Cournot games with limited demand: from multiple equilibria to stochastic equilibrium

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Abstract

We construct Cournot games with limited demand, resulting into capped sales volumes according to the respective production shares of the players. We show that such games admit three distinct equilibrium regimes, including an intermediate regime that allows for a range of possible equilibria. When information on demand is modeled by a delayed diffusion process, we also show that this intermediate regime collapses to a single equilibrium while the other regimes approximate the deterministic setting as the delay tends to zero. Moreover, as the delay approaches zero, the unique equilibrium achieved in the stochastic case provides a way to select a natural equilibrium within the range observed in the no lag setting. Numerical illustrations are presented when demand is modeled by an Ornstein-Uhlenbeck process and price is an affine function of output.

Keywords: Game theory; multiple equilibria; equilibrium selection; limited demand; delayed information; stochastic control.

Mathematics Subject Classification (2010): 91A05, 91A10, 91A18, 91B24.

1 Introduction

Cournot games with multiple equilibria under uncertain demand have attracted significant attention in recent years, see for example [4], [3], [2]. On the other hand, the stochastic control of diffusions with delayed information has recently been applied to newsvendor games, cf. [5], [6] in order to deal with partial information. Under model uncertainty, these results have been extended in [7] to the case of delayed coefficients in the equation driving the state process.

In this paper, we construct Cournot games in which limited demand can induce multiple equilibria in absence of lag. This range of equilibria collapses to a single equilibrium when the information on demand is delayed and modeled by a random process. We choose to model limited demand by sharing sales proportionally to production in case total output exceeds demand. This setting differs from the basic model in which all production is sold as long as prices can adjust to their market clearing levels. It also allows for unsold production to be salvaged at a fixed price.

In Section 2 we consider one-shot Cournot games. In this settting, Proposition 2.1 presents a general formula for the individual equilibrium production shares, and we derive an equation satisfied by the total production. When the market price $P(q)$ is of the familiar affine form $P(q) = a - bq$ for $a, b > 0$, we show the existence of three different regimes of equilibria, including an intermediate regime that includes multiple equilibria, cf. Proposition 2.2 and Figures 1 and 2.

Second, in Section 3 we consider a setup in which demand is modeled by a stochastic diffusion process $(D_t)_{t\in[0,T]}$ and information about demand is lagged by a time amount $\delta > 0$ in such a way that firms have to decide on their strategies for time t based on the delayed value $D_{t-\delta}$ of a random demand process. Using the maximum principle for the stochastic control of diffusions with delayed filtrations [5], [6], we derive necessary and sufficient conditions for optimality of controls in this delayed setting. In Proposition 3.2 we present a natural generalization of Proposition 2.1 to the case of delayed controls, based on the derivation of an equation for the total production.

A more explicit formulation is then obtained in Proposition 3.3 when the stochastic demand $(D_t)_{t\in[0,T]}$ is modeled exogenously according to an Ornstein-Uhlenbeck process. Finally, in the setting of affine prices we show in Corollary 3.4 that the ranges of equilibria obtained in Proposition 2.2 reduce to a unique equilibrium at every time, no matter how small the lag. The lagged case can be seen as a smoothing of the no lag case, as the solution of the deterministic no lag problem can be obtained as the limit of the lagged setup as δ tends to 0. In addition, as the delay approaches zero, this also provides an equilibrium selection mechanism in the intermediate regime, within the range observed in the no lag setting. As such, a small quantity of random noise can resolve the absence of uniqueness appearing in the no lag case.

It turns out that when demand is low, the equilibrium is unique in both the lag and no lag cases, and total output exceeds demand (overproduction). When demand is high enough, individual equilibrium outputs are also unique and in line with the usual Cournot equilibrium, and total output falls short of demand (underproduction) in both setups. Finally, when demand is within an intermediate range, the output exactly meeting demand belongs to the range of possible equilibria in the no lag case, while output falls short of demand in the lag case. In contrast, the Stackelberg setting studied in [8] does not give rise to multiple equilibria.

2 Cournot game with limited demand

We consider two competitors facing a demand $d \geqslant 0$ for goods priced $P(q) \geqslant 0$ where $q := q^{(1)} + q^{(2)}$ is the sum of the production outputs $q^{(1)} \geqslant 0$, $q^{(2)} \geqslant 0$ of Players 1 and 2. We choose to model limited demand by assuming that the profit $f^{(i)}(d, q^{(1)}, q^{(2)})$ of player $i = 1, 2$ is given by

$$\begin{aligned} f^{(i)}(d, q^{(1)}, q^{(2)}) &:= P(q^{(1)} + q^{(2)}) \min\left(q^{(i)}, \frac{q^{(i)} d}{q^{(1)} + q^{(2)}}\right) - c^{(i)} q^{(i)} \qquad (2.1) \\ &= P(q^{(1)} + q^{(2)}) \left(q^{(i)} \mathbf{1}_{\{q^{(1)}+q^{(2)} \leqslant d\}} + \frac{q^{(i)} d}{q^{(1)} + q^{(2)}} \mathbf{1}_{\{d < q^{(1)}+q^{(2)}\}}\right) - c^{(i)} q^{(i)}, \end{aligned}$$

where $c^{(i)} > 0$ is firm i's per unit production cost, $i = 1, 2$. In other words, when the total output $q^{(1)} + q^{(2)}$ becomes larger than d, the demand is shared between Players 1 and 2 according to their respective shares $q^{(1)}/(q^{(1)} + q^{(2)})$ and $q^{(2)}/(q^{(1)} + q^{(2)})$ of total production. The intuition is that a firm that produces more output generates more exposure, so if a buyer would randomly buy the product, the probability that firm i makes a sale is $q^{(i)}/(q^{(1)} + q^{(2)})$, $i = 1, 2$.

In (2.1) we assume the retail price $P(q^{(1)} + q^{(2)})$ to be a decreasing function of total output. When $q^{(1)} + q^{(2)} > d$, this means that excess production may force down retail prices, which may further increase consumer surplus. As in [5], we will assume that unsold items are salvaged at a constant unit price $S \geqslant 0$ which can be incorporated into the constants $c^{(1)}$, $c^{(2)}$ and the price function $P(q)$ without loss of generality.

A Nash equilibrium for the game (2.1) is a pair $(q^{(1)*}, q^{(2)*})$ such that

$$f^{(1)}(d, q^{(1)}, q^{(2)*}) \leqslant f^{(1)}(d, q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} \geqslant 0, \tag{2.2}$$

and

$$f^{(2)}(d, q^{(1)*}, q^{(2)}) \leqslant f^{(2)}(d, q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} \geqslant 0, \tag{2.3}$$

so that neither player has an incentive to deviate given the control of the other.
In Proposition 2.1 we present a general characterization of equilibrium for the game (2.1) under limited demand, when $f^{(i)}(d, q^{(1)}, q^{(2)})$, $i = 1, 2$, are concave functions of $(q^{(1)}, q^{(2)})$.

Proposition 2.1 *Any Cournot equilibrium action $(q^{(1)*}, q^{(2)*})$ for the game (2.1) satisfies*

$$q^{(i)*} = S^* \frac{\min(d, S^*)P(S^*) - c^{(i)}S^*}{2\min(d, S^*)P(S^*) - (c^{(1)} + c^{(2)})S^*}, \qquad 1 \leqslant i \neq j \leqslant 2, \tag{2.4}$$

provided that $S^* := q^{(1)*} + q^{(2)*}$ satisfies $S^* \neq d$ and solves the equation

$$S^* P'(S^*) + P(S^*) = (c^{(1)} + c^{(2)}) \max(1, S^*/d) + P(S^*)\mathbf{1}_{\{S^* > d\}}.$$

Proof. Consider the partial derivatives

$$\begin{aligned}
\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) &= \big(q^{(i)}P'(q^{(1)} + q^{(2)}) + P(q^{(1)} + q^{(2)})\big)\,\mathbf{1}_{\{q^{(1)}+q^{(2)} \leqslant d\}} \\
&\quad + \left(\frac{q^{(i)}P'(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}}d + \frac{q^{(j)}P(q^{(1)} + q^{(2)})}{(q^{(1)} + q^{(2)})^2}d\right)\mathbf{1}_{\{d < q^{(1)}+q^{(2)}\}} - c^{(i)} \\
&= \left(\frac{q^{(i)}P'(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}} + \frac{q^{(j)}P(q^{(1)} + q^{(2)})}{(q^{(1)} + q^{(2)})^2}\right)\min\big(q^{(1)} + q^{(2)}, d\big) \\
&\quad + \frac{q^{(i)}P(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}}\mathbf{1}_{\{q^{(1)}+q^{(2)} \leqslant d\}} - c^{(i)}, \qquad 1 \leqslant i \neq j \leqslant 2.
\end{aligned}$$

a) When $q^{(1)*} + q^{(2)*} < d$, the first order conditions become

$$q^{(i)*}P'(q^{(1)*} + q^{(2)*}) + P(q^{(1)*} + q^{(2)*}) - c^{(i)} = 0, \quad i = 1, 2,$$

and the equilibrium agrees with the "usual" Cournot equilibrium. Namely, we have

$$\begin{aligned} q^{(i)*} &= \frac{c^{(i)} - P(q^{(1)*} + q^{(2)*})}{P'(q^{(1)*} + q^{(2)*})} \\ &= (q^{(1)*} + q^{(2)*}) \frac{(q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2(q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})}, \end{aligned}$$

$i = 1, 2$, and by summation we find

$$(q^{(1)*} + q^{(2)*})P'(q^{(1)*} + q^{(2)*}) + 2P(q^{(1)*} + q^{(2)*}) = c^{(1)} + c^{(2)}.$$

b) On the other hand, if $d < q^{(1)*} + q^{(2)*}$ we have

$$q^{(i)*}P'(q^{(1)*} + q^{(2)*}) + \frac{q^{(j)*}}{q^{(1)*} + q^{(2)*}} P(q^{(1)*} + q^{(2)*}) = \frac{c^{(i)}}{d}(q^{(1)*} + q^{(2)*}), \qquad 1 \leqslant i \neq j \leqslant 2,$$

hence by summation we have

$$(q^{(1)*} + q^{(2)*})P'(q^{(1)*} + q^{(2)*}) + P(q^{(1)*} + q^{(2)*}) = \frac{c^{(1)} + c^{(2)}}{d}(q^{(1)*} + q^{(2)*}),$$

and

$$q^{(i)*} = (q^{(1)*} + q^{(2)*}) \frac{dP(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2dP(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})}, \qquad 1 \leqslant i \neq j \leqslant 2.$$

In general, we conclude that

$$S^* P'(S^*) + 2P(S^*) = (c^{(1)} + c^{(2)}) \max\left(1, \frac{S^*}{d}\right) + P(S^*)\mathbf{1}_{\{S^* > d\}},$$

and

$$q^{(i)*} = (q^{(1)*} + q^{(2)*}) \frac{\min(d, q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2\min(d, q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})},$$

$1 \leqslant i \neq j \leqslant 2$. □

The setting of Proposition 2.1 allows for negative values of the equilibrium production outputs $q^{(1)*}$ and $q^{(2)*}$ and it does not treat the case where production $q^{(1)*}$ and $q^{(2)*}$ meets the demand d, as the derivatives of the profit functions $f^{(i)}(d, q^{(1)}, q^{(2)})$, $i = 1, 2$, are not continuous at this point, and more information on the function $P(q)$ is needed. In Proposition 2.2 below we will refine the result of Proposition 2.1 by constraining production to be nonnegative when the market price $P(q)$ is an affine function of the total output q.

Affine prices

Until the end of this section we focus on the case

$$P(q^{(1)} + q^{(2)}) = a - b(q^{(1)} + q^{(2)}), \tag{2.5}$$

with $a > 0$ and $b \geqslant 0$. We set the equilibrium output $q^{(i)*} := 0$ if the production cost $c^{(i)}$ is larger than $a > 0$, henceforth we focus on the case where $0 < c^{(i)} < a$ for $i = 1, 2$. Similarly, negative prices cannot lead to an equilibrium. The next proposition shows the existence of three equilibrium regimes, depending on the level of the demand.

Proposition 2.2 *Assume that $P(q)$ is given by (2.5), and that*

$$a - 2c^{(1)} + c^{(2)} \geqslant 0 \quad \text{and} \quad a + c^{(1)} - 2c^{(2)} \geqslant 0,$$

and let

$$\underline{d} := \frac{a - c^{(1)} - c^{(2)}}{2b} \quad \text{and} \quad \overline{d} := \frac{2a - c^{(1)} - c^{(2)}}{3b}.$$

a) In case $d > \overline{d}$, we have the unique equilibrium

$$(q^{(1)*}, q^{(2)*}) = \left(\frac{a - 2c^{(1)} + c^{(2)}}{3b}, \frac{a + c^{(1)} - 2c^{(2)}}{3b} \right) = \left(2(\overline{d} - \underline{d}) - \frac{c^{(1)}}{b}, 2(\overline{d} - \underline{d}) - \frac{c^{(2)}}{b} \right), \tag{2.6}$$

with $q^{(1)*} + q^{(2)*} = \overline{d} < d$.

b) In case $d < \underline{d}$, we have the unique equilibrium

$$(q^{(1)*}(d), q^{(2)*}(d)) = \left(\frac{ad(c^{(2)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2}, \frac{ad(c^{(1)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2} \right), \tag{2.7}$$

with $d < q^{(1)*}(d) + q^{(2)*}(d) = \dfrac{ad}{c^{(1)} + c^{(2)} + 2bd}$.

c) In case $\underline{d} \leqslant d \leqslant \overline{d}$, a point $(q^{(1)*}(d), q^{(2)*}(d))$ is an equilibrium if and only if

$$\max \left(\left(2(d - \underline{d}) - \frac{c^{(i)}}{b} \right)^+, \frac{(a - c^{(i)} - bd)d}{a} \right) \leqslant q^{(i)*}(d) \leqslant \min \left(\frac{(c^{(j)} + bd)d}{a}, 2\underline{d} - d + \frac{c^{(j)}}{b} \right), \tag{2.8}$$

with $q^{(1)*}(d) + q^{(2)*}(d) = d$, $1 \leqslant i \neq j \leqslant 2$.

The proof of Proposition 2.2 is deferred to the appendix.

Remarks.

1) We check that when $d = \overline{d}$, the lower bound in (2.8) reads

$$\frac{a - 2c^{(i)} + c^{(j)}}{3b} \leqslant q^{(i)*},$$

which is consistent with the equality (2.6).

2) When $d = \underline{d}$, the upper bound in (2.8) is also consistent with the solution

$$(q^{*1}(\underline{d}), q^{*2}(\underline{d})) = \left(\frac{(c^{(2)} + b\underline{d})\underline{d}}{a}, \frac{(c^{(1)} + b\underline{d})\underline{d}}{a} \right)$$

given in (2.7).

3) When $\underline{d} < d < \overline{d}$, the condition $q^{(1)*}(d) + q^{(2)*}(d) = d$, $1 \leqslant i \neq j \leqslant 2$, shows that only one of the two controls $q^{(1)*}(d)$, $q^{(2)*}(d)$ can be chosen independently.

The following Figures 1 and 2 illustrate the three different regimes (a)-(b)-(c) of Proposition 2.2, including the existence of a range of equilibria that result from (2.8) in the regime (c) when $\underline{d} \leqslant d \leqslant \overline{d}$.

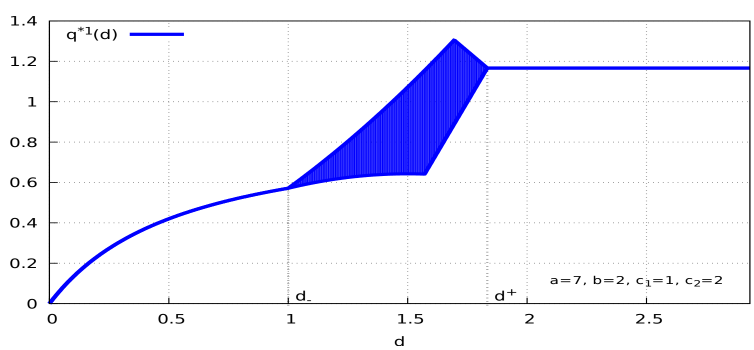

Figure 1: Equilibrium output $q^{(1)*}(d)$ as a function of d.

Figure 3 shows that we move from overproduction to underproduction as the value of d increases, while demand is exactly met in the regime (c) when $\underline{d} \leqslant d \leqslant \overline{d}$.

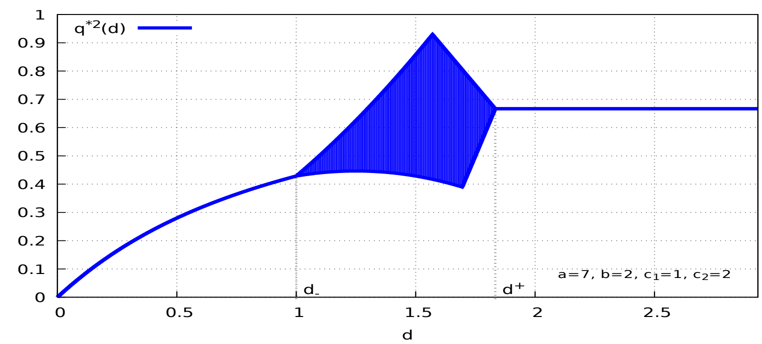

Figure 2: Equilibrium output $q^{(2)*}(d)$ as a function of d.

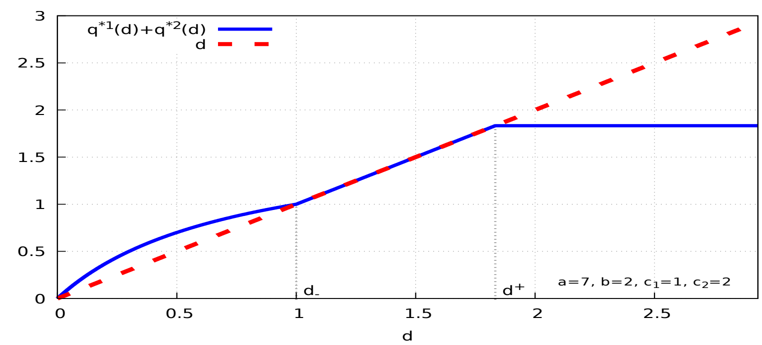

Figure 3: Total output $d \mapsto q^{(1)*}(d) + q^{(2)*}(d)$.

The parametric plot of Figure 4 shows the possible equilibrium quantities $(q^{(1)*}(d), q^{(2)*}(d))$ depending on the level of demand $d > 0$. We check that every value of $d \in [0, \underline{d}] \cup [\overline{d}, \infty)$ correspond to a unique choice of $(q^{(1)*}(d), q^{(2)*}(d))$, whereas the values $d \in (\underline{d}, \overline{d})$ yield a continuum of equilibria located on the isolines $q^{(1)*}(d) + q^{(2)*}(d) = d$.

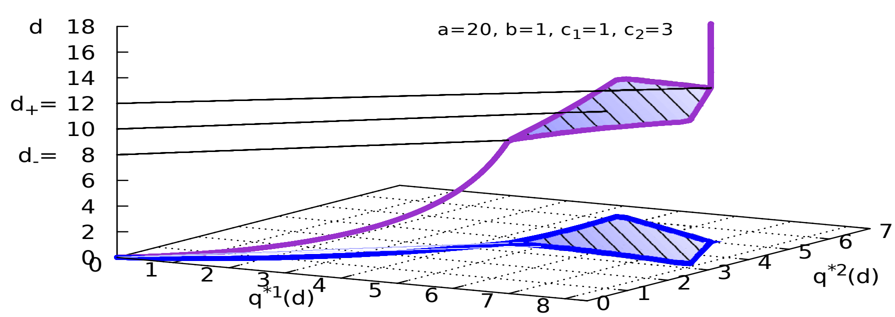

Figure 4: Parametric plot of $(q^{(1)*}(d), q^{(2)*}(d), d)$ with its projection onto the plane $d = 0$.

The next three Figures 5, 6 and 7 present the corresponding profits $f^{(i)}(d, q^{(1)}, q^{(2)})$ of players $i = 1, 2$ as given from (2.1), with $a = 7$, $b = 2$, $c^{(1)} = 1$, $c^{(2)} = 2$.

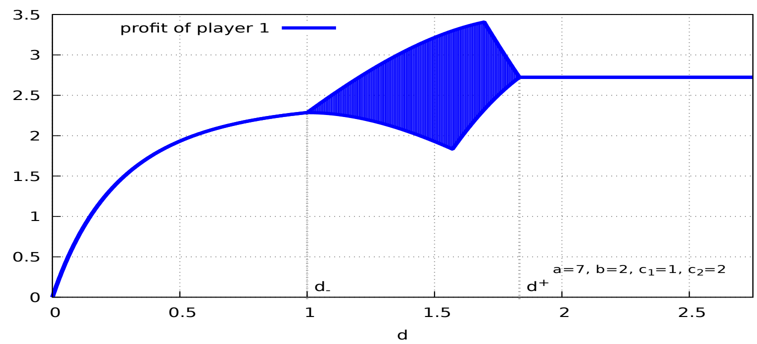

Figure 5: Profit of Player 1.

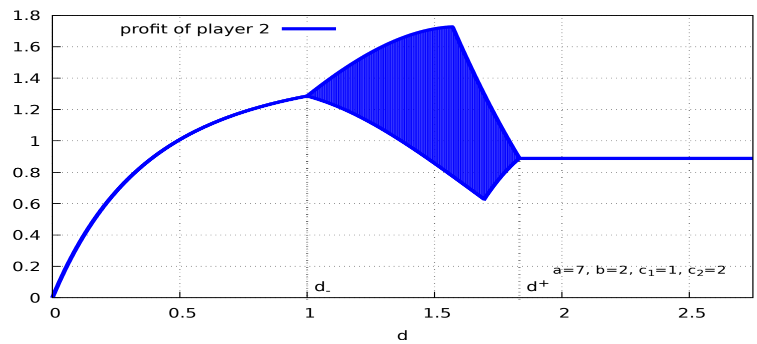

Figure 6: Profit of Player 2.

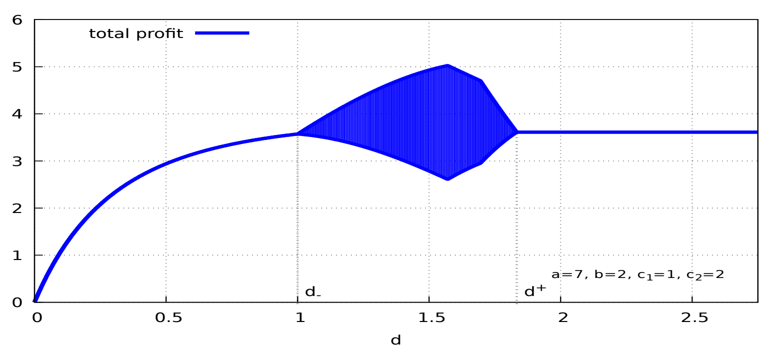

Figure 7: Total profit.

The setting of this section can be made time-dependent with demand $d_t \geqslant 0$ at time $t > 0$, where $q_t^{(1)} \geqslant 0$, $q_t^{(2)} \geqslant 0$ denote the production outputs of Players 1 and 2. In this case,

$$J^{(i)}(q^{(1)}, q^{(2)}) := \int_0^T f^{(i)}(d_t, q_t^{(1)}, q_t^{(2)}) \mathrm{d}t, \tag{2.9}$$

with $(q^{(1)}, q^{(2)}) := (q_t^{(1)}, q_t^{(2)})_{t\in[0,T]}$, denotes the total profit of Player $i = 1, 2$, and a Nash equilibrium for the game (2.9) is a pair $(q^{(1)*}, q^{(2)*}) = (q_t^{(1)*}, q_t^{(2)*})_{t\in[0,T]}$ such that

$$J^{(1)}(q^{(1)}, q^{(2)*}) \leqslant J^{(1)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} = (q_t^{(1)})_{t\in[0,T]}, \tag{2.10}$$

and

$$J^{(2)}(q^{(1)*}, q^{(2)}) \leqslant J^{(2)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} = (q_t^{(2)})_{t\in[0,T]}. \tag{2.11}$$

In the next section we will consider such a time-dependent setting in which competitors face uncertain stochastic demand and make optimal decisions based on historical data.

3 Cournot game with delayed information

Let $(B_t)_{t\in[0,T]}$ denote a standard Brownian motion that generates the $\mathbb{P}$-augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The information flow available to the players is represented by the delayed filtration $(\mathcal{E}_t)_{t\in[\delta,T]}$ defined by

$$\mathcal{E}_t := \mathcal{F}_{t-\delta}, \qquad t \in [\delta, T], \tag{3.1}$$

where $\delta > 0$ denotes the information delay or lag.

We let $\mathcal{A}^{(i)}$ denote a given set of admissible control processes for Player i, which are made of real-valued $(\mathcal{E}_t)_{t\in[\delta,T]}$-predictable processes, $i = 1, 2$. We endow $\mathcal{A}^{(i)}$ with the supremum norm and denote $\mathcal{A} := \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$.

In this section we introduce uncertainty in the demand, which is now modeled as the solution $(D_t)_{t\in[0,T]}$ of a stochastic differential equation of the form

$$\begin{cases} dD_t = \mu_t(D_t)\mathrm{d}t + \sigma_t(D_t)\mathrm{d}B_t, \\ D_0 = d_0 > 0, \end{cases} \tag{3.2}$$

where $\mu_\cdot(d)$, $\sigma_\cdot(d)$ are given predictable processes for each $d \in \mathbb{R}$, which do not depend on the production output control processes $(q^{(1)}, q^{(2)})$, and such that (3.2) admits a unique strong solution.

Given $f^{(i)} : \mathbb{R}^2 \to \mathbb{R}$, the profit function of Player i defined in (2.1), $i = 1, 2$, $t \in [0, T]$, let

$$J^{(i)}(q^{(1)}, q^{(2)}) := \mathbb{E}\left[\int_\delta^T e^{-rt} f^{(i)}\big(D_t, q_t^{(1)}, q_t^{(2)}\big)\mathrm{d}t\right], \tag{3.3}$$

denote the discounted expected profits of Players 1 and 2, where D_t denotes the solution of (3.2), provided the integrals and expectations exist.

A Nash equilibrium for the game (3.3) is a pair $(q^{(1)*}, q^{(2)*}) \in \mathcal{A}$ such that

$$J^{(1)}(q^{(1)}, q^{(2)*}) \leqslant J^{(1)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} \in \mathcal{A}^{(1)}, \tag{3.4}$$

and

$$J^{(2)}(q^{(1)*}, q^{(2)}) \leqslant J^{(2)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} \in \mathcal{A}^{(2)}, \tag{3.5}$$

so that neither player has an incentive to deviate given the control of the other.

Maximum principle

We rely on the framework developed in [6], under the following conditions on the set $\mathcal{A} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$ of control processes.

(A1) For all $u^{(i)} \in \mathcal{A}^{(i)}$ and all bounded $\beta^{(i)} \in \mathcal{A}^{(i)}$ there exists $\varepsilon > 0$ such that

$$u^{(i)} + s\beta^{(i)} \in \mathcal{A}^{(i)} \text{ for all } s \in (-\varepsilon, \varepsilon); \quad i = 1, 2.$$

(A2) For all $t_0 \in [\delta, T]$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables $\alpha^{(i)}$, the control process $\beta_t^{(i)}$ defined by

$$\beta_t^{(i)} = \mathbf{1}_{[t_0,T]}(t)\alpha^{(i)}(\omega), \qquad t \in [\delta, T],$$

belongs to $\mathcal{A}^{(i)}$; $i = 1, 2$.

Define the Hamiltonians $H_t^{(i)} : \mathbb{R}^5 \longrightarrow \mathbb{R}$ of this game by

$$H_t^{(i)}(d, q^{(1)}, q^{(2)}, y^{(i)}, z^{(i)}) := f^{(i)}(d, q^{(1)}, q^{(2)}) + y^{(i)}\mu_t(d) + z^{(i)}\sigma_t(d), \tag{3.6}$$

$i = 1, 2$. To these Hamiltonians we associate the backward SDEs

$$\begin{cases} dY_t^{(i)} = -\dfrac{\partial H_t^{(i)}}{\partial d}(D_t, q_t^{(1)}, q_t^{(2)}, Y_t^{(i)}, Z_t^{(i)})\mathrm{d}t + Z_t^{(i)}\mathrm{d}B_t, \qquad t \in [\delta, T], \\ \\ Y_T^{(i)} = 0 \end{cases} \tag{3.7}$$

in the adjoint processes $Y_t^{(i)}$ and $Z_t^{(i)}$, $i = 1, 2$, which can be solved under the assumptions (A1)-(A2) above. We will rely on the necessary maximum principle for

Forward-Backward Stochastic Differential Equation (FBSDE) games, cf. Theorem 3.1 in [1] and Theorem 2.2 in [6]. In Theorem 3.1 below we specialize Theorem 2.2 of [6] to the case of uncontrolled demand D_t as this is the case in (3.2), and we refer to Theorem 2.2 in [6] for the general case.

Theorem 3.1 *Under (A1) and (A2), suppose that* $(q^{(1)}, q^{(2)}) \in \mathcal{A}$ *and that the solutions* D_t, $Y_t^{(i)}$, $Z_t^{(i)}$ *of Equation* (3.7) *are square-integrable and differentiable in the directions of the control processes in* $\mathcal{A}$. *Then the following two statements are equivalent:*

(1) For all bounded $\beta^{(1)} \in \mathcal{A}^{(1)}$, $\beta^{(2)} \in \mathcal{A}^{(2)}$ *we have*

$$\frac{d}{ds} J^{(1)}(q^{(1)} + s\beta^{(1)}, q^{(2)})_{|s=0} = \frac{d}{ds} J^{(2)}(q^{(1)}, q^{(2)} + s\beta^{(2)})_{|s=0} = 0.$$

(2) We have

$$\mathbb{E}\left[\frac{\partial H_t^{(i)}}{\partial u^{(i)}}(D_t, u^{(1)}, u^{(2)}, Y_t^{(i)}, Z_t^{(i)}) \middle| \mathcal{E}_t \right]_{u^{(1)}=q_t^{(1)}, u^{(2)}=q_t^{(2)}} = 0, \quad i = 1, 2, \quad t \in [\delta, T]. \tag{3.8}$$

Cournot equilibrium

In particular, when $f^{(i)}$ is given by

$$f^{(i)}(d, q^{(1)}, q^{(2)}) = P(q^{(1)} + q^{(2)}) \left(q^{(i)} \mathbf{1}_{\{q^{(1)}+q^{(2)} \leqslant d\}} + \frac{q^{(i)} d^+}{q^{(1)} + q^{(2)}} \mathbf{1}_{\{d < q^{(1)}+q^{(2)}\}} \right) - c^{(i)} q^{(i)},$$

$i = 1, 2$, by (3.6) we find that (3.8) reads

$$\begin{aligned}
&\mathbb{E}\left[\frac{\partial H_t^{(i)}}{\partial r^{(i)}}(D_t, r^{(i)}, q_t^{(j)}, Y_t^{(i)}, Z_t^{(i)}) \middle| \mathcal{E}_t \right]_{r^{(i)}=q_t^{(i)}} \\
&= \mathbb{E}\left[\frac{\partial f^{(i)}}{\partial r^{(i)}}(D_t, r^{(1)}, r^{(2)}) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} \\
&= \mathbb{E}\Bigg[P'(r^{(1)} + r^{(2)}) \left(r^{(i)} \mathbf{1}_{\{D_t \geqslant r^{(1)}+r^{(2)}\}} + \frac{r^{(i)}}{r^{(1)} + r^{(2)}} D_t^+ \mathbf{1}_{\{D_t < r^{(1)}+r^{(2)}\}} \right) \\
&\quad + P(r^{(1)} + r^{(2)}) \left(\mathbf{1}_{\{D_t \geqslant r^{(1)}+r^{(2)}\}} + \frac{r^{(j)}}{(r^{(1)} + r^{(2)})^2} D_t^+ \mathbf{1}_{\{D_t < r^{(1)}+r^{(2)}\}} \right) \Bigg]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} - c^{(i)} \\
&= 0, \qquad 1 \leqslant i \neq j \leqslant 2.
\end{aligned}$$

Hence from the maximum principle we must have

$$\Big(P(q_t^{(1)}+q_t^{(2)})+q_t^{(i)}P'(q_t^{(1)}+q_t^{(2)})\Big)\,\mathbb{E}\left[\mathbf{1}_{\{q_t^{(1)}+q_t^{(2)}\leqslant D_t\}}\middle|\mathcal{E}_t\right] \tag{3.9}$$

$$+\left(q_t^{(j)}\frac{P(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})^2}+q_t^{(i)}\frac{P'(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}}\right)\mathbb{E}\left[D_t\mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\middle|\mathcal{E}_t\right]=c^{(i)},$$

$1\leqslant i\neq j\leqslant 2$.

The next Proposition 3.2 is the counterpart of Proposition 2.1 in the delayed setting. In the following we focus on nonnegative equilibrium outputs, $(q_t^{(1)}, q_t^{(2)})$ for all $t\in[\delta,T]$, which are economically relevant, although $\mathcal{A}^{(i)}$ contains functions that may take negative values.

Proposition 3.2 *Any Cournot equilibrium action* $(q^{(1)*}, q^{(2)*})$ *satisfies*

$$q_t^{(i)*}=S_t^*\frac{\mathbb{E}\left[P(S_t^*)\min(D_t,S_t^*)-c^{(i)}S_t^*\mid\mathcal{E}_t\right]}{\mathbb{E}\left[2P(S_t^*)\min(D_t,S_t^*)-(c^{(1)}+c^{(2)})S_t^*\mid\mathcal{E}_t\right]},\qquad i=1,2,\quad t\in[\delta,T], \tag{3.10}$$

where $S_t^*:=q_t^{(1)*}+q_t^{(2)*}$ *solves the equation*

$$S_t^*P'(S_t^*)\,\mathbb{E}\left[\min(D_t,S_t^*)\mid\mathcal{E}_t\right]+2P(S_t^*)\,\mathbb{E}\left[\min(D_t,S_t^*)\mid\mathcal{E}_t\right]$$
$$=(c^{(1)}+c^{(2)})S_t^*+P(S_t^*)\,\mathbb{E}\left[D_t\mathbf{1}_{\{D_t\leqslant S_t^*\}}\mid\mathcal{E}_t\right].$$

Proof. We apply the maximum principle Theorem 3.1 to the functional $J^{(i)}(q^{(1)},q^{(2)})$ given in (3.3). Namely, taking $f^{(i)}$ in (3.3) given by

$$f^{(i)}(d,q^{(1)},q^{(2)})=P(q^{(1)}+q^{(2)})\left(q^{(i)}\mathbf{1}_{\{q^{(1)}+q^{(2)}\leqslant d\}}+\frac{q^{(i)}d^+}{q^{(1)}+q^{(2)}}\mathbf{1}_{\{d<q^{(1)}+q^{(2)}\}}\right)-c^{(i)}q^{(i)},$$

$i=1,2$, by (3.6) we find that (3.9) reads

$$\mathbb{E}\left[\frac{\partial H_t^{(i)}}{\partial r^{(i)}}(D_t,r^{(i)},q_t^{(j)},Y_t^{(i)},Z_t^{(i)})\middle|\mathcal{E}_t\right]_{r^{(i)}=q_t^{(i)}}$$
$$=\mathbb{E}\left[\frac{\partial f^{(i)}}{\partial r^{(i)}}(D_t,r^{(i)},r^{(j)})\right]_{r^{(1)}=q_t^{(1)},r^{(2)}=q_t^{(2)}}$$
$$=\mathbb{E}\Bigg[P'(r^{(1)}+r^{(2)})\left(r^{(i)}\mathbf{1}_{\{D_t\geqslant r^{(1)}+r^{(2)}\}}+\frac{r^{(i)}}{r^{(1)}+r^{(2)}}D_t^+\mathbf{1}_{\{D_t<r^{(1)}+r^{(2)}\}}\right)$$
$$+P(r^{(1)}+r^{(2)})\left(\mathbf{1}_{\{D_t\geqslant r^{(1)}+r^{(2)}\}}+\frac{r^{(j)}}{(r^{(1)}+r^{(2)})^2}D_t^+\mathbf{1}_{\{D_t<r^{(1)}+r^{(2)}\}}\right)\Bigg]_{r^{(1)}=q_t^{(1)},r^{(2)}=q_t^{(2)}}$$

$$- c^{(i)}$$

$$= 0, \qquad 1 \leqslant i \neq j \leqslant 2.$$

Hence from the maximum principle we must have

$$\begin{aligned}
&\Big(P(q_t^{(1)}+q_t^{(2)}) + q_t^{(i)} P'(q_t^{(1)}+q_t^{(2)})\Big)\,\mathbb{E}\left[\mathbf{1}_{\{q_t^{(1)}+q_t^{(2)}\leqslant D_t\}}\middle|\mathcal{E}_t\right] \\
&+\left(q_t^{(j)}\frac{P(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})^2} + q_t^{(i)}\frac{P'(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}}\right)\mathbb{E}\left[D_t\mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\middle|\mathcal{E}_t\right] = c^{(i)},
\end{aligned} \tag{3.11}$$

$1 \leqslant i \neq j \leqslant 2$. By summing the above two equations we find that the sum $S_t^* := q_t^{(1)*} + q_t^{(2)*}$ satisfies the equation

$$\begin{aligned}
&\Big(2P(q_t^{(1)}+q_t^{(2)}) + (q_t^{(1)}+q_t^{(2)})P'(q_t^{(1)}+q_t^{(2)})\Big)\,\mathbb{E}\left[\mathbf{1}_{\{q_t^{(1)}+q_t^{(2)}\leqslant D_t\}}\middle|\mathcal{E}_t\right] \\
&+\left(\frac{P(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}} + P'(q_t^{(1)}+q_t^{(2)})\right)\mathbb{E}\left[D_t\mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\middle|\mathcal{E}_t\right] = c^{(1)}+c^{(2)}.
\end{aligned} \tag{3.12}$$

or equivalently,

$$\begin{aligned}
&2P(S_t^*)\,\mathbb{E}\left[\min(D_t,S_t^*)\mid\mathcal{E}_t\right] - (c^{(1)}+c^{(2)})S_t^* \\
&= \; P(S_t^*)\,\mathbb{E}\left[D_t\mathbf{1}_{\{D_t\leqslant S_t^*\}}\mid\mathcal{E}_t\right] - S_t^*P'(S_t^*)\,\mathbb{E}\left[\min(D_t,S_t^*)\mid\mathcal{E}_t\right].
\end{aligned} \tag{3.13}$$

By (3.11) and (3.12) we get

$$\begin{aligned}
&\Big(-P(q_t^{(1)}+q_t^{(2)}) - q_t^{(j)}P'(q_t^{(1)}+q_t^{(2)})\Big)\,\mathbb{E}\left[\mathbf{1}_{\{q_t^{(1)}+q_t^{(2)}\leqslant D_t\}}\middle|\mathcal{E}_t\right] \\
&+\left(q_t^{(j)}\frac{P(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})^2} - \frac{P(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}} - q_t^{(j)}\frac{P'(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}}\right)\mathbb{E}\left[D_t\mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\middle|\mathcal{E}_t\right] \\
&= -c^{(j)},
\end{aligned} \tag{3.14}$$

hence

$$\begin{aligned}
&q_t^{(j)}\frac{P(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})^2}\,\mathbb{E}\left[D_t\mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\middle|\mathcal{E}_t\right] - q_t^{(j)}P'(q_t^{(1)}+q_t^{(2)})\,\mathbb{E}\left[\min\left(1,\frac{D_t}{q_t^{(1)}+q_t^{(2)}}\right)\middle|\mathcal{E}_t\right] \\
&= -c^{(j)} + P(q_t^{(1)}+q_t^{(2)})\,\mathbb{E}\left[\min\left(1,\frac{D_t}{q_t^{(1)}+q_t^{(2)}}\right)\middle|\mathcal{E}_t\right],
\end{aligned} \tag{3.15}$$

and

$$q_t^{(j)} = \frac{c^{(j)} - P(q_t^{(1)} + q_t^{(2)})\,\mathbb{E}\left[\min\left(1, \frac{D_t}{q_t^{(1)}+q_t^{(2)}}\right)\Big|\mathcal{E}_t\right]}{P'(q_t^{(1)} + q_t^{(2)})\,\mathbb{E}\left[\min\left(1, \frac{D_t}{q_t^{(1)}+q_t^{(2)}}\right)\Big|\mathcal{E}_t\right] - \frac{P(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})^2}\,\mathbb{E}\left[D_t \mathbf{1}_{\{D_t<q_t^{(1)}+q_t^{(2)}\}}\Big|\mathcal{E}_t\right]}, \tag{3.16}$$

$1 \leqslant i \neq j \leqslant 2$. Hence, using (3.13) we find

$$q_t^{(i)*} = S_t^* \frac{\mathbb{E}\left[P(S_t^*)\min(D_t, S_t^*) - c^{(i)}S_t^* \mid \mathcal{E}_t\right]}{\mathbb{E}\left[2P(S_t^*)\min(D_t, S_t^*) - (c^{(1)} + c^{(2)})S_t^* \mid \mathcal{E}_t\right]}, \tag{3.17}$$

$i = 1, 2$. □

Ornstein-Uhlenbeck setting

In order to derive explicit solutions we henceforth assume that $\mu_t(D_t, q_t) = \alpha(\beta - D_t)$ where $\alpha, \beta \in \mathbb{R}$, and $\sigma_t(D_t, q_t) = \sigma(t)$, where σ is a deterministic function of $t \in \mathbb{R}$. Therefore the dynamics of D_t is given by

$$\begin{cases} \mathrm{d}D_t = \alpha(\beta - D_t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t \\ \\ D_0 = d_0 > 0, \end{cases}$$

whose solution $(D_t)_{t\in[\delta,T]}$ satisfies

$$D_t = D_{t-\delta}e^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + \int_{t-\delta}^t e^{\alpha(s-t)}\sigma(s)\mathrm{d}B_s, \qquad t \in [\delta, T]. \tag{3.18}$$

In the sequel we denote by $\Phi : \mathbb{R} \to [0, 1]$ the standard Gaussian cumulative distribution function.

Proposition 3.3 *In the Ornstein-Uhlenbeck setting, any Cournot equilibrium action* $(q_t^{(1)*}, q_t^{(2)*})$ *satisfies*

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right) + (c^{(j)} - c^{(i)})P(S_t^*) - c^{(j)}(P(S_t^*) - S_t^*P'(S_t^*))}{2P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right) - (P(S_t^*) - S_t^*P'(S_t^*))(c^{(1)} + c^{(2)})}, \tag{3.19}$$

at any $t \in [\delta, T]$, $1 \leqslant i \neq j \leqslant 2$, *where* $S_t^* := q_t^{(1)*} + q_t^{(2)*}$ *solves the equation*

$$(P(S_t^*) + S_t^*P'(S_t^*))\left(\varphi(D_{t-\delta})\Phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) - \varepsilon(t)\phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right)\right)$$

$$+S_t^* \left(2P(S_t^*) + S_t^* P'(S_t^*)\right) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) = (c^{(1)} + c^{(2)}) S_t^*, \qquad 1 \leqslant i \neq j \leqslant 2,$$

with

$$\varphi(d) := de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}),$$

and

$$\varepsilon^2(t) := \int_{t-\delta}^{t} e^{2\alpha(s-t)} \sigma^2(s) ds, \qquad t \in [\delta, T].$$

In contrast with Proposition 2.1, this result covers the case where production $q_t^{(1)*}$ and $q_t^{(2)*}$ meets the demand D_t, as the derivatives of the profit functions $f^{(i)}(D_t, q_t^{(1)}, q_t^{(2)})$, $i = 1, 2$, are smoothened by the introduction of a conditional expectation due delayed information. The proof of Proposition 3.3 is deferred to the appendix.

Affine prices

In the sequel we consider the affine price model $P(q) = a - bq$, which satisfies the equation $P(q) - qP'(q) = a$. If $c^{(i)} \geqslant a$ then $P(q^{(i)}) - c^{(i)} = a - bq^{(i)} - c^{(i)} \leqslant 0$, in which case we have $q^{(i)*} = 0$, henceforth we focus on the case where $0 < c^{(i)} < a$, $i = 1, 2$. Proposition 3.3 yields the following corollary. When δ tends to zero, the equation (3.20) below converges to (2.4) in Proposition 2.1, and it also recovers the result of Proposition 2.2, since $\lim_{\varepsilon \to 0} \Phi(x/\varepsilon) = \mathbf{1}_{[0,\infty)}(x)$, $x \in \mathbb{R}$.

Corollary 3.4 *In the affine price model the Cournot equilibrium action at any $t \in [\delta, T]$ is given by*

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) + (c^{(j)} - c^{(i)})P(S_t^*) - ac^{(j)}}{2P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) - a(c^{(1)} + c^{(2)})}, \tag{3.20}$$

$1 \leqslant i \neq j \leqslant 2$, where $S_t^* = q_t^{(1)*} + q_t^{(2)*}$ solves the equation

$$(a - 2bS_t^*) \left(\varphi(D_{t-\delta})\Phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) - \varepsilon(t)\phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right)\right) \tag{3.21}$$
$$+S_t^* \left(2a - 3bS_t^*\right) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) = (c^{(1)} + c^{(2)}) S_t^*.$$

Figure 8 presents a numerical solution of (3.21) showing the values of $(q_t^{(1)}, q_t^{(2)})$ as functions of the demand d.

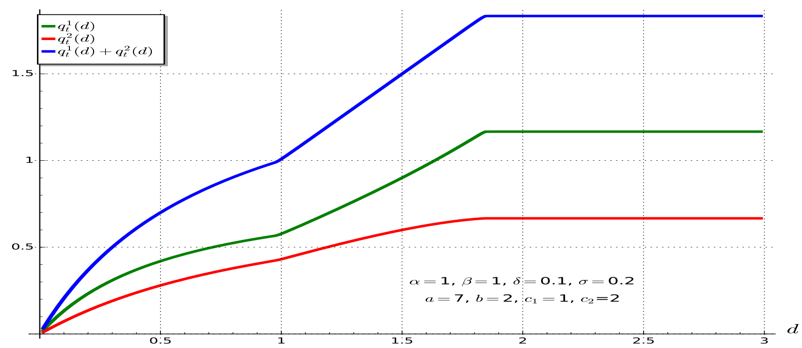

Figure 8: Graphs of $d \longmapsto q^{(1)*}(d), q^{(2)*}(d)$.

As the delay $\delta > 0$ tends to zero, the lagged equilibrium converge to a limiting equilibrium which belongs to the range of equilibria of Proposition 2.2, as illustrated in the following Figures 9-11. In the intermediate regime, this provides a way to select an equilibrium within the range of equilibria available in the no lag setting.

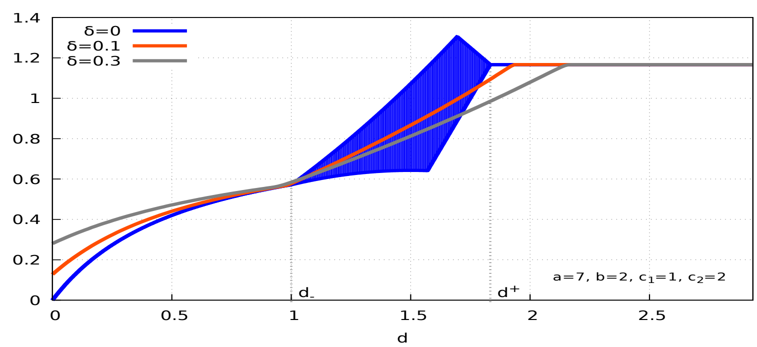

Figure 9: Equilibrium output $q_t^{(1)*}(d)$ as a function of d.

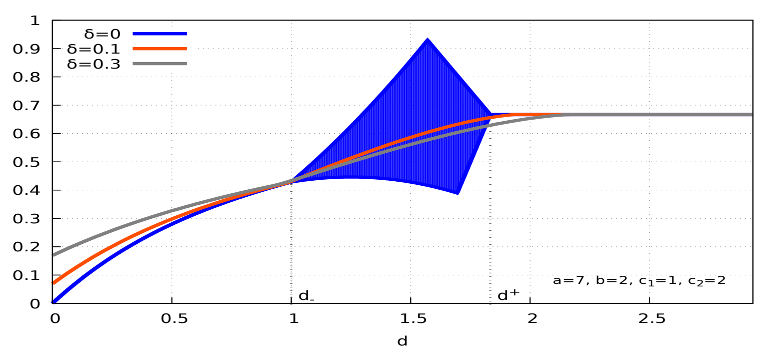

Figure 10: Equilibrium output $q_t^{(2)*}(d)$ as a function of d.

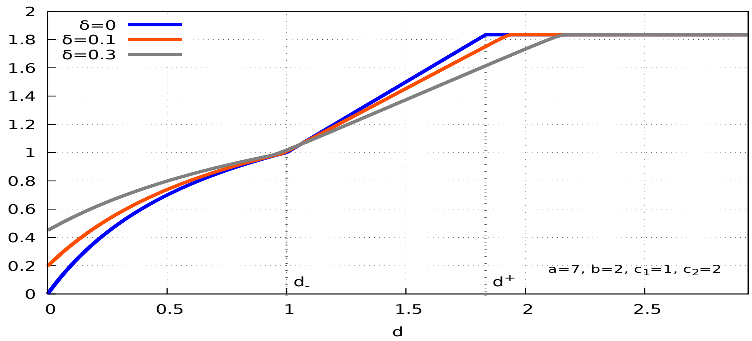

Figure 11: Total output $d \mapsto q_t^{(1)*}(d) + q_t^{(2)*}(d)$.

4 Conclusion

We have shown that the inclusion of limited demand in our model results in the existence of three different regimes, depending on the level of the demand, including an intermediate regime that shows a range of possible equilibria.

When demand is low, the market is inefficient as there will be overproduction, whereas when demand is high, there is unfulfilled demand since the usual Cournot equilibrium will materialise. When demand is at an intermediate level, it is exactly met by the combined production of the firms when there is no lag, but output falls short of demand in case of lag.

We have also shown that the introduction of a delayed stochastic demand reduces the range of equilibria to a single equilibrium in the intermediate regime. In addition, equilibria in other regimes converge to the equilibrium obtained in the non-delayed setting. In this sense, the lagged case can be seen as a smoothing of the no lag case. In order to treat the case of n-player oligopolies, we would replace (2.1) with

$$f^{(i)}(d, q^{(1)}, \dots, q^{(n)})$$
$$= P(q^{(1)} + \cdots + q^{(n)}) \left(q^{(i)} \mathbf{1}_{\{q^{(1)}+\cdots+q^{(n)} \leqslant d\}} + \frac{q^{(i)} d}{q^{(1)} + \cdots + q^{(n)}} \mathbf{1}_{\{d < q^{(1)}+\cdots+q^{(n)}\}} \right) - c^{(i)} q^{(i)}.$$

In this case, the optimal $q^{(i)*}$ depends on the sum $\sum_{\substack{j=1 \\ j \neq i}}^{n} q^{(j)*}$ and this will result into more first order conditions in the proof of Propositions 2.1 and 3.2. This requires different solution techniques and will be considered for future research.

5 Appendix: proofs of Propositions 2.2 and 3.3

Proof of Proposition 2.2.

We have

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)})$$
$$= \left(a - 2bq^{(i)} - bq^{(j)}\right) \mathbf{1}_{\{q^{(1)}+q^{(2)}<d\}} + \left(\frac{aq^{(j)}}{(q^{(1)}+q^{(2)})^2} - b\right) d\mathbf{1}_{\{d<q^{(1)}+q^{(2)}\}} - c^{(i)}, \tag{5.1}$$

$1 \leqslant i \neq j \leqslant 2$.

a) If $q^{(1)} + q^{(2)} < d$, (5.1) reads

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)}, \quad 1 \leqslant i \neq j \leqslant 2,$$

We distinguish two cases:

1. $q^{(i)*}, q^{(j)*} > 0$. In this case we must have

$$\begin{cases} \dfrac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)} = 0 \\ \\ \dfrac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(j)} - bq^{(i)} - c^{(j)} = 0, \end{cases}$$

hence

$$q^{(i)*} = \frac{a - 2c^{(i)} + c^{(j)}}{3b} \quad \text{and} \quad q^{(j)*} = \frac{a + c^{(i)} - 2c^{(j)}}{3b},$$

and

$$q^{(1)*} + q^{(2)*} = \frac{2a - c^{(1)} - c^{(2)}}{3b} < d.$$

2. $q^{(i)*} < d$ and $q^{(j)*} = 0$. In this case we may have an inequality

$$\begin{cases} \dfrac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)} = 0 \\ \\ \dfrac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(j)} - bq^{(i)} - c^{(j)} \leqslant 0, \end{cases}$$

i.e.

$$\begin{cases} a - 2bq^{(i)} - c^{(i)} = 0 \\ \\ a - bq^{(i)} - c^{(j)} \leqslant 0, \end{cases}$$

or $(a - c^{(j)})/b \leqslant q^{(i)*} = (a - c^{(i)})/(2b)$, which together with the condition $a + c^{(i)} - 2c^{(j)} \geqslant 0$, implies $a + c^{(i)} - 2c^{(j)} = 0$ and in this case, the equilibrium is already taken care of in point 1 above.

b) On the other hand, if $d < q^{(1)} + q^{(2)}$, (5.1) reads

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)}, \quad 1 \leqslant i \neq j \leqslant 2,$$

There are two kinds of candidate equilibria:

1. $q^{(i)*}, q^{(j)*} > 0$ and we must have:

$$\begin{cases} \dfrac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\dfrac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)} = 0 \\[2ex] \dfrac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \left(\dfrac{aq^{(i)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(j)} = 0, \end{cases}$$

In this case,

$$\begin{cases} q^{(1)}a - b(q^{(1)} + q^{(2)})^2 = \dfrac{c^{(2)}}{d}(q^{(1)} + q^{(2)})^2, \\[2ex] q^{(2)}a - b(q^{(1)} + q^{(2)})^2 = \dfrac{c^{(1)}}{d}(q^{(1)} + q^{(2)})^2, \end{cases}$$

hence

$$q^{(1)*}(d) = \frac{ad(c^{(2)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2}, \quad \text{and} \quad q^{(2)*}(d) = \frac{ad(c^{(1)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2},$$

and

$$d < q^{(1)*}(d) + q^{(2)*}(d) = \frac{ad}{c^{(1)} + c^{(2)} + 2bd}.$$

2. $q^{(i)*} > d$ and $q^{(j)*} = 0$ and instead we may have an inequality:

$$\begin{cases} \dfrac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\dfrac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)} = 0 \\[2ex] \dfrac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \left(\dfrac{aq^{(i)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(j)} \leqslant 0, \end{cases}$$

And therefore,

$$\begin{cases} \dfrac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = -bd - c^{(i)} = 0 \\[2ex] \dfrac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \dfrac{ad}{q^{(i)}} - bd - c^{(j)} \leqslant 0, \end{cases}$$

which is impossible.

c) Finally we consider what happens when $\underline{d} \leqslant d \leqslant \overline{d}$, i.e. d is between the above boundaries. The sum of derivatives

$$\frac{\partial f^{(1)}}{\partial q^{(1)}}(d, q^{(1)}, q^{(2)}) + \frac{\partial f^{(2)}}{\partial q^{(2)}}(d, q^{(1)}, q^{(2)}) = \left(a - 3b(q^{(i)} + q^{(j)})\right) \mathbf{1}_{\{q^{(1)}+q^{(2)}<d\}} + \left(\frac{a}{q^{(1)} + q^{(2)}} - b\right) d\mathbf{1}_{\{d<q^{(1)}+q^{(2)}\}} - c^{(i)} - c^{(j)} \tag{5.2}$$

is a decreasing function of $q^{(1)} + q^{(2)}$ which vanishes at $q^{(1)} + q^{(2)} = \overline{d}$, therefore it is strictly positive when $q^{(1)} + q^{(2)} < d \leqslant \overline{d}$, hence at least one of the individual derivatives must be strictly positive at such points. Therefore, $q^{(1)} + q^{(2)} < d$ cannot be an equilibrium.

In the case $q^{(1)} + q^{(2)} > d$ the summation of derivatives vanishes at $q^{(1)} + q^{(2)} = \underline{d}$ hence it is strictly negative when $q^{(1)} + q^{(2)} > d \geqslant \underline{d}$ and at least one of the derivatives must be negative. Therefore $q^{(1)} + q^{(2)} > d$ cannot be an equilibrium either.

Consider a point $(q^{(1)*}, q^{(2)*}) \in [0, d] \times [0, d]$ on the line $q^{(1)} + q^{(2)} = d$. This is an equilibrium if there exists no profitable deviations. We need to show that the value function

$$\begin{aligned} & f^{(i)}(d, q^{(1)}, q^{(2)}) \\ = \; & (a - b(q^{(i)} + q^{(j)*})) \left(q^{(i)} \mathbf{1}_{\{q^{(i)}+q^{(j)*}\leqslant d\}} + \frac{q^{(i)} d}{q^{(i)} + q^{(j)*}} \mathbf{1}_{\{d<q^{(i)}+q^{(j)*}\}}\right) - c^{(i)} q^{(i)}, \end{aligned}$$

$1 \leqslant i \neq j \leqslant 2$, is increasing in $q^{(i)} \in [0, q^{(i)*})$ and decreasing in $q^{(i)} \in (q^{(i)*}, \infty)$. Given that the partial derivative

$$\begin{aligned} & \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) \\ = \; & \left(a - 2bq^{(i)} - bq^{(j)*}\right) \mathbf{1}_{\{q^{(i)}+q^{(j)*}<d\}} + \left(\frac{aq^{(j)*}}{(q^{(i)} + q^{(j)*})^2} - b\right) d\mathbf{1}_{\{d<q^{(i)}+q^{(j)*}\}} - c^{(i)} \end{aligned}$$

is a decreasing function of $q^{(i)}$, it is necessary and sufficient to have the left derivative

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(i)*-}, q^{(j)*}) = a - bq^{(i)*} - bd - c^{(i)} = a + bq^{(j)*} - 2bd - c^{(i)} \geqslant 0, \tag{5.3}$$

and the right derivative satisfies

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(i)*+}, q^{(j)*}) = \frac{aq^{(j)*}}{d} - bd - c^{(i)} \leqslant 0. \tag{5.4}$$

Therefore $(q^{(1)*}, q^{(2)*}) \in (0, d) \times (0, d)$ on the line $q^{(1)} + q^{(2)} = d$ is an equilibrium if and only if

$$\left(\frac{-a + c^{(i)} + 2bd}{b}\right)^+ \leqslant q^{(j)*} \leqslant \frac{(c^{(i)} + bd)d}{a}, \qquad 1 \leqslant i \neq j \leqslant 2, \tag{5.5}$$

or equivalently

$$\left(\frac{-a + c^{(j)} + 2bd}{b}\right)^+ \leqslant d - q^{(j)*} \leqslant \frac{(c^{(j)} + bd)d}{a}, \qquad j = 1, 2, \tag{5.6}$$

which yields (2.8). Notice that from (2.8), if there exists $q^{(i)*}, q^{(j)*}$ that satisfy the inequalties, then $\underline{d} \leqslant d \leqslant \overline{d}$, or in other words, if d is outside these bounds, we do not have equilibria satisfying $q^{(i)*} + q^{(j)*} = d$. Finally, consider $(q^{(1)*}, q^{(2)*})$ on the line $q^{(1)} + q^{(2)} = d$ with $q^{(i)*} = d$ and $q^{(j)*} = 0$. In this case, the value function for i

$$f^{(i)}(d, q^{(1)}, q^{(2)}) = (a - bq^{(i)}) \left(q^{(i)} \mathbf{1}_{\{q^{(i)} \leqslant d\}} + d \mathbf{1}_{\{d < q^{(i)}\}}\right) - c^{(i)} q^{(i)},$$

is clearly decreasing in $q^{(i)} \in (d, \infty)$, and it suffices to show that it is increasing in $q^{(i)} \in [0, d)$, and from (5.3) for this we need

$$d \leqslant \frac{a - c^{(i)}}{2b}.$$

On the other hand, we check that the right derivative for j, (5.4) is negative if

$$\frac{a - c^{(j)}}{b} \leqslant d,$$

hence we need

$$\frac{a - c^{(j)}}{b} \leqslant d \leqslant \frac{a - c^{(i)}}{2b}.$$

However, $(q^{(i)*}, q^{(j)*}) = (d, 0)$ cannot be an equilibrium unless

$$d = \frac{a - c^{(j)}}{b} = \frac{a - c^{(i)}}{2b},$$

as we assume that $a + c^{(i)} - 2c^{(j)} \geqslant 0$. In the latter case we have $a + c^{(i)} - 2c^{(j)} = 0$ and the inequalities (2.8) read

$$d = \frac{-a + c^{(j)} + 2bd}{b} \leqslant q^{(i)*} \leqslant d, \text{ i.e. } q^{(i)*} = d, \quad \text{and} \quad 0 \leqslant q^{(j)*} \leqslant 0, \text{ i.e. } q^{(j)*} = 0,$$

which is compatible with the equilibrium $(q^{(i)*}, q^{(j)*}) = (d, 0)$. Notice that $(d, 0)$ is an equilibrium if and only if d is i's monopoly quantity.

Proof of Proposition 3.3.

From (3.9) and the fact that $q_t^{(1)} + q_t^{(2)}$ is $\mathcal{E}_t$-measurable we have

$$\begin{aligned}
&\left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(i)} P'(q_t^{(1)} + q_t^{(2)})\right) \mathbb{E}[\mathbf{1}_{\{q \leqslant D_t\}} \mid \mathcal{E}_t]_{q=q_t^{(1)}+q_t^{(2)}} \\
&+ \left(q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(i)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \mathbb{E}[D_t \mathbf{1}_{\{q > D_t\}} \mid \mathcal{E}_t]_{q=q_t^{(1)}+q_t^{(2)}} = c^{(i)},
\end{aligned} \tag{5.7}$$

$1 \leqslant i \neq j \leqslant 2$. From (3.18) we write $D_t = d_{t-\delta} e^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X$ where $X \simeq \mathcal{N}(0, \varepsilon^2(t))$ and

$$\varepsilon^2(t) = \mathbb{E}\left[\left(\int_{t-\delta}^t e^{\alpha(s-t)} \sigma(s) \mathrm{d}B_s \right)^2 \right] = \int_{t-\delta}^t e^{2\alpha(s-t)} \sigma^2(s) ds = \sigma^2 \frac{1 - e^{-2\alpha\delta}}{2\alpha},$$

hence by the independence between $\mathcal{E}_t := \mathcal{F}_{t-\delta}$ and $\int_{t-\delta}^t e^{\alpha(s-t)} \sigma(s) \mathrm{d}B_s$ in (3.18) we have

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\{q \leqslant D_t\}} \mid \mathcal{E}_t]_{q=q_t^{(1)}+q_t^{(2)}} &= \mathbb{P}(q \leqslant de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)_{d=d_{t-\delta},\, q=q_t^{(1)}+q_t^{(2)}} \\
&= \int_{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}^{\infty} e^{-x^2/(2\varepsilon^2(t))} \frac{dx}{\sqrt{2\pi\varepsilon^2(t)}} \\
&= \Phi\left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right).
\end{aligned}$$

For the second expectation we have

$$\begin{aligned}
&\mathbb{E}[D_t \mathbf{1}_{\{q > D_t\}}]_{q=q_t^{(1)}+q_t^{(2)}} \\
&= \mathbb{E}[(de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X) \mathbf{1}_{\{q > de^{-\alpha\delta} + \beta(1-e^{-\alpha\delta}) + X\}}]_{d=d_{t-\delta},\, q=q_t^{(1)}+q_t^{(2)}} \\
&= \varphi(D_{t-\delta})\, \mathbb{E}[\mathbf{1}_{q > \varphi(D_{t-\delta}) + X}]_{q=q_t^{(1)}+q_t^{(2)}} + \mathbb{E}[X \mathbf{1}_{q > \varphi(D_{t-\delta}) + X}]_{q=q_t^{(1)}+q_t^{(2)}} \\
&= \varphi(D_{t-\delta}) \mathbb{P}(X < q - \varphi(D_{t-\delta}))_{q=q_t^{(1)}+q_t^{(2)}} + \int_{-\infty}^{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})} x e^{-x^2/(2\varepsilon^2(t))} \frac{dx}{\sqrt{2\pi\varepsilon^2(t)}} \\
&= \varphi(D_{t-\delta}) \Phi\left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) - \varepsilon(t) \phi\left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right).
\end{aligned}$$

Hence (5.7) reads

$$\left(q_t^{(i)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(j)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \tag{5.8}$$

$$\times \left(\varepsilon(t)\phi\left(\frac{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}{\varepsilon(t)}\right)-\varphi(D_{t-\delta})\Phi\left(\frac{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}{\varepsilon(t)}\right)\right)$$
$$-\left(P(q_t^{(1)}+q_t^{(2)})+q_t^{(j)}P'(q_t^{(1)}+q_t^{(2)})\right)\Phi\left(\frac{\varphi(D_{t-\delta})-q_t^{(1)}-q_t^{(2)}}{\varepsilon(t)}\right)$$
$$=-c^{(j)},\qquad 1\leqslant i\neq j\leqslant 2.$$

By summing the above two equations we find that the sum $S_\varepsilon^* := q_t^{(1)}+q_t^{(2)}$ satisfies the equation

$$\left(\frac{P(q_t^{(1)}+q_t^{(2)})}{q_t^{(1)}+q_t^{(2)}}+P'(q_t^{(1)}+q_t^{(2)})\right)$$
$$\times \left(\varepsilon(t)\phi\left(\frac{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}{\varepsilon(t)}\right)-\varphi(D_{t-\delta})\Phi\left(\frac{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}{\varepsilon(t)}\right)\right) \tag{5.9}$$
$$-\left(2P(q_t^{(1)}+q_t^{(2)})+(q_t^{(1)}+q_t^{(2)})P'(q_t^{(1)}+q_t^{(2)})\right)\Phi\left(\frac{\varphi(D_{t-\delta})-q_t^{(1)}-q_t^{(2)}}{\varepsilon(t)}\right)$$
$$=-c^{(1)}-c^{(2)},\qquad 1\leqslant i\neq j\leqslant 2.$$

Combining (5.8) with (5.9), we get

$$\left(\frac{q_t^{(i)}P(q_t^{(1)}+q_t^{(2)})+q_t^{(j)}(q_t^{(1)}+q_t^{(2)})P'(q_t^{(1)}+q_t^{(2)})}{(q_t^{(1)}+q_t^{(2)})P(q_t^{(1)}+q_t^{(2)})+(q_t^{(1)}+q_t^{(2)})^2P'(q_t^{(1)}+q_t^{(2)})}\right) \tag{5.10}$$
$$\times\left(c^{(1)}+c^{(2)}-\left(2P(q_t^{(1)}+q_t^{(2)})+(q_t^{(1)}+q_t^{(2)})P'(q_t^{(1)}+q_t^{(2)})\right)\Phi\left(\frac{\varphi(D_{t-\delta})-q_t^{(1)}-q_t^{(2)}}{\varepsilon(t)}\right)\right)$$
$$+\left(P(q_t^{(1)}+q_t^{(2)})+q_t^{(j)}P'(q_t^{(1)}+q_t^{(2)})\right)\Phi\left(\frac{\varphi(D_{t-\delta})-q_t^{(1)}-q_t^{(2)}}{\varepsilon(t)}\right)$$
$$=c^{(j)},\qquad 1\leqslant i\neq j\leqslant 2,$$

or

$$-q_t^{(j)}(P(S_t^*)-S_tP'(S_t^*))\left(c^{(1)}+c^{(2)}-(2P(S_t^*)+S_t^*P'(S_t^*))\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right)\right)$$
$$+q_t^{(j)}P'(S_t^*)(S_t^*P(S_t^*)+S_t^{*2}P'(S_t^*))\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right)$$
$$=c^{(j)}(S_t^*P(S_t^*)+S_t^{*2}P'(S_t^*))-P(S_t^*)(S_t^*P(S_t^*)+S_t^{*2}P'(S_t^*))\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right)$$
$$-S_t^*P(S_t^*)\left(c^{(1)}+c^{(2)}-(2P(S_t^*)+S_t^*P'(S_t^*))\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right)\right),\qquad 1\leqslant i\neq j\leqslant 2,$$

which shows that

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right) + (c^{(j)} - c^{(i)})P(S_t^*) - c^{(j)}(P(S_t^*) - S_t^* P'(S_t^*))}{2P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta})-S_t^*}{\varepsilon(t)}\right) - (P(S_t^*) - S_t P'(S_t^*))(c^{(1)} + c^{(2)})}.$$

In the limit when d_t is large, (5.10) yields the system of equations

$$\begin{cases} 2bq_t^{(1)} + bq_t^{(2)} + c^{(1)} = a \\ \\ bq_t^{(1)} + 2bq_t^{(2)} + c^{(2)} = a, \end{cases}$$

with solution

$$\begin{cases} q_t^{(1)} = \dfrac{a - 2c^{(1)} + c^{(2)}}{3b} \\ \\ q_t^{(2)} = \dfrac{a + c^{(1)} - 2c^{(2)}}{3b}, \end{cases}$$

and sum

$$q_t^{(1)} + q_t^{(2)} = \frac{2a - c^{(1)} - c^{(2)}}{3b}, \qquad t \in [\delta, T],$$

which recover the no lag setting of Proposition 2.2.

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