforms on the Poisson space. This presentation relies on elementary combinatorics based on the Faà di Bruno formula, partitions and polynomials, which are used together with multiple stochastic integrals, finite difference operators and integration by parts.

Nicolas Privault
Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, e-mail: nprivault@ntu.edu.sg

1 Introduction

The cumulants $(\kappa_n^X)_{n\geq 1}$ of a random variable X have been defined in [33] and were originally called the "semi-invariants" of X due to the property $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$, $n \geq 1$, when X and Y are independent random variables. Precisely, given the moment generating function

$$\mathbb{E}[e^{tX}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[X^n], \tag{1}$$

of a random variable X, where t is in a neighborhood of 0, the *cumulants* of X are defined to be the coefficients $(\kappa_n^X)_{n\geq 1}$ appearing in the series expansion of the logarithmic moment generating function of X, i.e. we have

$$\log(\mathbb{E}[e^{tX}]) = \sum_{n=1}^{\infty} \kappa_n^X \frac{t^n}{n!}, \tag{2}$$

where t is in a neighborhood of 0. In relation with the Faà di Bruno formula, (1) and (2) yield the classical identity

$$\mathbb{E}[X^n] = \sum_{a=0}^{n} \sum_{P_1 \cup \cdots \cup P_a = \{1,\ldots,n\}} \kappa_{|P_1|}^X \cdots \kappa_{|P_a|}^X, \qquad n \in \mathbb{N}, \tag{3}$$

which links the moments $(\mathbb{E}[X^n])_{n\geq 1}$ of a random variable X with its cumulants $(\kappa_n^X)_{n\geq 1}$, cf. e.g. Theorem 1 of [16], and also [15] or § 2.4 and Relation (2.4.4) page 27 of [17].

The summation in (3) runs over the partitions $P_1, \ldots, P_a$ of the set $\{1, \ldots, n\}$, i.e. each sequence $P_1, \ldots, P_a$ is a family of nonempty and nonoverlapping subsets of $\{1, \ldots, n\}$ whose union is $\{1, \ldots, n\}$, and $|P_i|$ denotes the cardinal of P_i, cf. § 2.2 of [21] for a complete review of the notion of set partition. For example when X is centered Gaussian we have $\kappa_n^X = 0$, $n \neq 2$, and (3) reads as Wick's theorem for the computation of Gaussian moments of X counting the pair partitions of $\{1, \ldots, n\}$, cf. [10].

In this survey we derive moment identities for Poisson stochastic integrals with random integrands, cf. Theorem 1 below, with application to invariance of Poisson random measures. Our method relies on the tools from combinatorics appearing in [3], i.e. the Faà di Bruno formula and related Stirling numbers, partitions and polynomials, in relation with Poisson random measures, integration by parts on Poisson probability spaces and multiple stochastic integrals. Such moment identities have been recently extended to point processes with Papangelou intensities in [6] for the moments and in [5] for the factorial moments of such point processes.

The outline of this survey is as follows. Section 2 starts with preliminaries on combinatorics and the Faà di Bruno formula, providing the needed combinatorial background to rederive the classical identity (3). Then in Section 3 we introduce the Poisson random measures and integration by parts on Poisson probability spaces, along with the tools of $\mathscr{S}$ and $\mathscr{U}$ transforms in view of applications to moment identities. Single and joint moment identities themselves are then detailed in Section 4, in relation with set-indexed adaptedness and invariance of Poisson measures.

Our computation of Poisson moments will proceed from the Bismut-Girsanov approach to the stochastic calculus of variations (Malliavin calculus), via the use of functional $\mathscr{S}$ and $\mathscr{U}$-transforms, cf. Sections 3.3 and 3.4. As an illustration we start with some informal remarks on that approach in the framework of the Malliavin calculus on the Wiener space. Given $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $F(\omega)$ is a random functional of the Brownian path $B_t(\omega)=\omega(t)$, $t\in\mathbb{R}_+$, we start from the Girsanov identity

$$\mathbb{E}[F\xi(f)] = \mathbb{E}\left[F\left(\omega(\cdot)+\int_0^{\cdot} f(s)\mathrm{d}s\right)\right], \tag{4}$$

where $f\in L^2(\mathbb{R}_+)$ and $\xi(f)=X_\infty$ is the terminal value of the (martingale) solution of the stochastic differential equation

$$\mathrm{d}X_t = f(t)X_t\mathrm{d}B_t, \qquad t\in\mathbb{R}_+. \tag{5}$$

By iterations the solution of (5) can be written as the series

$$\begin{aligned}\xi(f) &= X_\infty\\ &= 1+\int_0^\infty f(t)X_t\mathrm{d}B_t\\ &= 1+\sum_{n=1}^\infty\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f(t_1)\cdots f(t_n)\mathrm{d}B_{t_1}\cdots\mathrm{d}B_{t_n}\\ &= 1+\sum_{n=1}^\infty\frac{1}{n!}I_n(f^{\otimes n}),\end{aligned}$$

of multiple stochastic integrals

$$I_n(f^{\otimes n}) = n!\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f(t_1)\cdots f(t_n)\mathrm{d}B_{t_1}\cdots\mathrm{d}B_{t_n}, \qquad n\geq 1.$$

We can then rewrite (4) as

$$\begin{aligned}\mathbb{E}[F\xi(f)] &= \mathbb{E}[F]+\sum_{n=1}^\infty\frac{1}{n!}\mathbb{E}[FI_n(f^{\otimes n})] \qquad (6)\\ &= \mathbb{E}\left[F\left(\omega(\cdot)+\int_0^{\cdot} f(s)\mathrm{d}s\right)\right]\\ &= \mathbb{E}[F]+\sum_{n=1}^\infty\frac{1}{n!}\frac{\partial^n}{\partial\varepsilon^n}\mathbb{E}\left[F\left(\omega(\cdot)+\varepsilon\int_0^{\cdot} f(s)\mathrm{d}s\right)\right]_{\varepsilon=0}.\end{aligned}$$

By successive differentiations this yields the iterated integration by parts formula

$$\mathbb{E}[FI_n(f^{\otimes n})] = \mathbb{E}[\nabla_f^n F], \tag{7}$$

where ∇_f is the gradient operator defined by

$$\nabla_f F := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F\left(\omega(\cdot) + \varepsilon \int_0^{\cdot} f(s) \mathrm{d}s \right) - F(\omega(\cdot)) \right).$$

On the other hand, on the Wiener space the above Girsanov shift acts on the paths $(\omega(t))_{t \in \mathbb{R}_+}$ of the underlying Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ as

$$\omega(\cdot) \longmapsto \omega(\cdot) + \varepsilon \int_0^{\cdot} f(s) \mathrm{d}s,$$

which yields

$$\mathbb{E}[\nabla_f^n F] = \mathbb{E}\left[\int_0^\infty \cdots \int_0^\infty f(s_1) \cdots f(s_n) D_{s_1} \cdots D_{s_n} F \mathrm{d}s_1 \cdots \mathrm{d}s_n \right], \tag{8}$$

where $D_s F$ is the Malliavin gradient which satisfies

$$\nabla_f F = \int_0^\infty D_s F f(s) \mathrm{d}s,$$

hence by (7) and (8) we obtain the iterated integration by parts identity

$$\mathbb{E}\left[I_k(f^{\otimes k}) F \right] = \mathbb{E}\left[\int_0^\infty \cdots \int_0^\infty f(s_1) \cdots f(s_k) D_{s_1} \cdots D_{s_k} F \mathrm{d}s_1 \cdots \mathrm{d}s_k \right], \qquad k \geq 1, \tag{9}$$

which will be the basis for our computation of moments. On the Wiener space the operator D also satisfies the identity

$$D_t I_n(g^{\otimes n}) = n g(t) I_{n-1}(g^{\otimes (n-1)}), \qquad t \in \mathbb{R}_+, \tag{10}$$

which can be used to recover (9) as the Stroock [32] formula, cf. Corollary 1 below for the Poisson case.

However, when carrying over this approach to the probability space of a Poisson random measure it turns out that there is no differential operator ∇_f that can satisfy both relations (8) and (10) above. In the sequel we will develop the above approach on the Poisson space via the use of finite difference operators.

2 Combinatorics

In this section we provide the necessary combinatorial background for the derivation of cumulant-type moment identities. We refer the reader to [21] and references therein, cf. also [22], for additional background on combinatorial probability and for the relationships between the moments and cumulants of random variables.

2.1 *Faà di Bruno formula and Bell polynomials*

Faà di Bruno formula

The Faà di Bruno formula plays a fundamental role in the combinatorics of moments, cumulants, and factorial moments. Namely, instead of the multinomial identity

$$\left(\sum_{l=1}^{n} x_l\right)^k = k! \sum_{\substack{d_1+\cdots+d_n=k\\ d_1\geq 0,\ldots,d_n\geq 0}} \frac{x_1^{d_1}}{d_1!}\cdots\frac{x_n^{d_n}}{d_n!}, \tag{11}$$

we will use the combinatorial identity

$$\left(\sum_{n=1}^{\infty} x_n\right)^k = \sum_{n=k}^{\infty} \sum_{\substack{d_1+\cdots+d_k=n\\ d_1\geq 1,\ldots,d_k\geq 1}} x_{d_1}\cdots x_{d_k}, \tag{12}$$

or

$$\left(\sum_{n=1}^{\infty} x_{1,n}\right)\cdots\left(\sum_{n=1}^{\infty} x_{k,n}\right) = \sum_{n=k}^{\infty} \sum_{\substack{d_1+\cdots+d_k=n\\ d_1\geq 1,\ldots,d_k\geq 1}} x_{1,d_1}\cdots x_{k,d_k}. \tag{13}$$

The above identity (12) is equivalent to the Faà di Bruno formula, i.e. given $g(x)$ and $f(y)$ two functions given by their series expansions

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}$$

with $g(0) = 0$ and

$$f(y) = \sum_{k=0}^{\infty} a_k \frac{y^k}{k!},$$

the series expansion of $f(g(x))$ is given by

$$f(g(x)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\sum_{n=1}^{\infty} b_n \frac{x^n}{n!}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k!} \sum_{n=k}^{\infty} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} b_{d_1}\cdots b_{d_k} \frac{x^{d_1}}{d_1!}\cdots\frac{x^{d_k}}{d_k!}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{a_k}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{b_{d_1}}{d_1!}\cdots\frac{b_{d_k}}{d_k!}. \qquad (14)$$

In the sequel we will often rewrite (12) using sums over partitions $P_1^n,\ldots,P_k^n$ of $\{1,\ldots,n\}$ into subsets of cardinals $|P_1^n|,\ldots,|P_k^n|$, as

$$\frac{n!}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{b_{d_1}}{d_1!}\cdots\frac{b_{d_k}}{d_k!} = \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} b_{|P_1^n|}\cdots b_{|P_k^n|}.$$

Bell polynomials

The Faà di Bruno formula (14) can be rewritten as

$$f(g(x)) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} a_k B_{n,k}(b_1,\ldots,b_{n-k+1}), \qquad (15)$$

where $B_{n,k}(b_1,\ldots,b_{n-k+1})$ is the Bell polynomial of order (n,k) defined by

$$B_{n,k}(b_1,\ldots,b_{n-k+1}) := \frac{1}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{n!}{d_1!\cdots d_k!} b_{d_1}\cdots b_{d_k}$$

$$= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} b_{|P_1^n|}\cdots b_{|P_k^n|}$$

$$= n! \sum_{\substack{r_1+2r_2+\cdots+(n-k+1)r_{n-k+1}=n \\ r_1+r_2+\cdots+r_{n-k+1}=k \\ r_1\geq 0,\ldots,r_{n-k+1}\geq 0}} \prod_{l=1}^{n-k+1} \left(\frac{1}{r_l!} \left(\frac{b_l}{l!} \right)^{r_l} \right)$$

$$= \frac{n!}{k!} \sum_{\substack{r_1+2r_2+\cdots+(n-k+1)r_{n-k+1}=n \\ r_1+r_2+\cdots+r_{n-k+1}=k \\ r_1\geq 0,\ldots,r_{n-k+1}\geq 0}} \frac{k!}{r_1!\cdots r_{n-k+1}!} \left(\frac{b_1}{1!} \right)^{r_1} \cdots \left(\frac{b_{n-k+1}}{(n-k+1)!} \right)^{r_{n-k+1}},$$

cf. e.g. Definition 2.4.1 of [21], with $B_{n,0}(b_1,\ldots,b_n)=0$, $n\geq 1$, and $B_{0,0}=1$. In particular when $f(y)=e^y$ we have $a_k=1$, $k\geq 0$, and (15) rewrites as

$$\exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(b_1,\ldots,b_n). \qquad (16)$$

where

$$
\begin{aligned}
A_n(b_1,\ldots,b_n) &= \sum_{k=0}^{n} B_{n,k}(b_1,\ldots,b_{n-k+1}) \qquad (17)\\
&= \sum_{k=0}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} b_{|P_1^n|}\cdots b_{|P_k^n|}\\
&= n!\sum_{k=0}^{n} \sum_{\substack{r_1+2r_2+\cdots+(n-k+1)r_{n-k+1}=n\\ r_1+r_2+\cdots+r_{n-k+1}=k\\ r_1\geq 0,\ldots,r_{n-k+1}\geq 0}} \prod_{l=1}^{n-k+1}\left(\frac{1}{r_l!}\left(\frac{b_l}{l!}\right)^{r_l}\right)\\
&= n!\sum_{\substack{r_1+2r_2+\cdots+nr_n=n\\ r_1\geq 0,\ldots,r_n\geq 0}} \prod_{l=1}^{n}\left(\frac{1}{r_l!}\left(\frac{b_l}{l!}\right)^{r_l}\right)
\end{aligned}
$$

is the (complete) Bell polynomial of degree n. Relation (16) is a common formulation of the Faà di Bruno formula and it will be used in the proof of Proposition 5 below on the $\mathscr{U}$-transform on the Poisson space.

2.2 Stirling inversion

The Stirling numbers will be used for the construction of multiple stochastic integrals and their relations to the Charlier polynomials in Section 3.2. Let

$$
\begin{aligned}
S(n,k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n\\
&= \frac{1}{k!}\sum_{\substack{d_1+\cdots+d_k=n\\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{n!}{d_1!\cdots d_k!} \qquad (18)\\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbf{1},
\end{aligned}
$$

denote the Stirling number of the second kind with $S(n,0)=0$, $n\geq 1$, and $S(0,0)=1$, cf. page 824 of [1], i.e. $S(n,k)$ is the number of partitions of a set of n objects into k nonempty subsets, cf. also Relation (3) page 59 of [3], with

$$
B_{n,k}(x,\ldots,x) = x^k S(n,k), \qquad 0\leq k\leq n.
$$

Let also

$$
s(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(k-i)^n
$$

denote the (signed) Stirling number of the first kind, cf. e.g. page 824 of [1], i.e. $(-1)^{n-k}s(n,k)$ is the number of permutations of n elements which contain exactly k

permutation cycles.

The following Stirling transform Lemma 1, cf. e.g. Relation (3) page 59 of [3], also relies on the Faà di Bruno formula applied to

$$f(t)=\frac{t^k}{k!} \quad \text{and} \quad a_n=\mathbf{1}_{\{n=k\}}$$

and

$$g(t)=\log(1+t) \quad \text{and} \quad b_k=\mathbf{1}_{\{n=k\}}.$$

Lemma 1. *Assume that the function $f(t)$ has the series expansion*

$$f(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}a_n, \qquad t\in\mathbb{R}.$$

Then we have

$$f(e^t-1)=\sum_{k=0}^{\infty}\frac{t^k}{k!}c_k, \qquad t\in\mathbb{R},$$

with

$$c_n=\sum_{k=0}^{n}a_kS(n,k),$$

and the inversion formula

$$a_n=\sum_{k=0}^{n}c_ks(n,k), \qquad n\in\mathbb{N}.$$

Proof. Applying the Faà di Bruno identity (14) to $g(t)=e^t-1$ and using (18) we have

$$\begin{aligned} f(e^t-1) &= \sum_{k=0}^{\infty}a_k\frac{(e^t-1)^k}{k!}=\sum_{k=0}^{\infty}a_k\sum_{n=k}^{\infty}\frac{t^n}{n!}S(n,k) \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}a_kS(n,k)=\sum_{n=0}^{\infty}\frac{t^n}{n!}c_n, \qquad t\in\mathbb{R}, \end{aligned}$$

with

$$c_n=\sum_{k=0}^{n}a_kS(n,k).$$

Conversely we have

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty}\frac{c_k}{k!}(\log(1+t))^k=\sum_{k=0}^{\infty}c_k\sum_{n=k}^{\infty}\frac{t^n}{n!}s(n,k) \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}c_ks(n,k)=\sum_{n=0}^{\infty}\frac{t^n}{n!}a_n, \qquad t\in\mathbb{R}, \end{aligned}$$

with

$$a_n = \sum_{k=0}^{n} c_k s(n,k).$$

□

As a consequence of Lemma 1, the Stirling transform

$$a_n = \sum_{k=0}^{n} c_k s(n,k), \qquad n \in \mathbb{N},$$

can be inverted as

$$c_n = \sum_{k=0}^{n} a_k S(n,k), \qquad n \in \mathbb{N},$$

i.e. we have the inversion formula

$$\sum_{k=l}^{n} S(n,k)s(k,l) = \mathbf{1}_{\{n=l\}}, \qquad n,l \in \mathbb{N}, \tag{19}$$

for Stirling numbers, cf. e.g. page 825 of [1]. As particular cases of the Stirling transform of Lemma 1 we find that

$$\begin{aligned} \frac{1}{k!}(e^{\lambda}-1)^k &= \frac{1}{k!}\left(\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\right)^k = \frac{1}{k!}\sum_{n=k}^{\infty}\frac{\lambda^n}{n!}\sum_{\substack{d_1+\cdots+d_k=n\\ d_1\geq 1,\ldots,d_k\geq 1}}\frac{n!}{d_1!\cdots d_k!} \\ &= \sum_{n=k}^{\infty}\frac{\lambda^n}{n!}B_{n,k}(1,\ldots,1) = \sum_{n=k}^{\infty}\frac{\lambda^n}{n!}S(n,k), \qquad k \geq 1. \end{aligned} \tag{20}$$

We also have

$$\begin{aligned} \frac{1}{k!}(\log(1+t))^k &= \frac{(-1)^k}{k!}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}t^n\right)^k \\ &= (-1)^k\sum_{n=k}^{\infty}\frac{t^n}{n!}B_{n,k}\left(-1,\frac{1}{2},-\frac{1}{3},\ldots,\frac{(-1)^{n-k+1}}{n-k+1}\right) \\ &= \frac{(-1)^k}{k!}\sum_{n=k}^{\infty}(-1)^n\frac{t^n}{n!}\sum_{\substack{d_1+\cdots+d_k=n\\ d_1\geq 1,\ldots,d_k\geq 1}}\frac{n!}{d_1\cdots d_k} \\ &= \sum_{n=k}^{\infty}\frac{t^n}{n!}s(n,k), \qquad k \geq 1, \end{aligned}$$

which shows the relation

$$s(n,k) = \frac{n!}{k!}\sum_{d_1+\cdots+d_k=n}\frac{(-1)^{n-k}}{d_1\cdots d_k}. \tag{21}$$

In particular, taking $c_k = x^k$ and letting $a_n = x_{(n)}$ defined by the falling factorial

$$x_{(n)} := x(x-1)\cdots(x-n+1), \qquad k,n \geq 0,$$

i.e.

$$f(e^t-1) = e^{xt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} x^k,$$

and by Lemma 1 we get

$$f(t) = (1+x)^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} x_{(n)}, \tag{22}$$

which will be used in Lemma 2 below on the Charlier polynomials.

By Stirling inversion we also find the expansion of the falling factorial

$$x_{(n)} = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k) x^k \tag{23}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)\, x(x-1)\cdots(x-k+1),$$

cf. e.g. [9] or page 72 of [8].

2.3 *Charlier and Touchard polynomials*

Charlier polynomials

The Charlier polynomials $C_n(x,\lambda)$ or order $n \in \mathbb{N}$ with parameter $\lambda > 0$ are essential in the construction of multiple Poisson stochastic integrals in Section 3.2. They can be defined through their generating function

$$\psi_\lambda(x,t) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(x,t) = e^{-\lambda t}(1+\lambda)^x, \qquad x,t \in \mathbb{R}_+, \tag{24}$$

$\lambda \in (-1,1)$, cf. e.g. § 4.3.3 of [30].

Lemma 2. *We have*

$$C_n(x,\lambda) = \sum_{k=0}^{n} x^k \sum_{l=0}^{n} \binom{n}{l} (-\lambda)^{n-l} s(l,k), \qquad x,\lambda \in \mathbb{R}. \tag{25}$$

Proof. We check that defining $C_n(x,t)$ by (25) yields

$$
\begin{aligned}
\psi_\lambda(x,t) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(x,t) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} x^k \sum_{l=k}^{n} \binom{n}{l} (-t)^{n-l} s(l,k) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{l=0}^{n} \binom{n}{l} (-t)^{n-l} \sum_{k=0}^{l} x^k s(l,k) \\
&= \sum_{l=0}^{\infty} x_{(l)} \sum_{n=l}^{\infty} \frac{\lambda^n}{n!} \frac{n!}{(n-l)!l!} (-t)^{n-l} \\
&= \sum_{l=0}^{\infty} x_{(l)} \frac{\lambda^l}{l!} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (-t)^n \\
&= e^{-\lambda t} \sum_{l=0}^{\infty} x_{(l)} \frac{\lambda^l}{l!} \\
&= e^{-\lambda t} (1+\lambda)^x,
\end{aligned}
$$

$\lambda, t > 0$, $x \in \mathbb{N}$, where we applied (22) and (23). □

As a consequence of Lemma 2 and (23), the Charlier polynomial $C_n(x,\lambda)$ can be rewritten in terms of the falling factorial $x_{(n)}$ as

$$
C_n(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} (-\lambda)^{n-l} \sum_{k=0}^{l} x^k s(l,k) = \sum_{l=0}^{n} \binom{n}{l} (-\lambda)^{n-l} x_{(l)}, \quad x,\lambda \in \mathbb{R}. \tag{26}
$$

Lemma 3. *We have the orthogonality relation*

$$
e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} C_n(k,\lambda) C_m(k,\lambda) = n! \lambda^n \mathbf{1}_{\{n=m\}}. \tag{27}
$$

Proof. We have

$$
\begin{aligned}
e^{\lambda ab} &= e^{-\lambda(1+a+b)} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (1+a)^k (1+b)^k \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \psi_a(k,\lambda) \psi_b(k,\lambda) \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{n!} \frac{b^m}{m!} C_n(k,\lambda) C_m(k,\lambda),
\end{aligned}
$$

which shows that

$$
\sum_{p=0}^{\infty} \lambda^p \frac{(ab)^p}{p!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^n}{n!} \frac{b^m}{m!} C_n(k,\lambda) C_m(k,\lambda)
$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(ab)^n}{(n!)^2} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (C_n(k,\lambda))^2,$$

with

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} C_n(k,\lambda) C_m(k,\lambda) = 0$$

for $n \neq m$, and

$$n!\lambda^n = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (C_n(k,\lambda))^2,$$

for $n = m$. □

Touchard polynomials

The Touchard polynomials can be used to express the moments of a Poisson random variable as a function of its intensity parameter. They can be defined by their generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n(\lambda), \qquad t \in \mathbb{R},$$

and from (16) or (20) they satisfy

$$\begin{aligned} T_n(\lambda) :&= A_n(\lambda, \ldots, \lambda) = \sum_{k=0}^{n} B_{n,k}(\lambda, \ldots, \lambda) \\ &= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \lambda^k = \sum_{k=0}^{n} \lambda^k S(n,k), \end{aligned} \tag{28}$$

cf. e.g. Proposition 2 of [4] or § 3.1 of [20]. Relation (28) above will be used in the proof of the combinatorial Lemma 7 below.

2.4 Moments and cumulants of random variables

Given the identity (1) defining the moment generating function of X, we can write

$$\mathbb{E}[e^{tX}] = 1 + t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] + o(t^2),$$

which allows us to rewrite the cumulant generating function (2) as

$$\log(\mathbb{E}[e^{tX}]) = \log\left(1 + t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] + o(t^2)\right)$$

$$
\begin{aligned}
&= t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] - \frac{1}{2}\left(t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2]\right)^2 + o(t^2) \\
&= t\mathbb{E}[X] + \frac{t^2}{2}\mathbb{E}[X^2] - \frac{t^2}{2}(\mathbb{E}[X])^2 + o(t^2) \\
&= t\mathbb{E}[X] + \frac{t^2}{2}\mathrm{var}[X] + o(t^2),
\end{aligned}
$$

hence $\kappa_1^X = \mathbb{E}[X]$ and $\kappa_2^X = \mathrm{var}[X]$. More generally, as a consequence of (16), the moment generating function of X expands using the complete Bell polynomials $A_n(b_1, \ldots, b_n)$ of (17) as

$$
\begin{aligned}
\mathbb{E}[e^{tX}] &= \exp(\log(\mathbb{E}[e^{tX}])) \\
&= \exp\left(\sum_{n=1}^{\infty} \kappa_n^X \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} A_n(\kappa_1^X, \ldots, \kappa_n^X),
\end{aligned}
$$

which shows by comparison with (1) that

$$
\begin{aligned}
\mathbb{E}[X^n] &= A_n(\kappa_1^X, \kappa_2^X, \ldots, \kappa_n^X) \\
&= \sum_{k=0}^{n} \frac{n!}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{\kappa_{d_1}^X}{d_1!} \cdots \frac{\kappa_{d_k}^X}{d_k!} \\
&= \sum_{k=0}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \kappa_{|P_1^n|}^X \cdots \kappa_{|P_k^n|}^X, \qquad (29)
\end{aligned}
$$

and allows us to recover (3).

The identity (29) can also be recovered from the Thiele [33] recursion formula

$$
\mathbb{E}[X^n] = \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-l-1)!} \kappa_{n-l}^X \mathbb{E}[X^l] = \sum_{l=1}^{n} \frac{(n-1)!}{(n-l)!(l-1)!} \kappa_l^X \mathbb{E}[X^{n-l}] \qquad (30)
$$

between moments and cumulants of random variables, cf. e.g. § 1.3.2 of [22]. Indeed, assuming at the order $n \geq 1$ that

$$
\mathbb{E}[X^n] = \sum_{a=0}^{n} \frac{n!}{a!} \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{\kappa_{l_1}^X}{l_1!} \cdots \frac{\kappa_{l_a}^X}{l_a!} = \sum_{a=0}^{n} \sum_{P_1^n\cup\cdots\cup P_a^n=\{1,\ldots,n\}} \kappa_{|P_1^n|}^X \cdots \kappa_{|P_a^n|}^X,
$$

and using (30), we have, at the order $n+1$,

$$\begin{aligned}
\mathbb{E}[X^{n+1}] &= \sum_{k=1}^{n+1} \binom{n}{k-1} \kappa_k^X \mathbb{E}[X^{n+1-k}] \\
&= \sum_{k=1}^{n+1} \frac{n!}{(k-1)!} \kappa_k^X \sum_{a=0}^{n+1-k} \frac{1}{a!} \sum_{\substack{l_1+\cdots+l_a=n+1-k \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{\kappa_{l_1}^X}{l_1!} \cdots \frac{\kappa_{l_a}^X}{l_a!} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} \kappa_k^X \sum_{a=0}^{n+1-k} \sum_{P_1^{n+1-k}\cup\cdots\cup P_a^{n+1-k}=\{1,\ldots,n+1-k\}} \kappa_{|P_1^{n+1-k}|}^X \cdots \kappa_{|P_a^{n+1-k}|}^X \\
&= \sum_{a=0}^{n} \sum_{k=1}^{n+1-a} \binom{n}{k-1} \kappa_k^X \sum_{P_1^{n+1-k}\cup\cdots\cup P_a^{n+1-k}=\{1,\ldots,n+1-k\}} \kappa_{|P_1^{n+1-k}|}^X \cdots \kappa_{|P_a^{n+1-k}|}^X \\
&= \sum_{a=0}^{n} \sum_{P_1^{n+1}\cup\cdots\cup P_{a+1}^{n+1}=\{1,\ldots,n+1\}} \kappa_{|P_1^{n+1}|}^X \cdots \kappa_{|P_{a+1}^{n+1}|}^X \qquad (31) \\
&= \sum_{a=1}^{n+1} \sum_{P_1^{n+1}\cup\cdots\cup P_a^{n+1}=\{1,\ldots,n+1\}} \kappa_{|P_1^{n+1}|}^X \cdots \kappa_{|P_a^{n+1}|}^X \\
&= \sum_{a=0}^{n+1} \frac{(n+1)!}{a!} \sum_{\substack{l_1+\cdots+l_a=n+1 \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{\kappa_{l_1}^X}{l_1!} \cdots \frac{\kappa_{l_a}^X}{l_a!},
\end{aligned}$$

where in (31) the set P_{a+1}^{n+1} of cardinal $|P_{a+1}^{n+1}| = k$ is built by combining $\{n+1\}$ with $k-1$ elements of $\{1,\ldots,n\}$.

The cumulant formula (29) can also be inverted to compute the cumulant κ_n^X from the moments μ_n^X of X by the inversion formula

$$\kappa_n^X = \sum_{a=1}^{n} (a-1)!(-1)^{a-1} \sum_{P_1^n\cup\cdots\cup P_a^n=\{1,\ldots,n\}} \mu_{|P_1^n|}^X \cdots \mu_{|P_a^n|}^X, \qquad n \geq 1, \qquad (32)$$

where the sum runs over the partitions $P_1^n,\ldots,P_a^n$ of $\{1,\ldots,n\}$ with cardinal $|P_i^n|$ by the Faà di Bruno formula, cf. Theorem 1 of [16], and also [15] or § 2.4 and Relation (2.4.3) page 27 of [17].

Example - Gaussian cumulants

When X is centered we have $\kappa_1^X = 0$ and $\kappa_2^X = \mathbb{E}[X^2] = \mathrm{var}[X]$, and X becomes Gaussian if and only if $\kappa_n^X = 0$, $n \geq 3$, i.e. $\kappa_n^X = \mathbf{1}_{\{n=2\}}\sigma^2$, $n \geq 1$, or

$$(\kappa_1^X, \kappa_2^X, \kappa_3^X, \kappa_4^X, \ldots) = (0, \sigma^2, 0, 0, \ldots).$$

When X is centered Gaussian we have $\kappa_n^X = 0$, $n \neq 2$, and (29) can be read as Wick's theorem for the computation of Gaussian moments of $X \simeq \mathcal{N}(0,\sigma^2)$ by counting the pair partitions of $\{1,\ldots,n\}$, cf. [10], as

$$\mathbb{E}[X^n] = \sigma^n \sum_{k=1}^{n} \sum_{\substack{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\} \\ |P_1^n|=2,\ldots,|P_k^n|=2}} \kappa^X_{|P_1^n|} \cdots \kappa^X_{|P_a^n|} = \begin{cases} \sigma^n (n-1)!!, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{33}$$

where the double factorial

$$(n-1)!! = \prod_{1 \le 2k \le n} (2k-1) = 2^{-n/2} \frac{n!}{(n/2)!}$$

counts the number of pair-partitions of $\{1,\ldots,n\}$ when n is even.

Example - Poisson cumulants

In the particular case of a Poisson random variable $Z \simeq \mathscr{P}(\lambda)$ with intensity $\lambda > 0$ we have

$$\mathbb{E}[e^{tZ}] = \sum_{n=0}^{\infty} e^{nt} \mathbb{P}(Z=n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{\lambda(e^t-1)}, \qquad t \in \mathbb{R}_+,$$

hence $\kappa_n^Z = \lambda$, $n \ge 1$, or

$$(\kappa_1^Z, \kappa_2^Z, \kappa_3^Z, \kappa_4^Z, \ldots) = (\lambda, \lambda, \lambda, \lambda, \ldots),$$

and by (29) we have

$$\begin{aligned} \mathbb{E}_\lambda[Z^n] &= A_n(\lambda,\ldots,\lambda) = \sum_{k=0}^{n} B_{n,k}(\lambda,\ldots,\lambda) \\ &= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \lambda^k = \sum_{k=0}^{n} \lambda^k S(n,k) \\ &= T_n(\lambda), \end{aligned}$$

i.e. the n-th Poisson moment with intensity parameter $\lambda > 0$ is given by $T_n(\lambda)$ where T_n is the Touchard polynomial of degree n.

In the case of centered Poisson random variables we note that Z and $Z - \mathbb{E}[Z]$ have same cumulants of order $k \ge 2$, hence in case $Z - \mathbb{E}[Z]$ is a centered Poisson random variable with intensity $\lambda > 0$ we have

$$\mathbb{E}[(Z - \mathbb{E}[Z])^n] = \sum_{a=1}^{n} \sum_{\substack{P_1^n \cup \cdots \cup P_a^n = \{1,\ldots,n\} \\ |P_1^n| \ge 2,\ldots,|P_a^n| \ge 2}} \lambda^a = \sum_{k=0}^{n} \lambda^k S_2(n,k), \qquad n \ge 0,$$

where $S_2(n,k)$ is the number of ways to partition a set of n objects into k nonempty subsets of size at least 2, cf. [25].

Example - compound Poisson cumulants

Consider the compound Poisson random variable

$$\beta_1 Z_{\alpha_i} + \cdots + \beta_p Z_{\alpha_p} \tag{34}$$

with Lévy measure

$$\alpha_i \delta_{\beta_1} + \cdots + \alpha_p \delta_{\beta_p},$$

where $\beta_1,\ldots,\beta_p \in \mathbb{R}$ are constant parameters and $Z_{\alpha_1},\ldots,Z_{\alpha_p}$ is a sequence of independent Poisson random variables with respective parameters $\alpha_1,\ldots,\alpha_p \in \mathbb{R}_+$. The moment generating function of (34) is given by

$$\mathbb{E}[e^{t(\beta_1 Z_{\alpha_i}+\cdots+\beta_p Z_{\alpha_p})}] = e^{\alpha_1(e^{t\beta_1}-1)+\cdots+\alpha_p(e^{t\beta_p}-1)},$$

which shows that the cumulant of order $k \geq 1$ of (34) is given by

$$\alpha_1 \beta_1^k + \cdots + \alpha_p \beta_p^k.$$

As a consequence of the identity (29) the moment of order n of (34) is given by

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^p \beta_i Z_{\lambda\alpha_i}\right)^n\right] \qquad (35)\\
&= \sum_{m=0}^n \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} (\alpha_1\beta_1^{|P_1^n|}+\cdots+\alpha_p\beta_p^{|P_1^n|})\cdots(\alpha_1\beta_1^{|P_m^n|}+\cdots+\alpha_p\beta_p^{|P_m^n|})\\
&= \sum_{m=0}^n \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^p \beta_{i_1}^{|P_1^n|}\alpha_{i_1}\cdots\beta_{i_m}^{|P_m^n|}\alpha_{i_m},
\end{aligned}$$

where the above sum runs over all partitions $P_1^n,\ldots,P_m^n$ of $\{1,\ldots,n\}$.

Example - infinitely divisible cumulants

On the other hand, in case X is the infinitely divisible Poisson stochastic integral

$$X = \int_0^\infty h(t)\mathrm{d}N_t$$

with respect to a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ and $h \in \bigcap_{p=1}^\infty L^p(\mathbb{R}_+)$, the logarithmic generating function

$$\log \mathbb{E}\left[\exp\left(\int_0^\infty h(t)\mathrm{d}N_t\right)\right] = \lambda \int_0^\infty (e^{h(t)}-1)\mathrm{d}t = \lambda \sum_{n=1}^\infty \frac{1}{n!}\int_0^\infty h^n(t)\mathrm{d}t = \sum_{n=1}^\infty \kappa_n^X \frac{t^n}{n!},$$

shows that the cumulants of $\int_0^\infty h(t)\mathrm{d}N_t$ are given by

$$\kappa_n^X = \lambda \int_0^\infty h^n(t)\mathrm{d}t, \qquad n \geq 1, \tag{36}$$

and (29) becomes the moment identity

$$\mathbb{E}\left[\left(\int_0^\infty h(t)\mathrm{d}N_t\right)^n\right] = \sum_{k=1}^n \lambda^k \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_0^\infty h^{|P_1^n|}(t)\mathrm{d}t \cdots \int_0^\infty h^{|P_k^n|}(t)\mathrm{d}t, \tag{37}$$

where the sum runs over all partitions $P_1^n,\ldots,P_k^n$ of $\{1,\ldots,n\}$, cf. [2] for the non-compensated case and [28], Proposition 3.2 for the compensated case.

3 Analysis of Poisson random measures

In this section we introduce the basic definitions and notations relative to Poisson random measures, and we derive the functional transform identities that will be useful for the computation of moments in Section 4.

3.1 Poisson point processes

From now on we consider a Poisson point process η on the space $\mathbf{N}_\sigma(\mathbb{X})$ of all σ-finite measures on a measure space $(\mathbb{X},\mathscr{X})$ equipped with a σ-finite intensity measure $\mu(\mathrm{d}x)$, see [12] and [13] for further details and additional notation. The random measure η in $\mathbf{N}_\sigma(\mathbb{X})$ will be represented as

$$\eta = \sum_{n=1}^{\eta(\mathbb{X})} \delta_{x_n},$$

where $(x_n)_{n=1}^{\eta(\mathbb{X})}$ is a (random) sequence in $\mathbb{X}$, δ_x denotes the Dirac measure at $x\in\mathbb{X}$, and $\eta(\mathbb{X})\in\mathbb{N}\cup\{\infty\}$ denote the cardinality of η identified with the sequence $(x_n)_n$.

Recall that the probability law $\mathbb{P}_\eta$ of η is the Poisson probability measure with intensity $\mu(\mathrm{d}x)$ on $\mathbb{X}$ is the only probability measure on $\mathbf{N}_\sigma(\mathbb{X})$ satisfying

i) For any measurable subset $A\in\mathscr{X}$ of $\mathbb{X}$ such that $\mu(A)<\infty$, the number $\eta(A)$ of configuration points contained in A is a Poisson random variable with intensity $\mu(A)$, *i.e.*

$$\mathbb{P}_\eta(\{\eta \in \mathbf{N}_\sigma(\mathbb{X}) \ : \ \eta(A) = n\}) = e^{-\mu(A)} \frac{(\mu(A))^n}{n!}, \qquad n \in \mathbb{N}.$$

ii) In addition, if $A_1, \ldots, A_n$ are disjoint subsets of $\mathbb{X}$ with $\mu(A_k) < \infty$, $k = 1, \ldots, n$, the $\mathbb{N}^n$-valued random vector

$$\eta \longmapsto (\eta(A_1), \ldots, \eta(A_n)), \qquad \eta \in \mathbf{N}_\sigma(\mathbb{X}),$$

is made of independent random variables for all $n \geq 1$.

When $\mu(\mathbb{X}) < \infty$ the expectation under the Poisson measure $\mathbb{P}_\eta$ can be written as

$$\mathbb{E}[F(\eta)] = e^{-\mu(\mathbb{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} f_n(x_1, \ldots, x_n) \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n) \tag{38}$$

for a random variable F of the form

$$F(\eta) = \sum_{n=0}^{\infty} \mathbf{1}_{\{\eta(\mathbb{X})=n\}} f_n(x_1, \ldots, x_n) \tag{39}$$

where for each $n \geq 1$, f_n is a symmetric integrable function of $\eta = \{x_1, \ldots, x_n\}$ when $\eta(\mathbb{X}) = n$, cf. *e.g.* § 6.1 of [24].

The next lemma is well known.

Lemma 4. *Given μ and ν two intensity measures on $\mathbb{X}$, the Poisson random measure $\eta_{\mu+\nu}$ with intensity $\mu + \nu$ decomposes into the sum*

$$\eta_{\mu+\nu} \simeq \eta_\mu \oplus \eta_\nu, \tag{40}$$

of a Poisson random measure η_μ with intensity $\mu(\mathrm{d}x)$ and an independent Poisson random measure η_ν with intensity $\nu(\mathrm{d}x)$.

Proof. Taking F a random variable of the form (39) we have

$$\mathbb{E}[F(\eta_{\mu+\nu})] = e^{-\mu(\mathbb{X})-\nu(\mathbb{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} f_n(\{x_1, \ldots, x_n\}) \prod_{k=1}^{n} (\mu(\mathrm{d}x_k) + \nu(\mathrm{d}x_k)),$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} f_n(\{s_1, \ldots, s_n\}) \prod_{k=1}^{n} (\mu(\mathrm{d}s_k) + \nu(\mathrm{d}s_k)) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} \int_{\mathbb{X}^n} f_n(\{s_1, \ldots, s_n\}) \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_k) \nu(\mathrm{d}s_{k+1}) \cdots \nu(\mathrm{d}s_n) \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{1}{(n-l)!l!} \int_{\mathbb{X}^n} f_n(\{s_1, \ldots, s_l, \ldots, s_n\}) \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_l) \nu(\mathrm{d}s_{l+1}) \cdots \nu(\mathrm{d}s_n) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=0}^{\infty} \frac{1}{l!} \int_{\mathbb{X}^{l+m}} f_{l+m}(\{s_1, \ldots, s_l, \ldots, s_{l+m}\}) \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_l) \nu(\mathrm{d}s_{l+1}) \cdots \nu(\mathrm{d}s_{l+m})
\end{aligned}$$

$$= e^{\mu(\mathbb{X})} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \mathbb{E}\left[\varepsilon^+_{\mathfrak{s}_m} F(\eta_\mu)\right] \nu(\mathrm{d}s_1)\cdots\nu(\mathrm{d}s_m) \tag{41}$$
$$= e^{\mu(\mathbb{X})+\nu(\mathbb{X})}\mathbb{E}[F(\eta_\mu \oplus \eta_\nu)],$$

where $\varepsilon^+_{\mathfrak{s}_m}$ is the addition operator defined on any random variable $F : \mathbf{N}_\sigma(\mathbb{X}) \to \mathbb{R}$ by

$$\varepsilon^+_{\mathfrak{s}_m} F(\eta) = F(\eta + \delta_{s_1} + \cdots + \delta_{s_m}), \quad \eta \in \mathbf{N}_\sigma(\mathbb{X}), \quad s_1, \ldots, s_m \in \mathbb{X}, \tag{42}$$

and

$$\mathfrak{s}_m := (s_1, \ldots, s_m) \in \mathbb{X}^m, \quad m \geq 1.$$

□

In the course of the proof of Lemma 4 we have shown in (41) that

$$\mathbb{E}[F(\eta_{\mu+\nu})] = e^{-\mu(\mathbb{X})} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \mathbb{E}\left[\varepsilon^+_{\mathfrak{s}_m} F(\eta_\nu)\right] \nu(\mathrm{d}s_1)\cdots\nu(\mathrm{d}s_m) = \mathbb{E}[F(\eta_\mu \oplus \eta_\nu)],$$

where $\varepsilon^+_{\mathfrak{s}_k}$ is defined in (42).

In particular, by applying Lemma 4 above to $\mu(\mathrm{d}x)$ and $\nu(\mathrm{d}x) = f(x)\mu(\mathrm{d}x)$ with $f(x) \geq 0$ $\mu(\mathrm{d}x)$-a.e. we find that the Poisson random measure η with intensity $(1+f)\mathrm{d}\mu$ decomposes into the sum

$$\eta_{(1+f)\mathrm{d}\mu} \simeq \eta_{\mathrm{d}\mu} \oplus \eta_{f\mathrm{d}\mu},$$

of a Poisson random measure $\eta_{\mathrm{d}\mu}$ with intensity $\mu(\mathrm{d}x)$ and an independent Poisson random measure $\eta_{f\mathrm{d}\mu}$ with intensity $f(x)\mu(\mathrm{d}x)$.

In addition we have, using the shortand notation $\mathbb{E}_\mu$ to denote the Poisson probability measure with intensity μ,

$$\mathbb{E}_{(1+f)\mathrm{d}\mu}[F] = e^{-\mu(\mathbb{X})} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_m} F\right] f(s_1)\cdots f(s_n)\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_m). \tag{43}$$

The above identity extends to $f \in L^2(\mathbb{X})$ with $f > -1$, and when $f(x) \in (-1,0)$, Relation (43) can be interpreted as a thinning of $\eta_{(1+f)\mathrm{d}\mu}$.

Mecke identity

The following version of Mecke's identity [19], cf. also Relation (7) in [12], allows us to compute the first moment of the first order stochastic integral of a random integrand. In the sequel we use the expression "measurable process" to denote a real-valued measurable function from $\mathbb{X} \times \mathbf{N}_\sigma(\mathbb{X})$ into $\mathbb{R}$.

Proposition 1. *For $u : \mathbb{X} \times \mathbf{N}_\sigma(\mathbb{X}) \longrightarrow \mathbb{R}$ a measurable process we have*

$$\mathbb{E}_\mu \left[\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x) \right] = \mathbb{E}_\mu \left[\int_{\mathbb{X}} u(x,\eta+\delta_x)\mu(\mathrm{d}x) \right], \tag{44}$$

provided

$$\mathbb{E}_\mu \left[\int_{\mathbb{X}} |u(x,\eta+\delta_x)|\mu(\mathrm{d}x) \right] < \infty.$$

Proof. The proof is done when $\mu(\mathbb{X}) < \infty$. We take $u(x,\eta)$ written as

$$u(x,\eta) = \sum_{n=0}^{\infty} \mathbf{1}_{\{\eta(\mathbb{X})=n\}} f_n(x;x_1,\ldots,x_n),$$

where $(x_1,\ldots,x_n) \longmapsto f_n(x;x_1,\ldots,x_n)$ is a symmetric integrable function of $\eta = \{x_1,\ldots,x_n\}$ when $\eta(\mathbb{X}) = n$, for each $n \geq 1$. We have

$$\begin{aligned}
&\mathbb{E}_\mu \left[\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x) \right] \\
&= e^{-\mu(\mathbb{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \int_{\mathbb{X}^n} f_n(x_i;x_1,\ldots,x_n)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n) \\
&= \sum_{n=1}^{\infty} \frac{e^{-\mu(\mathbb{X})}}{(n-1)!} \int_{\mathbb{X}^n} f_n(x;x_1,\ldots,x_{i-1},x,x_i,\ldots,x_{n-1})\mu(\mathrm{d}x)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{n-1}) \\
&= e^{-\mu(\mathbb{X})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \int_{\mathbb{X}} f_{n+1}(x;x,x_1,\ldots,x_n)\mu(\mathrm{d}x)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n) \\
&= \mathbb{E}_\mu \left[\int_{\mathbb{X}} u(x,\eta+\delta_x)\mu(\mathrm{d}x) \right].
\end{aligned}$$

□

3.2 *Multiple stochastic integrals*

In this section we define the multiple Poisson stochastic integral using Charlier polynomials. We denote by "$\circ$" the symmetric tensor product of functions in $L^2(\mathbb{X})$, i.e. given $f_1,\ldots,f_d \in L^2(\mathbb{X})$ and $k_1,\ldots,k_d \geq 1$,

$$f_1^{\circ k_1} \circ \cdots \circ f_d^{\circ k_d}$$

denotes the symmetrization in $n = k_1 + \cdots + k_d$ variables of

$$f_1^{\otimes k_1} \otimes \cdots \otimes f_d^{\otimes k_d},$$

cf. Relation (26) in [12].

Definition 1. Consider $A_1,\ldots,A_d$ mutually disjoint subsets of $\mathbb{X}$ with finite μ-measure and $n = k_1 + \cdots + k_d$, where $k_1,\ldots,k_d \geq 1$. The multiple Poisson stochastic integral of the function

$$\mathbf{1}_{A_1}^{\circ k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\circ k_d}$$

is defined by

$$I_n(\mathbf{1}_{A_1}^{\otimes k_1} \otimes \cdots \otimes \mathbf{1}_{A_d}^{\otimes k_d})(\eta) := \prod_{i=1}^{d} C_{k_i}(\eta(A_i), \mu(A_i)). \tag{45}$$

Note that by (26), Relation (45) actually coincides with Relation (25) in [12] and this recovers the fact that

$$\eta^{(k)}(A) := \#(\{(i_1,\ldots,i_k) \in \{1,\ldots,\eta(A)\}^k \ : \ i_l \neq i_m, \ 1 \leq l \neq m \leq k\})$$

defined in Relation (9) of [12] coincides with the falling factorial $(\eta(A))_{(k)}$ for $A \in \mathscr{X}$ such that $\mu(A) < \infty$.

See also [31] and [7] for a more general framework for the expression of multiple stochastic integrals with respect to Lévy processes based on the combinatorics of the Möbius inversion formula.

From (27) and Definition 1 it can be shown that the multiple Poisson stochastic integral satisfies the isometry formula

$$\mathbb{E}[I_n(f_n) I_m(g_m)] = \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(\mathbb{X}^n)}, \tag{46}$$

cf. Lemma 4 in [12], which allows one to extend the definition of I_n to any symmetric function $f_n \in L^2(\mathbb{X}^n)$, cf. also (51) below.

The generating series

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(\eta(A), \mu(A)) = e^{-\lambda \mu(A)} (1+\lambda)^{\eta(A)} = \psi_\lambda(\eta(A), \mu(A)),$$

cf. (24), admits a multivariate extension using multiple stochastic integrals.

Proposition 2. *For $f \in L^2(\mathbb{X}) \cap L^1(\mathbb{X})$ we have*

$$\xi(f) := \sum_{k=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}) = \exp\left(-\int_{\mathbb{X}} f(x) \mu(\mathrm{d}x) \right) \prod_{x \in \eta} (1 + f(x)). \tag{47}$$

Proof. From (46) and an approximation argument it suffices to consider simple functions of the form

$$f = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k},$$

by the multinomial identity (11) we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!} I_n\left(\left(\sum_{k=1}^{m} a_k \mathbf{1}_{A_k}\right)^{\otimes n}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d_1+\cdots+d_m=n} \frac{n!}{d_1!\cdots d_m!} a_1^{d_1}\cdots a_m^{d_m} I_n\left(\mathbf{1}_{A_1}^{\otimes d_1}\circ\cdots\circ\mathbf{1}_{A_m}^{\otimes d_m}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d_1+\cdots+d_m=n} \frac{n!}{d_1!\cdots d_m!} a_1^{d_1}\cdots a_m^{d_m} \prod_{i=1}^{m} C_{d_i}(\eta(A_i),\mu(A_i)) \\
&= \prod_{i=1}^{m}\sum_{n=0}^{\infty} \frac{a_i^n}{n!} C_n(\eta(A_i),\mu(A_i)) \\
&= \prod_{i=1}^{m}\left(e^{-a_i\mu(A_i)}(1+a_i)^{\eta(A_i)}\right) \\
&= \exp\left(-\sum_{i=1}^{m} a_i\mu(A_i)\right)\prod_{i=1}^{m}(1+a_i)^{\eta(A_i)} \\
&= \exp\left(\sum_{i=1}^{m} a_i(\eta(A_i)-\mu(A_i))\right)\prod_{i=1}^{m}((1+a_i)^{\eta(A_i)}e^{-a_i\eta(A_i)}).
\end{aligned}
$$

□

The relation between $\xi(f)$ in (47) and the exponential functional in Lemma 5 of [12] is given by

$$
\exp\left(\int_{\mathbb{X}}(e^{f(x)}-1)\mu(\mathrm{d}x)\right)\xi(e^f-1) = \exp\left(\int_{\mathbb{X}} f(x)\eta(\mathrm{d}x)\right),
$$

provided $e^f - 1 \in L^1(\mathbb{X}) \cap L^2(\mathbb{X})$.

3.3 $\mathscr{S}$-transform

Given $f \in L^1(\mathbb{X},\mu) \cap L^2(\mathbb{X},\mu)$ with $f(x) > -1$ $\mu(\mathrm{d}x)$-a.e. we define the measure $\mathbb{Q}_f$ by its Girsanov density

$$
\frac{\mathrm{d}\mathbb{Q}_f}{\mathrm{d}\mathbb{P}_\eta} = \xi(f) = \exp\left(-\int_{\mathbb{X}} f(x)\mu(\mathrm{d}x)\right)\prod_{x\in\mathbb{X}}(1+f(x)), \tag{48}
$$

where $\mathbb{P}_\eta$ is the Poisson probability measure with intensity $\mu(\mathrm{d}x)$. From (38), for F a bounded random variable we have the relation

$$
\mathbb{E}_\mu[F\xi(f)] = \mathbb{E}_\mu\left[F\exp\left(-\int_{\mathbb{X}} f(x)\mu(\mathrm{d}x)\right)\prod_{x\in\eta}(1+f(x))\right]
$$

$$= \exp\left(-\int_{\mathbb{X}}(1+f(x))\mu(\mathrm{d}x)\right)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{X}^n}F(\{s_1,\ldots,s_n\})\prod_{k=1}^{n}(1+f(s_k))\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_n)$$
$$= \mathbb{E}[F(\eta_{(1+f)\mathrm{d}\mu})],$$

which shows the following proposition.

Proposition 3. *Under the probability $\mathbb{Q}_f$ defined by (48), the random measure η is Poisson with intensity $(1+f)\mathrm{d}\mu$, i.e.*

$$\mathbb{E}_\mu[F\xi(f)] = \mathbb{E}_{(1+f)\mathrm{d}\mu}[F]$$

for all sufficiently integrable random variables F.

The $\mathscr{S}$-transform (or Segal-Bargmann transform, see [14] for references) on the Poisson space is defined on bounded random variables F by

$$f \longmapsto \mathscr{S}F(f) := \mathbb{E}_{f\mathrm{d}\mu}[F] = \mathbb{E}_\mu[F\xi(f)] = \mathbb{E}_\mu\left[F\exp\left(-\int_{\mathbb{X}}f(x)\mu(\mathrm{d}x)\right)\prod_{x\in\eta}(1+f(x))\right],$$

for f bounded and vanishing outside a set of finite σ-measure in $\mathscr{X}$ and Lemma 4 and Proposition 3 show that

$$\mathscr{S}F(f) = \mathbb{E}[F(\eta_{\mathrm{d}\mu}\oplus\eta_{f\mathrm{d}\mu})] \tag{49}$$
$$= e^{-\int_{\mathbb{X}}fd\mu}\mathbb{E}_\mu[F] + e^{-\int_{\mathbb{X}}fd\mu}\sum_{k=1}^{\infty}\frac{1}{k!}\int_{\mathbb{X}^k}f(s_1)\cdots f(s_k)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k),$$

where $\eta_{f\mathrm{d}\mu}$ is a Poisson random measure with intensity $f\mathrm{d}\mu$ independent of $\eta_{\mathrm{d}\mu}$, by Lemma 4. In the next proposition we use the finite difference operator

$$D_x := \varepsilon^+_x - I, \qquad x\in\mathbb{X},$$

i.e.

$$D_xF(\eta) = F(\eta+\delta_x) - F(\eta),$$

and apply a binomial transformation to get rid of the exponential term in (49). In the next proposition we let

$$D^k_{\mathfrak{s}_k} = D_{s_1}\cdots D_{s_k}, \qquad s_1,\ldots,s_k\in\mathbb{X},$$

and

$$\varepsilon^+_{\mathfrak{s}_k} = \varepsilon^+_{s_1}\cdots\varepsilon^+_{s_k}, \qquad s_1,\ldots,s_k\in\mathbb{X},$$

as in (42), where

$$\mathfrak{s}_k = (s_1,\ldots,s_k)\in\mathbb{X}^k, \quad k\geq 1.$$

Proposition 4. *For any bounded random variable F and f bounded and vanishing outside a set of finite μ-measure in $\mathbb{X}$ we have*

$$\mathscr{S}F(f)=\mathbb{E}_\mu[F(\eta)\xi(f)]=\sum_{k=0}^\infty\frac{1}{k!}\int_{\mathbb{X}^k}f(s_1)\cdots f(s_k)\mathbb{E}_\mu\left[D^k_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k). \tag{50}$$

Proof. We apply a binomial transformation to the expansion (49). We have

$$\begin{aligned}
&\mathscr{S}F(f)\\
&=e^{-\int_{\mathbb{X}}fd\mu}\sum_{k=0}^\infty\frac{1}{k!}\int_{\mathbb{X}^k}f(s_1)\cdots f(s_k)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)\\
&=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(\int_{\mathbb{X}}fd\mu\right)^n\sum_{k=0}^\infty\frac{1}{k!}\int_{\mathbb{X}^k}f(s_1)\cdots f(s_k)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)\\
&=\sum_{m=0}^\infty\sum_{k=0}^m\frac{(-1)^{m-k}}{(m-k)!}\left(\int_{\mathbb{X}}fd\mu\right)^{m-k}\frac{1}{k!}\int_{\mathbb{X}^k}f(s_1)\cdots f(s_k)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)\\
&=\sum_{m=0}^\infty\frac{1}{m!}\sum_{k=0}^m\binom{m}{k}(-1)^{m-k}\int_{\mathbb{X}^m}f(s_1)\cdots f(s_m)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_m)\\
&=\sum_{m=0}^\infty\frac{1}{m!}\int_{\mathbb{X}^m}f(s_1)\cdots f(s_m)\mathbb{E}_\mu\left[D^m_{\mathfrak{s}_m}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_m).
\end{aligned}$$

□

By identification of terms in the expansions (47) and (50) we obtain the following result, which is equivalent by (46) and duality to the Stroock [32] formula, cf. also Theorem 2 in [12].

Corollary 1. *Given F a bounded random variable, for all $n\geq 1$ and f bounded and vanishing outside a set of finite μ-measure in $\mathscr{X}$ we have*

$$\mathbb{E}_\mu\left[I_n(f^{\otimes n})F\right]=\int_{\mathbb{X}^n}f(s_1)\cdots f(s_n)\mathbb{E}_\mu\left[D^n_{\mathfrak{s}_n}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_n). \tag{51}$$

Proof. We note that (47) yields

$$\mathscr{S}F(f)=\mathbb{E}_{f\mathrm{d}\mu}[F]=\mathbb{E}_\mu[F\xi(f)]=\sum_{n=0}^\infty\frac{1}{n!}\mathbb{E}_\mu[FI_n(f^{\otimes n})],$$

and by Proposition 4 we have

$$\mathscr{S}F(f)=\mathbb{E}_{f\mathrm{d}\mu}[F]=\sum_{n=0}^\infty\frac{1}{n!}\int_{\mathbb{X}^n}f(s_1)\cdots f(s_n)\mathbb{E}_\mu\left[D^n_{\mathfrak{s}_n}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_n),$$

and identify the respective terms of orders $n\geq 1$ in order to show (51). □

When $k=1$ we have the integration by parts formula

$$\mathbb{E}_\mu[I_1(f)F]=\mathbb{E}_\mu\left[\int_{\mathbb{X}}f(s)D_sF\mu(\mathrm{d}s)\right].$$

Note that with the pathwise extension $I_k((Ff)^{\otimes k}) = F^k I_k(f^{\otimes k})$ of the multiple stochastic integral, (51) can be rewritten as the identity

$$\mathbb{E}_\mu[I_k((Ff)^{\otimes k})] = \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} f(s_1)\cdots f(s_k) D_{s_1}\cdots D_{s_k}(F^k)\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)\right],$$

cf. also Proposition 4.1 of [26].

3.4 $\mathscr{U}$-transform

The Laplace transform on the Poisson space (also called $\mathscr{U}$-transform, cf. e.g. § 2 of [11]), is defined using the exponential functional of Lemma 5 of [12] by

$$f \longmapsto \mathscr{U}F(f) := \mathbb{E}_\mu\left[F e^{\int_{\mathbb{X}} f \mathrm{d}\eta}\right] = e^{\int_{\mathbb{X}}(e^f-1)\mathrm{d}\mu}\mathbb{E}_\mu[F\xi(e^f-1)],$$

for f bounded and vanishing outside a set of finite μ-measure in $\mathscr{X}$, and will be useful for the derivation of general moment identities in Section 4.

Proposition 5. *Let F a bounded random variable. We have*

$$\mathscr{U}F(f) = \sum_{n=0}^\infty \frac{1}{n!}\sum_{k=0}^n \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_{\mathbb{X}^k} f^{|P_1^n|}(s_1)\cdots f^{|P_k^n|}(s_k)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k), \tag{52}$$

$f \in L^2(\mathbb{X},\mu)$.

Proof. Using the Faà di Bruno identity (13) or (16) we have

$$\begin{aligned}
\sum_{n=0}^\infty \frac{1}{n!}\mathbb{E}_\mu\left[F\left(\int_{\mathbb{X}} f\mathrm{d}\eta\right)^n\right] &= \mathbb{E}_\mu\left[Fe^{\int_{\mathbb{X}} f\mathrm{d}\eta}\right] = e^{\int_{\mathbb{X}}(e^f-1)\mathrm{d}\mu}\mathbb{E}_\mu\left[F\xi(e^f-1)\right] \\
&= e^{\int_{\mathbb{X}}(e^f-1)\mathrm{d}\mu}\sum_{k=0}^\infty \frac{1}{k!}\int_{\mathbb{X}^k}(e^{f(s_1)}-1)\cdots(e^{f(s_k)}-1)\mathbb{E}_\mu\left[D^k_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k) \qquad (53) \\
&= \sum_{n=0}^\infty \frac{1}{n!}\sum_{k=0}^n \binom{n}{k}\int_{\mathbb{X}^n}(e^{f(s_1)}-1)\cdots(e^{f(s_n)}-1)\mathbb{E}_\mu\left[D^k_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_n) \\
&= \sum_{k=0}^\infty \frac{1}{k!}\int_{\mathbb{X}^k}(e^{f(s_1)}-1)\cdots(e^{f(s_n)}-1)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k) \\
&= \sum_{k=0}^\infty \frac{1}{k!}\int_{\mathbb{X}^k}\left(\sum_{n=1}^\infty \frac{f^n(s_1)}{n!}\right)\cdots\left(\sum_{n=1}^\infty \frac{f^n(s_k)}{n!}\right)\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k) \\
&= \sum_{k=0}^\infty \frac{1}{k!}\sum_{n=1}^\infty \sum_{\substack{d_1+\cdots+d_k=n \\ d_1,\ldots,d_k\geq 1}} \int_{\mathbb{X}^k}\frac{f^{d_1}(s_1)}{d_1!}\cdots\frac{f^{d_k}(s_k)}{d_k!}\mathbb{E}_\mu\left[\varepsilon^+_{\mathfrak{s}_k}F\right]\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1,\ldots,d_k\geq 1}} \frac{n!}{d_1!\cdots d_k!} \int_{\mathbb{X}^k} f^{d_1}(s_1)\cdots f^{d_k}(s_k) \mathbb{E}_\mu \left[\varepsilon^+_{\mathfrak{s}_k} F \right] \mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k),$$

where we applied the Faà di Bruno identity (13). □

In particular, by (53) we have

$$\begin{aligned} &\mathbb{E}_\mu \left[I_m(g_m) e^{\int_{\mathbb{X}} f \mathrm{d}\eta} \right] \qquad (54) \\ &= \frac{1}{m!} \int_{\mathbb{X}^m} (e^{f(s_1)}-1)\cdots(e^{f(s_m)}-1) \mathbb{E}_\mu \left[D^+_{\mathfrak{s}_m} I_m(g_m) \right] \mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_m) \\ &= \int_{\mathbb{X}^m} (e^{f(s_1)}-1)\cdots(e^{f(s_m)}-1) g_m(s_1,\ldots,s_m) \mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_m), \end{aligned}$$

cf. Proposition 3.2 of [11].

4 Moment identities and invariance

The following cumulant-type moment identities have been extended to the Poisson stochastic integrals of random integrands in [28] through the use of the Skorohod integral on the Poisson space, cf. [23], [27]. These identities and their consequences on invariance have been recently extended to point processes with Papangelou intensities in [6], via simpler proofs based on an induction argument.

4.1 Moment identities for random integrands

The moments of Poisson stochastic integrals of deterministic integrands have been derived in [2] by direct iterated differentiation of the Lévy-Khintchine formula or moment generating function

$$\mathbb{E}_\mu \left[\exp\left(\int_{\mathbb{X}} f(x) \eta(\mathrm{d}x) \right) \right] = \exp\left(\int_{\mathbb{X}} (e^{f(x)}-1) \mu(\mathrm{d}x) \right),$$

for f bounded and vanishing outside a set of finite μ-measure in $\mathscr{X}$. We can also note that

$$\begin{aligned} &\mathbb{E}_\mu \left[\exp\left(\int_{\mathbb{X}} f(x) \eta(\mathrm{d}x) \right) \right] = \exp\left(\int_{\mathbb{X}} (e^{f(x)}-1) \mu(\mathrm{d}x) \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{\mathbb{X}} (e^{f(x)}-1) \mu(\mathrm{d}x) \right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} (e^{f(x_1)}-1)\cdots(e^{f(x_n)}-1) \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} (e^{f(x_1)} - 1) \cdots (e^{f(x_n)} - 1) \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \left(\sum_{k=1}^{\infty} \frac{f^k(x_1)}{k!} \right) \cdots \left(\sum_{k=1}^{\infty} \frac{f^k(x_n)}{k!} \right) \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\substack{d_1+\cdots+d_n=k \\ d_1,\ldots,d_n \geq 1}} \int_{\mathbb{X}^n} \frac{f^{d_1}(x_1)}{d_1!} \cdots \frac{f^{d_n}(x_k)}{d_n!} \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \frac{1}{n!} \sum_{\substack{d_1+\cdots+d_n=k \\ d_1,\ldots,d_n \geq 1}} \frac{k!}{d_1! \cdots d_n!} \int_{\mathbb{X}^n} f^{d_1}(x_1) \cdots f^{d_n}(x_n) \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n),
\end{aligned}
$$

where we applied the Faà di Bruno identity (13), and shows that

$$
\mathbb{E}_\mu \left[\left(\int_{\mathbb{X}} f(x) \eta(\mathrm{d}x) \right)^n \right] = \sum_{P_1^n,\ldots,P_a^n} \int_{\mathbb{X}^a} f^{|P_1^n|}(s_1) \mu(\mathrm{d}s_1) \cdots \int_{\mathbb{X}^a} f^{|P_a^n|}(s_a) \mu(\mathrm{d}s_a), \quad (55)
$$

which recovers in particular (37).

The next Lemma 5 is a moment formula for deterministic Poisson stochastic integrals applies in particular under a change of measure given by a density F.

Lemma 5. *Let $n \geq 1$, $f \in \bigcap_{p=1}^n L^p(\mathbb{X}, \mu)$, and consider F a bounded random variable. We have*

$$
\begin{aligned}
&\mathbb{E}_\mu \left[F \left(\int_{\mathbb{X}} f \mathrm{d}\eta \right)^n \right] \\
&= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \int_{\mathbb{X}^k} f^{|P_1^n|}(s_1) \cdots f^{|P_k^n|}(s_k) E \left[\varepsilon_{s_1}^+ \cdots \varepsilon_{s_k}^+ F \right] \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_k).
\end{aligned}
$$

Proof. We apply Proposition 5 on the $\mathscr{U}$-transform, which reads

$$
\begin{aligned}
&\mathbb{E}_\mu \left[F \exp \left(\int_{\mathbb{X}} f \mathrm{d}\eta \right) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_\mu \left[F \left(\int_{\mathbb{X}} f \mathrm{d}\eta \right)^n \right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1,\ldots,d_k \geq 1}} \frac{n!}{d_1! \cdots d_k!} \int_{\mathbb{X}^k} f^{d_1}(s_1) \cdots f^{d_k}(s_k) \mathbb{E} \left[\varepsilon_{s_1}^+ \cdots \varepsilon_{s_k}^+ F \right] \mu(\mathrm{d}s_1) \cdots \mu(\mathrm{d}s_k).
\end{aligned}
$$

□

Lemma 5 with $F = 1$ recovers the identity (37), and associated with the complete Bell polynomials $A_n(b_1, \ldots, b_n)$ as in (29) it can be used to compute the moments of stochastic integrals of deterministic integrands with respect to Lévy processes, cf. [18] for the case of subordinators.

Relation (54) yields

$$\mathbb{E}[FZ^n] = \sum_{k=0}^{n} S(n,k) \int_{A^k} \mathbb{E}\left[\varepsilon_{s_1}^+ \cdots \varepsilon_{s_k}^+ F\right] \mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k), \qquad n \in \mathbb{N}, \tag{56}$$

and when f is a deterministic function, Relation (53) shows that

$$\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} f(x)\eta(\mathrm{d}x)\right)^n\right] = \sum_{P_1^n,\ldots,P_a^n} \int_{\mathbb{X}^a} f^{|P_1^n|}(s_1)\mu(\mathrm{d}s_1)\cdots\int_{\mathbb{X}^a} f^{|P_a^n|}(s_a)\mu(\mathrm{d}s_a),$$

which recovers (55).

Based on the following version of (56)

$$\mathbb{E}_\mu\left[F\eta(A)^n\right] = \sum_{k=0}^{n} S(n,k)\mathbb{E}\left[\int_{\mathbb{X}^k} \varepsilon_{s_1}^+\cdots\varepsilon_{s_k}^+ F\mathbf{1}_A(s_1)\cdots\mathbf{1}_A(s_k)\mu(\mathrm{d}s_1)\cdots\mu(\mathrm{d}s_k)\right] \tag{57}$$

and an induction argument we obtain the following Lemma 6, which can be seen as an elementary joint moment identity obtained by iteration of Lemma 57.

Lemma 6. *For $A_1,\ldots,A_p$ mutually disjoint bounded measurable subsets of $\mathbb{X}$ and $F_1,\ldots,F_p$ bounded random variables we have*

$$\begin{aligned}
&\mathbb{E}_\mu\left[(F_1\eta(A_1))^{n_1}\cdots(F_p\eta(A_p))^{n_p}\right] \\
&= \sum_{k_1=0}^{n_1}\cdots\sum_{k_p=0}^{n_p} S(n_1,k_1)\cdots S(n_p,k_p) \\
&\quad \times\mathbb{E}_\mu\Bigg[\int_{\mathbb{X}^{k_1+\cdots+k_p}} \varepsilon_{x_1}^+\cdots\varepsilon_{x_{k_1+\cdots+k_p}}^+ (F_1^{n_1}\cdots F_p^{n_p}(\mathbf{1}_{A_1^{k_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{k_p}})(x_1,\ldots,x_{k_1+\cdots+k_p}) \\
&\quad \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{k_1+\cdots+k_p})\Bigg].
\end{aligned}$$

Lemma 6 allows us to recover the following moment identity, which can also be used for the computation of moments under a probability with density F with respect to $\mathbb{P}_\eta$.

Theorem 1. *Given F a random variable and $u : \mathbb{X}\times\mathbf{N}_\sigma(\mathbb{X}) \longrightarrow \mathbb{R}$ a measurable process we have*

$$\begin{aligned}
&\mathbb{E}_\mu\left[F\left(\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)\right)^n\right] \qquad (58)\\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon_{s_1}^+\cdots\varepsilon_{s_k}^+(Fu_{s_1}^{|P_1^n|}\cdots u_{s_k}^{|P_k^n|})\mu(ds_1)\cdots\mu(ds_k)\right],
\end{aligned}$$

provided all terms in the above summations are $\mathbb{P}_\eta\otimes\mu^{\otimes k}$- integrable, $k = 1,\ldots,n$.

Proof. We use the argument of Proposition 4.2 in [5] in order to extend Lemma 6 to (58). We start with $u : \mathbb{X}\times\mathbf{N}_\sigma(\mathbb{X}) \longrightarrow \mathbb{R}$ a simple measurable process of the form $u(x,\eta) = \sum_{i=1}^p F_i(\eta)\mathbf{1}_{A_i}(x)$ with disjoint sets $A_1,\ldots,A_p$. Using Lemma 6 we have

$$\mathbb{E}_\mu\left[\left(\sum_{i=1}^p F_i \int_{\mathbb{X}} \mathbf{1}_{A_i}(x)\ \eta(\mathrm{d}x)\right)^n\right] = \mathbb{E}_\mu\left[\left(\sum_{i=1}^p F_i \eta(A_i)\right)^n\right]$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!}\mathbb{E}_\mu\left[(F_1\eta(A_1))^{n_1}\cdots(F_p\eta(A_p))^{n_p}\right]$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!}\sum_{k_1=0}^{n_1}\cdots\sum_{k_p=0}^{n_p} S(n_1,k_1)\cdots S(n_p,k_p)$$
$$\mathbb{E}_\mu\left[\int_{\mathbb{X}^{k_1+\cdots+k_p}} \varepsilon^+_{x_1}\cdots\varepsilon^+_{x_{k_1+\cdots+k_p}}\left(F_1^{n_1}\cdots F_p^{n_p}\mathbf{1}_{A_1^{k_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{k_p}}(x_1,\ldots,x_{k_1+\cdots+k_p})\right)\right.$$
$$\left.\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{k_1+\cdots+k_p})\right]$$
$$= \sum_{m=0}^n \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{k_1+\cdots+k_p=m\\ 1\leq k_1\leq n_1,\ldots,1\leq k_p\leq n_p}} S(n_1,k_1)\cdots S(n_p,k_p)$$
$$\mathbb{E}_\mu\left[\int_{\mathbb{X}^m} \varepsilon^+_{x_1}\cdots\varepsilon^+_{x_m}\left(F_1^{n_1}\cdots F_p^{n_p}\mathbf{1}_{A_1^{k_1}}\otimes\cdots\otimes\mathbf{1}_{A_p^{k_p}}(x_1,\ldots,x_m)\right)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_m)\right]$$
$$= \sum_{m=0}^n \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\cdots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}} S(n_1,|I_1|)\cdots S(n_p,|I_p|)\frac{|I_1|!\cdots|I_p|!}{m!}$$
$$\mathbb{E}_\mu\left[\int_{\mathbb{X}^m} \varepsilon^+_{x_1}\cdots\varepsilon^+_{x_m}\left(F_1^{n_1}\cdots F_p^{n_p}\prod_{j\in I_1}\mathbf{1}_{A_1}(x_j)\cdots\prod_{j\in I_p}\mathbf{1}_{A_p}(x_j)\right)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_m)\right] \tag{59}$$
$$= \sum_{m=0}^n \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^p \mathbb{E}_\mu\left[\int_{\mathbb{X}^m} \varepsilon^+_{x_1}\cdots\varepsilon^+_{x_m}\left(F_{i_1}^{|P_1^n|}\mathbf{1}_{A_{i_1}}(x_1)\cdots F_{i_m}^{|P_m^n|}\mathbf{1}_{A_{i_m}}(x_m)\right)\right.$$
$$\left.\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_m)\right], \tag{60}$$

where in (59) we made changes of variables in the integral and, in (60), we used the combinatorial identity of Lemma 7 below with $\alpha_{i,j} = \mathbf{1}_{A_i}(x_j)$, $1\leq i\leq p, 1\leq j\leq m$, and $\beta_i = F_i$. The proof is concluded by using the disjunction of the A_i's in (62), as follows:

$$\mathbb{E}_\mu\left[\left(\sum_{i=1}^p F_i \int_{\mathbb{X}} \mathbf{1}_{A_i}(x)\ \eta(\mathrm{d}x)\right)^n\right]$$
$$= \sum_{m=0}^n \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^m} \varepsilon^+_{x_1}\cdots\varepsilon^+_{x_m}\left(\sum_{i=1}^p\left(F_i^{|P_1^n|}\mathbf{1}_{A_i}(x_1)\right)\cdots\sum_{i=1}^p\left(F_i^{|P_m^n|}\mathbf{1}_{A_i}(x_m)\right)\right)\right.$$
$$\left.\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_m)\right] \tag{61}$$

$$= \sum_{m=0}^{n} \sum_{P_1^n \cup \cdots \cup P_m^n = \{1,\ldots,n\}} \mathbb{E}_\mu \left[\int_{\mathbb{X}^m} \varepsilon_{x_1}^+ \cdots \varepsilon_{x_m}^+ \left(\left(\sum_{i=1}^{p} F_i \mathbf{1}_{A_i}(x_1) \right)^{|P_1^n|} \cdots \left(\sum_{i=1}^{p} F_i \mathbf{1}_{A_i}(x_m) \right)^{|P_m^n|} \right) \right.$$
$$\left. \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_m) \right]. \tag{62}$$

The general case is obtained by approximating $u(x,\eta)$ with simple processes. □

The next lemma has been used above in the proof of Theorem 1, cf. Lemma 4.3 of [5], and its proof is given for completeness.

Lemma 7. *Let $m,n,p \in \mathbb{N}$, $(\alpha_{i,j})_{1\le i\le p, 1\le j\le m}$ and $\beta_1,\ldots,\beta_p \in \mathbb{R}$. We have*

$$\sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\le n_1,\ldots,|I_p|\le n_p}} S(n_1,|I_1|)\cdots S(n_p,|I_p|) \times$$
$$\times \frac{|I_1|!\cdots|I_p|!}{m!} \beta_1^{n_1} \Big(\prod_{j\in I_1} \alpha_{1,j} \Big) \cdots \beta_p^{n_p} \Big(\prod_{j\in I_p} \alpha_{p,j} \Big)$$
$$= \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^{p} \beta_{i_1}^{|P_1^n|} \alpha_{i_1,1} \cdots \beta_{i_m}^{|P_m^n|} \alpha_{i_m,m}. \tag{63}$$

Proof. Observe that (18) ensures

$$S(n,|I|)\beta^n \Big(\prod_{j\in I} \alpha_j \Big) = \sum_{\bigcup_{a\in I} P_a=\{1,\ldots,n\}} \prod_{j\in I} (\alpha_j \beta^{|P_j|})$$

for all α_j, $j\in I$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$. We have

$$\sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\le n_1,\ldots,|I_p|\le n_p}} S(n_1,|I_1|)\cdots S(n_p,[I_p|)$$
$$\frac{|I_1|!\cdots|I_p|!}{m!} \beta_1^{n_1} \Big(\prod_{j\in I_1} \alpha_{1,j} \Big) \cdots \beta_p^{n_p} \Big(\prod_{j\in I_p} \alpha_{p,j} \Big)$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\le n_1,\ldots,|I_p|\le n_p}} \frac{|I_1|!\cdots|I_p|!}{m!}$$
$$\left(\sum_{\bigcup_{a\in I_1} P_a^1=\{1,\ldots,n_1\}} \prod_{j_1\in I_1} (\alpha_{1,j_1} \beta_1^{|P_{j_1}^1|}) \right) \cdots \left(\sum_{\bigcup_{a\in I_p} P_a^p=\{1,\ldots,n_p\}} \prod_{j_p\in J_p} (\alpha_{p,j_p} \beta_p^{|P_{j_p}^p|}) \right)$$
$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\ge 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\le n_1,\ldots,|I_p|\le n_p}} \sum_{\bigcup_{a\in I_1} P_a^1=\{1,\ldots,n_1\}} \cdots \sum_{\bigcup_{a\in I_p} P_a^p=\{1,\ldots,n_p\}}$$
$$\frac{|I_1|!\cdots|I_p|!}{m!} \prod_{l=1}^{p} \prod_{j_l\in I_l} (\alpha_{l,j_l} \beta_l^{|P_{j_l}^l|})$$

$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{I_1\cup\cdots\cup I_p=\{1,\ldots,m\}\\ |I_1|\leq n_1,\ldots,|I_p|\leq n_p}} \sum_{\bigcup_{a\in I_1} P_a^1=\{1,\ldots,n_1\}} \cdots \sum_{\bigcup_{a\in I_p} P_a^p=\{1,\ldots,n_p\}} \frac{|I_1|!\cdots|I_p|!}{m!} \prod_{l=1}^p \prod_{j_l\in I_l} \alpha_{l,j_l} \prod_{l=1}^p \prod_{j_l\in I_l} \beta_l^{|P_{j_l}^l|}$$

$$= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{k_1+\cdots+k_p=m\\ 1\leq k_1\leq n_1,\ldots,1\leq k_p\leq n_p}} \sum_{i_1,\ldots,i_m=1}^p \sum_{P_1^1\cup\cdots\cup P_{k_1}^1=\{1,\ldots,n_1\}} \cdots \sum_{P^p_{k_1+\cdots+k_{p-1}+1}\cup\cdots\cup P^p_{k_1+\cdots+k_p}=\{1,\ldots,n_p\}} \prod_{j=1}^m \left(\alpha_{i_j,j}\beta_{i_j}^{|P_j^{i_1}|+\cdots+|P_j^{i_m}|}\right)$$

$$= \sum_{P_1\cup\cdots\cup P_m=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^p \beta_{i_1}^{|P_1|}\alpha_{i_1,1}\cdots\beta_{i_m}^{|P_m|}\alpha_{i_m,m},$$

by a reindexing of the summations and the fact that the reunions of the partitions $P_1^j,\ldots,P_{|I_j|}^j$, $1\leq j\leq p$, of disjoint p subsets of $\{1,\ldots,m\}$ run the partition of $\{1,\ldots,m\}$ when we take into account the choice of the p subsets and the possible length k_j, $1\leq j\leq p$, of the partitions. □

As noted in [5], the combinatorial identity of Lemma 7 also admits a probabilistic proof. Namely given $Z_{\lambda\alpha_1},\ldots,Z_{\lambda\alpha_p}$ independent Poisson random variables with parameters $\lambda\alpha_1,\ldots,\lambda\alpha_p$ we have

$$\begin{aligned}
&\sum_{m=0}^n \lambda^m \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{\substack{k_1+\cdots+k_p=m\\ k_1\leq n_1,\ldots,k_p\leq n_p}} S(n_1,k_1)\cdots S(n_p,k_p)\beta_1^{n_1}\alpha_1^{k_1}\cdots\beta_p^{n_p}\alpha_p^{k_p}\\
&= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!} \sum_{k_1=0}^{n_1} S(n_1,k_1)(\lambda\alpha_1)^{k_1}\cdots \sum_{k_p=0}^{n_p} S(n_p,k_p)(\lambda\alpha_p)^{k_p}\beta_1^{n_1}\cdots\beta_p^{n_p}\\
&= \sum_{\substack{n_1+\cdots+n_p=n\\ n_1,\ldots,n_p\geq 0}} \frac{n!}{n_1!\cdots n_p!}\mathbb{E}[Z_{\lambda\alpha_1}^{n_1}\cdots Z_{\lambda\alpha_p}^{n_p}]\beta_1^{n_1}\cdots\beta_p^{n_p}\\
&= \mathbb{E}\left[\left(\sum_{i=1}^p \beta_i Z_{\lambda\alpha_i}\right)^n\right]\\
&= \sum_{m=0}^n \lambda^m \sum_{P_1^n\cup\cdots\cup P_m^n=\{1,\ldots,n\}} \sum_{i_1,\ldots,i_m=1}^p \beta_{i_1}^{|P_1^n|}\alpha_{i_1}\cdots\beta_{i_m}^{|P_m^n|}\alpha_{i_m},
\end{aligned} \tag{64}$$

since the moment of order n_i of $Z_{\lambda\alpha_i}$ is given by (28) as

$$\mathbb{E}\left[Z_{\lambda\alpha_i}^{n_i}\right] = \sum_{k=0}^{n_i} S(n_i,k)(\lambda\alpha_i)^k.$$

The above relation (64) being true for all λ, this implies (63). Next we specialize the above results to processes of the form $u = \mathbf{1}_A$ where $A(\eta)$ is a random set.

Proposition 6. *For any bounded variable F and random set $A(\eta)$ we have*

$$\mathbb{E}_\mu\left[F\,(\eta(A))^n\right] = \sum_{k=0}^n S(n,k)\mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{s_1}\cdots\varepsilon^+_{s_k}(F\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right].$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}_\mu\left[F\,(\eta(A))^n\right] = \mathbb{E}_\mu\left[F\left(\int_{\mathbb{X}} \mathbf{1}_{A(\eta)}(x)\eta(\mathrm{d}x)\right)^n\right]\\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{s_1}\cdots\varepsilon^+_{s_k}(F\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right]\\
&= \sum_{k=0}^n S(n,k)\mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{s_1}\cdots\varepsilon^+_{s_k}(F\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right].
\end{aligned}$$

□

We also have

$$\begin{aligned}
&\mathbb{E}_\mu\left[F\,(\eta(A))^n\right]\\
&= \sum_{k=0}^n S(n,k) \sum_{\Theta\subset\{1,\ldots,k\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_\Theta(F\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right]\\
&= \sum_{k=0}^n S(n,k)\sum_{l=0}^k \binom{k}{l}\mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_{s_1}\cdots D_{s_l}(F\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right].
\end{aligned}$$

When $\mu(A(\eta))$ is deterministic this yields

$$\begin{aligned}
&\mathbb{E}_\mu\left[(\eta(A))^n\right] = \mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} \mathbf{1}_{A(\eta)}(x)\eta(\mathrm{d}x)\right)^n\right]\\
&= \sum_{k=0}^n S(n,k)\mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{s_1}\cdots\varepsilon^+_{s_k}(\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right]\\
&= \sum_{k=0}^n S(n,k) \sum_{\Theta\subset\{1,\ldots,k\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_\Theta(\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_k)\right]\\
&= \sum_{k=0}^n S(n,k)\sum_{l=0}^k \binom{k}{l}\mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_{s_1}\cdots D_{s_l}(\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_k))\mu(ds_1)\cdots\mu(ds_l)\right]\\
&= \sum_{k=0}^n S(n,k)\sum_{l=0}^k \binom{k}{l}\mathbb{E}_\mu\left[(\mu(A))^{k-l}\int_{\mathbb{X}^k} D_{s_1}\cdots D_{s_l}(\mathbf{1}_{A(\eta)}(s_1)\cdots\mathbf{1}_{A(\eta)}(s_l))\mu(ds_1)\cdots\mu(ds_l)\right].
\end{aligned}$$

4.2 Joint moment identities

In this section we derive a joint moment identity for Poisson stochastic integrals with random integrands, which has been applied to mixing of interacting transformations in [29].

Proposition 7. *Let $u : \mathbb{X} \times \mathbf{N}_\sigma(\mathbb{X}) \longrightarrow \mathbb{R}$ be a measurable process and let $n = n_1 + \cdots + n_p$, $p \geq 1$. We have*

$$\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} u_1(x,\eta)\eta(\mathrm{d}x)\right)^{n_1}\cdots\left(\int_{\mathbb{X}} u_p(x,\eta)\eta(\mathrm{d}x)\right)^{n_p}\right] \tag{65}$$
$$= \sum_{k=1}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{x_1,\ldots,x_k}\left(\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\eta)\right)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\right],$$

where the sum runs over all partitions $P_1^n,\ldots,P_k^n$ of $\{1,\ldots,n\}$ and the power $l^n_{i,j}$ is the cardinal

$$l^n_{i,j} := |P_j^n \cap (n_1+\cdots+n_{i-1}, n_1+\cdots+n_i]|, \qquad i=1,\ldots,k, \quad j=1,\ldots,p,$$

for any $n \geq 1$ such that all terms in the right hand side of (65) are integrable.

Proof. We will show the modified identity

$$\mathbb{E}_\mu\left[F\left(\int_{\mathbb{X}} u_1(x,\eta)\eta(\mathrm{d}x)\right)^{n_1}\cdots\left(\int_{\mathbb{X}} u_p(x,\eta)\eta(\mathrm{d}x)\right)^{n_p}\right] \tag{66}$$
$$= \sum_{k=1}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon^+_{x_1,\ldots,x_k}\left(F\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\eta)\right)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\right],$$

for F a sufficiently integrable random variable, where $n = n_1 + \cdots + n_p$. For $p = 1$ the identity is Theorem 1. Next we assume that the identity holds at the rank $p \geq 1$. Replacing F with $F\left(\int_{\mathbb{X}} u_{p+1}(x,\eta)\eta(\mathrm{d}x)\right)^{n_{p+1}}$ in (66) we get

$$\mathbb{E}_\mu\left[F\left(\int_{\mathbb{X}} u_1(x,\eta)\eta(\mathrm{d}x)\right)^{n_1}\cdots\left(\int_{\mathbb{X}} u_{p+1}(x,\eta)\eta(\mathrm{d}x)\right)^{n_{p+1}}\right]$$
$$= \sum_{k=1}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_{\mathbb{X}^k} \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)$$
$$\mathbb{E}_\mu\left[\varepsilon^+_{x_1,\ldots,x_k}\left(F\left(\int_{\mathbb{X}} u_{p+1}(x,\eta)\eta(\mathrm{d}x)\right)^{n_{p+1}}\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\eta)\right)\right]$$
$$= \sum_{k=1}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_{\mathbb{X}^k} \mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} \varepsilon^+_{x_1,\ldots,x_k} u_{p+1}(x,\eta)\eta(\mathrm{d}x) + \sum_{i=1}^{k} \varepsilon^+_{x_1,\ldots,x_k} u_{p+1}(x_i,\eta)\right)^{n_{p+1}}\right.$$
$$\left.\varepsilon^+_{x_1,\ldots,x_k}\left(F\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\eta)\right)\right]\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)$$

$$= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \sum_{a_0+\cdots+a_k = n_{p+1}} \frac{n_{p+1}!}{a_0!\cdots a_k!} \int_{\mathbb{X}^k} \mathbb{E}_\mu \left[\left(\left(\int_{\mathbb{X}} \varepsilon^+_{x_1,\ldots,x_k} u_{p+1}(x,\eta)\eta(\mathrm{d}x) \right)^{a_0} \right. \right.$$

$$\left. \varepsilon^+_{x_1,\ldots,x_k} \left(F \prod_{j=1}^{k} \left(u_{p+1}^{a_j}(x_j,\eta) \prod_{i=1}^{p} u_i^{l_{i,j}^n}(x_j,\eta) \right) \right) \right] \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)$$

$$= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \sum_{a_0+\cdots+a_k = n_{p+1}} \frac{n_{p+1}!}{a_0!\cdots a_k!} \sum_{j=1}^{a_0} \int_{\mathbb{X}^{k+a_0}} \mathbb{E}_\mu \left[\sum_{Q_j^{a_0} \cup \cdots \cup Q_j^{a_0} = \{1,\ldots,a_0\}} \right.$$

$$\left. \varepsilon^+_{x_1,\ldots,x_{k+a_0}} \left(F \prod_{q=k+1}^{k+a_0} u_{p+1}^{|Q_q^{a_0}|}(x_q,\eta) \prod_{j=1}^{k} \left(u_{p+1}^{a_j}(x_j,\eta) \prod_{i=1}^{p} u_i^{l_{i,j}^n}(x_j,\eta) \right) \right) \right] \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{k+a_0})$$

$$= \sum_{k=1}^{n+n_{p+1}} \sum_{P_1^{n+n_{p+1}} \cup \cdots \cup P_k^{n+n_{p+1}} = \{1,\ldots,n+n_{p+1}\}}$$

$$\mathbb{E}_\mu \left[\int_{\mathbb{X}^k} \varepsilon^+_{x_1,\ldots,x_k} \left(F \prod_{l=1}^{k} \prod_{i=1}^{p+1} u_i^{l_{i,j}^{n+n_{p+1}}}(x_l,\eta) \right) \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k) \right],$$

where the summation over the partitions $P_1^{n+n_{p+1}}, \ldots, P_k^{n+n_{p+1}}$ of $\{1,\ldots,n+n_{p+1}\}$, is obtained by combining the partitions of $\{1,\ldots,n\}$ with the partitions $Q_j^{a_0},\ldots,Q_j^{a_0}$ of $\{1,\ldots,a_0\}$ and $a_1,\ldots,a_k$ elements of $\{1,\ldots,n_{p+1}\}$ which are counted according to $n_{p+1}!/(a_0!\cdots a_k!)$, with

$$l_{p+1,j}^{n+n_{p+1}} = l_{i,j}^n + a_j, \quad 1 \le j \le k, \qquad l_{p+1,j}^{n+n_{p+1}} = l_{i,j}^n + |Q_q^{a_0}|, \quad k+1 \le j \le k+a_0,.$$

□

Note that when $n = 1$, (65) coincides with the classical Mecke [19] identity of Proposition 1.

When $n_1 = \cdots = n_p = 1$ and $p = n$, the result of Proposition 7 reads

$$\mathbb{E}_\mu \left[\int_{\mathbb{X}} u_1(x,\eta)\eta(\mathrm{d}x) \cdots \int_{\mathbb{X}} u_n(x,\eta)\eta(\mathrm{d}x) \right]$$
$$= \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \mathbb{E}_\mu \left[\int_{\mathbb{X}^k} \varepsilon^+_{x_1,\ldots,x_k} \left(\prod_{j=1}^{k} \prod_{i \in P_j^n} u_i(x_j,\eta) \right) \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k) \right],$$

where the sum runs over all partitions $P_1^n,\ldots,P_k^n$ of $\{1,\ldots,n\}$, which coincides with the Poisson version of Theorem 3.1 of [6].

4.3 Invariance and cyclic condition

Using the relation $\varepsilon_x^+ = D_x + I$, the result

$$\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)\right)^n\right] = \sum_{k=1}^n \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} \varepsilon_{s_1}^+\cdots\varepsilon_{s_k}^+ (u_{s_1}^{|P_1^n|}\cdots u_{s_k}^{|P_k^n|})\mu(ds_1)\cdots\mu(ds_k)\right]$$

of Theorem 1 can be rewritten as

$$\begin{aligned}
&\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)\right)^n\right] \\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \sum_{l=0}^k \binom{k}{l} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_{s_1}\cdots D_{s_l}(u_{s_1}^{|P_1^n|}\cdots u_{s_k}^{|P_k^n|})\mu(ds_1)\cdots\mu(ds_k)\right] \\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \sum_{l=0}^k \binom{k}{l} \\
&\quad\times \mathbb{E}_\mu\left[\int_{\mathbb{X}^l} D_{s_1}\cdots D_{s_l}\left(u_{s_1}^{|P_1^n|}\cdots u_{s_l}^{|P_l^n|}\int_{\mathbb{X}} u_{s_{l+1}}^{|P_1^n|}\mu(ds_{l+1})\cdots\int_{\mathbb{X}} u_{s_k}^{|P_k^n|}\mu(ds_k)\right)\mu(ds_1)\cdots\mu(ds_l)\right].
\end{aligned}$$

Next is an immediate corollary of Theorem 1.

Corollary 2. *Suppose that*

a) we have

$$D_{s_1}\cdots D_{s_k}(u_{s_1}\cdots u_{s_k}) = 0, \qquad s_1,\ldots,s_k\in\mathbb{X}, \quad k=1,\ldots,n, \tag{67}$$

b) $\int_{\mathbb{X}} u_s^k \mu(ds)$ *is deterministic for all* $k=1,\ldots,n$.

Then $\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)$ *has cumulants* $\int_{\mathbb{X}} u^k(x,\eta)\mu(\mathrm{d}x)$, $k=1,\ldots,n$.

Proof. We have

$$\begin{aligned}
&\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)\right)^n\right] \\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \sum_{l=0}^{k-1} \binom{k}{l} \mathbb{E}_\mu\left[\int_{\mathbb{X}^k} D_{s_1}\cdots D_{s_l}(u_{s_1}^{|P_1^n|}\cdots u_{s_k}^{|P_k^n|})\mu(ds_1)\cdots\mu(ds_k)\right] \\
&= \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \sum_{l=0}^{k-1} \binom{k}{l} \mathbb{E}_\mu\left[\int_{\mathbb{X}^{k-1}} D_{s_1}\cdots D_{s_l}\left(u_{s_1}^{|P_1^n|}\cdots u_{s_{k-1}}^{|P_{k-1}^n|}\int_{\mathbb{X}} u_s^{|P_k^n|}\mu(\mathrm{d}s)\right)\right. \\
&\quad \left.\mu(ds_1)\cdots\mu(ds_{k-1})\right],
\end{aligned}$$

hence by a decreasing induction on k we can show that the formula

$$\mathbb{E}_\mu\left[\left(\int_{\mathbb{X}} u(x,\eta)\eta(\mathrm{d}x)\right)^n\right] = \sum_{k=0}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_{\mathbb{X}^k} u_{s_1}^{|P_1^n|}\cdots u_{s_k}^{|P_k^n|}\mu(ds_1)\cdots\mu(ds_k)$$
$$= \sum_{k=0}^{n} \sum_{P_1^n\cup\cdots\cup P_k^n=\{1,\ldots,n\}} \int_{\mathbb{X}} u_{s_1}^{|P_1^n|}\mu(ds_1)\cdots\int_{\mathbb{X}} u_{s_k}^{|P_k^n|}\mu(ds_k)$$

holds for the moment of order n and for the moments of lower orders $1,\ldots,n-1$.
□

Note that from the relation

$$D_\Theta(u(x_1,\eta)\cdots u(x_k,\eta)) = \sum_{\Theta_1\cup\cdots\cup\Theta_k=\Theta} D_{\Theta_1}u(x_1,\eta)\cdots D_{\Theta_k}u(x_k,\eta), \qquad (68)$$

where the above sum runs over all (possibly empty) subsets $\Theta_1,\ldots,\Theta_k$ of Θ, in particular when $\Theta=\{1,\ldots,k\}$ we get

$$D_{s_1}\cdots D_{s_k}(u(x_1,\eta)\cdots u(x_k,\eta)) = D_\Theta(u(x_1,\eta)\cdots u(x_k,\eta))$$
$$= \sum_{\Theta_1\cup\cdots\cup\Theta_k=\{1,\ldots,k\}} D_{\Theta_1}u(x_1,\eta)\cdots D_{\Theta_k}u(x_k,\eta),$$

where the sum runs over the (possibly empty) subsets $\Theta_1,\ldots,\Theta_k$ of $\{1,\ldots,k\}$. This shows that we can replace (67) with the condition

$$D_{\Theta_1}u(x_1,\eta)\cdots D_{\Theta_k}u(x_k,\eta)=0, \qquad (69)$$

for all $x_1,\ldots,x_k\in\mathbb{X}$ and all (non empty) subsets $\Theta_1,\ldots,\Theta_k\subset\{x_1,\ldots,x_n\}$, such that $\Theta_1\cup\cdots\cup\Theta_n=\{1,\ldots,n\}$, $k=1,2,\ldots,n$. See Proposition 3.3 of [5] for examples of random mappings that satisfy Condition (69).

References

1. Abramowitz, M. and Stegun, I. (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Volume 55. New York: Dover Publications. 9th Edition.
2. Bassan, B. and Bona, E. (1990). Moments of stochastic processes governed by Poisson random measures. *Comment. Math. Univ. Carolin. 31*(2), 337–343.
3. Bernstein, M. and Sloane, N. J. A. (1995). Some canonical sequences of integers. *Linear Algebra Appl. 226/228*, 57–72.
4. Boyadzhiev, K. (2009). Exponential polynomials, Stirling numbers, and evaluation of some gamma integrals. *Abstr. Appl. Anal.*, Art. ID 168672, 18.
5. Breton, J.-C. and Privault, N. (2014). Factorial moments of point processes. *Stoch. Proc. Appl. 124*, 3412–3428.
6. Decreusefond, L. and Flint, I. (2014). Moment formulae for general point processes. *J. Funct. Anal. 267*, 452–476.

7. Farré, M., Jolis, M., and Utzet, F. (2010). Multiple Stratonovich integral and Hu-Meyer formula for Lévy processes. *Ann. Probab. 38*(6), 2136–2169.
8. Fristedt, B. and Gray, L. (1997). *A modern approach to probability theory*. Probability and its Applications. Boston, MA: Birkhäuser Boston Inc.
9. Gerstenkorn, T. (1983). Relations between the crude, factorial and inverse factorial moments. In *Transactions of the ninth Prague conference on information theory, statistical decision functions, random processes, Vol. A (Prague, 1982)*, pp. 247–252. Dordrecht: Reidel.
10. Isserlis, L. (1918). On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika 12*(1-2), 134–139.
11. Ito, Y. (1988). Generalized Poisson functionals. *Probab. Theory Related Fields 77*, 1–28.
12. Last, G. (2014). Stochastic analysis for Poisson processes. This volume.
13. Last, G. and M. Penrose (2011). Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probab. Theory Related Fields 150*(3-4), 663–690.
14. Lee, Y.-J., Privault, N., and Shih, H.-H. (2012). Clark-Ocone formula by the S-transform on the Poisson white noise space. *Communications on Stochastic Analysis 6*, 513–524.
15. Leonov, V. and Shiryaev, A. (1959). On a method of calculation of semi-invariants. *Theory Probab. Appl. 4*, 319–329.
16. Lukacs, E. (1955). Applications of Faà di Bruno's formula in mathematical statistics. *Am. Math. Mon. 62*, 340–348.
17. Lukacs, E. (1970). *Characteristic Functions*. Hafner Publishing Co., New York. Second edition, revised and enlarged.
18. Madan, D. and Yor, M. (2012). Moments of Wiener integrals for subordinators. *Electron. Commun. Probab. 17*, no. 55, 8.
19. Mecke, J. (1967). Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie Verw. Geb. 9*, 36–58.
20. Nardo, E. D. and Senato, D. (2001). Umbral nature of the Poisson random variables. In *Algebraic combinatorics and computer science*, pp. 245–266. Springer Italia, Milan.
21. Peccati, G. and Taqqu, M. (2011). *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation*. Bocconi & Springer Series. Springer.
22. Pitman, J. (2006). *Combinatorial stochastic processes*, Volume 1875 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002.
23. Privault, N. (2009a). Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris 347*, 1071–1074.
24. Privault, N. (2009b). *Stochastic Analysis in Discrete and Continuous Settings*, Volume 1982 of *Lecture Notes in Math.* Berlin, 309 pp.: Springer-Verlag.
25. Privault, N. (2011). Generalized Bell polynomials and the combinatorics of Poisson central moments. *Electron. J. Combin. 18*(1), Research Paper 54, 10.
26. Privault, N. (2012a). Girsanov identities for Poisson measures under quasi-nilpotent transformations. *Ann. Probab. 40*(3), 1009–1040.
27. Privault, N. (2012b). Invariance of Poisson measures under random transformations. *Ann. Inst. H. Poincaré Probab. Statist. 48*(4), 947–972.
28. Privault, N. (2012c). Moments of Poisson stochastic integrals with random integrands. *Probability and Mathematical Statistics 32*(2), 227–239.
29. Privault, N. (2013). Mixing of Poisson random measures under interacting transformations. arXiv:1301.0672.
30. Roman, S. (1984). *The umbral calculus*, Volume 111 of *Pure and Applied Mathematics*. New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers].
31. Rota, G.-C. and Wallstrom, T. (1997). Stochastic integrals: a combinatorial approach. *Ann. Probab. 25*(3), 1257–1283.
32. Stroock, D. (1987). Homogeneous chaos revisited. In *Séminaire de Probabilités XXI*, Volume 1247 of *Lecture Notes in Mathematics*. Springer Verlag.
33. Thiele, T. (1899). On semi invariants in the theory of observations. *Kjöbenhavn Overs.*, 135–141.