Conditionally Gaussian stochastic integrals

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Abstract - We derive conditional Gaussian type identities of the form

$$E\left[\exp\left(i\int_0^T u_t dB_t\right) \Big| \int_0^T |u_t|^2 dt\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2 dt\right),$$

for Brownian stochastic integrals, under conditions on the process $(u_t)_{t\in[0,T]}$ specified using the Malliavin calculus. This applies in particular to the quadratic Brownian integral $\int_0^t AB_s dB_s$ under the matrix condition $A^\dagger A^2 = 0$, using a characterization of Yor [6].

Intégrales stochastiques conditionnellement gaussiennes

Résumé - Nous obtenons des identités gaussiennes conditionnelles de la forme

$$E\left[\exp\left(i\int_0^T u_t dB_t\right) \Big| \int_0^T |u_t|^2 dt\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2 dt\right),$$

pour les intégrales stochastiques browniennes, sous des conditions sur le processus $(u_t)_{t\in[0,T]}$ exprimées à l'aide du calcul de Malliavin. Ces résultats s'appliquent en particulier à l'intégrale brownienne quadratique $\int_0^t AB_s dB_s$ sous la condition matricielle $A^\dagger A^2 = 0$, en utilisant une caractérisation de Yor [6].

Key words: Quadratic Brownian functionals, multidimensional Brownian motion, moment identities, characteristic functions.
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1 Introduction

Let $(B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. When A is a $d\times d$ skew-symmetric matrix, the identity

$$E\left[\exp\left(i\int_0^T AB_s dB_s\right)\Big| B_t\right] = E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2\, ds\right)\Big| B_t\right], \qquad (1)$$

$0\le t\le T$, has been proved in Theorem 2.1 of [1], extending a formula of [7] for the computation of the characteristic function of Lévy's stochastic area in case $d=2$.

This approach is connected to a result of Yor [6] stating that when $A^\dagger A^2=0$, the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ of $t\mapsto \int_0^t AB_s dB_s$ is generated by k independent Brownian motions, where k is the number of distinct eigenvalues of $A^\dagger A$.

In this Note we derive conditional versions of the identity (1) for the stochastic integral $\int_0^T u_t dB_t$ of an $(\mathcal{F}_t)$-adapted process $(u_t)_{t\in[0,T]}$ in Theorem 1, under conditions formulated in terms of the Malliavin calculus, using the cumulant-moment formula of [3], [4]. In particular we provide conditions for $\displaystyle\int_0^T u_t dB_t$ to be Gaussian $\mathcal{N}\left(0,\displaystyle\int_0^T |u_t|^2 dt\right)$-distributed given $\displaystyle\int_0^T |u_t|^2 dt$, cf. Theorem 2. This holds for example when $(u_t)_{t\in[0,T]}=(AB_t)_{t\in[0,T]}$ under Yor's condition $A^\dagger A^2=0$, cf. Corollary 3. We also consider a weakening of this condition to $A^\dagger A^2$ skew-symmetric, provided that $A^\dagger A$ is proportional to a projection, cf. Corollary 6.

2 Conditional characteristic functions

Let D denote the Malliavin gradient with domain $\mathbb{D}_{2,1}$ on the d-dimensional Wiener space, cf. § 1.2 of [2] for definitions. Taking $H=L^2([0,T];\mathbb{R}^d)$ for some $T>0$ and u in the domain $\mathbb{D}_{k,1}(H)$ of D in $L^k(\Omega;H)$, we let

$$(Du)^k u_t := \int_0^T\cdots\int_0^T (D_{t_k}u_t)^\dagger (D_{t_{k-1}}u_{t_k})^\dagger\cdots(D_{t_1}u_{t_2})^\dagger u_{t_1} dt_1\cdots dt_k, \quad t\in[0,T], \quad k\ge 1.$$

Theorem 1. *Let $u\in\bigcap_{k\ge1}\mathbb{D}_{k,1}(H)$ be an $(\mathcal{F}_t)$-adapted process such that*

$$\langle u_t, (Du)^k u_t\rangle_{\mathbb{R}^d} = 0, \qquad t\in[0,T], \quad k\ge 1.$$

We have

$$E\left[\exp\left(i\int_0^T u_t dB_t\right)\Big| (|u_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2 dt\right), \qquad (2)$$

provided that $\displaystyle\frac{1}{2}\int_0^T |u_t|^2 dt$ is exponentially integrable.

Proof. For any $F \in \mathbb{D}_{2,1}$ and $k \geq 1$, let

$$\Gamma_k^u F := \mathbb{1}_{\{k\geq 2\}} F \int_0^T \langle u_t, (Du)^{k-2} u_t \rangle_{\mathbb{R}^d} dt + \int_0^T \langle D_t F, (Du)^{k-1} u_t \rangle_{\mathbb{R}^d} dt.$$

Recall that for any $u \in \mathbb{D}_{2,1}(H)$ such that $\Gamma_{l_1}^u \cdots \Gamma_{l_k}^u \mathbb{1}$ has finite expectation for all $l_1 + \cdots + l_k \leq n$, $k = 1, \ldots, n$, by Theorem 1 of [3] or Proposition 4.3 of [4] we have

$$E\left[F \left(\int_0^T u_t dB_t \right)^n \right] = n! \sum_{a=1}^n \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F \right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}, \tag{3}$$

for $F \in \mathbb{D}_{2,1}$. Next, for any $f \in \mathcal{C}_b^1(\mathbb{R})$ and $k \geq 1$ we have

$$\begin{aligned}
&\Gamma_k^u f\left(\int_a^b |u_t|^2 dt \right) \\
&= \mathbb{1}_{\{k=2\}} \int_0^T |u_t|^2 dt f\left(\int_a^b |u_t|^2 dt \right) + f'\left(\int_a^b |u_t|^2 dt \right) \int_0^T \left\langle D_t \int_a^b |u_s|^2 ds, (Du)^{k-1} u_t \right\rangle_{\mathbb{R}^d} dt, \\
&= \mathbb{1}_{\{k=2\}} \int_0^T |u_t|^2 dt f\left(\int_a^b |u_t|^2 dt \right) + 2 f'\left(\int_a^b |u_t|^2 dt \right) \int_a^b \langle u_s, (Du)^k u_s \rangle_{\mathbb{R}^d} ds, \\
&= \mathbb{1}_{\{k=2\}} \int_0^T |u_t|^2 dt f\left(\int_a^b |u_t|^2 dt \right), \qquad 0 \leq a \leq b.
\end{aligned}$$

By induction this yields

$$\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F = \mathbb{1}_{\{l_1=\cdots=l_a=2\}} \left(\int_0^T |u_t|^2 dt \right)^a F, \qquad l_1, \ldots, l_a \geq 1, \quad a \geq 1, \tag{4}$$

for any random variable F of the form

$$F = f\left(\int_{a_1}^{b_1} |u_t|^2 dt, \ldots, \int_{a_m}^{b_m} |u_t|^2 dt \right), \quad 0 \leq a_i \leq b_i \leq T, \quad i = 1, \ldots, m,$$

where $f \in \mathcal{C}_b^1(\mathbb{R}^m)$, and by (3) and (4) we find

$$E\left[\left(\int_0^T u_t dB_t \right)^{2n} F \right] = \frac{(2n)!}{2^n n!} E\left[\left(\int_0^T |u_t|^2 dt \right)^n F \right], \tag{5}$$

and $E\left[\left(\int_0^T u_t dB_t \right)^{2n+1} F \right] = 0$ for all $n \in \mathbb{N}$. □

The following result is obtained by an argument similar to the proof of Theorem 1.

Theorem 2. *Let $u \in \bigcap_{k\geq 1} I\!\!D_{k,1}(H)$ be an $(\mathcal{F}_t)$-adapted process such that*

$$\langle u, (Du)^k u\rangle_H = 0, \qquad k \geq 1.$$

We have

$$E\left[\exp\left(i\int_0^T u_t dB_t\right) \middle| \int_0^T |u_t|^2 dt\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2 dt\right),$$

provided that $\frac{1}{2}\int_0^T |u_t|^2 dt$ *is exponentially integrable.*

In the particular case where $u_t = R_t h$, $t \in [0,T]$, $h \in H$, where R is a random, adapted (or quasi-nilpotent) isometry of H, we find that $\int_0^T |u_t|^2 dt = \int_0^T |h(t)|^2 dt$ is deterministic, hence

$$\langle u, (Du)^k u\rangle_H = \frac{1}{2}\langle (Du)^{k-1}u, D\langle u, u\rangle_H\rangle_H = 0, \qquad k \geq 1,$$

and Theorem 2 shows that $\int_0^T (R_t h) dB_t$ has a centered Gaussian distribution with variance $\int_0^T |h(t)|^2 dt$, as in Theorem 2.1-(b) of [5].

Theorems 1 and 2 also apply when $\int_0^T |u_t|^2 dt$ is random, for example when $(u_t)_{t\in[0,T]}$ takes the form $u_t = g(B_t)$, $t \in [0,T]$, where $g \in \mathcal{C}_b^1(\mathbb{R}^d;\mathbb{R}^d)$ satisfies the condition $\langle g(x), ((\nabla g(x))^\dagger)^k g(x)\rangle_{\mathbb{R}^d} = 0$, $x \in \mathbb{R}^d$, $k \geq 1$. Next, we check that this condition is satisfied on concrete examples based on [6], when g is a linear mapping of the form $g(x) = Ax$, $x \in \mathbb{R}^d$.

Vanishing of $A^\dagger A^2$

Applying Theorem 1 to the adapted process $(u_t)_{t\in[0,T]} := (AB_t)_{t\in[0,T]}$ under Yor's [6] condition $A^\dagger A^2 = 0$, by the relation $D_t B_s = \mathbb{1}_{[0,s]}(t) I_{\mathbb{R}^d}$ we obtain the vanishing

$$\begin{aligned}\langle u_t, (Du)^k u_t\rangle_{\mathbb{R}^d} &= \int_0^T \cdots \int_0^T \langle u_t, (D_{t_k}u_t)^\dagger (D_{t_{k-1}}u_{t_k})^\dagger \cdots (D_{t_1}u_{t_2})^\dagger u_{t_1}\rangle_{\mathbb{R}^d} dt_1 \cdots dt_k \\ &= \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} \langle AB_t, (A^\dagger)^k AB_t\rangle_{\mathbb{R}^d} dt_1 \cdots dt_k \\ &= 0, \qquad t \in [0,T], \quad k \geq 1.\end{aligned}$$

This yields the next corollary of Theorem 1, in which the condition $A^\dagger A^2 = 0$ includes 2-nilpotent matrices as a particular case.

Corollary 3. *Assume that $A^\dagger A^2 = 0$. We have*

$$E\left[\exp\left(i\int_0^T AB_t dB_t\right)\Bigg| (|AB_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\,dt\right). \tag{6}$$

Note that the filtration of $(|AB_t|)_{t\in[0,T]}$ coincides with the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ generated by k independent Brownian motions where k is the number of nonzero eigenvalues of $A^\dagger A$, cf. Corollary 2 of [6].

3 Skew-symmetric $A^\dagger A^2$

When $A^\dagger A$ has only one nonzero eigenvalue, i.e. $A^\dagger A$ is proportional to a projection, the condition $A^\dagger A^2 = 0$ can be relaxed using stochastic calculus, by only assuming that $A^\dagger A^2$ is skew-symmetric. We start with the following variation of Corollary 2 of [6].

Lemma 4. *Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue λ_1. Then the processes*

$$Y_t^1 := \frac{1}{\sqrt{\lambda_1}}\int_0^t \frac{AB_s}{|AB_s|}dAB_s, \quad \text{and} \quad Y_t^2 := \int_0^t \frac{AB_s}{|AB_s|}dB_s, \quad t\in[0,T], \tag{7}$$

are independent standard Brownian motions.

Proof. Since $A^\dagger A$ is symmetric it can be written as $A^\dagger A = R^\dagger C R$, where R is orthogonal and C is diagonal, therefore since $(RB_t)_{t\in[0,T]}$ is also a standard Brownian motion we can assume that $A^\dagger A$ has the form $A^\dagger A = (\lambda_k \mathbb{1}_{\{1\le k=l\le r\}})_{1\le k,l\le d}$ with $\lambda_i > 0$, $1\le i\le r$. Clearly $(Y_t^2)_{t\in[0,T]}$ is a standard Brownian motion, and

$$d\langle Y^1, Y^2\rangle_t = \frac{\langle A^\dagger A^2 B_t, B_t\rangle}{|AB_t|^2\sqrt{\lambda_1}}dt = 0.$$

In addition we have $dY_t^1 = \dfrac{\lambda_1^{-1/2}}{|AB_t|}\displaystyle\sum_{i=1}^r \lambda_i B_t^i dB_t^i$ and

$$d\langle Y^1, Y^1\rangle_t = \frac{(\lambda_1 B_t^1)^2 + \cdots + (\lambda_r B_t^r)^2}{\lambda_1(\lambda_1 (B_t^1)^2 + \cdots + \lambda_r (B_t^r)^2)}dt,$$

hence $(Y_t^1)_{t\in[0,T]}$ is also a standard Brownian motion when $\lambda_1 = \cdots = \lambda_r$. □

The following result relaxes the vanishing hypothesis of Corollary 3.

Corollary 5. *Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue λ_1. Then we have*

$$E\left[\exp\left(i\int_0^T AB_t dB_t\right)\Bigg| (|AB_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\,dt\right). \tag{8}$$

Proof. We let $S_t := |AB_t|^2$, $t \in [0,T]$, and note that by Corollary 2 of [6], the filtration generated by $(|AB_t|)_{t\in[0,T]}$ coincides with the filtration $(\mathcal{F}_t^1)_{t\in[0,T]}$ of $(Y_t^1)_{t\in[0,T]}$. Next, Itô's formula shows that

$$S_t = 2\int_0^t AB_s dAB_s + \mathrm{Tr}\left(A^\dagger A\right) t = 2\int_0^t \sqrt{\lambda_1 S_s} dY_s^1 + r\lambda_1 t, \qquad t \in [0,T],$$

hence $(|AB_t|)_{t\in[0,T]}$ is $(\mathcal{F}_t^1)_{t\in[0,T]}$-adapted and therefore independent of $(Y^2)_{t\in[0,T]}$, hence

$$\int_0^T AB_t dB_t = \int_0^T |AB_t|\, dY_t^2$$

is centered Gaussian with variance $\int_0^T |AB_t|^2\, dt$ given $\mathcal{F}_T^1$, which yields (8). □

Commutation with orthogonal matrices

Under the assumptions of Corollaries 3 or 5 it follows that

$$E\left[\exp\left(i\int_0^T AB_t dB_t\right)\middle|\, |AB_t|\right] = E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\, dt\right)\middle|\, |AB_t|\right], \tag{9}$$

since $(|AB_t|)_{t\in[0,T]}$ and $(Y_t^1)_{t\in[0,T]}$ generate the same filtration on $(\mathcal{F}_t^1)_{t\in[0,T]}$.

Corollary 6. *Assume that either $A^\dagger A^2 = 0$, or $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue. If in addition A commutes with orthogonal matrices, then we have*

$$E\left[\exp\left(i\int_0^T AB_s dB_s\right)\middle|\, AB_t\right] = E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2\, ds\right)\middle|\, AB_t\right], \tag{10}$$

$0 \le t \le T$.

Proof. We check that for any $d \times d$ orthogonal matrix R we have

$$E\left[\exp\left(i\int_0^T AB_t dB_t\right)\middle|\, AB_t = Rx\right] = E\left[\exp\left(i\int_0^T AB_t dB_t\right)\middle|\, AB_t = x\right],$$

$x \in \mathbb{R}^d$, which shows that

$$E\left[\exp\left(i\int_0^T AB_t dB_t\right)\middle|\, AB_t\right] = E\left[\exp\left(i\int_0^T AB_t dB_t\right)\middle|\, |AB_t|\right]$$

and similarly for the right hand side, and we conclude by (9). □

Skew-symmetric orthogonal A

We note that when A is skew-symmetric and orthogonal the condition $A^\dagger A^2$ skew-symmetric is satisfied as in this case we have $(A^\dagger A^2)^\dagger = A^\dagger A^\dagger A = A^\dagger = -A = -A^\dagger A^2$, and (10) can be written as

$$E\left[\exp\left(i\int_0^T AB_s dB_s\right)\middle| B_t\right] = E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2 \, ds\right)\middle| B_t\right], \qquad (11)$$

$0 \leq t \leq T$. This holds in particular when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, in which case $A^\dagger A = I_{\mathbb{R}^2}$ has the unique eigenvalue $\lambda_1 = 1$ and $A^\dagger A^2 = A$ is skew-symmetric, in which case we recover the result of [7] which has been used to show that (11) holds when A is skew-symmetric and not necessarily orthogonal in Theorem 2.1 of [1].

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