Cumulant operators for Lie-Wiener-Itô-Poisson stochastic integrals

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Abstract

The classical combinatorial relations between moments and cumulants of random variables are generalized into covariance-moment identities for stochastic integrals and divergence operators. This approach is based on cumulant operators defined by the Malliavin calculus in a general framework that includes Itô-Wiener and Poisson stochastic integrals as well as the Lie-Wiener path space. In particular, this allows us to recover and extend various characterizations of Gaussian and infinitely divisible distributions.

Key words: Moments; cumulants; stochastic integrals; Malliavin calculus; Wiener space; path space; Lie groups; Poisson space.
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1 Introduction

This paper is concerned with the relations between integration by parts on probability spaces and the combinatorics of moments and cumulants of stochastic integrals.

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More precisely, given $(M_t)_{t\in\mathbb{R}_+}$ a normal martingale (e.g., a standard Brownian motion or a compensated Poisson process), the stochastic integral $\int_0^\infty u_t dM_t$ of a square-integrable adapted process $(u_t)_{t\in\mathbb{R}_+}$ is known to be a centered random variable whose second moment is given by the Itô isometry

$$E\left[\left(\int_0^\infty u_t dM_t\right)^2\right] = E\left[\int_0^\infty |u_t|^2 dt\right]. \tag{1.1}$$

Although the Itô isometry still requires the computation of an expectation on the right hand side, its main interest is to remove the stochastic integration present on the left hand side.

Our goal in this paper is to derive extensions of (1.1) to moments of all orders, by removing any stochastic integral (or "noise") term from the left hand side. This is achieved by the covariance-moment identity (3.1) below which is based on the Skorohod integral on the Lie-Wiener and Poisson spaces.

Clearly, the moment

$$E\left[\left(\int_0^\infty u_t dM_t\right)^n\right]$$

can be evaluated by decomposing the power $\left(\int_0^\infty u_t dM_t\right)^n$ into a sum of multiple stochastic integrals having zero expectation plus a remainder term, based on the Itô rule and combinatorics of the underlying martingale $(M_t)_{t\in\mathbb{R}_+}$. In this sense, our aim is to compute the expectation of this remainder term. For this, we will rely on the integration by parts formulas of the Malliavin calculus on the Wiener and Poisson spaces. In particular, we will represent $\int_0^\infty u_t dM_t$ using the Skorohod integral operator $\delta(u)$.

The moment formulas obtained in this paper are based on a "cumulant operator" Γ_k and stated in Proposition 4.3 in the general case, followed by Propositions 5.7 and 6.3, respectively, in the Lie-Wiener and Poisson cases. A different type of cumulant operator has been defined in [8] using the inverse L^{-1} of the Ornstein-Uhlenbeck operator L in order to derive expressions for the cumulants of random variables on the

Wiener space. Our approach is different as it is specially suited to the moments of stochastic integrals on both the Lie-Wiener and Poisson spaces.

In Theorem 5.1 of [18], moment formulas for have been obtained in the Poisson case by carrying out repeated integration by parts and removing all stochastic integral terms by means of using finite difference operators, leading to another type of cumulant operators in Proposition 3.1 of [21]

We proceed as follows. After a review of definitions and results on moments and cumulants in Section 2, the main results of the paper are presented in Section 3. The general integration by parts setting under minimal conditions is treated in Section 4, and the results are then specialized to the Lie-Wiener and Poisson cases in Sections 5 and 6 respectively. The appendix Section 7 contains several technical results and additional notation.

2 Moments and cumulants

We refer the reader to [10] and references therein for the relationships between the moments and cumulants of random variables recalled in this section. Given the cumulants $(\kappa_n^X)_{n\geq 1}$ of a random variable X, defined from the generating function

$$\log E[e^{tX}] = \sum_{n=1}^{\infty} \kappa_n^X \frac{t^n}{n!},$$

for t in a neighborhood of 0, the moments of X can be recovered by the combinatorial identity

$$E[X^n] = \sum_{a=1}^{n} \sum_{P_1,\ldots,P_a} \kappa_{|P_1|}^X \cdots \kappa_{|P_a|}^X = A_n(\kappa_1^X, \ldots, \kappa_n^X), \tag{2.1}$$

where the sum runs over the partitions $P_1, \ldots, P_a$ of $\{1, \ldots, n\}$ with cardinal $|P_i|$ by the Faà di Bruno formula, cf. § 2.4 and Relation (2.4.4) page 27 of [7], and

$$A_n(x_1, \ldots, x_n) = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \prod_{l=1}^{n} \left(\frac{1}{r_l!} \left(\frac{x_l}{l!} \right)^{r_l} \right)$$

is the Bell polynomial of degree n.

Gaussian cumulants

When X is centered, we have $\kappa_1^X = 0$ and $\kappa_2^X = E[X^2] = \mathrm{var}[X]$, and X becomes Gaussian if and only if $\kappa_n^X = 0$, $n \geq 3$. Consequently, (2.1) can be read as Wick's theorem for the computation of Gaussian moments of $X \simeq \mathcal{N}(0, \sigma^2)$ by counting the pair partitions of $\{1, \ldots, n\}$, cf. [5], as

$$E[X^n] = \sigma^n \sum_{a=1}^{n} \sum_{|P_1|=2,\ldots,|P_a|=2} \kappa_{|P_1|}^X \cdots \kappa_{|P_a|}^X = \begin{cases} \sigma^n (n-1)!!, & n \text{ even}, \\ \\ 0, & n \text{ odd}, \end{cases} \tag{2.2}$$

where the double factorial $(n-1)!! = \prod_{1 \leq 2k \leq n} (2k-1) = 2^{-n/2} n!/(n/2)!$ counts the number of pair partitions of $\{1, \ldots, n\}$ when n is even, Relation (2.2) clearly applies when X is given by the Wiener integral

$$X = \int_0^\infty h(s) dB_s$$

of the deterministic function $h \in L^2(\mathbb{R}_+)$ with respect to a standard Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, i.e., X has the Gaussian cumulants

$$\kappa_n(h) = \mathbf{1}_{\{n=2\}} \int_0^\infty |h(s)|^2 ds, \qquad n \geq 1. \tag{2.3}$$

Infinitely divisible cumulants

On the other hand, when X is the infinitely divisible Poisson stochastic integral

$$X = \int_0^\infty h(t)(dN_t - \lambda dt)$$

with respect to a standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ with intensity $\lambda > 0$, we have

$$\log E\left[\exp\left(\int_X h(t)(dN_t - \lambda dt)\right)\right] = \lambda \int_0^\infty (e^{h(t)} - h(t) - 1) dt = \lambda \sum_{n=1}^{\infty} \kappa_n(h) \frac{t^n}{n!},$$

where

$$\kappa_n(h) = \mathbf{1}_{\{n \geq 2\}} \int_X h^n(t) dt, \qquad n \geq 1, \tag{2.4}$$

is the normalized centered Poisson cumulant, and (2.1) becomes the moment identity

$$E\left[\left(\int_X h(t)(dN_t - \lambda dt)\right)^n\right] = \sum_{a=1}^{n} \lambda^a \sum_{|P_1| \geq 2,\ldots,|P_a| \geq 2} \int_X h^{|P_1|}(t) dt \cdots \int_X h^{|P_a|}(t) dt, \tag{2.5}$$

where the sum runs over the partitions $P_1, \ldots, P_a$ of $\{1, \ldots, n\}$ of size at least equal to 2, cf. [1] for the non-compensated case and [21], Proposition 3.2 for the compensated case. In the particular case of a Poisson random variable $Z \simeq \mathcal{P}(\lambda)$ with intensity $\lambda > 0$ we have

$$E[Z^n] = \sum_{a=1}^{n} \sum_{|P_1|\geq 1, \ldots, |P_a|\geq 1} \lambda^a = \sum_{k=0}^{n} \lambda^k S(n,k), \tag{2.6}$$

where $S(n,k)$ is the Stirling number of the second kind, i.e., the number of ways to partition a set of n objects into k non-empty subsets. Note that (2.4) and (2.5) immediately extend to Poisson random measures over a metric space X with arbitrary σ-finite intensity measure on X.

3 Main results

In this paper, we work in the general setting of an arbitrary probability space $(\Omega, \mathcal{F}, \mu)$ on which is defined a Skorohod type stochastic integral operator (or divergence) δ which coincides with the stochastic integral with respect to an underlying martingale $(M_t)_{t\in\mathbb{R}_+}$ on the square-integrable processes, which are adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(M_t)_{t\in\mathbb{R}_+}$.

The operator δ is adjoint of a closable linear operator

$$D : \mathcal{S} \longrightarrow L^2(\Omega; H),$$

defined on a dense linear subspace $\mathcal{S}$ of $L^2(\Omega, \mathcal{F}, \mu)$, where H itself is a linear space dense in $L^2(\mathbb{R}_+; \mathbb{R}^d)$ for some $d \geq 1$, and endowed with the inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$ of $L^2(\mathbb{R}_+; \mathbb{R}^d)$, with the duality relation

$$\lambda E[\langle DF, u \rangle] = E[F\delta(u)], \qquad F \in \mathrm{Dom}(D), \quad u \in \mathrm{Dom}(\delta),$$

for some $\lambda > 0$. We show in particular that the above formulas (2.2) and (2.5) both stem from the general covariance-moment identity

$$E[F\delta(u)^n] = n! \sum_{a=1}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1, \ldots, l_a\geq 1}} \frac{E\left[\Gamma^u_{l_1} \cdots \Gamma^u_{l_a} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}, \tag{3.1}$$

cf. Proposition 4.3 below, for $u \in \mathrm{Dom}(\delta)$ a possibly anticipating process and F a sufficiently D-differentiable random variable, where

$$\Gamma_k^u : \mathcal{S} \longrightarrow L^2(\Omega; H), \qquad k \geq 1,$$

is a cumulant operator defined from D in (3.2) and Definition 4.1 below.

When $\Gamma_l^u \mathbf{1}$ is deterministic for all $l \geq 1$, Relation (3.1) shows that the cumulant $\kappa_l^{\delta(u)}$ of $\delta(u)$ is given by

$$\kappa_l^{\delta(u)} = \lambda(l-1)! \Gamma_l^u \mathbf{1}, \qquad l \geq 1,$$

cf. Relation (4.13) below. This will allow us to recover various results on invariance of the Lie-Wiener and Poisson measures, cf. Propositions 5.8 and 6.5 below.

The canonical example for this setting is when (Ω, μ) is the d-dimensional Wiener space with the Wiener measure μ. In addition to the Wiener space, our framework covers both the Lie-Wiener space, for which the operators D and δ can be defined on the path space over a Lie group, cf. [4], [11], [24], and the discrete path setting of the Poisson process, cf. [2], [3], [12].

In all of the above cases, the Skorohod integral operator δ coincides, on the square-integrable $\mathcal{F}_t$-adapted processes, with the Itô or Poisson stochastic integral with respect to the underlying martingale. In particular, Relation (3.1) yields an extension of the Gaussian moment identity (2.2) to the case where X is given by the Itô-Wiener stochastic integral

$$X = \int_0^\infty u_t dB_t$$

of a square-integrable process $(u_t)_{t \in \mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+)$ adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ generated by $(B_t)_{t \in \mathbb{R}_+}$, cf. (3.9) below and Section 5. A similar extension follows for adapted stochastic integrals with respect to the compensated Poisson process $(N_t - \lambda t)_{t \in \mathbb{R}_+}$ in Section 6.

Being based on the divergence operator δ, our results also include the case where the process u is anticipating with respect to the Brownian or Poisson filtrations and are stated in a general framework that covers path spaces over Lie groups as well as (discrete) stochastic integrals with respect to the standard Poisson process. In this general framework, the operator Γ_k^u in (3.1) is shown to be given by

$$\Gamma_k^u F = F\langle (\nabla u)^{k-2} u, u\rangle + F\langle \nabla^* u, \nabla((\nabla u)^{k-2} u)\rangle + \langle (\nabla u)^{k-1} u, DF\rangle, \qquad (3.2)$$

$k \geq 2$, where D denotes the Malliavin gradient operator on the Lie-Wiener or Poisson spaces ∇ is a covariant derivative operator acting on the stochastic process u, cf. Condition **(H3)** in Section 4. The composition $(\nabla u)^l$ is defined in the sense of a matrix power with continuous indices, cf. (7.3) below, and Γ_k^u satisfies the product rule

$$\Gamma_k^u(FG) = G\Gamma_k^u F + F\langle (\nabla u)^{k-1} u, DG\rangle, \qquad k \geq 1. \qquad (3.3)$$

For any integer $n \geq 2$ we let

$$\kappa_n(u) = \begin{cases} \mathbf{1}_{\{n=2\}} \displaystyle\int_0^\infty u_t^2 dt, & \text{on the Lie-Wiener space, and} \\ \kappa_n(u) = \displaystyle\int_0^\infty u_t^n dt, & \text{on the Poisson space,} \end{cases}$$

denote the natural extensions of $\kappa_n(h)$ in (2.3) and (2.4) from deterministic $h \in L^p(\mathbb{R}_+)$ to random u. We show that $\langle (\nabla u)^n u, u\rangle$ in (3.2) can be computed by the relation

$$\langle (\nabla u)^n u, u\rangle = \frac{1}{(n+1)!}\kappa_{n+2}(u) + \sum_{i=2}^{n+1} \frac{1}{i!} \left\langle (\nabla u)^{n+1-i} u, D\kappa_i(u)\right\rangle, \qquad (3.4)$$

$n \geq 0$, on both the Lie-Wiener and Poisson spaces, cf. Relation (5.5) in Lemma 5.3 on the Lie-Wiener space and Relation (6.4) in Lemma 6.2 below on the Poisson space. Applying (3.4) to (3.2) shows that Γ_k^u is given by

$$\Gamma_k^u \mathbf{1} = \frac{\kappa_k(u)}{(k-1)!} + \langle \nabla^* u, \nabla((\nabla u)^{k-2} u)\rangle + \sum_{i=0}^{k-1} \frac{1}{i!} \left\langle (\nabla u)^{k-1-i} u, D\kappa_i(u)\right\rangle, \qquad (3.5)$$

$k \geq 2$, cf. Lemma 5.4 on the Lie-Wiener space and Proposition 6.3 on the Poisson space, with

$$\Gamma_k^u F = F\Gamma_k^u \mathbf{1} + \langle (\nabla u)^{k-1} u, DF\rangle, \qquad (3.6)$$

from (3.3). Next, we discuss two consequences of (3.5).

1) When the process u is adapted with respect to the Brownian or Poisson filtration, the term

$$\langle \nabla^* u, \nabla((\nabla u)^{l-2} u) \rangle = 0, \qquad l \geq 2, \tag{3.7}$$

vanishes, since (5.14) below vanishes by (5.15) and Lemma 2.3 of [19] on the Wiener space and by (6.10) below on the Poisson space. Hence, (3.5) reduces to

$$\Gamma_l^u \mathbf{1} = \frac{\kappa_l(u)}{(l-1)!} + \sum_{i=0}^{l-1} \frac{1}{i!} \left\langle (\nabla u)^{l-1-i} u, D\kappa_i(u) \right\rangle, \tag{3.8}$$

$l \geq 2$, cf. (5.16) and (6.9) below, while $\Gamma_l^u F$ can be computed by (3.6).

For example, in case u is a sufficiently differentiable adapted process on the Wiener space, (3.1) shows the moment identity

$$E\left[\left(\int_0^\infty u_t dB_t\right)^n\right] = \sum_{a=1}^n \lambda^a \sum_{P_1,\ldots,P_a} (|P_1|-1)! \cdots (|P_a|-1)! E\left[\Gamma_{|P_1|}^u \cdots \Gamma_{|P_a|}^u \mathbf{1}\right], \tag{3.9}$$

where (3.8) reads

$$\Gamma_k^u \mathbf{1} = \mathbf{1}_{\{k=2\}} \langle u, u \rangle + \mathbf{1}_{\{k\geq 3\}} \frac{1}{2} \left\langle (\nabla u)^{k-3} u, D\langle u, u \rangle \right\rangle$$

and $\Gamma_k^u F$ is given by (3.6), cf. Lemma 5.4 below. On the Lie-Wiener space, this applies in particular when $u = Rh$ is given from a random adapted isometry $R : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$, as noted after the proof of Proposition 5.5 below, cf. [25] Theorem 2.1$-b$) on the Wiener space.

On the Poisson space of Section 6, when u is adapted with respect to the filtration generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ we find the moment identity

$$E\left[\left(\int_0^\infty u_t(dN_t - \lambda dt)\right)^n\right] = n! \sum_{a=1}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u \mathbf{1}\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)},$$

from (3.1), where (3.8) reads

$$\Gamma_k^u \mathbf{1} = \frac{1}{(k-1)!} \int_0^\infty u_t^k dt + \sum_{i=0}^{k-1} \frac{1}{i!} \left\langle (\nabla u)^{k-1-i} u, D \int_0^\infty u_t^i dt \right\rangle,$$

with $\Gamma_k^u F$ given by (3.6), cf. Proposition 6.3 below.

2) If, in addition to (3.7), $\kappa_l(u)$ is deterministic for all $l \geq 2$, then we have

$$\Gamma_l^u \mathbf{1} = \frac{\kappa_l(u)}{(l-1)!}, \tag{3.10}$$

$l \geq 2$, cf. (5.10) and (6.11) below, and in this case, it follows that

$$\Gamma_{l_a}^u \cdots \Gamma_{l_1}^u \mathbf{1} = \frac{\kappa_{l_a}(u)}{(l_a-1)!} \cdots \frac{\kappa_{l_1}(u)}{(l_1-1)!}, \qquad l_1, \ldots, l_a \geq 1,$$

and (3.1) recovers the classical combinatorial identity (2.1). In this case, the Skorohod integral $\delta(u)$ has a centered infinitely divisible distribution whose cumulant of order $n \geq 1$ is given by

$$\kappa_n(u) = (n-1)!\Gamma_n^u \mathbf{1},$$

and we have the coincidence $\kappa_n(u) = \kappa_n^{\delta(u)}$ between $\kappa_n(u)$ and the cumulant $\kappa_n^{\delta(u)}$ of $\delta(u)$, cf. Propositions 5.8 and 6.5 below, respectively, on the Lie-Wiener and Poisson spaces.

For example, when both $h \in H$ and ∇h are deterministic, which will be in particular the case in Sections 5 and 6 on the Lie-Wiener and Poisson spaces, the cumulant $\kappa_n^{\delta(h)}$ of $\delta(h)$ is given by

$$\kappa_n^{\delta(h)} = (n-1)!\langle (\nabla h)^{n-2} h, h \rangle, \qquad n \geq 2, \tag{3.11}$$

cf. Corollary 4.4 below, and (3.1) shows that

$$E[\delta(h)^n] = \sum_{a=1}^n \lambda^a \sum_{|P_1|\geq 2, \ldots, |P_a| \geq 2} (|P_1|-1)! \cdots (|P_a|-1)! \langle (\nabla h)^{|P_1|-2} h, h \rangle \cdots \langle (\nabla h)^{|P_a|-2} h, h \rangle. \tag{3.12}$$

As a consequence of (3.1) and (3.10), when h is deterministic, we also find the covariance-moment identity

$$E[F\delta(h)^n] = n! \sum_{n=1}^a \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1, \ldots, l_a \geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}$$

$$= n! \sum_{a=1}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1, \ldots, l_a \geq 1}} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\}} \frac{E\left[D_{(\nabla h)^{l_{i_1}-1}h} \cdots D_{(\nabla h)^{l_{i_k}-1}h} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \prod_{j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}} \kappa_{l_j}(h).$$

(3.13)

cf. (4.15) below, in particular (5.12) holds on the Lie-Wiener path space, and (6.12) holds on the Poisson space.

On the Wiener space we have $\nabla = D$ and the identity

$$\langle D^* u, D((Du)^{k-2}v)\rangle = \mathrm{trace}\,((Du)^{k-1}Dv) + \sum_{i=2}^{k-1} \frac{1}{i}\langle (Du)^{k-1-i}v, D\,\mathrm{trace}\,(Du)^i\rangle,$$

for sufficiently smooth processes u, $k \geq 2$, cf. Lemma 4 in [20], which shows that (3.7) vanishes under the quasi-nilpotence condition

$$\mathrm{trace}\,(Du)^n = 0, \qquad n \geq 2, \tag{3.14}$$

which is satisfied when the process u is adapted with respect to the Brownian filtration, cf. Corollary 5.8 below and Lemma 2.3 of [19]. However, adaptedness of the process u is not necessary for the condition (3.7) to hold, as shown in the example (5.11) below.

The classical cumulant formula (2.1) can be inverted to compute the cumulant κ_n^X from the moments μ_n^X of X by the inversion formula

$$\kappa_n^X = \sum_{a=1}^{n} (a-1)!(-1)^{a-1} \sum_{P_1,\ldots,P_a} \mu_{|P_1|}^X \cdots \mu_{|P_a|}^X, \qquad n \geq 1, \tag{3.15}$$

where the sum runs over the partitions $P_1, \ldots, P_a$ of $\{1, \ldots, n\}$ with cardinal $|P_i|$ by the Faà di Bruno formula, cf. [6] or § 2.4 and Relation (2.4.3) page 27 of [7]. Hence, (3.1) can be used to compute the cumulants of $\delta(u)$ via (3.15), cf. Relation (4.13) below.

Note that another type of cumulant operators Γ_k has been recursively defined in [8] using the inverse L^{-1} of the Ornstein-Uhlenbeck operator $L = \delta D$ on the Wiener space, with the direct relation

$$\kappa_{k+1}^F = E[\Gamma_k F], \qquad k \geq 1.$$

Our representation formula is different as it relies on the representation of F as the stochastic integral $F = \delta(u)$, while applying to both the Lie-Wiener and Poisson spaces. On the other hand it does not involve the inverse operator L^{-1} which is better suited to multiple stochastic integrals since they form a sequence of eigenvectors for L.

4 The general case

In this section, we consider a closable gradient operator $D : \mathcal{S} \longrightarrow L^2(\Omega; H)$ initially defined on a dense linear subspace $\mathcal{S}$ of $L^2(\Omega, \mathcal{F}, \mu)$, and extended to its closed domain $\mathrm{Dom}(D) \subset L^2(\Omega)$, and we work under the following general assumptions **(H1)-(H4)** on the Skorohod integral operator δ and the covariant derivative ∇.

(H1) The operator D satisfies the chain rule of derivation

$$D_t g(F) = g'(F) D_t F, \qquad t \in \mathbb{R}_+, \quad F \in \mathrm{Dom}(D), \tag{4.1}$$

for g in the space $\mathcal{C}_b^1(\mathbb{R})$ continuously differentiable functions on $\mathbb{R}$ with bounded derivative, with $D_t F = (DF)(t)$, $t \in \mathbb{R}_+$.

(H2) there exists a closable divergence (or Skorohod integral) operator

$$\delta : \mathcal{S} \otimes H \longmapsto L^2(\Omega),$$

acting on stochastic processes, with domain $\mathrm{Dom}(\delta) \subset L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d)$ and adjoint of D, with the duality relation

$$\lambda E[\langle DF, u \rangle] = E[F\delta(u)], \qquad F \in \mathrm{Dom}(D), \quad u \in \mathrm{Dom}(\delta), \tag{4.2}$$

where $\lambda > 0$ is a parameter that can represent the variance or the intensity of the underlying process.

(H3) There exists a closable covariant derivative operator

$$\nabla : \mathcal{S} \otimes H \longrightarrow H \otimes H$$

with domain $\mathrm{Dom}(\nabla) \subset L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d)$ that satisfies the commutation relations

$$D_t \delta(h) = h(t) + \delta(\nabla_t^\dagger h), \qquad t \in \mathbb{R}_+, \quad h \in H, \tag{4.3}$$

, where $\dagger$ denotes matrix transposition in $\mathbb{R}^d \otimes \mathbb{R}^d$, the relations

$$\nabla_s (F \otimes h(t)) = h(t) D_s F + F \nabla_s h(t), \qquad t \in \mathbb{R}_+,$$

and $\nabla_s h(t) = 0$, $0 \leq t < s$, $h \in H$.

We refer to the Appendix Section 7 for additional notational conventions and the definition of the Sobolev spaces $\mathbb{D}_{p,k}$ and $\mathbb{D}_{p,k}(H)$ which satisfy $\mathbb{D}_{2,1} \subset \mathrm{Dom}(D)$, $\mathbb{D}_{2,1}(H) \subset \mathrm{Dom}(\delta)$ and $\mathbb{D}_{2,1}(H) \subset \mathrm{Dom}(\nabla)$. Note that as a consequence of the chain rule **(H1)** and duality **(H2)** we have the divergence relation

$$F \delta(u) = \delta(uF) + \langle DF, u \rangle, \tag{4.4}$$

for $F \in \mathbb{D}_{2,1}$ such that $uF \in L^2(\Omega; H)$, cf., e.g., Proposition 1.3.3 of [9]. Given $F \in \mathcal{S}$ we let $D_h F := \langle h, DF \rangle$, $h \in H$, and define the Lie bracket $\{f, g\}$ of $f, g \in H$ by

$$D_{\{f,g\}} F = D_f D_g F - D_g D_f F, \qquad F \in \mathcal{S}.$$

In addition to **(H2)**, **(H1)** and **(H3)** we will assume that

(H4) The connection defined by ∇ has a vanishing torsion, i.e.

$$\{f, g\} = \nabla_f g - \nabla_g f, \qquad f, g \in H, \tag{4.5}$$

As a consequence of Assumption **(H4)** we can extend the commutation relation (4.3) to random processes as in Lemma 4.5 below. This framework includes both the Lie-Wiener and Poisson cases that will be detailed in Sections 5 and 6. In both cases, the operator δ coincides with the stochastic integral over square-integrable adapted processes.

Definition 4.1 *Given $k \geq 1$ and $u \in \mathbb{D}_{k,2}(H)$, the cumulant operator*

$$\Gamma_k^u : \mathbb{D}_{2,1} \longrightarrow L^2(\Omega)$$

is defined by $\Gamma_1^u \mathbf{1} = 0$ *and*

$$\Gamma_k^u \mathbf{1} = \langle (\nabla u)^{k-2} u, u \rangle + \langle \nabla^* u, \nabla ((\nabla u)^{k-2} u) \rangle, \qquad k \geq 2, \tag{4.6}$$

and is extended to all $F \in \mathbb{D}_{2,1}$ *by the formula*

$$\Gamma_k^u F := F \Gamma_k^u \mathbf{1} + \langle (\nabla u)^{k-1} u, DF \rangle, \qquad k \geq 1. \tag{4.7}$$

By (4.6) we also have the product rule

$$\Gamma_k^u (FG) = G \Gamma_k^u F + F \langle (\nabla u)^{k-1} u, DG \rangle,$$

which implies

$$\Gamma_k^u (FG) = G \Gamma_k^u F + F \Gamma_k^u G - FG \Gamma_k^u \mathbf{1},$$

and in particular

$$\Gamma_k^u F = F \Gamma_k^u \mathbf{1} + \langle (\nabla u)^{k-1} u, DF \rangle.$$

First, we state the next Lemma 4.2 which is used in the proof of Proposition 4.3 below and follows from Lemma 2.2 of [19]. It can be seen as a generalization to random u of the recurrence relation

$$E[X^n] = \sum_{l=0}^{n-1} \binom{n-1}{l} \kappa_{n-l}^X E[X^l], \qquad n \geq 1,$$

between the moments and cumulants of a given random variable X, cf., e.g., Relation (5) of [22].

Lemma 4.2 *For any* $n \geq 1$, $u \in \mathbb{D}_{n,1}(H)$ *and* $F \in \mathbb{D}_{2,1}$ *such that* $(\nabla u)^k u \in \mathbb{D}_{2,1}(H)$, $k = 1, \ldots, n-2$, *we have*

$$E[F \delta(u)^n] = \lambda \sum_{l=0}^{n-1} \frac{(n-1)!}{l!} E\left[\delta(u)^l \Gamma_{n-l}^u F\right] = \lambda \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} E\left[\delta(u)^{n-k} \Gamma_k^u F\right], \tag{4.8}$$

where $\Gamma_k^u F$, $k \geq 1$, *is defined in* (4.6).

The proof of Lemma 4.2 is postponed to the end of this section. We note that in addition to $\Gamma_1^u \mathbf{1} = 0$, in this general framework, for $k = 2$ we always have

$$\Gamma_2^u \mathbf{1} = \langle u, u \rangle + \langle \nabla^* u, \nabla u \rangle, \tag{4.9}$$

by (4.6), which from (4.8) yields the Skorohod isometry

$$E[\delta(u)^2] = \lambda E[\Gamma_2^u \mathbf{1}] = \lambda E[\langle u, u\rangle] + \lambda E[\langle \nabla^* u, \nabla u\rangle].$$

As an application of Lemma 4.2 by induction, we obtain the following Proposition 4.3 which establishes the covariance-moment Relation (3.1) and can be seen as a nonlinear (polynomial) extension of the integration by parts formula (or duality)

$$E\left[F\delta(u)\right] = \lambda E\left[\langle u, DF\rangle\right] = \lambda E\left[\Gamma_1^u F\right], \tag{4.10}$$

between D and δ, where $\Gamma_1^u F = \langle u, DF\rangle$, $F \in \mathbb{D}_{2,1}$, $u \in H$.

Proposition 4.3 *Let $F \in \mathbb{D}_{2,1}$ and $u \in \mathbb{D}_{2,1}(H)$, $n \geq 1$, and assume that*

$$\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F \in \mathbb{D}_{2,1}, \tag{4.11}$$

for all $l_1 + \cdots + l_a \leq n$, $a = 1, \ldots, n$. Then we have

$$E[F\delta(u)^n] = n! \sum_{a=1}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}. \tag{4.12}$$

Proof. For $n = 1$ we check that (4.11) holds from (4.10). Next, for $n \geq 1$, by (4.8) we have

$$\begin{aligned}
E[F\delta(u)^{n+1}] &= \lambda \sum_{k=1}^{n+1} \frac{n!}{(n+1-k)!} E\left[\delta(u)^{n+1-k}\Gamma_k^u F\right] \\
&= \lambda n! \sum_{k=1}^{n+1} \sum_{a=1}^{n+1-k} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n+1-k \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u \Gamma_k^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= n! \sum_{a=1}^{n+1} \lambda^{a+1} \sum_{\substack{l_1+\cdots+l_a=n+1-l_{a+1} \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{l_{a+1}=1}^{n+1} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_{a+1}}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= n! \sum_{a=1}^{n+1} \lambda^{a+1} \sum_{\substack{l_1+\cdots+l_{a+1}=n+1 \\ l_1\geq 1,\ldots,l_{a+1}\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_{a+1}}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \\
&= (n+1)! \sum_{a=1}^{n+1} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n+1 \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_{a+1})},
\end{aligned}$$

showing that

$$E[F\delta(u)^n] = n! \sum_{a=1}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma^u_{l_1}\cdots\Gamma^u_{l_a}F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)},$$

or

$$E[F\delta(u)^n] = n! \sum_{a=1}^{n} a!\lambda^a \sum_{0=k_0<k_1<\cdots<k_{a-1}<k_a=n} \frac{E\left[\Gamma^u_{k_1-k_0}\cdots\Gamma^u_{k_a-k_{a-1}}F\right]}{k_1\cdots k_a}.$$

□

In Proposition 4.3, Condition (4.11) is satisfied if $F \in I\!D_{n,n}$ and $u \in I\!D_{n,n}(H)$ for all $n \geq 1$. As examples of application of (4.12), for $n = 2$ we have $E[\delta(u)^2] = \lambda^2 E[\Gamma^u_2 \mathbf{1}]$ and

$$\begin{aligned} E[F\delta(u)^2] &= \lambda^2 E\left[\Gamma^u_1\Gamma^u_1 F\right] + \lambda E\left[\Gamma^u_2 F\right] \\ &= \lambda^2 E\left[\langle u, D\langle u, DF\rangle\rangle\right] + \lambda E\left[\langle(\nabla u)u, DF\rangle + F\langle u, u\rangle + F\langle\nabla^* u, \nabla u\rangle\right]. \end{aligned}$$

For $n = 3$ we find

$$E[F\delta(u)^3] = 2\lambda^2 E\left[\Gamma^u_1\Gamma^u_2 F\right] + 3\lambda^2 E\left[\Gamma^u_2\Gamma^u_1 F\right] + \lambda E\left[\Gamma^u_3 F\right].$$

We note that when $\Gamma^u_l \mathbf{1}$ is deterministic for all $l \geq 2$, the cumulant $\kappa^{\delta(u)}_l$ of $\delta(u)$ is given by

$$\kappa^{\delta(u)}_l = \lambda(l-1)!\Gamma^u_l \mathbf{1}, \qquad l \geq 1. \tag{4.13}$$

Indeed, Proposition 4.3 yields the moment identity

$$E[\delta(u)^n] = \sum_{a=1}^{n} \lambda^a \sum_{P_1,\ldots,P_a} (|P_1|-1)!\cdots(|P_a|-1)! E\left[(\Gamma^u_{|P_1|}\mathbf{1})\cdots(\Gamma^u_{|P_a|}\mathbf{1})\right], \tag{4.14}$$

which recovers the cumulant $\kappa^{\delta(u)}_l$ in (4.13) in the same way as in the classical case by application of the inversion formula (3.15).

In addition, when both $h \in H$ and ∇h are deterministic, which will be the case in Sections 5 and 6 on the Lie-Wiener and Poisson spaces, we obtain the following consequence of Relation (4.13).

Corollary 4.4 *Assume that both $h \in H$ and ∇h are deterministic. The cumulant $\kappa_k^{\delta(h)}$ of $\delta(h)$ is given by*

$$\kappa_k^{\delta(h)} = (k-1)!\langle(\nabla h)^{k-2}h, h\rangle, \qquad k \geq 2,$$

and by (4.7) *we have*

$$\Gamma_k^u F = \frac{1}{(k-1)!} F\kappa_k^{\delta(h)} + \langle(\nabla h)^{k-1}h, DF\rangle, \qquad F \in \mathcal{S}, \quad k \geq 1.$$

Proof. Relation (4.6) shows that $\Gamma_k^h \mathbf{1} = \langle(\nabla h)^{k-2}h, h\rangle$, $k \geq 2$, hence from Relation (4.13) the cumulant $\kappa_k^{\delta(h)}$ of $\delta(h)$ is given by $\kappa_1^{\delta(h)} = 0$ and

$$\kappa_k^{\delta(h)} = (k-1)!\Gamma_k^h \mathbf{1} = (k-1)!\langle(\nabla h)^{k-2}h, h\rangle,$$

$k \geq 2$. □

By Proposition 4.3 and Corollary 4.4, when both $h \in H$ and ∇h are deterministic, we also get the covariance-moment identity

$$E[F\delta(h)^n] = n! \sum_{a=0}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^h \cdots \Gamma_{l_a}^h F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \tag{4.15}$$

$$= n! \sum_{a=0}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\}} \frac{E\left[D_{(\nabla h)^{l_{i_1}-1}h} \cdots D_{(\nabla h)^{l_{i_k}-1}h} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_k)} \prod_{j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}} \frac{\kappa_{l_j}^{\delta(h)}}{(l_j-1)!}.$$

These results will be specialized to the Lie-Wiener, Wiener, and Poisson cases in the next sections 5 and 6 respectively.

Proof of Lemma 4.2. The proof of this key lemma is a combination of arguments from Lemma 3.1 of [15], Lemma 2.3 of [19] and Proposition 1 of [20], extended to include the parameter $\lambda > 0$ and the role of the covariant derivative operator ∇. First, we note that for $F \in \mathbb{D}_{2,1}$, $u \in \mathbb{D}_{n+1,2}(H)$, and all $i, l \in \mathbb{N}$ we have

$$E[F\delta(u)^l\langle(\nabla u)^i u, \delta(\nabla^* u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1}u, \delta(\nabla^* u)\rangle] \tag{4.16}$$
$$= \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1}u, u\rangle] + \lambda E[\delta(u)^l\langle(\nabla u)^{i+1}u, DF\rangle] + \lambda E[F\delta(u)^l\langle\nabla^* u, D((\nabla u)^i u)\rangle]$$
$$+\lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, \nabla((\nabla u)^{i+1}u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, D((\nabla u)^{i+1}u)\rangle].$$

Indeed, the duality (4.2) between D and δ, the chain rule of derivation (4.1) and Lemma 4.5 show that

$$\begin{aligned}
& E[F\delta(u)^l\langle(\nabla u)^i u, \delta(\nabla^* u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle] \\
&= \lambda E[\langle\nabla^* u, D(F\delta(u)^l(\nabla u)^i u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle] \\
&= \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, D\delta(u)\rangle] \\
&\quad -\lambda l E[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle] + \lambda E[\delta(u)^l\langle\nabla^* u, D(F(\nabla u)^i u)\rangle] \\
&= \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, u\rangle] + \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, \delta(\nabla^* u)\rangle] \\
&\quad +\lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, \nabla((\nabla u)^{i+1} u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, D((\nabla u)^{i+1} u)\rangle] \\
&\quad -\lambda l E[F\delta(u)^{l-1}\langle(\nabla^* u)^{i+1} u, \delta(\nabla^* u)\rangle] + \lambda E[\delta(u)^l\langle\nabla^* u, D(F(\nabla u)^i u)\rangle] \\
&= \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, u\rangle] \\
&\quad +\lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, \nabla((\nabla u)^{i+1} u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, D((\nabla u)^{i+1} u)\rangle] \\
&\quad +\lambda E[\delta(u)^l\langle\nabla^* u, D(F(\nabla u)^i u)\rangle] \\
&= \lambda l E[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, u\rangle] \\
&\quad +\lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, \nabla((\nabla u)^{i+1} u)\rangle] - \lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, D((\nabla u)^{i+1} u)\rangle] \\
&\quad +\lambda E[\delta(u)^l\langle(\nabla u)^{i+1} u, DF\rangle] + \lambda E[F\delta(u)^l\langle\nabla^* u, D((\nabla u)^i u)\rangle].
\end{aligned}$$

Next, since $(\nabla u)^{k-1} u \in \mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}$, by (4.16) and Lemma 4.5 we get

$$\begin{aligned}
& E\left[F\delta(u)^l\langle(\nabla u)^i u, D\delta(u)\rangle\right] - \lambda l E\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, D\delta(u)\rangle\right] \\
&= \lambda E\left[F\delta(u)^l\langle(\nabla u)^i u, u\rangle\right] + \lambda E\left[F\delta(u)^l\langle(\nabla u)^i u, \delta(\nabla^* u)\rangle\right] \\
&\quad +\lambda E[F\delta(u)^l\langle\nabla^* u, \nabla((\nabla u)^i u)\rangle] - \lambda E[F\delta(u)^l\langle\nabla^* u, D((\nabla u)^i u)\rangle] \\
&\quad -\lambda l E\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, u\rangle\right] - \lambda l E\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} u, \delta(\nabla^* u)\rangle\right] \\
&\quad -\lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, \nabla((\nabla u)^{i+1} u)\rangle] + \lambda l E[F\delta(u)^{l-1}\langle\nabla^* u, D((\nabla u)^{i+1} u)\rangle] \\
&= \lambda E\left[F\delta(u)^l\langle(\nabla u)^i u, u\rangle\right] + \lambda E[F\delta(u)^l\langle\nabla^* u, \nabla((\nabla u)^i u)\rangle] + \lambda E[\delta(u)^l\langle(\nabla u)^{i+1} u, DF\rangle],
\end{aligned}$$

and applying this formula to $l = n - k$ and $i = k - 1$ via a telescoping sum yields

$$\begin{aligned}
& E[F\delta(u)^n\delta(u)] = \lambda E[F\langle u, D\delta(u)^n\rangle] + \lambda E[\delta(u)^n\langle u, DF\rangle] \\
&= \lambda n E[F\delta(u)^{n-1}\langle u, D\delta(u)\rangle] + \lambda E[\delta(u)^n\langle u, DF\rangle]
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left(E\left[F\delta(u)^{n-k} \langle (\nabla u)^{k-1} u, D\delta(u) \rangle \right] - (n-k) E\left[F\delta(u)^{n-k-1} \langle (\nabla u)^{k} u, D\delta(u) \rangle \right] \right) \\
&\quad + \lambda E[\delta(u)^n \langle u, DF \rangle] \\
&= \lambda \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left(E\left[F\delta(u)^{n-k} \langle (\nabla u)^{k-1} u, u \rangle \right] + E\left[F\delta(u)^{n-k} \langle \nabla^* u, \nabla((\nabla u)^{k-1} u) \rangle \right] \right) \\
&\quad + \lambda \sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k} \langle (\nabla u)^{k} u, DF \rangle \right].
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&E[F\delta(u)^{n+1}] \\
&= \lambda \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left(E\left[F\delta(u)^{n-k} \langle (\nabla u)^{k-1} u, u \rangle \right] + E\left[F\delta(u)^{n-k} \langle \nabla^* u, \nabla((\nabla u)^{k-1} u) \rangle \right] \right) \\
&\quad + \lambda \sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k} \langle (\nabla u)^{k} u, DF \rangle \right] \\
&= \lambda \sum_{k=0}^{n} \binom{n}{k} E\left[\delta(u)^{n-k} \left(\frac{1}{k!} \Gamma_{k+1}^{u} \mathbf{1} + \langle (\nabla u)^{k} u, DF \rangle \right) \right] \\
&= \lambda \sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k} \Gamma_{k+1}^{u} F \right].
\end{aligned}
$$

□

The next commutation relation has been used in the proof of Lemma 4.2.

Lemma 4.5 *Let $u \in \mathrm{Dom}(\nabla)$ such that $\nabla_t u \in \mathrm{Dom}(\delta)$, $t \in \mathbb{R}_+$. We have*

$$
\langle h, D\delta(u) \rangle = \langle h, u \rangle + \langle h, \delta(\nabla u) \rangle + \langle \nabla^* u, \nabla h \rangle, \qquad h \in H.
$$

Proof. The argument is done for $u \in \mathcal{S} \otimes H$ of the form $u = F \otimes g \in \mathcal{S}$ and $h \in H$, and extended by closability. By (4.3) and (4.4) we have, under Condition **(H4)**, and using the notation $D_h F := \langle h, DF \rangle$,

$$
\begin{aligned}
\langle h, D\delta(u) \rangle &= \langle h, D(F\delta(g) - D_g F) \rangle \\
&= \delta(g) D_h F + F D_h \delta(g) - D_h D_g F \\
&= \delta(g) D_h F + F\langle g, h \rangle + F\langle h, \delta(\nabla g) \rangle - D_h D_g F \\
&= \langle h, \delta(g DF) \rangle + F\langle g, h \rangle + \langle h, \delta(F \nabla_t g) \rangle
\end{aligned}
$$

$$
\begin{aligned}
& +D_g D_h F - D_h D_g F + \langle DF \otimes h, \nabla g\rangle \\
= \ & \langle h, u\rangle + \langle h, \delta(\nabla u)\rangle + D_g D_h F - D_h D_g F + \langle DF \otimes h, \nabla g\rangle \\
= \ & \langle h, u\rangle + \langle h, \delta(\nabla u)\rangle + D_{\{g,h\}} F + \langle DF, \nabla_h g\rangle \\
= \ & \langle h, u\rangle + \langle h, \delta(\nabla u)\rangle + \langle D^* u, \nabla h\rangle \\
= \ & \langle h, u\rangle + \langle h, \delta(\nabla u)\rangle + \langle \nabla^* u, \nabla h\rangle,
\end{aligned}
$$

where at the last step, we used the relation $\nabla_s h(t) = 0$, $0 \leq t < s$, $h \in H$. □

5 The Lie-Wiener path space

In this section, we specialize the results of Section 4 to the setting of path spaces over Lie groups, which includes the classical Wiener space. Let G denote either $\mathbb{R}^d$ or a compact connected d-dimensional Lie group with associated Lie algebra $\mathcal{G}$ identified to $\mathbb{R}^d$ and equipped with an Ad-invariant scalar product on $\mathbb{R}^d \simeq \mathcal{G}$, also denoted by $\langle \cdot, \cdot \rangle$, with $H = L^2(\mathbb{R}_+; \mathcal{G})$. The commutator in $\mathcal{G}$ is denoted by $[\cdot, \cdot]$ and we let $\mathrm{ad}\,(u)v = [u, v]$, $u, v \in \mathcal{G}$, with $\mathrm{Ad}\, e^u = e^{\mathrm{ad}\, u}$, $u \in \mathcal{G}$. The Brownian motion $(\gamma(t))_{t\in\mathbb{R}_+}$ on G is constructed from a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ with variance $\lambda > 0$ via the Stratonovich differential equation

$$
\begin{cases}
d\gamma(t) = \gamma(t) \odot dB_t \\
\gamma(0) = \mathrm{e},
\end{cases}
$$

where e is the identity element in G. Let $\mathbb{P}(\mathsf{G})$ denote the space of continuous G-valued paths starting at e, with the image measure of the Wiener measure by the mapping $I : (B_t)_{t\in\mathbb{R}_+} \longmapsto (\gamma(t))_{t\in\mathbb{R}_+}$. Here, we take

$$
\mathcal{S} = \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \quad : \quad f \in \mathcal{C}_b^\infty(\mathsf{G}^n)\},
$$

and

$$
\mathcal{U} = \left\{ \sum_{i=1}^n u_i F_i \quad : \quad F_i \in \mathcal{S},\ u_i \in L^2(\mathbb{R}_+; \mathcal{G}),\ i = 1, \ldots, n,\ n \geq 1 \right\}.
$$

Next is the definition of the right derivative operator D, which satisfies Condition **(H1)**.

Definition 5.1 *For F of the form*

$$F = f(\gamma(t_1), \ldots, \gamma(t_n)) \in \mathcal{S}, \qquad f \in \mathcal{C}_b^\infty(\mathsf{G}^n), \tag{5.1}$$

we let $DF \in L^2(\Omega \times \mathbb{R}_+; \mathcal{G})$ be defined as

$$\langle DF, v \rangle = \frac{d}{d\varepsilon} f\left(\gamma(t_1) e^{\varepsilon \int_0^{t_1} v_s ds}, \ldots, \gamma(t_n) e^{\varepsilon \int_0^{t_n} v_s ds} \right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+, \mathcal{G}).$$

Given F of the form (5.1) we also have

$$D_t F = \sum_{i=1}^n \partial_i f(\gamma(t_1), \ldots, \gamma(t_n)) \mathbf{1}_{[0,t_i]}(t), \qquad t \geq 0.$$

The operator D is known to admit an adjoint δ that satisfies Condition **(H2)**, i.e.

$$E[F\delta(v)] = \lambda E[\langle DF, v \rangle], \quad F \in \mathcal{S}, \ v \in L^2(\mathbb{R}_+; \mathcal{G}), \tag{5.2}$$

cf., e.g., [4]. In addition, recall that when $(u_t)_{t\in\mathbb{R}_+}$ is square-integrable and adapted to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, $\delta(u)$ coincides with the Itô integral of $u \in L^2(\Omega; H)$ with respect to the underlying Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t, \tag{5.3}$$

as a consequence of, e.g., Lemma 4.1 of [14].

Definition 5.2 *Let the operator $\nabla : I\!D_{2,1}(H) \longrightarrow L^2(\Omega; H \otimes H)$ be defined as*

$$\nabla_s u_t = D_s u_t + \mathbf{1}_{[0,t]}(s) \mathrm{ad}\, u_t \in \mathcal{G} \otimes \mathcal{G}, \qquad s, t \in \mathbb{R}_+, \tag{5.4}$$

$u \in I\!D_{2,1}(H)$.

It is known that D and ∇ satisfy Condition **(H3)** and the commutation relation (4.3) as a consequence, cf. [4]. From Lemma 5.6 below, for all deterministic $h \in H$ we have $(\nabla^* h)h = (D^* h)h = 0$, and hence

$$\langle (\nabla h)u, h \rangle = \langle u, (\nabla^* h)h \rangle = 0,$$

and in particular,

$$\langle (\nabla h)^k h, h \rangle = 0, \qquad k \geq 1,$$

which shows, by Corollary 4.4, that (3.12) recovers (2.1). On the other hand, it is known that ∇ satisfies Conditions **(H3)** and **(H4)**, cf. Theorem 2.3$-i$) of [4].

As another consequence of Lemma 5.6 we have the following result which shows that (3.5) holds on the Lie-Wiener path space, cf. also Lemma 3 in [20].

Lemma 5.3 *Letting $k \geq 1$ and $u \in \mathbb{D}_{2,1}(H)$, we have*

$$\langle (\nabla u)^k v, u \rangle = \frac{1}{2} \langle (\nabla u)^{k-1} v, D\langle u, u \rangle \rangle, \qquad v \in H. \tag{5.5}$$

Proof. By Lemma 5.6 below and the relation $D\langle u, u\rangle = 2(D^* u)u$ we have

$$\langle (\nabla u) v, u \rangle = \langle (\nabla^* u) u, v \rangle = \langle (D^* u) u, v \rangle = \frac{1}{2} \langle v, D\langle u, u \rangle \rangle.$$

□

In the following Proposition 5.5 we compute the cumulant operator Γ_k^u appearing in the relation

$$E[F\delta(u)^n] = n! \sum_{a=0}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}, \tag{5.6}$$

$n \geq 1$, $F \in \mathbb{D}_{n,n}$, $u \in \mathbb{D}_{n,n}(H)$, which is a consequence of Proposition 4.3 and shows that (3.5) holds on the Lie-Wiener path space. Due to Relations (4.7) and (4.9) it is sufficient to compute $\Gamma_k^u \mathbf{1}$ for $k \geq 3$. The next lemma has been used in the proof of Lemma 5.3 above.

Lemma 5.4 *Letting $k \geq 3$ and $u \in \mathbb{D}_{k,k}(H)$, we have*

$$\Gamma_k^u \mathbf{1} = \frac{1}{2} \langle (\nabla u)^{k-3} u, D\langle u, u \rangle \rangle + \langle \nabla^* u, \nabla((\nabla u)^{k-2} u) \rangle. \tag{5.7}$$

Proof. This is a direct application of Relation (5.5) to (4.6). □

As a consequence of Lemma 5.4 we have the following result.

Proposition 5.5 *Let $u \in \mathbb{D}_{\infty,2}(H)$, and assume that*

$$\langle \nabla^* u, \nabla((\nabla u)^{k-2} u) \rangle = 0, \qquad k \geq 2, \tag{5.8}$$

and the cumulant $\kappa_2(u) = \langle u, u \rangle$ is deterministic. Then, $\delta(u)$ is a centered Gaussian random variable with variance $\langle u, u \rangle$.

Proof. By (5.7) and (5.8) we get $\Gamma_2^u \mathbf{1} = \langle u, u \rangle$, and

$$\Gamma_k^u \mathbf{1} = \frac{1}{2} \langle (\nabla u)^{k-3} u, D\langle u, u \rangle \rangle = 0, \qquad k \geq 3, \tag{5.9}$$

for any $u \in I\!\!D_{k,1}(H)$. Consequently, Relations (4.13) and (5.9) show that $\delta(u)$ has cumulants

$$\Gamma_l^u \mathbf{1} = \kappa_l(u) = \mathbf{1}_{\{l=2\}} \langle u, u \rangle, \qquad l \geq 1. \tag{5.10}$$

□

In particular, the Skorohod integral $\delta(Rh)$ on the Wiener space has a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H with quasi-nilpotent gradient, cf. Corollary 5.8 below, which extends by a direct argument to the Lie-Wiener space the sufficient conditions found in [25] Theorem 2.1$-b$). An example of anticipating process u satisfying (5.8) is provided in [19] on the Lie-Wiener space by letting

$$u = \sum_{k=0}^{\infty} A_k e_k \in I\!\!D_{2,1}(H) \tag{5.11}$$

where $(A_k)_{k\in\mathbb{N}}$ is a sequence of $\sigma(\delta(f_k) \ : \ k \in \mathbb{N})$-measurable scalar random variables such that $\|u\|_H = 1$, a.s., and $(e_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ are orthonormal sequences that are also mutually orthogonal in H, and such that $(e_k(t))_{k\in\mathbb{N}, t\in\mathbb{R}_+}$ is made of commuting elements in $\mathcal{G}$, by noting that $\nabla u_{t_3} \nabla_{t_1} u = D u_{t_3} D_{t_1} u = 0$, $t_1, t_3 \in \mathbb{R}_+$, and $(\nabla u) u = (Du) u$.

Note that Condition (5.15) is satisfied in particular when $(u_t)_{t\in\mathbb{R}_+}$ is adapted to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, cf. Lemma 3.5 of [19]. In addition, if h is deterministic, (4.15) shows that we have

$$\begin{aligned} & E[F\delta(h)^n] \qquad\qquad (5.12) \\ = \;& n! \sum_{a=1}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\}} \frac{E\left[D_{(\nabla h)^{l_{i_1}-1}h} \cdots D_{(\nabla h)^{l_{i_k}-1}h} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \prod_{j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}} \frac{\kappa_{l_j}(h)}{(l_j-1)!} \\ = \;& n! \sum_{a=1}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{\substack{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\} \\ l_j=2, \ j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}}} \frac{E\left[D_{(\nabla h)^{l_{i_1}-1}h} \cdots D_{(\nabla h)^{l_{i_k}-1}h} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \prod_{j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}} \frac{\langle h, h\rangle}{(l_j-1)!}. \end{aligned}$$

Lemma 5.6 *For $u \in \mathbb{D}_{2,1}(H)$, we have*

$$(\nabla^* u)u = (D^* u)u.$$

Proof. By Relation (7.5) in the appendix we have

$$\begin{aligned}(\nabla^* u)u_s &= \int_0^\infty (\nabla_s^\dagger u_t)u_t dt = \int_0^\infty (D_s u_t)^\dagger u_t dt + \int_0^\infty \mathbf{1}_{[0,t]}(s)(\mathrm{ad}\, u_t)^\dagger u_t dt \\ &= \int_0^\infty (D_s u_t)^\dagger u_t dt - \int_s^\infty (\mathrm{ad}\, u_t) u_t dt = \int_0^\infty (D_s u_t)^\dagger u_t dt - \int_s^\infty [u_t, u_t] dt \\ &= \int_0^\infty (D_s u_t^\dagger) u_t dt = (D^* u)u_s, \qquad s \in \mathbb{R}_+.\end{aligned}$$

□

Wiener space

Here we consider the case where $\mathsf{G} = \mathbb{R}^d$ and $(\gamma(t))_{t\in\mathbb{R}_+} = (B_t)_{t\in\mathbb{R}_+}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space $W = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$, in which case ∇ equals the Malliavin derivative which will be denoted by $\hat{D}$. In this case, we let $\hat{\delta} = \delta$ denote the Skorohod integral operator adjoint of $\hat{D}$, which coincides by (5.3) with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.,

$$\hat{\delta}(u) = \int_0^\infty u_t dB_t,$$

when $(u_t)_{t\in\mathbb{R}_+}$ is square-integrable and adapted with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, cf., e.g., Proposition 4.3.4 of [16], and references therein. In the Wiener case, the relation $\nabla = \hat{D}$ implies that (4.9) reads

$$\Gamma_2^u \mathbf{1} = \langle u, u\rangle + \mathrm{trace}\,(\hat{D}u)^2.$$

The following result shows how Γ_k^u in (5.6) can be computed on the Wiener space.

Proposition 5.7 *Letting $u \in \mathbb{D}_{2,2}(H)$, for all $k \geq 3$ we have*

$$\Gamma_k^u \mathbf{1} = \frac{1}{2}\langle (\hat{D}u)^{k-3}u, \hat{D}\langle u, u\rangle\rangle + \mathrm{trace}\,(\hat{D}u)^k + \sum_{i=2}^{k-1} \frac{1}{i}\langle (\hat{D}u)^{k-1-i}u, \hat{D}\,\mathrm{trace}\,(\hat{D}u)^i\rangle. \quad (5.13)$$

Proof. It suffices to use Lemma 5.4 and the relation

$$\langle \hat{D}^* u, \hat{D}((\hat{D}u)^k v)\rangle = \operatorname{trace}((\hat{D}u)^{k+1}\hat{D}v) + \sum_{i=2}^{k+1} \frac{1}{i}\langle (\hat{D}u)^{k+1-i} v, \hat{D}\operatorname{trace}(\hat{D}u)^i\rangle, \quad (5.14)$$

$u \in \mathbb{D}_{2,2}(H)$, $v \in \mathbb{D}_{2,1}(H)$, $k \in \mathbb{N}$, cf. Lemma 4 in [20]. □

As a consequence of Proposition 5.5 we have the following corollary.

Corollary 5.8 *Let $u \in \mathbb{D}_{\infty,2}(H)$, and assume*

1) the quasi-nilpotence condition

$$\operatorname{trace}(\hat{D}u)^n = 0, \qquad n \geq 2, \quad (5.15)$$

2) the cumulant $\kappa_2(u) = \langle u, u\rangle$ is deterministic.

Then, $\delta(u)$ is a centered Gaussian random variable with variance $\langle u, u\rangle$.

Proof. By (5.13) and (5.15) we get (5.8) and we conclude from Proposition 5.5. □

Under the quasi-nilpotence condition (5.15) we get $\Gamma_2^u \mathbf{1} = \langle u, u\rangle$ and

$$\Gamma_k^u \mathbf{1} = \frac{1}{2}\langle (\hat{D}u)^{k-3} u, \hat{D}\langle u, u\rangle\rangle, \quad (5.16)$$

$k \geq 3$, for any $u \in \mathbb{D}_{2,1}(H)$, which shows (3.7) on the Wiener space as a consequence of Proposition 5.7, and by Lemma 2.3 of [19], Condition (5.15) is satisfied in particular when $(u_t)_{t\in\mathbb{R}_+}$ is adapted to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. When $h \in H$ is a deterministic function, we have $\nabla h = \hat{D}h = 0$, and hence, (5.10) shows that

$$\Gamma_2^h \mathbf{1} = \mathbf{1}_{\{k=2\}}\kappa_2(h) = \mathbf{1}_{\{k=2\}}\langle h, h\rangle,$$

and $\Gamma_k^h F = 0$, $k \geq 3$, and hence, from (5.12), for $n \geq 2$ we find

$$E\left[F\left(\int_0^\infty h(t)dB_t\right)^n\right] = n!\sum_{a=0}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n\\ 1\leq l_1\leq 2,\ldots,1\leq l_a\leq 2}} \frac{E\left[\Gamma_{l_1}^h \cdots \Gamma_{l_a}^h F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}$$

$$= n!\sum_{a=0}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n\\ 1\leq l_1\leq 2,\ldots,1\leq l_a\leq 2}} \sum_{\substack{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\}\\ l_j=2,\ j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}}} \left(\frac{\langle h, h\rangle}{2}\right)^{a-k} E\left[D_{(\nabla h)^{l_{i_1}-1}h}\cdots D_{(\nabla h)^{l_{i_k}-1}h}F\right]$$

$$
\begin{aligned}
&= n!\sum_{a=0}^{n}\lambda^a\sum_{k=0}^{a}\binom{a}{k}\frac{\langle h,h\rangle^{a-k}}{2^{a-k}}E[\langle h^{\otimes k},\hat{D}^kF\rangle]\sum_{\substack{l_1+\cdots+l_{a-k}=n-k\\ l_1=\cdots=l_{a-k}=2}}1\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{\langle h,h\rangle^{(n-k)/2}}{2^{(n-k)/2}}E[\langle h^{\otimes k},\hat{D}^kF\rangle]\sum_{a=0}^{n}\lambda^{a+k}(n-k)!\sum_{\substack{l_1+\cdots+l_a=n-k\\ l_1=\cdots=l_a=2}}1,\\
&= \begin{cases}\displaystyle\sum_{k=0}^{n/2}\binom{n}{2k}(n-2k-1)!!\lambda^{n/2+k}\langle h,h\rangle^{n/2-k}E[\langle h^{\otimes 2k},\hat{D}^{2k}F\rangle], & n \text{ even},\\ \displaystyle\sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}(n-2k-2)!!\lambda^{(n-1)/2+k}\langle h,h\rangle^{(n-1)/2-k}E[\langle h^{\otimes 2k+1},\hat{D}^{2k+1}F\rangle], & n \text{ odd}.\end{cases}\\
&= \sum_{k=0}^{n}\binom{n}{k}\lambda^kE[\langle h^{\otimes k},\hat{D}^kF\rangle]E\left[\left(\int_0^\infty h(t)dB_t\right)^{n-k}\right]\\
&= \sum_{k=0}^{n}\binom{n}{k}E[FI_k(h^{\otimes k})]E\left[\left(\int_0^\infty h(t)dB_t\right)^{n-k}\right],
\end{aligned}
$$

where $I_k(f_k)$ denotes the multiple stochastic integral of the symmetric function f_k of k variables with respect to Brownian motion. This formula recovers the identity

$$
\begin{aligned}
E\left[Fe^{\delta(h)}\right] &= \sum_{n=0}^{\infty}\frac{1}{n!}E[F\delta(h)^n]\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\lambda^k\binom{n}{k}E[\langle h^{\otimes k},\hat{D}^kF\rangle]E[\delta(h)^{n-k}]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}E[\langle h^{\otimes k},\hat{D}^kF\rangle]\sum_{n=k}^{\infty}\frac{1}{(n-k)!}E[\delta(h)^{n-k}]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}E[\langle h^{\otimes k},\hat{D}^kF\rangle]\sum_{l=0}^{\infty}\frac{\lambda^l}{(2l)!}(2l-1)!!\langle h,h\rangle^l\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}E[\langle h^{\otimes k},\hat{D}^kF\rangle]\sum_{l=0}^{\infty}\frac{\lambda^l}{2^l l!}\langle h,h\rangle^l\\
&= e^{\lambda\langle h,h\rangle/2}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}E[\langle h^{\otimes k},\hat{D}^kF\rangle],
\end{aligned}
$$

which can be found independently by the Stroock [23] formula

$$
F=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(E[\hat{D}^nF]),
$$

as follows:

$$E\left[Fe^{\delta(h)}\right] = e^{\lambda\langle h,h\rangle/2}\sum_{n=0}^{\infty} E\left[FI_n(h^{\otimes n})\right] = e^{\lambda\langle h,h\rangle/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}E\left[\langle h^{\otimes n}, \hat{D}^n F\rangle\right],$$

cf. e.g. Proposition 4.2.5 in [16].

6 The Poisson case

In this section we show that the general framework of Section 4 also includes other infinitely divisible distributions as we apply it to the standard Poisson process on $\mathbb{R}_+$. Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with intensity $\lambda > 0$, jump times $(T_k)_{k\geq 1}$, and generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}_\infty, P)$, with $T_0 = 0$. The gradient $\tilde{D}$ defined as

$$\tilde{D}_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t)\frac{\partial f}{\partial x_k}(T_1, \ldots, T_n), \tag{6.1}$$

for $F \in \mathcal{S} := \{F = f(T_1, \ldots, T_n) \quad : \quad f \in \mathcal{C}_b^1(\mathbb{R}^n)\}$, has the derivation property and therefore satisfies Condition **(H1)**, cf. [2], § 7 of [16]. Here, we let

$$\mathcal{U} = \left\{\sum_{i=1}^{n} u_i F_i \quad : \quad F_i \in \mathcal{S},\ u_i \in \mathcal{C}_c^1(\mathbb{R}_+),\ i = 1, \ldots, n,\ n \geq 1\right\},$$

and we have $H = L^2(\mathbb{R}_+)$. The operator $\tilde{D}$ has an adjoint $\tilde{\delta}$ which coincides with the compensated Poisson stochastic integral on square-integrable processes $(u_t)_{t\in\mathbb{R}_+}$ adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(N_t)_{t\imath\mathbb{R}_+}$, i.e., we have

$$\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - \lambda t),$$

and in particular $\tilde{\delta}$ satisfies Condition **(H2)**. The next definition of covariant derivative in the jump case, cf. [13], is the counterpart of Definition 5.2.

Definition 6.1 *Let the operator $\tilde{\nabla}$ be defined as*

$$\tilde{\nabla}_s u_t := \tilde{D}_s u_t - \dot{u}_t \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}, \tag{6.2}$$

where $\dot{u}_t$ denotes the time derivative of $t \longmapsto u_t$ with respect to t.

In particular, for $u, v \in \mathcal{U}$ we have

$$(\tilde{\nabla} u) v_s = \int_0^\infty v_t \tilde{\nabla}_t u_s dt = -\dot{u}_s \int_0^\infty v_t \mathbf{1}_{[0,s]}(t) dt = -\dot{u}_s \int_0^s v_t dt \qquad s \in \mathbb{R}_+. \tag{6.3}$$

The operator $\tilde{D}$ defines the Sobolev spaces $\tilde{\mathbb{D}}_{p,1}$ and $\tilde{\mathbb{D}}_{p,1}(H)$, $p > 1$, respectively, by the Sobolev norms

$$\|F\|_{\tilde{\mathbb{D}}_{p,1}} = \|F\|_{L^p(\Omega)} + \|\tilde{D} F\|_{L^p(\Omega, H)}, \qquad F \in \mathcal{S},$$

and

$$\|u\|_{\tilde{\mathbb{D}}_{p,1}(H)} = \|u\|_{L^p(\Omega,H)} + \|\tilde{D} u\|_{L^p(\Omega, H\otimes H)} + \left(E\left[\left(\int_0^\infty t |\dot{u}_t|^2 dt \right)^{p/2} \right] \right)^{1/p},$$

$u \in \mathcal{U}$. In addition, the operators $\tilde{\nabla}$, $\tilde{\delta}$ and $\tilde{D}$ satisfy the commutation relation

$$\tilde{D}_t \tilde{\delta}(u) = u_t + \tilde{\delta}(\tilde{\nabla}_t u),$$

which is (4.3) in Condition **(H3)**, for $u \in \tilde{\mathbb{D}}_{2,1}(H)$ such that $\tilde{\nabla}_t u \in \tilde{\mathbb{D}}_{2,1}(H)$, $t \in \mathbb{R}_+$, cf. Relation (3.6) and Proposition 3.3 in [13], or Lemma 7.6.6 page 276 of [16]. Condition **(H4)** is satisfied from Proposition 3.1 of [13] or Proposition 7.6.3 of [16]. The following lemma shows that (3.4) holds on the Poisson space, which allows one to compute Γ_k^u by (3.2).

Lemma 6.2 *Letting $k \geq 1$ and $F \in \mathbb{D}_{k,k}$, $u \in \mathbb{D}_{k,k}(H)$, we have*

$$\langle (\tilde{\nabla} u)^n u, u \rangle = \frac{1}{(n+1)!} \int_0^\infty u_s^{n+2} ds + \sum_{i=2}^{n+1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{n+1-i} u, \tilde{D} \int_0^\infty u_t^i dt \right\rangle. \tag{6.4}$$

Proof. By Relation (4.10) in Lemma 4.7 of [19], for all $n \in \mathbb{N}$ and $u \in \tilde{\mathbb{D}}_{2,1}(H)$ such that $u \in \bigcap_{k=1}^{2n+2} L^k(\mathbb{R}_+)$ a.s., we have

$$\begin{aligned}(\tilde{\nabla}^* u)^n u_t &= \int_0^\infty \cdots \int_0^\infty u_{t_n} \tilde{\nabla}_t u_{t_1} \tilde{\nabla}_{t_1} u_{t_2} \cdots \tilde{\nabla}_{t_{n-1}} u_{t_n} dt_1 \cdots dt_n \\ &= \frac{1}{(n+1)!} u_t^{n+1} + \sum_{i=2}^{n+1} \frac{1}{i!} (\tilde{\nabla}^* u)^{n+1-i} \tilde{D}_t \int_0^\infty u_s^i ds,\end{aligned} \tag{6.5}$$

$t \in \mathbb{R}_+$, and by integration with respect to $t \in \mathbb{R}_+$ we get (6.4). □

When f and h are smooth deterministic functions, (6.3) extends to $k \geq 2$ as

$$\begin{aligned}
(\tilde{\nabla} h)^k f_s &= (-1)^k \dot{h}(s) \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_k} h(s) \tilde{\nabla}_{t_{k-1}} h(t_k) \cdots \tilde{\nabla}_{t_1} h(t_2) f(t_1) dt_1 \cdots dt_k \\
&= (-1)^k \dot{h}(s) \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[0,s]}(t_k) \mathbf{1}_{[0,t_k]}(t_{k-1}) \dot{h}(t_k) \cdots \mathbf{1}_{[0,t_2]}(t_1) \dot{h}(t_2) f(t_1) dt_1 \cdots dt_k \\
&= (-1)^k \dot{h}(s) \int_0^s \dot{h}(t_k) \int_0^{t_k} \cdots \dot{h}(t_2) \int_0^{t_2} f(t_1) dt_1 \cdots dt_k, \qquad s \in \mathbb{R}_+, \qquad (6.6)
\end{aligned}$$

which complements (6.5), and recovers (6.5) for deterministic functions as

$$\begin{aligned}
\langle (\tilde{\nabla} h)^k f, h \rangle &= (-1)^k \int_0^\infty h(s) \dot{h}(s) \int_0^s \int_0^{t_k} \cdots \int_0^{t_2} \dot{h}(t_k) \cdots \dot{h}(t_2) f(t_1) dt_1 \cdots dt_k \\
&= \frac{1}{2} (-1)^{k-1} \int_0^\infty h^2(t_k) \dot{h}(t_k) \int_0^{t_k} \cdots \int_0^{t_2} \dot{h}(t_{k-1}) \cdots \dot{h}(t_2) f(t_1) dt_1 \cdots dt_k \\
&= \frac{1}{(k+1)!} \int_0^\infty h^{k+1}(t_1) f(t_1) dt_1, \qquad s \in \mathbb{R}_+.
\end{aligned}$$

As a direct consequence of (4.6) and (6.4), in the next Proposition 6.3 we compute the cumulant operator Γ_k^u appearing in the moment identity (3.1) on the Poisson space, i.e. we have

$$E[F\delta(u)^n] = n! \sum_{a=1}^n \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1 \geq 1, \ldots, l_a \geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}, \qquad (6.7)$$

for $n \geq 1$, $u \in \tilde{I\!D}_{n,2}(H)$ and $F \in I\!D_{2,1}$, with $u \in \bigcap_{k=2}^n L^{2k}(\Omega, L^k(\mathbb{R}_+))$, and $(\tilde{\nabla} u)^k u \in \tilde{I\!D}_{2,1}(H)$, $k = 1, \ldots, n-2$, provided

$$\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F \in I\!D_{2,1},$$

for all $l_1 + \cdots + l_a \leq n$, $a = 1, \ldots, n$. Again, due to Relation (4.7) it suffices to give the value of $\Gamma_k^u \mathbf{1}$ in order to compute (6.7). The next proposition is a corollary of Proposition 4.3 and provides the expression of (4.6) in the Poisson case, cf. Proposition 5.7 for the Lie-Wiener case.

Proposition 6.3 *Let* $n \geq 1$, $u \in \tilde{I\!D}_{n,2}(H)$, *with* $u \in \bigcap_{k=2}^n L^{2k}(\Omega, L^k(\mathbb{R}_+))$, *and* $(\tilde{\nabla} u)^k u \in \tilde{I\!D}_{2,1}(H)$, $k = 1, \ldots, n-2$. *We have*

$$\Gamma_k^u \mathbf{1} = \frac{1}{(k-1)!} \int_0^\infty u_s^k ds + \sum_{i=2}^{k-1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{k-1-i} u, \tilde{D} \int_0^\infty u_t^i dt \right\rangle + \langle \tilde{\nabla}^* u, \tilde{\nabla}((\tilde{\nabla} u)^{k-2} u) \rangle, \qquad (6.8)$$

$k \geq 2$.

The next proposition shows that (3.8) holds for adapted processes on the Poisson space.

Proposition 6.4 *Let $u \in \tilde{\mathbb{D}}_{\infty,1}(H)$ be adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then, $\Gamma_k^u \mathbf{1}$ in (6.7) is given by*

$$\Gamma_k^u \mathbf{1} = \frac{1}{(k-1)!}\int_0^\infty u_s^k ds + \sum_{i=2}^{k-1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{k-1-i} u, \tilde{D} \int_0^\infty u_t^i dt \right\rangle, \tag{6.9}$$

$k \geq 2$, which shows that (3.8) holds on the Poisson space.

Proof. Let $u, v \in \tilde{\mathbb{D}}_{\infty,1}(H)$ be two processes adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, such that $(\tilde{\nabla} u)^n u \in \mathbb{D}_{2,1}(H)$, $n \geq 1$. By Lemma 4.4 of [19] we have

$$\langle \tilde{\nabla}^* u, \tilde{\nabla}((\tilde{\nabla} u)^k v)\rangle = 0, \qquad k \in \mathbb{N}. \tag{6.10}$$

Hence, when the process u is adapted, this yields (6.9) by (6.8) and (6.10). □

In particular, if h is a deterministic function, we have

$$\Gamma_k^h \mathbf{1} = \frac{1}{(k-1)!}\kappa_k(h) = \frac{1}{(k-1)!}\int_0^\infty h^k(s) ds, \tag{6.11}$$

$k \geq 2$, and consequently, we have the following result.

Proposition 6.5 *Let $(u_t)_{t\in\mathbb{R}_+}$ be a process in $\tilde{\mathbb{D}}_{\infty,1}(H)$ such that*

$$\langle \tilde{\nabla}^* u, \tilde{\nabla}((\tilde{\nabla} u)^k u)\rangle = 0, \qquad k \geq 0, \tag{6.10}$$

and $\int_0^\infty u_t^i dt$ is deterministic for all $i \geq 2$. Then, $\tilde{\delta}(u)$ has a centered infinitely divisible distribution with cumulants $\kappa_i(u) = \int_0^\infty u_t^i dt$, $i \geq 2$.

Examples of processes satisfying the conditions of Proposition 6.5 can be constructed by composition of a function of $\mathbb{R}_+$ with an adapted process $(u_t)_{t\in\mathbb{R}_+}$ such that $t \longmapsto u_t$ is a.s. measure-preserving on $\mathbb{R}_+$, cf. [19]. On the Poisson space, Relation (4.15) holds as

$$E[F\delta(h)^n] = n! \sum_{a=0}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \frac{E\left[\Gamma_{l_1}^u \cdots \Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \tag{6.12}$$

$$= n! \sum_{a=0}^{n} \lambda^a \sum_{\substack{l_1+\cdots+l_a=n \\ l_1\geq 1,\ldots,l_a\geq 1}} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,a\}} \frac{E\left[\tilde{D}_{(\tilde{\nabla} h)^{l_{i_1}-1}h} \cdots \tilde{D}_{(\tilde{\nabla} h)^{l_{i_k}-1}h} F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)} \prod_{j\in\{1,\ldots,a\}\setminus\{i_1,\ldots,i_k\}} \frac{\kappa_{l_j}(h)}{(l_j-1)!}$$

with $(\tilde{\nabla} h)^k h$ given by (6.6).

7 Appendix

In this appendix, for completeness, we gather some notation and conventions used in this paper, cf. [19] for details. Given X a real separable Hilbert space, the definition of D is naturally extended to X-valued random variables by letting

$$DF = \sum_{k=1}^{n} x_i \otimes DF_i \tag{7.1}$$

for $F \in X \otimes \mathcal{S} \subset L^2(\Omega; X)$ of the form

$$F = \sum_{k=1}^{n} x_i \otimes F_i$$

$x_1, \ldots, x_n \in X$, $F_1, \ldots, F_n \in \mathcal{S}$. When D maps $\mathcal{S}$ to $\mathcal{S} \otimes H$, as on the Lie-Wiener space, iterations of this definition starting with $X = \mathbb{R}$, then $X = H$, and successively replacing X with $X \otimes H$ at each step, allow one to define

$$D^n : X \otimes \mathcal{S} \longmapsto L^2(\Omega; X \hat{\otimes} H^{\hat{\otimes} n})$$

for all $n \geq 1$, where $\hat{\otimes}$ denotes the completed symmetric tensor product of Hilbert spaces. In that case, we let $\mathbb{D}_{p,k}(X)$ denote the completion of the space $X \otimes \mathcal{S}$ of X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(\Omega, X \hat{\otimes} H^{\hat{\otimes} l})}, \qquad p \geq 1, \tag{7.2}$$

with

$$\mathbb{D}_{\infty,k}(X) = \bigcap_{k\geq 1} \mathbb{D}_{p,k}(X),$$

and $\mathbb{D}_{p,k} = \mathbb{D}_{p,k}(\mathbb{R})$, $p \in [1, \infty]$, $k \geq 1$. Note that for all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, the gradient operator D is continuous from $\mathbb{D}_{p,k}(X)$ into

$\mathbb{D}_{q,k-1}(X\hat{\otimes}H)$ and the Skorohod integral operator δ adjoint of D is continuous from $\mathbb{D}_{p,k}(H)$ into $\mathbb{D}_{q,k-1}$. Given $u \in \mathbb{D}_{2,1}(H)$ we also identify

$$\nabla u = ((s,t) \longmapsto \nabla_t u_s)_{s,t\in\mathbb{R}_+} \in H\hat{\otimes}H$$

to the random operator ∇u on H almost surely defined by

$$(\nabla u)v_s := \int_0^\infty (\nabla_t u_s)v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in H, \tag{7.3}$$

in which $a \otimes b \in X\hat{\otimes}H$ is identified to the

$$(a \otimes b)c = a\langle b, c\rangle, \qquad a \otimes b \in X\hat{\otimes}H, \quad c \in H.$$

More generally, for $u \in \mathbb{D}_{2,1}(H)$ and $v \in H$ we have

$$(\nabla u)^k v_s = \int_0^\infty \cdots \int_0^\infty (\nabla_{t_k} u_s \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}) v_{t_1} dt_1 \cdots dt_k, \qquad s,t \in \mathbb{R}_+. \tag{7.4}$$

We also define the adjoint $\nabla^* u$ of ∇u on H which satisfies

$$\langle (\nabla u)v, h\rangle = \langle v, (\nabla^* u)h\rangle, \qquad v, h \in H,$$

and is given by

$$(\nabla^* u)v_s = \int_0^\infty (\nabla_s^\dagger u_t)v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H). \tag{7.5}$$

Although D is originally defined for scalar random variables, its definition extends pointwise to $u \in \mathbb{D}_{2,1}(H)$ by (7.1), i.e.,

$$D(u) := ((s,t) \longmapsto D_t u_s)_{s,t\in\mathbb{R}_+} \in H\hat{\otimes}H, \tag{7.6}$$

and the operators Du and D^*u are constructed in the same way as ∇u and $\nabla^* u$ in (7.3) and (7.5).

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