Stochastic deformation of integrable dynamical systems and random time symmetry

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Abstract

We present a deformation of a class of elementary classical integrable systems using stochastic diffusion processes. This deformation applies to the solution of the associated classical Newtonian, Hamiltonian, Lagrangian and variational problems and to the Hamilton-Jacobi method of characteristics. The underlying stochastic action functionals involve dual random times, whose expectations are connected to the new variables of the system after a canonical transformation.

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1 Introduction

One of the aims of this work is to answer the apparently elementary question of how to define a "stochastic integrable dynamical system". Stochastic analysis, initiated by Norbert Wiener's mathematical construction of Brownian motion in the 1920s, is

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nowadays a very elaborated subject due to its rapid development since 1980, cf. [9], [15], [20]. It is used always more in applications ranging from physics to life sciences and economics.

The theory of Stochastic Differential Equations (SDEs), due to Kyosi Itô, is a cornerstone of modern stochastic analysis. One way to look at SDEs is to see them as some probabilistic deformations of ordinary differential equations (ODEs). In particular, as for ODEs, regularity conditions on the coefficients of the SDE insuring the existence and uniqueness of the solutions have been known for a long time (Lipschitz conditions are sufficient). When those coefficients are linear, explicit formulas inspired by the corresponding classical ODE are available and, in this sense, these equations are integrable as the ones resulting, obviously, from diffeomorphic transformations of those. But, interestingly, no rigorous and general concept of integrability seems to have been defined for SDEs. In [14], however, it is shown how a stochastic theorem of Noether allows one to define the probabilistic counterparts of first integrals for special systems and, the associated relations between diffusions, called "stochastic quadratures", see also [17] in the jump case.

This contrasts strongly with the historical development of ODEs, where the concept of integrability appeared quite early with Liouville and others. One reason for this difference lies in the motivations of the scientists involved then and now. In the early stages of the ODE theory, the central problem to be solved was the N body problem, the most famous problem of classical dynamical system theory. In fact, the various approaches of classical mechanics, as it is taught nowadays, have precisely been successively elaborated with this specific target in mind.

Nothing analogue to the N body problem has accompanied the development of stochastic analysis. This explains, in parts, why this theory is often regarded as abstract and difficult by theoretical physicists. It is, indeed, quite hard to understand even in what sense stochastic analysis could be a dynamical theory at all. A way to strongly reduce the gap with the theory of dynamical systems has been initiated in the mid eighties and developed under the name of Euclidean Quantum Mechanics (also referred to by some authors as “Stochastic Deformation”, cf. [13]). A first dynamical attempt was due to Nelson twenty years earlier, however his non-Euclidean framework [16], founded on the Schrödinger equation and not on heat equations like ours, never sucessfully considered the notion of integrability.

In this Euclidean context, indeed, the basic tools of classical mechanics, the Hamiltonian, Lagrangian and variational ones were deformed so as to accomodate with the continuous but nowhere differentiable trajectories solving some SDE. Those underlying stochastic processes enjoy very special properties, as could be expected, and are known as “Bernstein processes” or “reciprocal processes”. Since then, they have reappeared in many different contexts, and under different names, often without reference to their motivational origin, cf. [22]. An unusual property of Bernstein processes is their invariance under time reversal, even when their infinitesimal coefficients are explicitly time dependent. This particularity lies, in fact, at the heart of their original construction, cf. [2], [4], [3]. The expression “stochastic deformation” mentioned above is indeed appropriate in this original perspective; those Bernstein processes are deformations of classical trajectories in the same sense as quantum mechanics can be regarded as a deformation of classical mechanics, especially along the line of Feynman’s path integrals.

We are going to use those ingredients in order to answer our “elementary” questions, in the simplest possible context of the underlying classical theory, where the notion of integrability is completely obvious. We shall see that, even in this simple case, the answer requires some ingenuity, and certainly not the use of well known methods of the traditional theory of stochastic processes. On the other hand, the result is convincing since it provides not only the deformation of the classical answer but, as well, that of all the independent approaches to obtain this answer in classical mechanics. Through the whole work, the term “classical” will therefore refer to a result known to be true for smooth trajectories, i.e. before applying the stochastic deformation advocated here.

The outline of this paper is as follows. In Section 2 we review the theory of one dimensional classical integrable systems. In Section 3 we introduce a stochastic deformation of such a system, and in Section 4 we present the deformation of its Characteristics. Section 5 is concerned with the random time reversal of the construction, and in Section 6 we consider the free case $V = 0$ as an example with explicit calculations.

2 One dimensional classical integrability

A remarkable particularity of one dimensional conservative Newtonian dynamical systems is that, in the terms used by the old masters of classical mechanics, they are "integrable by quadrature". This means that the solutions of their equations of motion can be expressed in terms of their coefficients by solving algebraic equations and integration.

More precisely, for an elementary (mass 1) particle with position $x(\tau)$ in a continuous potential V, the Newton and energy conservation laws during a time interval $I = (s, u)$ take the form

$$\ddot{x}(\tau) = -\nabla V(x(\tau)), \tag{2.1}$$

and

$$\frac{1}{2}|\dot{x}(\tau)|^2 + V(x(\tau)) = \alpha, \qquad \tau \in I, \tag{2.2}$$

for α a constant.

Next we review several classical approaches to the solution of the above problem. In the sequel, the notation $F[x(\cdot)]$ will denote a functional of the whole trajectory $x(\cdot)$ and the function $F(x)$ of a real variable will refer to the value of the same functional evaluated at the initial (or final) condition $x(t) = x \in \mathbb{R}$.

Integration by quadrature

Let now $t \in I = (s, t)$ be fixed and let $x = x(t)$. From Condition (2.2) we can solve (2.1) using the following reduction procedure:

$$\tau - t = \begin{cases} \displaystyle\int_x^{x(\tau)} \frac{d\xi}{\sqrt{2(\alpha - V(\xi))}} & \text{if } \dot{x}(t) > 0, \quad t \le \tau \le \tau_p \le u, \\ \\ \displaystyle -\int_x^{x(\tau)} \frac{d\xi}{\sqrt{2(\alpha - V(\xi))}} & \dot{x}(t) < 0, \quad t \le \tau \le \tau_n \le u, \end{cases} \tag{2.3}$$

where $\tau_p = \sup\{r \in [t, u) \ : \ \dot{x}(r') > 0, \ r' \in [t, r]\}$ and $\tau_n = \sup\{r \in [t, u) \ : \ \dot{x}(r') < 0, \ r' \in [t, r]\}$, cf. [6], where each of τ_p and τ_n can be infinite when I is.

In addition it is well known, cf. [1], that the solution trajectory

$$\begin{aligned} x(\cdot) \ : \ I &\longrightarrow [0, y] \\ \tau &\longmapsto x(\tau), \end{aligned}$$

where $y > 0$, is a critical point of the "reduced" action functional built from the Poincaré one-form:

$$\mathcal{W}_\alpha[x(\cdot)] = \int_x^y p(\tau) dx(\tau), \tag{2.4}$$

on the smooth and energy conditioned path space

$$\Omega^\alpha_{x,t} = \big\{ \, I \ni \tau \longmapsto x(\tau) \in [0, y], \ \text{of class } \mathcal{C}^2, \text{ s.t } x(t) = x \text{ and (2.2) holds} \, \big\},$$

where $p = \dot{x} = \partial L / \partial \dot{x}$ is the momentum for the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} |\dot{x}|^2 - V(x) \tag{2.5}$$

of our elementary Newtonian system (2.1).

An important observation, for what follows, is that since the ODE (2.1) is autonomous, it is time reversible in the sense that if

$$x(\cdot) \ : \ I \longrightarrow [0, y]$$

$$\tau \longmapsto x(\tau)$$

is a (smooth) solution, then the same is true of its time reversal $\hat{x}(\tau) = x(u+s-\tau)$ with

$$\dot{\hat{x}}(\tau) = -\dot{x}(u+s-\tau), \qquad \tau \in I, \tag{2.6}$$

when I is bounded. The same elementary dynamical system can, of course, be analysed using the method of Characteristics (or of Hamilton-Jacobi), cf. [1], [8].

Hamilton-Jacobi equation

Another approach to the problem (2.1)-(2.2) is, indeed, to compute the action functional $\mathcal{S}(x,t)$ of the system by solving the final value Hamilton-Jacobi (HJ) nonlinear partial differential equation

$$-\frac{\partial \mathcal{S}}{\partial t} + H\left(x, -\frac{\partial \mathcal{S}}{\partial x}\right) = 0, \tag{2.7}$$

where

$$H(x,p) = \frac{1}{2}|p|^2 + V(x) \tag{2.8}$$

is the associated Hamiltonian, and to recover the momentum and position of the system from the knowledge of $\mathcal{S}(x,t)$, as illustrated below. Note that two Hamilton-Jacobi equations, mutually adjoint in time, should in fact be associated with any elementary dynamical system, cf. [3] for more on this topic.

When $V(x)$ is bounded and time independent, the separation of time and space variables allows us to look for a one-parameter family $(\mathcal{S}_\alpha)_{\alpha\in\mathrm{R}}$ of solutions of (2.7) of the form

$$\mathcal{S}_\alpha(x,t) = \mathcal{W}_\alpha(x) - \alpha(\tau - t), \tag{2.9}$$

where the Hamilton characteristic function $\mathcal{W}_\alpha$ solves the reduced Hamilton-Jacobi equation

$$H\left(x, -\frac{\partial \mathcal{W}_\alpha}{\partial x}\right) = \alpha, \tag{2.10}$$

which is another formulation of the underlying conservation of energy principle, for our $L(x,\dot{x})$ defined in (2.5).

The reduced Hamilton-Jacobi equation (2.10) can be solved by

$$\mathcal{W}_\alpha(x) = \int_x^y \sqrt{2(\alpha - V(\xi))}\, d\xi, \tag{2.11}$$

which is called Hamilton's characteristic function.

Next, consider the canonical transformation of variables generated by $\mathcal{W}_\alpha(x)$, i.e.

$$\frac{\partial \mathcal{W}_\alpha}{\partial \alpha}(x) = \int_x^y \frac{d\xi}{\sqrt{2(\alpha - V(\xi))}} \tag{2.12}$$

and

$$\frac{\partial \mathcal{W}_\alpha}{\partial x}(x) = -\sqrt{2(\alpha - V(x))}. \tag{2.13}$$

Clearly, by (2.2) we have

$$p(x) = -\frac{\partial \mathcal{W}_\alpha}{\partial x}(x),$$

and this connects Hamilton's characteristic function (2.11) to the action functional $\mathcal{W}_\alpha(x)$ since we have

$$\frac{d}{dt}\mathcal{W}_\alpha(x(t)) = \dot{x}(t)\frac{\partial \mathcal{W}_\alpha}{\partial x}(x(t)) = -\dot{x}(t)p(x(t)), \tag{2.14}$$

as in (2.4). In addition, by (2.3) we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{W}_\alpha}{\partial \alpha}(x(t)) &= \frac{d}{dt}\int_{x(t)}^y \frac{d\xi}{\sqrt{2(\alpha - V(\xi))}} \\ &= -\frac{\dot{x}(t)}{\sqrt{2(\alpha - V(\xi))}} \\ &= -1, \end{aligned}$$

hence

$$m_\alpha(x(t)) := \frac{\partial \mathcal{W}_\alpha}{\partial \alpha}(x(t))$$

satisfies the following "time equation"

$$\begin{cases} \dfrac{d}{dt}m_\alpha(x(t)) = -1, \\ \\ m_\alpha(y) = 0. \end{cases} \tag{2.15}$$

By Relation (2.12) this equation has the same content as our integration by quadrature (2.3) (modulo the sign of $\dot{x}(t)$) and, in this sense, deserves its name.

Method of characteristics

Our classical dynamical problem (2.1) is solved by the Hamilton-Jacobi method of characteristics as, taking the gradient of (2.10) and using (2.13), we recover the starting Newton equation (2.1), i.e.

$$\frac{d}{dt}p(t) = p(t)\frac{\partial p}{\partial x}(t) = -\nabla V(x(t)).$$

In the general case, i.e. when the separation (2.9) of the space and time variables is not possible in the HJ Equation (2.7), one should look for critical points of the action functional

$$\mathcal{S}[x(\cdot)] := \int_s^u L(x(\tau), \dot{x}(\tau))\, d\tau, \tag{2.16}$$

instead of those of (2.4), where the Lagrangian $L(x, \dot{x})$ is defined on the unconditioned space

$$\Omega = \left\{ x \in \mathcal{C}^2(I;\mathbb{R}) \ : \ x(s) \text{ and } x(u) \text{ are fixed} \right\}$$

of smooth paths. Then (cf. [1]), the critical points of $\mathcal{S}[x(\cdot)]$ solve the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{2.17}$$

which, for our elementary system, coincides of course with (2.1).

The relation between the reduced action (2.4) and (2.16) rests on the fact that $\mathcal{S}$ can be extended to an Hamiltonian functional

$$\begin{aligned} \mathcal{S}_H[x(\cdot), p(\cdot)] &= \int_s^u p(\tau)dx(\tau) - \int_s^u H(x(\tau), p(\tau))d\tau \\ &= \int_s^u (p(\tau)\dot{x}(\tau) - H(x(\tau), p(\tau)))d\tau, \end{aligned} \tag{2.18}$$

on the smooth paths in the phase space of our system, and on the remarkable observation, relying on the properties of the Legendre transform, that the critical points of (2.18) coincide with those of (2.16) (cf. [1], [3]), so that (2.16) and (2.18) can be identified along those smooth extremals.

We close this section by recalling a fundamental invariance principle in the classical calculus of variations, which states that, for any (differentiable) function $m(q,\tau)$, and a given $L_M = L_M(q,\dot{q})$, the modified Lagrangian

$$L := L_M(x(\tau),\dot{x}(\tau)) + \frac{dm}{d\tau}(x(\tau),\tau) \tag{2.19}$$

is dynamically equivalent to L_M. The extra time derivative

$$\frac{dm}{d\tau}(x(\tau),\tau) = \frac{\partial m}{\partial \tau}(x(\tau),\tau) + \dot{x}(\tau)\cdot\nabla m(x,\tau)$$

is called a "null Lagrangian" (cf. [8]). The equivalence class of Lagrangians associated with this invariance is precisely at the origin of the theory of canonical transformations (symplectomorphisms) in Hamiltonian mechanics [1], [8].

3 Stochastic deformation

Our plan is now to deform stochastically the framework described in Section 2 so that all the above-mentioned statements can be generalized to Brownian-like trajectories. This means, in particular, that our new path spaces will be much larger than in Section 2, and will contain all continuous paths. It is also clear that the generic non-differentiability of those paths requires the introduction of a number of additional regularizations (in the sense used in Quantum Field Theory, for instance), as most of the expressions of Section 2 do not make sense anymore along such irregular paths.

By "stochastic deformation" we mean, here, a program of minimal deformation of the tools introduced in Section 2 according to a (positive) deformation parameter $\hbar$. By "minimal" we mean a program aiming to preserve the essential part of the various classical structure involved in Section 2.

Let $t \in I$ be fixed. Any smooth trajectory $\tau \mapsto x(\tau)$ considered in Section 2 will now be deformed into the solution $(Z_\tau)_{\tau\in[t,u]}$ of an Itô stochastic differential equation of the form

$$\begin{cases} dZ(\tau) = B(Z(\tau),\tau)\,d\tau + \hbar^{1/2}\,dW_\tau, & \tau \geq t, \\ \\ Z(t) = x, \end{cases} \tag{3.1}$$

where $(W_\tau)_{\tau\in\mathrm{R}_+}$ is a standard Brownian motion and $B(z,\tau)$ is a drift such that Equation (3.1) has a unique solution. The deformations considered here belong to the class of Bernstein or reciprocal processes, cf. Section 5 and [3], [4], [22].

The solution $(Z(\tau))_{\tau\in[t,u]}$ of (3.1) is a real-valued stochastic process defined on a probability space $(\Omega,\mathcal{G},P)$, equipped with an increasing family $(\mathcal{P}_\tau)_{\tau\in I}\subset\mathcal{G}$ of σ-algebras (i.e. a "filtration") generated by the Brownian motion $(W_\tau)_{\tau\in I}$, i.e. $\mathcal{P}_r$ represents the past information generated by $(Z(\tau))_{\tau\in[t,u]}$ up to time $r\in[t,u]$.

Any such diffusion $(Z(\tau))_{\tau\in[t,u]}$ admits an infinitesimal generator D_τ of the form

$$D_\tau f(Z(\tau),\tau)=\frac{\partial f}{\partial\tau}(Z(\tau),\tau)+B(z,\tau)\cdot\nabla f(Z(\tau),\tau)+\frac{\hbar}{2}\Delta f(Z(\tau),\tau), \tag{3.2}$$

interpreted here as a deformation (in the parameter $\hbar$) of the time derivative $\frac{d}{d\tau}$ along the classical flow associated with $B(z,\tau)$. Here, unlike in the classical theory of stochastic processes, cf. [9], we include the time partial derivative $\frac{\partial}{\partial\tau}$ in the definition of the infinitesimal generator D_τ. In particular, (3.2) applied to $f(\tau,z)=z$ yields

$$D_\tau\, Z(\tau)=B(Z(\tau),\tau). \tag{3.3}$$

Recall, cf. [22], [4], that the stochastic deformation of the action (2.16) is given by

$$\mathcal{S}_L(x,t)=E_{xt}\left[\int_t^{\hat\tau}L(Z(\tau),D_\tau\, Z(\tau))\,d\tau\right] \tag{3.4}$$

with Lagrangian

$$L(q,\dot q)=\frac{1}{2}|\dot q|^2+V(q), \tag{3.5}$$

where $\hat\tau$ denotes any Markov time (also called stopping time) with finite expectation and E_{xt} denotes the conditional expectation given $Z(t)=x$. Note that the left-hand side of (3.5) depends implicitly on the Markov time $\hat\tau$. Such Markov times will be needed afterwards. The relevant diffusion process Z will be defined, indeed, in spatial intervals and $\hat\tau$ will represent, typically, an exit time from such intervals, relevant for the boundary conditions of the PDE solved by processes that are critical for the given functional.

A process $Z = (Z(\tau))_{\tau\in[t,u]}$ in the domain of a functional $F[Z]$ of the form (3.4) is called critical for F, cf. [22], [4], [3], if the directional derivative of F at Z along any direction δZ vanishes in the sense that

$$\delta F = E_{xt}\big[\boldsymbol{\nabla} F[Z](\delta Z)\big] = 0, \tag{3.6}$$

where the dependence of F in the process $(Z(\tau))_{\tau\in I}$ is denoted by $F[Z]$. In (3.6), $\boldsymbol{\nabla} F$ denotes the a.s. Gâteaux derivative

$$\boldsymbol{\nabla} F[Z](\delta Z) = \lim_{\varepsilon\to 0} \frac{F[Z+\varepsilon\delta Z] - F[Z]}{\varepsilon} \tag{3.7}$$

and δZ is an absolutely continuous function in the Cameron-Martin space [15] equipped with the scalar product

$$\langle \varphi_1 | \varphi_2 \rangle := \int_0^\infty \dot\varphi_1(\tau)\dot\varphi_2(\tau)d\tau.$$

Consequently the probability measure induced by the shift $(Z(\tau) + \varepsilon\delta Z(\tau))_{\tau\in I}$ is absolutely continuous with respect to the law of $(Z_\tau)_{\tau\in I}$, and the variation $F[Z+\varepsilon\delta Z] - F[Z]$ of $F[Z]$ is well defined in (3.7).

For the special case of a functional such as (3.4), the computation of the directional derivative (3.6) has been done explicitly in [4], [3]. It amounts, as in the classical case, to a time integration by parts under the integral sign, using Itô's formula, and relies on the fact that δZ is differentiable and arbitrary in the Cameron-Martin space. In Stochastic Analysis (and in Feynman's approach) it is related to an integration by parts formula on a path space, cf. [15]. Interestingly, such integration by parts formulas have been regarded as a fundamental (but informal) tool since the inception of Feynman's space-time approach. They played again a central role 40 years later, in Malliavin's stochastic calculus of variations, cf. [3] for more details. These formulas are available, in fact, even when the underlying processes are not Markovian, cf. [21].

In particular it has been shown in [4] that the critical points $Z[\cdot]$ of $\mathcal{S}_L(x,t)$ in the class (3.1) are characterized by the drift

$$B(z,\tau) = \hbar\nabla \log \eta(z,\tau),$$

where $\eta(z,\tau)$ is any positive solution of the backward heat equation:

$$\hbar\frac{\partial \eta}{\partial \tau}(z,\tau) = -\frac{\hbar^2}{2}\Delta\eta(z,\tau) + V(z)\eta(z,\tau), \qquad \tau \in I, \tag{3.8}$$

and that they solve almost surely the stochastic deformation

$$D_\tau \left(\frac{\partial L}{\partial D_\tau Z(\tau)}\right) - \frac{\partial L}{\partial Z(\tau)} = 0, \qquad \tau \in I, \tag{3.9}$$

of the Euler-Lagrange equation (2.17), which can be rewritten here as the stochastic deformation

$$D_\tau D_\tau Z(\tau) = \nabla V(Z(\tau)) \tag{3.10}$$

of the Newton equation (2.1).

Next, for any constant $\alpha \in \mathbb{R}$ we introduce a deformation of the energy conservation law (2.2) as

$$\frac{1}{2}|B_\alpha|^2(z,\tau) + \frac{\hbar}{2}\nabla B_\alpha(z,\tau) - V(z) = \alpha, \tag{3.11}$$

cf. also Relation (4.3) below. We shall use later the fact that (3.11) is a Riccati equation for $B_\alpha(z,t)$.

Note that the sign of the potential $V(z)$ in the deformed energy conservation relation (3.11) is different from that of (2.2), which is to be expected when deforming, with well defined probability measures, the structure of classical mechanics. This is another expression of the familiar fact that Feynman's path integral is not well defined probabilistically, but the Feynman-Kac formula is, cf. [3] for instance. For simplicity, we shall consider here the class of real-valued bounded continuous potentials $V(z)$ such that $|V(z)| < \alpha$, $z \in \mathbb{R}$. Relation (3.11) shows that the following finite and positive kinetic energy condition follows:

$$0 < \frac{1}{2}|B_\alpha|^2 + \frac{\hbar}{2}\nabla B_\alpha < 2\alpha. \tag{3.12}$$

Notice that, although for smooth trajectories ($\hbar = 0$) the positivity is, of course, trivial, here it is not so because of the deformation term $\hbar^{1/2} dW_t$. In particular, a divergence of B_α cannot be excluded anymore in the calculations.

In Theorem 3.3 below we will state the stochastic deformation of the variational principle relative to the reduced action (2.4) under Condition (3.11), using the reduced action defined in (3.13) below.

We will need the following Lemmas 3.1 and 3.2. Here, $\circ$ denotes the Stratonovich differential, cf. [9], [10], and $\hat{\tau}$ denotes any integrable Markov time with respect to $(\mathcal{P}_r)_{r\in I}$.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$ and assume that the equation*

$$D_\tau m(z) = \beta, \qquad z \in \mathbb{R}, \quad \tau \in I,$$

has a smooth solution $m_\beta(z)$ for some $\beta \in \mathbb{R}$, $\beta \neq 0$. Then the trajectories $(Z_\alpha(\tau))_{\tau\in[t,u]}$ solving the deformed Newton equation (3.10) *under the deformed energy conservation relation* (3.11) *are critical points of the deformed reduced action*

$$E_{xt}\left[\int_t^{\hat{\tau}} B_\alpha(Z(\tau)) \circ dZ(\tau)\right] \tag{3.13}$$

whose Lagrangian is

$$L_M = |B_\alpha|^2 + \frac{\hbar}{2}\nabla B_\alpha,$$

among all the solutions $(Z_\alpha(\tau))_{\tau\in[t,u]}$ of (3.1) *with drift $B_\alpha(z,\tau)$ satisfying* (3.11).

Proof. Using Itô-Stratonovich calculus, Condition (3.11) and Relation (3.5), we rewrite (3.4) as

$$\begin{aligned}
\mathcal{S}_L(x,t) &= E_{xt}\left[\int_t^{\hat{\tau}} L(Z(\tau), D_\tau Z(\tau)) d\tau\right] \\
&= E_{xt}\left[\int_t^{\hat{\tau}} B_\alpha(Z(\tau),\tau) \circ dZ(\tau)\right] \\
&\quad - E_{xt}\left[\int_t^{\hat{\tau}} \left(\frac{1}{2}|B_\alpha(Z(\tau),\tau)|^2 + \frac{\hbar}{2}\nabla B_\alpha(Z(\tau),\tau) - V(Z(\tau))\right) d\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&= E_{xt}\left[\int_t^{\hat{\tau}} B_\alpha(Z(\tau),\tau)\circ dZ_\alpha(\tau)\right] - \alpha E_{xt}[\hat{\tau}-t] \\
&= E_{xt}\left[\int_t^{\hat{\tau}} \big(L_M(Z_\alpha(\tau),\tau) - \frac{\alpha}{\beta} D_\tau m_\beta(Z_\alpha(\tau))\big) d\tau\right],
\end{aligned}
\tag{3.14}
$$

on the restricted class of stationary diffusions $(Z_\alpha(\tau))_{\tau\in I}$ with drift $B_\alpha(z,\tau)$ satisfying (3.11). The conclusion follows from Lemma 3.2 below which shows that $L_M - \frac{\alpha}{\beta} D_\tau m_\beta$ is dynamically equivalent to L_M, and from the relation

$$
E_{xt}\left[\int_t^{\hat{\tau}} B_\alpha(Z(\tau),\tau)\circ dZ_\alpha(\tau)\right] = E_{xt}\left[\int_t^{\hat{\tau}} L_M(Z_\alpha(\tau),\tau)d\tau\right].
$$

□

In the next lemma we check that the stochastic actions built on a Lagrangian L_M and its Gauge transformation of the form $L_M + D_\tau m$ are dynamically equivalent as a deformed analog of (2.19) above. As a consequence, we will describe the dynamics of the system via the stochastic Euler-Lagrange equation using anyone of those Lagrangians.

Lemma 3.2. *Given $m(z)$ in the domain of D_τ, any smooth Lagrangian*

$$
L_M(Z(\tau), D_\tau Z(\tau))
$$

is dynamically equivalent to

$$
L(Z(\tau), D_\tau Z(\tau)) = L_M(Z(\tau), D_\tau Z(\tau)) + D_\tau m(Z(\tau))
$$

in the sense that L_M and $L_M + D_\tau m$ both satisfy the same stochastic Euler-Lagrange equation (3.9)*.*

Proof. Let the Lagrangian $\mathcal{L}(Z(\tau), D_\tau Z(\tau))$ be defined by

$$
\mathcal{L}(Z(\tau), D_\tau Z(\tau)) = D_\tau m(Z(\tau)).
$$

Using (3.2), this means that

$$
\mathcal{L}(Z(\tau), D_\tau Z(\tau)) = D_\tau Z(\tau)\cdot\nabla m(Z(\tau)) + \frac{\hbar}{2}\Delta m(Z(\tau)),
$$

is linear in $D_\tau Z$, hence

$$D_\tau \frac{\partial \mathcal{L}}{\partial D_\tau Z(\tau)} - \frac{\partial \mathcal{L}}{\partial Z(\tau)} = 0.$$

This shows that the critical processes for $L = L_M + \mathcal{L}$ and L_M solve the same stochastic Euler-Lagrange equation (3.9), hence $m(Z(\tau))$ does not bring any new dynamical constraint. □

In the sequel we let

$$\tau^y = \inf \left\{ \tau \in [t,u] \ : \ Z(\tau) = y \,|\, Z(t) = x \right\} \tag{3.15}$$

denote the first exit time from $[0,y]$ at $y > 0$ after time t.

Theorem 3.3. *The trajectories $(Z(\tau))_{\tau\in[t,u]} \subset [0,y]$ solving the deformed Newton equation* (3.10) *are critical points of the deformed reduced action*

$$E_{xt}\left[\int_t^{\tau^y} B_\alpha(Z(\tau),\tau) \circ dZ(\tau) \right] \tag{3.16}$$

among all the solutions $(Z_\alpha(\tau))_{\tau\in[t,u]}$ of (3.1) *with drift $B_\alpha(z,\tau)$ satisfying the deformed energy conservation relation* (3.11)*. In addition, the critical diffusion $(Z_\alpha(\tau))_{\tau\in[t,u]}$ solve the a.s. deformation*

$$\begin{cases} D_\tau m(Z_\alpha(\tau)) = -1, & \tau \in [t,\tau^y], \\ m(y) = 0, \end{cases} \tag{3.17}$$

of the time equation (2.15)*, where $m(x)$ is the function*

$$m(x) := E_{xt}[\tau^y] - t, \qquad x \in [0,y]. \tag{3.18}$$

Proof. First, we note that by Itô's calculus we have

$$m(Z_\alpha(\tau)) + \tau = m(Z_\alpha(t)) + t + \hbar^{1/2} \int_t^\tau \nabla m(Z_\alpha(r)) dW_r + \int_t^\tau (1 + D_r m(Z_\alpha(r))) dr, \tag{3.19}$$

hence

$$1 + D_\tau m(Z_\alpha(\tau)) = 0, \qquad \tau \in [t,u],$$

since $(\tau + m(Z_\alpha(\tau)))_{\tau\in[t,u]}$ is a $\mathcal{P}_\tau$-martingale by (3.18), which shows the first part of (3.17). Next we note that by (3.19) and Dynkin's formula [12] as in [3] we have

$$\begin{aligned} E_{xt}[m(Z_\alpha(\tau))] &= m(x) + E_{xt}\left[\int_t^{\hat{\tau}} D_\tau m(Z_\alpha(\tau))d\tau\right] \\ &= m(x) - E_{xt}[\hat{\tau} - t], \qquad \tau \in [t,u], \end{aligned} \tag{3.20}$$

with

$$m(y) = E_{yt}[\tau^y] - t = 0,$$

which shows the second part of (3.17). To conclude with the proof of (3.16) we apply Lemma 3.1 with $\hat{\tau} = \tau^y$. □

What has been already said for (smooth) critical points of stochastic actions of the form (3.4) and the linearization (3.26) of the Riccati equation expressing the energy conservation law, shows how to build explicitly the stationary diffusions $(Z_\alpha(\tau))_{\tau\in[t,u]}$.

Starting from a (smooth) positive stationary solution of the backward heat equation (3.8), of the form

$$\eta(x,t) = g_\alpha(x)e^{-\alpha t/\hbar}, \tag{3.21}$$

for

$$-\frac{\hbar^2}{2}\Delta g(x) + V(x)g(x) = -\alpha g(x), \tag{3.22}$$

the drift and probability density of $(Z_\alpha(\tau))_{\tau\in[t,u]}$ are respectively given by

$$B_\alpha(x) = \hbar\frac{\nabla g_\alpha}{g_\alpha}(x), \tag{3.23}$$

and

$$P(Z_\alpha(\tau) \in dx) = |g_\alpha(x)|^2 dx, \qquad \tau \in I, \tag{3.24}$$

cf. [22].

However, such an arbitrary $(Z_\alpha(\tau))_{\tau\in[t,u]}$ suffers from a serious flaw, in our dynamical system perspective, since it is clearly allowed to reach any border of the spatial interval $[0,y]$, at the bilateral random time

$$\tau_{[0,y]} = \inf\big\{\, \tau \geq t \;:\; Z_\alpha(\tau) = 0 \text{ or } Z_\alpha(\tau) = y \,|\, Z_\alpha(t) = x \,\big\}. \tag{3.25}$$

In fact, the variational argument used in Theorem 3.3 shows that it is the first exit time τ^y at y after t, and not $\tau_{[0,y]}$, which is dynamically relevant. Hence, in order to make sense in our framework, a diffusion $(Z_\alpha(\tau))_{\tau\in[t,u]}$ given by (3.23) with (3.24) has to satisfy the unilateral boundary condition required by Theorem 3.3 and (3.17).

Such critical processes will be constructed by conditioning the diffusion $(Z_\alpha(\tau))_{\tau\in[t,u]}$ to reach y only. J. L. Doob has shown long ago how to do this (cf. [5]), and the solution is nowadays called a Doob transform of the original process.

It is well-known that the solution $B_\alpha(x)$ of the Riccati equation (3.11) can be written as

$$B_\alpha(x) = \hbar\nabla\log(c_+ g_\alpha^+(x) + c_- g_\alpha^-(x)), \tag{3.26}$$

where g_α^+ and g_α^- are two (positive) linearly independent solutions of the time independent heat equation (3.22) and $c_+, c_- \in \mathbb{R}$ are arbitrary constants, not both equal to 0.

For a given, strictly positive, $g_\alpha(z)$ as in (3.21), not vanishing at $z=0$, let

$$g_\alpha^+(z) = g_\alpha(z)\int_0^z |g_\alpha(\xi)|^{-2}d\xi, \qquad z\in[0,y], \tag{3.27}$$

which satisfies the same PDE (3.22) as $g_\alpha(x)$. Next, defining

$$q_\alpha^+(z) = \frac{g_\alpha^+(z)}{g_\alpha(z)}\frac{g_\alpha(y)}{g_\alpha^+(y)}, \tag{3.28}$$

it is easy to check that $q_\alpha^+(z)$ is positive and solves, a.s.,

$$\begin{cases} D_\tau q_\alpha^+(Z_\alpha(\tau)) = B_\alpha(Z_\alpha(\tau),\tau)\cdot\nabla q_\alpha^+(Z_\alpha(\tau)) + \dfrac{\hbar}{2}\Delta q_\alpha^+(Z_\alpha(\tau)) = 0, \\ \\ q_\alpha^+(0) = 0, \quad q_\alpha^+(y) = 1, \end{cases} \tag{3.29}$$

hence $(q_\alpha^+(Z_\alpha(\tau)))_{\tau[t,u]}$ is a positive, forward martingale which allows us to produce a new diffusion $(Z_\alpha^+(\tau))_{\tau\in[t,u]}$ starting from the given one $(Z_\alpha(\tau))_{\tau\in[t,u]}$. According to Doob [5], this process has the same diffusion coefficient $\hbar^{1/2}$ as $(Z_\alpha(\tau))_{\tau\in[t,u]}$, and a new drift $B^+_\alpha(z)$ of the form

$$B^+_\alpha(z) = \hbar\frac{\nabla g_\alpha}{g_\alpha}(z) + \hbar\frac{\nabla q^+_\alpha}{q^+_\alpha}(z) = \hbar\frac{\nabla g^+_\alpha}{g^+_\alpha}(z) \tag{3.30}$$

In particular, $B^+_\alpha(z)$ is singular at $z = 0$ from (3.27), and this implies that $(Z^+_\alpha(\tau))_{\tau\in I}$, in contrast with $(Z_\alpha(\tau))_{\tau\in I}$, cannot reach the origin anymore, i.e. 0 becomes an "entrance boundary".

Since $g^+_\alpha(z)$ is solution of the same stationary free heat equation (3.22) as $g_\alpha(z)$, it is clear by (3.8) and (3.10) that $(Z^+_\alpha(\tau))_{\tau\in[t,u]}$ solves the Newton equation

$$D_\tau D_\tau Z^+_\alpha(\tau) = \nabla V(Z^+_\alpha(\tau)), \tag{3.31}$$

as well as the deformed time equation (3.17) of Theorem 3.3, i.e.

$$\begin{cases} D_\tau m(Z^+_\alpha(\tau)) = -1, \\ \\ m(y) = 0, \end{cases} \tag{3.32}$$

where

$$m(x) = E_{xt}[\tau^y - t], \tag{3.33}$$

under the boundary condition (3.17). The construction of $(Z^+_\alpha(\tau))_{\tau\in[t,u]}$ from the initial diffusion $(Z_\alpha(\tau))_{\tau\in[t,u]}$ has clearly introduced an asymmetry between the two boundaries of our space interval $[0, y]$. But we could have chosen as well to condition $(Z_\alpha(\tau))_{\tau\in[t,u]}$ on reaching only the lower border 0, and the analysis would have been quite similar.

Let us now summarize the above results regarding the counterpart $(Z^-_\alpha(\tau))_{\tau\in[t,u]}$ of $(Z^+_\alpha(\tau))_{\tau\in[t,u]}$. Let us define, if $g_\alpha(y) \neq 0$,

$$q^-_\alpha(x) = \frac{g_\alpha(0)}{g^-_\alpha(0)}\frac{g^-_\alpha(x)}{g_\alpha(x)}, \qquad x \in [0, y], \tag{3.34}$$

where

$$g^-_\alpha(x) = g_\alpha(x)\int_x^y |g_\alpha(\xi)|^{-2} d\xi, \qquad x \in [0, y]. \tag{3.35}$$

Then $(q_\alpha^-(Z_\alpha^-(\tau)))_{\tau\in[t,u]}$ is another positive martingale solving as well (3.29), but with permuted boundary conditions:

$$q_\alpha^-(0) = 1, \quad \text{and} \quad q_\alpha^-(y) = 0. \tag{3.36}$$

The corresponding Doob transformed process $(Z_\alpha^-(\tau))_{\tau\in[t,u]}$, which cannot reach y anymore, will have the drift

$$B_\alpha^-(x) = \hbar \frac{\nabla g_\alpha^-}{g_\alpha^-}(x) \tag{3.37}$$

which is singular at $x = y$ by (3.35).

4 Deformation of characteristics

In this section we turn to the solution of the above problem by stochastic deformation of the method of characteristics summarized in Section 2.

We start by defining the deformation $W_\alpha^+(x)$ of Hamilton's characteristic function (2.11), as the reduced version of the action functional $\mathcal{S}_L(x,t)$ introduced in (3.4).

Definition 4.1. *Given $g_\alpha^+(x)$ a smooth, positive solution of the form* (3.27) *of the stationary free heat equation* (3.22)*, let the deformed Hamilton characteristic function be defined as*

$$W_\alpha^+(x) = -\hbar \log g_\alpha^+(x), \qquad x \in [0,y]. \tag{4.1}$$

It follows from (4.1) and Definition (3.30) that the drift of $(Z_\alpha^+(\tau))_{\tau\in[t,u]}$ can be written as well as

$$B_\alpha^+(x) = -\nabla W_\alpha^+(x). \tag{4.2}$$

Using (3.21) we obtain the deformation (4.3) below of the classical equation (2.10). The name of Bellman, associated with this deformation, has its origin in stochastic control theory [7].

Proposition 4.1. *The action $W_\alpha^+(z)$ solves the reduced Hamilton-Jacobi-Bellman (HJB) equation*

$$\frac{1}{2}|\nabla W_\alpha^+|^2(z) - \frac{\hbar}{2}\Delta W_\alpha^+(z) - V(z) = \alpha. \tag{4.3}$$

Proof. Using Relation (4.2) we check that the HJB equation (4.3) coincides with the deformed energy conservation law (3.11). □

It follows as well from (4.2), using the generator (3.2) of $(Z_\alpha^+(\tau))_{\tau\in I}$, that

$$D_t W_\alpha^+(Z_\alpha^+(t)) = -|B_\alpha^+(Z_\alpha^+(t))|^2 - \frac{\hbar}{2}\nabla B_\alpha^+(Z_\alpha^+(t)),$$

and therefore, after integration with respect to $E_{tx}\int_t^{\tau^y}$ on both sides and using Dynkin's formula and (4.2) when $D_t W_\alpha^+(x)$ is sufficiently integrable, we get

$$W_\alpha^+(x) = E_{tx}\left[\int_t^{\tau^y} B_\alpha^+(Z_\alpha^+(\tau)) \circ dZ_\alpha^+(\tau)\right] + E_{xt}\big[W_\alpha^+(Z_\alpha^+(\tau^y))\big].$$

On the other hand, we can always adjust the terminal condition in (3.27) in such a way that $g_\alpha^+(y) = 1$, so that by (4.1) we have

$$W_\alpha^+(Z_\alpha^+(\tau^y)) = W_\alpha^+(y) = 0,$$

hence the action $W_\alpha^+(x)$ is written as

$$W_\alpha^+(x) = E_{tx}\left[\int_t^{\tau^y} B_\alpha^+(Z_\alpha^+(\tau)) \circ dZ_\alpha^+(\tau)\right]. \tag{4.4}$$

Then $W_\alpha^+(x)$ coincides with the deformed reduced action (3.16) of our Theorem 3.3, i.e. the first term of the action $\mathcal{S}_L(x,t)$ in (3.14).

In complete analogy with the smooth case of Section 2, the gradient of the HJB equation (4.3) now reduces, using (4.2), to

$$D_\tau D_\tau Z_\alpha^+(\tau) = \nabla V(Z_\alpha^+(\tau)) \tag{4.5}$$

which is our almost sure Newton law by analogy with (2.1). In this sense, the reduced HJB equation (4.3) allows us to solve our stochastic dynamical problem (3.10) in the same sense as the Hamilton-Jacobi method of characteristics solves the classical problem (2.1). This observation remains true in the general, time dependent, case mentioned in Section 3, cf. [23].

Defining now, inspired by what was done before (2.15),

$$m_\alpha^+(x) = \frac{\partial W_\alpha^+}{\partial \alpha}(x), \tag{4.6}$$

it is easy to check from (4.1) and by differentiation of (4.3) that, along the conditioned diffusion $(Z_\alpha^+(\tau))_{\tau\in[t,u]}$, we have

$$\begin{cases} D_\tau m_\alpha^+(Z_\alpha^+(\tau)) = -1, \qquad \tau \in [t,u], \\ \\ m^+(y) = 0, \end{cases} \tag{4.7}$$

which recovers (3.32) with its proper unilateral boundary condition.

On the other hand, the diffusion $(Z_\alpha^-(\tau))_{\tau\in[t,u]}$ with drift (3.37) will solve, a.s., the Newton equation

$$D_\tau D_\tau Z_\alpha^-(\tau) = \nabla V(Z_\alpha^-(\tau)),$$

and the time equation

$$\begin{cases} D_\tau m_\alpha^-(Z_\alpha^-(\tau)) = -1 \\ \\ m_\alpha^-(0) = 0, \end{cases} \tag{4.8}$$

where

$$m_\alpha^-(x) = \frac{\partial W_\alpha^-}{\partial \alpha}(x), \tag{4.9}$$

and

$$W_\alpha^-(x) = -\hbar \log g_\alpha^-(x), \tag{4.10}$$

which is another solution of the same reduced HJB equation (4.3). The relation between the two conditioned processes can be exposed in a deeper way, however, through time reversal, as developed in the next section.

5 Random time reversal

As noted in Section 3, up to now our construction uses a single increasing filtration $(\mathcal{P}_\tau)_{\tau\in I}$, where $\mathcal{P}_\tau$ represents the past information about our deformed mechanical system up to time τ. But any critical diffusions of the action $\mathcal{S}_L(x,t)$, associated with a (backward) heat equation as described in the proof of Theorem 3.3 is, in fact, time-symmetrizable, even when its drift $B(z,\tau)$ is time dependent. This is specific to the class of Bernstein (or reciprocal) processes.

It has indeed been shown, cf. [11], [22], [4], [3], that for $s \leq \tau \leq u$, and any Borel set $A \subset \mathbb{R}$ we have

$$P(Z_\tau \in A) = \int_A \eta^*(z,\tau)\eta(z,\tau)dz, \tag{5.1}$$

where, besides the positive solution $\eta(z,\tau)$ of the backward heat equation mentioned before, appears a positive solution $\eta^*(z,\tau)$ of the PDE

$$-\hbar\frac{\partial\eta^*}{\partial\tau}(z,\tau) = -\frac{\hbar^2}{2}\Delta\eta^*(z,\tau) + V(z)\eta^*(z,\tau), \tag{5.2}$$

adjoint to (3.8) with respect to the time parameter. This solution $\eta^*(z,\tau)$ is associated with another, decreasing, filtration $(\mathcal{F}_\tau)_{\tau\in I}$, containing the future information about our system. With respect to $(\mathcal{F}_\tau)_{\tau\in I}$, the process $(Z_\tau)_{\tau\in I}$ has, of course, the same diffusion coefficient $\hbar^{1/2}$ and a different drift given by

$$D^*_\tau Z(\tau) = B^*(Z(\tau),\tau) = -\hbar\frac{\nabla\eta^*}{\eta^*}(Z(\tau),\tau), \tag{5.3}$$

where D^*_τ is the (backward) infinitesimal generator defined by

$$D^*_\tau = \frac{\partial}{\partial\tau} + B^*\cdot\nabla - \frac{\hbar}{2}\Delta. \tag{5.4}$$

The change of sign in (5.4) with respect to (3.2) comes from the transition from forward to backward Itô calculus, cf. [9], [10].

The original construction of Bernstein processes was quite distinct from the traditional one of diffusion processes. It can precisely be regarded as a deformation of the solution of the classical variational problem (2.16), where the data of $x(s)$ and $x(u)$, in Ω, is replaced by the one of two (nodeless) probability densities at time s and u. More precisely, given a time interval $[s,u]$, two strictly positive boundary probability densities $p_s(dx)$ and $p_u(dx)$, and the integral kernel of the heat equation (5.2) (with $V(x)$ such that this kernel is strictly positive), one shows that there is a unique diffusion process $(Z(\tau))_{\tau\in[s,u]}$ having those boundary probabilities. Its probability density at time $\tau \in [s,u]$ is of the form (5.1), i.e.

$$p_\tau(z)dz = \eta^*(z,\tau)\eta(z,\tau)dz.$$

Moreover, this process is critical for the action (3.4)-(3.5), cf. [22], [4], [3] and references therein.

Denoting by E^{xt} the expectation conditioned by the future condition $Z_t = x$, it has been shown that the same process $(Z(\tau))_{\tau\in I}$ is also a critical point of the action time reversed of (3.4), namely, for any Markov time $\hat{\tau}^*$ with finite expectation we have

$$\mathcal{S}_L^*(x,t) = E^{xt}\left[\int_{\hat{\tau}^*}^{t} L(Z(\tau), D_\tau^* Z(\tau))d\tau\right]. \tag{5.5}$$

The proof relies on the fact that, in contrast with what happens for regular diffusion processes, the time reversal

$$\hat{Z}(\tau) := Z(u+s-\tau), \qquad \tau \in I,$$

is, here, well defined and has generator D_τ^* instead of D_τ, with same Lagrangian

$$L = \frac{1}{2}|\dot{q}|^2 + V(q).$$

The diffusions critical for the associated actions are, indeed, invariant under time reversal on I by construction, cf. [22]. This property is due to the special product form (5.1) for $\eta(z,\tau)$ solving the backward heat equation on I and $\eta^*(z,\tau)$ its time adjoint equation (5.2). The two drifts are related through their probability density $(\eta^*\eta)(z,\tau)$ via

$$B^*(z,\tau) = B(z,\tau) - \hbar\nabla\log(\eta^*\eta)(z,\tau). \tag{5.6}$$

As observed in (3.21), however, we only need to consider here the stationary diffusions $(Z_\alpha(\tau))_{\tau\in[t,u]}$, and therefore stationary solutions of the backward heat equation and its time adjoint (5.2), i.e.

$$\begin{cases} \eta(x,t) = g_\alpha(x)e^{-\alpha t/\hbar} \\ \\ \eta^*(x,t) = g_\alpha(x)e^{\alpha t/\hbar}, \end{cases} \tag{5.7}$$

so that the relation between drifts reduces here to the very classical looking one

$$B_\alpha^*(x) = -B_\alpha(x), \tag{5.8}$$

cf. (2.6) in the classical case.

Now we can take the time reversal of Equation (3.29), the equation solved by the positive martingale $q_\alpha^+(Z_\alpha(\tau))$, using $D_t \mapsto -D_t^*$, then

$$D_t^* q_\alpha^{+*}(Z_\alpha(t)) = -D_t q_\alpha^{+*}(Z_\alpha(t)) = 0, \tag{5.9}$$

so that

$$q_\alpha^{+*}(x) = q_\alpha^-(x), \qquad x \in [0, y]. \tag{5.10}$$

It can be shown (cf. [19]) that the various boundary conditions needed for the above construction are compatible if the relation

$$q_\alpha^{+*}(x) = 1 - q_\alpha^+(x) \tag{5.11}$$

holds between $q_\alpha^+(x)$ and its time reversal $q_\alpha^{+*}(x) = q_\alpha^-(x)$. In this case, there is a dual probabilistic interpretation of these two martingales as

$$q_\alpha^+(Z_\alpha(t)) = P(\tau^y < \tau^0 \,|\, Z_\alpha(t)), \tag{5.12}$$

and

$$q_\alpha^{+*}(Z_\alpha(t)) = q_\alpha^-(Z_\alpha(t)) = P(\tau_*^y < \tau_*^0 \,|\, Z_\alpha(t)). \tag{5.13}$$

In the first expression (5.12) the time τ^y, resp. τ^0, denotes the first (random) exit time of $(Z_\alpha(\tau))_{\tau\in[t,u]}$ from $[0, y]$ at the point y, resp. 0, after the time t. In the second expression (5.13), the time τ_*^y, resp. τ_*^0, is the last entrance time of $(Z_\alpha(\tau))_{\tau\in I}$ in $[0, y]$ at the point y, resp. 0, before the time t.

If needed, the expectation of the bilateral exit time $\tau_{[0,y]}$, defined in (3.25) as

$$\tau_{[0,y]} = \min(\tau^0, \tau^y), \tag{5.14}$$

i.e. the first time that $(Z_\alpha(\tau))_{\tau\in[t,u]}$ reaches either boundary 0 or y, can be expressed as a weighted sum of the solutions of Equations (4.7) and (4.8) by

$$m_\alpha(x) = E_{x,t}\left[\tau_{[0,y]}\right] = q_\alpha^+(x)m_\alpha^+(x) + q_\alpha^-(x)m_\alpha^-(x), \tag{5.15}$$

which solves as well the time equation

$$D_t m_\alpha(Z_\alpha(t)) = -1, \tag{5.16}$$

but with the bilateral boundary conditions

$$m_\alpha(0) = m_\alpha(y) = 0. \tag{5.17}$$

This mean time $m(x)$, however, does not seem to be appropriate to our deformation of classical dynamical systems. On the other hand, using τ^0, τ^y and their time reversed versions, we have indeed achieved a stochastic deformation of our one dimensional integrable system such that all the classical tools available to describe the solution with smooth trajectories are deformed as well along the way.

Let us conclude this section by observing that the method used here is considerably more general than our results. For instance, Bernstein processes of the Lévy type (i.e. with jumps) have been constructed [18] and should allow for the same kind of analysis.

6 Example: the free case $V = 0$

We start from any positive solution $g_\alpha(x)$ of (3.22) with $V = 0$, without restrictions on the boundary $\partial\Lambda = \{0, y\}$ of $\Lambda = [0, y]$ other than not being zero, for example

$$g_\alpha(x) = \exp\left(-\frac{\sqrt{2\alpha}}{\hbar}(y-x)\right), \qquad x \in [0, y],$$

where $\alpha > 0$ by (3.12). Next we choose the constants in (3.26) as

$$c_+ = \frac{g_\alpha(y)}{g_\alpha^+(y)} \qquad \text{and} \qquad c_- = \frac{g_\alpha(0)}{g_\alpha^-(0)},$$

where, for any $x \in \Lambda$,

$$g_\alpha^+(x) = \frac{\sinh(x\sqrt{2\alpha}/\hbar)}{\sinh(y\sqrt{2\alpha}/\hbar)}, \qquad g_\alpha^-(x) = \frac{\sinh((y-x)\sqrt{2\alpha}/\hbar)}{\sinh(y\sqrt{2\alpha}/\hbar)},$$

are the two linearly independent solutions of (3.22) required by our construction.

The process $(Z_\alpha(\tau))_{\tau\in[t,u]}$ of Theorem 3.3 is a Brownian motion stated at x at time t, with constant drift $B_\alpha(z) = \sqrt{2\alpha}$ given by (3.23) and variance $\hbar$, i.e.

$$\begin{cases} dZ_\alpha(\tau) = \sqrt{2\alpha}d\tau + \hbar^{1/2}dW_\tau \\ Z_\alpha(t) = x, \end{cases}$$

Its deformed energy fulfills the conservation law (3.11) and $(Z_\alpha(\tau))_{\tau\in[t,u]}$ is a critical point of the action (3.4) with Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}|\dot{q}_t|^2,$$

therefore it solves the a.s. Euler-Lagrange equation (3.9) with $V=0$ and $D_\tau D_\tau Z_\alpha(\tau) = 0$. Note that this process can reach both sides of $\partial\Lambda$ and therefore is not appropriate dynamically. The positive expression

$$q_\alpha^+(x) = \frac{g_\alpha^+(x)g_\alpha(y)}{g_\alpha(x)g_\alpha^+(y)} = \frac{1-e^{-2x\sqrt{2\alpha}/\hbar}}{1-e^{-2y\sqrt{2\alpha}/\hbar}}, \tag{6.1}$$

of (3.28), in our example, yields a positive martingale in $Z_\alpha(\tau)$ solving (3.29) and its boundary conditions $q_\alpha^+(0) = 0$, $q_\alpha^+(y) = 1$. By Doob's transform it produces a new diffusion $(Z_\alpha^+(\tau))_{\tau\in[t,u]}$ with drift

$$B_\alpha^+(x) = \hbar\frac{\nabla g_\alpha^+(x)}{g_\alpha^+(x)} = \sqrt{2\alpha}\coth(x\sqrt{2\alpha}/\hbar),$$

given by (3.30), which is singular at $x=0$, meaning that the origin cannot be reached by $(Z_\alpha^+)_{\tau\in[t,u]}$. In addition it solves as well the above Euler-Lagrange equation and the time equation

$$D_t m_\alpha^+(Z_\alpha^+(t)) = -1,$$

together with its unilateral boundary condition (3.17).

Explicitly, the solution of

$$\begin{cases} D_t m_\alpha^+(Z_\alpha^+(t)) = B_\alpha^+ \cdot \nabla m_\alpha^+(Z_\alpha^+(t)) + \dfrac{\hbar}{2}\Delta m_\alpha^+(Z_\alpha^+(t)) = -1, \\ \\ m_\alpha^+(Z_\alpha^+(\tau^y)) = m_\alpha^+(y) = 0. \end{cases}$$

is given by

$$m_\alpha^+(x) = \frac{y}{\sqrt{2\alpha}}\left(\frac{1+e^{-2y\sqrt{2\alpha}/\hbar}}{1-e^{-2y\sqrt{2\alpha}/\hbar}}\right) - \frac{x}{\sqrt{2\alpha}}\left(\frac{1+e^{-2x\sqrt{2\alpha}/\hbar}}{1-e^{-2x\sqrt{2\alpha}/\hbar}}\right). \tag{6.2}$$

Alternatively, one can compute the deformed Hamilton characteristic functions W_α^+ of (4.1), solving by construction the (free) reduced Hamilton-Jacobi-Bellman equation (4.3). We also check, using (4.1), that

$$m_\alpha^+(x) = \frac{\partial W_\alpha^+}{\partial \alpha}(x) = -\frac{\hbar}{g_\alpha^+(x)}\frac{\partial g_\alpha^+}{\partial \alpha}(x)$$

coincides with (6.2). The symmetric conditioning operation can be done in relation with our starting diffusion $Z_\alpha(\tau)$, to produce a Doob transformed process $(Z_\alpha^-(\tau))_{\tau\in[t,u]}$ in terms of the new positive martingale

$$q_\alpha^-(x) = \frac{g_\alpha(0)g_\alpha^-(x)}{g_\alpha^-(0)g_\alpha(x)} = \frac{e^{-2x\sqrt{2\alpha}/\hbar} - e^{-2y\sqrt{2\alpha}/\hbar}}{1-e^{-2y\sqrt{2\alpha}/\hbar}} \tag{6.3}$$

solving as well (3.29) but with the permuted boundary conditions $q_\alpha^-(0) = 1$, $q_\alpha^-(y) = 1$. It can also be observed that the relation (5.11) holds true. The new process $(Z_\alpha^-(\tau))_{\tau\in[t,u]}$ has the drift (3.37)

$$B_\alpha^-(x) = \hbar\frac{\nabla g_\alpha^-(x)}{g_\alpha^-(x)} = -\sqrt{2\alpha}\frac{e^{-2x\sqrt{2\alpha}/\hbar} + e^{-2y\sqrt{2\alpha}/\hbar}}{e^{2x\sqrt{2\alpha}/\hbar} - e^{-2y\sqrt{2\alpha}/\hbar}},$$

singular at the right boundary point y which cannot be reached by $(Z_\alpha^-(\tau))_{\tau\in[t,u]}$.

The associated time equations and boundary conditions are

$$\begin{cases} D_t m_\alpha^-(Z_\alpha^-(t)) = B_\alpha^- \cdot \nabla m_\alpha^-(x) + \dfrac{\hbar}{2}\Delta m_\alpha^-(x) = -1 \\ \\ m_\alpha^-(Z_\alpha^-(\tau^y)) = m_\alpha^-(y) = 0. \end{cases}$$

whose solution is

$$m_\alpha^-(x) = \frac{x}{\sqrt{2\alpha}}\left(\frac{1+e^{2(x-y)\sqrt{2\alpha}/\hbar}}{1-e^{2(x-y)\sqrt{2\alpha}/\hbar}}\right) - \frac{y}{\sqrt{2\alpha}}\left(\frac{e^{2x\sqrt{2\alpha}/\hbar}-1}{(1-e^{2(x-y)\sqrt{2\alpha}/\hbar})(e^{2y\sqrt{2\alpha}/\hbar}-1)}\right), \quad (6.4)$$

which coincides with (4.9) for $W_\alpha^-(x) = -\hbar \log g_\alpha^-(x)$, another solution of the free reduced Hamilton-Jacobi-Bellman equation (4.3).

If needed, the bilateral expected time for leaving the interval $\Lambda = [0, y]$ can be computed via (6.1), (6.2), (6.3) and (6.4), according to (5.15).

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