Critical Exponents for Semilinear PDEs with Bounded Potentials

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Abstract

Using heat kernel estimates obtained in [18] and the Feynman-Kac formula, we investigate finite-time blow-up and stability of semilinear partial differential equations of the form $\frac{\partial w_t}{\partial t}(x) = \Delta w_t(x) - V(x)w_t(x) + v_t(x)G(w_t(x))$, $w_0(x) \geq 0$, $x \in \mathbb{R}^d$, where v and G are positive measurable functions subject to certain growth conditions, and V is a positive bounded potential. We recover the results of [19] and [14] by probabilistic arguments and in the quadratic decay case $V(x) \sim_{+\infty} a(1+|x|^2)^{-1}$, $a > 0$, we find two critical exponents $\beta_*(a)$, $\beta^*(a)$ with $0 < \beta_*(a) \leq \beta^*(a) < 2/d$, such that any nontrivial positive solution blows up in finite time if $0 < \beta < \beta_*(a)$, whereas if $\beta^*(a) < \beta$, then nontrivial positive global solutions may exist.

Key words: Semilinear partial differential equations, Feynman-Kac representation, critical exponent, finite time blow-up, global solution.

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1 Introduction

Consider a semilinear Cauchy problem of the form

$$\frac{\partial u_t}{\partial t}(x) = Au_t(x) + u_t^{1+\beta}(x), \qquad u_0(x) = \varphi(x), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $\beta > 0$ is constant, $\varphi \geq 0$ is bounded and measurable, and A is the generator of a strong Markov process in $\mathbb{R}^d$. It is well known that, for any non-trivial initial value φ, there exists a number $T_\varphi \in (0, \infty]$ such that (1.1) has a unique mild solution u which is bounded on $[0, T] \times \mathbb{R}^d$ for any $0 < T < T_\varphi$, and if $T_\varphi < \infty$, then $\|u_t(\cdot)\|_\infty \to \infty$ as $t \uparrow T_\varphi$. When $T_\varphi = \infty$ the function u is called a global solution of (1.1), and when $T_\varphi < \infty$ one says that u blows up in finite time or that u is nonglobal.

The blow-up behaviors of semilinear equations of the above type have been intensely studied mainly in the analytic literature; see [1, 3, 7, 12, 13] for surveys. In the case of the fractional power $A = -(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \le 2$, it has been proved that, for $d \le \alpha/\beta$, any nontrivial positive solution of (1.1) is nonglobal, whereas if $d > \alpha/\beta$, then the solution of (1.1) is global provided the initial value satisfies $\varphi \le \gamma G_r^\alpha$ for some $r > 0$ and some sufficiently small constant $\gamma > 0$, where G_r^α, $r > 0$, are the transition densities of the stable motion with generator $-(-\Delta)^{\alpha/2}$, see [2, 4, 10, 11, 15].

Critical exponents for blow-up of the semilinear equation

$$\frac{\partial u_t}{\partial t}(x) = \Delta u_t(x) - V(x)u_t(x) + u_t^{1+\beta}(x), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $\varphi \ge 0$ and V is a bounded potential, have been studied in [14, 18, 19], where it is proved that if $d \ge 3$ and

$$0 \le V(x) \le \frac{a}{1+|x|^b}, \quad x \in \mathbb{R}^d, \tag{1.3}$$

for some $a > 0$ and $b \in [2, \infty)$, then $b > 2$ implies finite time blow-up of (1.2) for all $0 < \beta < 2/d$, whereas if $b = 2$, then there exists $\beta_*(a) < 2/d$ such that blow-up occurs if $0 < \beta < \beta_*(a)$. It is also proved that if

$$V(x) \ge \frac{a}{1+|x|^b}, \quad x \in \mathbb{R}^d, \tag{1.4}$$

for some $a > 0$ and $0 \le b < 2$, then (1.2) admits a global solution for all $\beta > 0$ and all non-negative initial values satisfying $\varphi(x) \le c/(1+|x|^\sigma)$ for a sufficiently small constant $c > 0$ and all σ obeying $\sigma \ge b/\beta$.

In this note we give conditions for finite time blow-up and for existence of non-trivial global solutions of the semilinear problem

$$\frac{\partial u_t}{\partial t}(x) = \Delta u_t(x) - V(x)u_t(x) + v_t(x)G(u_t(x)), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{1.5}$$

where V, φ are as above, and v, G are positive measurable function subject to certain growth conditions. Using heat kernel estimates obtained in [18] and the Feynman-Kac representation of (1.5) we prove that, for dimensions $d \ge 3$, condition (1.3) with $b > 2$ entails finite time blow-up of any nontrivial positive solution of (1.5) provided

$$G(z) \ge \kappa z^{1+\beta}, \quad z > 0 \quad \text{and} \quad v_t(x) \ge t^\zeta \mathbf{1}_{B_{t^{1/2}}}(x), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,$$

where $\kappa > 0$ and β, ζ are positive constants satisfying $0 < \beta < 2(1+\zeta)/d$. (Here and in the sequel, $B_r(x)$ denotes the open ball of radius r centered at x).

We also prove that Eq. (1.5) admits nontrivial global solutions if (1.4) holds with $b < 2$ and $v_t(x)G(z) \leq \kappa t^\zeta z^{1+\beta}$, $t \geq 0$, $z \geq 0$, for some positive constants κ, ζ and β.

As to the critical value $b = 2$, we investigate Equation (1.2) with a potential satisfying either (1.3) or (1.4), and with more general nonlinearities. We prove that, in dimensions $d \geq 3$, there exist critical exponents $\beta_*(a)$, $\beta^*(a)$, both decreasing in $a > 0$, given by

$$0 < \beta_*(a) := \frac{2(1+\zeta) - 4ac}{d + 2ac} \leq \beta^*(a) := \frac{2(1+\zeta)}{d + \min(1, a(d+4)^{-2}/64)} < \frac{2(1+\zeta)}{d},$$

where $c > 0$ is independent of a, and such that

a) If $0 \leq V(x) \leq \dfrac{a}{1+|x|^2}$, then (1.2) blows up in finite time provided $0 < \beta < \beta_*(a)$.

b) If $V(x) \geq \dfrac{a}{1+|x|^2}$, then (1.2) admits a global solution for all $\beta > \beta^*(a)$.

We remark that the blow-up behavior of (1.2) with potentials of the class we are considering here remains unknown when $\beta_*(a) \leq \beta \leq \beta^*(a)$, but notice that in the (unbounded) case $V(x) = a|x|^{-2}$, it can be deduced from [1], [8] and [5] that (1.2) admits a unique critical exponent $\beta(a) < 2/d$, given by

$$\beta(a) = \frac{2}{1 + d/2 + \sqrt{a + (d-2)^2/4}}.$$

Namely, if $V(x) = a|x|^{-2}$, then no global nontrivial solution of (1.2) exists if $\beta < \beta(a)$, whereas global solutions exist if $\beta(a) < \beta$. However, the approaches of the papers quoted above are specially suitable for the potential $V(x) = a|x|^{-2}$ and do not apply to our potentials, which are bounded on $\mathbb{R}^d$ in the subcritical case.

In the case of the one-dimensional equation

$$\frac{\partial u_t}{\partial t}(x) = -(-\Delta)^{\alpha/2} u_t(x) - V(x)u_t(x) + \kappa t^\zeta G(u_t(x)), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R}, \ (1.6)$$

where $G(z)$ satisfies a suitable growth condition with respect to $z^{1+\beta}$, we show that, for every $\alpha \in (1, 2]$ and $\zeta \geq 0$, any nontrivial solution of (1.6) blows up in finite time whenever $0 < \beta < 1 + \alpha\zeta$ and $V : \mathbb{R} \to \mathbb{R}_+$ is integrable. The same happens when $\beta = 1 + \alpha\zeta$ and the L^1-norm of V is sufficiently small. We were not able to investigate here the blow-up properties of (1.6) in the general case $d \geq 1$. From the perspective of our present methods, such investigation requires to derive sharp heat kernel estimates for the operator $\Delta_\alpha - V$, which is a topic of current research.

Let us remark that the heat kernel bounds from [18] play a major role in our arguments. In Section 2 we briefly recall such estimates, and derive some other ones that we will need in the sequel. These estimates are used to obtain semigroup bounds in Section 3. In Section 4 we investigate finite time blow-up of solutions using the Feynman-Kac approach developed in [2] (see also [9]). Section 5 is devoted to proving results on existence of global solutions.

We end this section by introducing some notations and basic facts we shall need.

Let $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ denote the fractional power of the d-dimensional Laplacian, $0 < \alpha \leq 2$. We write $(S_t^\alpha)_{t\geq 0}$ for the semigroup generated by $\Delta_\alpha - V$, i.e.

$$S_t^\alpha \varphi(y) = \int_{\mathbb{R}^d} \varphi(x) p_t^\alpha(x,y)dx = f_t(y),$$

where f_t denotes the solution of

$$\frac{\partial f_t}{\partial t}(x) = \Delta_\alpha f_t(x) - V(x) f_t(x), \qquad f_0(x) = \varphi(x),$$

and $p_t^\alpha(x,y)$, $t > 0$, are the transition densities of the Markov process in $\mathbb{R}^d$ having $\Delta_\alpha - V$ as its generator. Recall that from the Feynman-Kac formula we have

$$p_t^\alpha(x,y) = G_t^\alpha(x-y) E_x \left[\exp\left(-\int_0^t V(W_s^\alpha)\, ds \right) \Big| W_t^\alpha = y \right], \tag{1.7}$$

where $(W_s^\alpha)_{s\in\mathbb{R}_+}$ is a symmetric α-stable motion, and G_t^α, $t > 0$ are the corresponding α-stable transition densities. In case $\alpha = 2$ we will omit the index α and write

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbb{R}^d, \quad t > 0,$$

for the standard Gaussian kernel, and

$$p_t(x,y) = G_t(x-y) E_x \left[\exp\left(-\int_0^t V(W_s)\, ds \right) \Big| W_t = y \right], \qquad t > 0,$$

where $(W_s)_{s\in\mathbb{R}_+}$ is a Brownian motion.

2 Heat kernel bounds of $\Delta - V$

Recall that from Theorem 1.1 in [18] we have:

Theorem 2.1 *Let $d \geq 3$, $b \geq 0$, $a > 0$, and assume that*

$$V(x) \geq \frac{a}{1+|x|^b}, \qquad x \in \mathbb{R}^d.$$

There exist constants $c_1, c_2, c_3 > 0$, and $\alpha_1(a) > 0$, such that for all $x, y \in \mathbb{R}^d$ and $t > 0$ there holds

$$p_t(x,y) \le \begin{cases} c_2 G_t(c_3(x-y)) \exp\left(-c_1 \left(\frac{t^{1/2}}{1+|x|^{b/2}}\right)^{1-b/2} - c_1 \left(\frac{t^{1/2}}{1+|y|^{b/2}}\right)^{1-b/2}\right) & \text{if } b < 2, \\ c_2 G_t(c_3(x-y)) \max\left(\frac{t^{1/2}}{1+|x|}, 1\right)^{-\alpha_1(a)} \max\left(\frac{t^{1/2}}{1+|y|}, 1\right)^{-\alpha_1(a)} & \text{if } b = 2, \\ c_2 G_t(c_3(x-y)) & \text{if } b > 2. \end{cases}$$

We also recall the following estimates, cf. Theorem 1.2 in [18].

Theorem 2.2 *Let $d \ge 3$ and assume that, for some $b \ge 0$ and $a > 0$,*

$$0 \le V(x) \le \frac{a}{1+|x|^b}, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

There exist constants $c_4, c_5, c_6 > 0$, and $\alpha_2(a) > 0$, such that for all $t > 0$ and $x, y \in \mathbb{R}^d$ there holds

$$p_t(x,y) \ge \begin{cases} c_6 e^{-2c_5 t} G_t(c_4(x-y)) & \text{if } b < 2, \\ c_6 t^{-\alpha_2(a)} G_t(c_4(x-y)) & \text{if } b = 2, \\ c_6 G_t(c_4(x-y)) & \text{if } b > 2. \end{cases}$$

Remark 2.3 Notice that from Proposition 2.1 of [17] we have

$$\alpha_1(a) = \min(1, a(d+4)^{-2}/64), \qquad a > 0.$$

Moreover, from the arguments in [18], pp. 391-392, it follows that $\alpha_2 = ca$ for some $c > 0$ independent of a.

Let $B_r \subset \mathbb{R}^d$ denote the open ball of radius $r > 0$, centered at the origin. Notice that, under (2.1), Lemma 4.5 and Lemma 5.1 of [18] imply the more precise statement: for $t \ge 1$ and $x, y \in \mathbb{R}^d$,

$$p_t(x,y) \ge \begin{cases} c_6 e^{-2c_5 t} \mathbf{1}_{B_{a_1 t^{1/2}}}(x) \mathbf{1}_{B_{a_1 t^{1/2}}}(y), & \text{if } 0 \le b < 2, \\ c_6 t^{-\alpha_2(a) - d/2} \mathbf{1}_{B_{a_2 t^{1/2}}}(x) \mathbf{1}_{B_{a_2 t^{1/2}}}(y), & \text{if } b = 2, \end{cases}$$

where c_5, c_6, a_1, a_2 are positive constants and $\alpha_2(a) = ca$ is a linear function of a.

We complete the above results with the following estimate, which yields an extension of Theorem 2.2 to the case $\alpha \in (1, 2]$, though only in dimension $d = 1$.

Theorem 2.4 *Let $d = 1$ and $\alpha \in (1, 2]$, and assume that $V(x)$ is integrable on $\mathbb{R}$. Then, for all $x, y \in \mathbb{R}$,*

$$p_t^\alpha(x,y) \geq e^{-Ct^{1-1/\alpha}} G_t^\alpha(x-y) \mathbf{1}_{B_{t^{1/\alpha}}}(x) \mathbf{1}_{B_{t^{1/\alpha}}}(y), \quad t > 0, \tag{2.2}$$

where $C > 0$ is a constant.

Proof. Using (1.7) and Jensen's inequality we have

$$p_t^\alpha(x,y) \geq G_t^\alpha(x-y) \exp\left(-E_x \left[\int_0^t V(W_s^\alpha)\, ds \,\Big|\, W_t^\alpha = y \right] \right).$$

From the scaling property of stable densities we obtain, for $y \in B_{t^{1/\alpha}}$ and $x \in B_{t^{1/\alpha}}$,

$$\begin{aligned} \frac{G_s^\alpha(z-x) G_{t-s}^\alpha(z-y)}{G_t^\alpha(y-x)} &= \frac{s^{-1/\alpha}(t-s)^{-1/\alpha} G_1^\alpha(s^{-1/\alpha}(z-x)) G_1^\alpha((t-s)^{-1/\alpha}(z-y))}{t^{-1/\alpha} G_1^\alpha(t^{-1/\alpha}(y-x))} \\ &\leq C_\alpha \frac{s^{-1/\alpha}(t-s)^{-1/\alpha}}{t^{-1/\alpha}}, \qquad 0 < s < t, \end{aligned}$$

for some $C_\alpha > 0$. Hence

$$\begin{aligned} E_x \left[\int_0^t V(W_s^\alpha)\, ds \,\Big|\, W_t^\alpha = y \right] &= \int_{\mathbb{R}} \int_0^t V(z) \frac{G_s^\alpha(z-x) G_{t-s}^\alpha(z-y)}{G_t^\alpha(y-x)}\, dz\, ds \\ &\leq C_\alpha \int_{\mathbb{R}} V(z)\, dz \int_0^t \frac{s^{-1/\alpha}(t-s)^{-1/\alpha}}{t^{-1/\alpha}}\, ds \\ &= C_\alpha t^{1-1/\alpha} \int_{\mathbb{R}} V(z)\, dz \int_0^1 s^{-1/\alpha}(1-s)^{-1/\alpha}\, ds. \end{aligned} \tag{2.3}$$

□

3 Semigroup bounds

In this section we establish some bounds for the semigroup $(S_t)_{t \in \mathbb{R}_+}$ of generator $\Delta - V$. The following proposition will be used in the proof of Theorem 5.2.

Proposition 3.1 *Let $a_1, a_2, \sigma > 0$ and $0 \leq b \leq 2$, and assume that*

$$V(x) \geq \frac{a_1}{1 + |x|^b} \qquad \text{and} \qquad 0 \leq \varphi(x) \leq \frac{a_2}{1 + |x|^\sigma}, \qquad x \in \mathbb{R}^d.$$

i) If $b < 2$ then for all $\varepsilon \in (0, 1)$ we have

$$\|S_t \varphi\|_\infty \leq c_\varepsilon t^{-\sigma(1-\varepsilon)/b}, \qquad t > 0,$$

for some $c_\varepsilon > 0$.

ii) If $b = 2$ then for all $\varepsilon \in (0,1)$ there exists $c_\varepsilon > 0$ such that

$$\|S_t\varphi\|_\infty \leq c_\varepsilon t^{-(1-\varepsilon)\alpha_1(a_1)-d/2}, \qquad t > 0,$$

provided $\sigma > d$.

Proof. $i)$ If $b < 2$, applying Theorem 2.1 we obtain

$$\begin{aligned} S_t\varphi(y) &= \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)dx \\ &\leq c_2 \int_{\mathbb{R}^d} \varphi(x) \exp\left(-c_1\left(\frac{t^{1/2}}{1+|x|^{b/2}}\right)^{1-b/2}\right) G_t(c_3(x-y))dx \\ &\leq c_2 \exp\left(-c_1\left(\frac{t^{1/2}}{1+t^{(1-\varepsilon)/2}}\right)^{1-b/2}\right) \int_{\{|x|\leq t^{(1-\varepsilon)/b}\}} \varphi(x)G_t(c_3(x-y))dx \\ &\quad +c_2 \int_{\{|x|>t^{(1-\varepsilon)/b}\}} \varphi(x)G_t(c_3(x-y))dx, \end{aligned}$$

hence

$$S_t\varphi(y) \leq a_2 \exp\left(-c_1\left(\frac{t^{1/2}}{1+t^{(1-\varepsilon)/2}}\right)^{1-b/2}\right) + \frac{a_2c_2}{1+t^{(1-\varepsilon)\sigma/b}}.$$

$ii)$ Let now $b = 2$ and $\varepsilon \in (0,1)$. From Theorem 2.1 we know that

$$\begin{aligned} S_t\varphi(y) &\leq c_2 \int \varphi(x) \max\left(\frac{t^{1/2}}{1+|x|},1\right)^{-\alpha_1(a_1)} \max\left(\frac{t^{1/2}}{1+|y|},1\right)^{-\alpha_1(a_1)} G_t(c_3(x-y))dx \\ &\leq c_2 \int_{\{|x|<t^{\varepsilon/2}\}} \varphi(x) \max\left(\frac{t^{1/2}}{1+|x|},1\right)^{-\alpha_1(a_1)} G_t(c_3(x-y))dx \\ &\quad +c_2 \int_{\{|x|>t^{\varepsilon/2}\}} \varphi(x) \max\left(\frac{t^{1/2}}{1+|x|},1\right)^{-\alpha_1(a_1)} G_t(c_3(x-y))dx \\ &\leq c_2 \int_{\{|x|<t^{\varepsilon/2}\}} \varphi(x) \left(\frac{t^{1/2}}{1+t^{\varepsilon/2}}\right)^{-\alpha_1(a_1)} G_t(c_3(x-y))dx + c_2 \int_{\{|x|>t^{\varepsilon/2}\}} \varphi(x)G_t(c_3(x-y))dx \\ &\leq c_2 t^{-(1-\varepsilon)\alpha_1(a_1)/2} \int_{\{|x|<t^{\varepsilon/2}\}} \varphi(x)G_t(c_3(x-y))dx + \frac{c_2}{(4\pi)^{d/2}} t^{-d/2} \int_{\{|x|>t^{\varepsilon/2}\}} \varphi(x)dx \\ &\leq \frac{c_2}{(4\pi)^{d/2}} t^{-(1-\varepsilon)\alpha_1(a_1)/2-d/2} \int_{\{|x|<t^{\varepsilon/2}\}} \varphi(x)dx + c_7 t^{-(\sigma-d)\varepsilon/2-d/2}. \end{aligned}$$

Hence for some $c_\varepsilon > 0$ we have

$$S_t\varphi(y) \leq c_\varepsilon t^{-(1-\varepsilon)\alpha_1(a_1)/2-d/2}, \qquad y \in \mathbb{R}^d,\ t > 1,$$

provided $(1-\varepsilon)\alpha_1(a_1) \leq (\sigma-d)\varepsilon$. □

The following lemma will be used in the proof of Theorem 4.1.

Lemma 3.2 *Let $d \geq 3$, $b \geq 2$, and let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be bounded and measurable. Assume that*

$$0 \leq V(x) \leq \frac{a}{1+|x|^b}.$$

Then, for all $t \geq 1$ and $y \in \mathbb{R}^d$ we have

$$S_t\varphi(y) \geq c_0 t^{-\alpha_2 - d/2} \mathbf{1}_{B_{t^{1/2}}}(y) \int_{B_{t^{1/2}}} \varphi(x)\, dx,$$

where $\alpha_2 = 0$ if $b > 2$, and $\alpha_2(a) = ca$ for some $c > 0$ when $b = 2$.

Proof. Let $y \in B_{t^{1/2}}$. Due to Theorem 2.2 and self-similarity of Gaussian densities we have

$$\begin{aligned} S_t\varphi(y) &= \int_{\mathbb{R}^d} \varphi(x) p_t(x,y)\, dx \\ &\geq c_2 t^{-\alpha_2(a)} \int_{B_{t^{1/2}}} \varphi(x) G_t(c_4(x-y))\, dx \\ &\geq c_2 t^{-\alpha_2(a) - d/2} \int_{B_{t^{1/2}}} \varphi(x) G_1(c_4 t^{-1/2}(x-y))\, dx \\ &\geq c_0 t^{-\alpha_2(a) - d/2} \int_{B_{t^{1/2}}} \varphi(x)\, dx. \end{aligned}$$

□

The next lemma, which will be needed in the proof of Theorem 4.1 below, provides lower bounds on certain balls for the distributions of the bridges of the Markov process $(X_t)_{t\in\mathbb{R}_+}$ generated by $\Delta - V$.

Lemma 3.3 *Assume that $d \geq 3$ and let $(X_t)_{t\in\mathbb{R}_+}$ denote the Markov process with generator $\Delta - V$. If for some $b \geq 2$,*

$$0 \leq V(x) \leq \frac{a}{1+|x|^b}, \qquad x \in \mathbb{R}^d,$$

Then there exists $c_8 > 0$ such that for all $t \geq 2$, $y \in B_{t^{1/2}}$, $x \in B_1$ and $s \in [1, t/2]$,

$$\mathbb{P}_x(X_s \in B_{s^{1/2}} \mid X_t = y) \geq c_8 t^{-2\alpha_2(a)},$$

where $\alpha_2(a) = 0$ when $b > 2$ and $\alpha_2(a) = ca$ when $b = 2$.

Proof. Since $V(x) \geq 0$, the Feynman-Kac formula (1.7) yields $p_t(x,y) \leq G_t(y-x)$, $t > 0$, $x, y \in \mathbb{R}^d$. An application of Theorem 2.2 and of the Markov property of $(X_s)_{s\in\mathbb{R}_+}$ give

$$\begin{aligned}
\mathbb{P}_x(X_s \in B_{s^{1/2}} \mid X_t = y) &\geq \int_{B_{s^{1/2}}} \frac{p_{t-s}(y,z)p_s(z,x)}{p_t(y,x)}\, dz \\
&= \frac{1}{c_6^2 s^{\alpha_2(a)}(t-s)^{\alpha_2(a)}} \int_{B_{s^{1/2}}} \frac{G_{t-s}(c_4(y-z))G_s(c_4(z-x))}{G_t(c_4(y-x))}\, dz \\
&\geq c_8 t^{-2\alpha_2(a)},
\end{aligned}$$

where we used Lemma 2.2 of [2] to obtain the last inequality. □

We conclude this section with the following lemma, which will be used in the proof of Theorem 5.2.

Lemma 3.4 *Let $d \geq 3$ and $V(x) \geq 0$, $x \in \mathbb{R}^d$. Assume that*

$$V(x) \geq \frac{a}{1+|x|^b}$$

holds for all $|x|$ greater than some $r_0 > 0$, where $a > 0$ and $0 \leq b < 2$. There exists $\gamma > 0$ such that for all bounded measurable $D \subset \mathbb{R}^d$,

$$S_t\mathbf{1}_D(x) \leq c_D t^{-(1+\gamma)}, \quad x \in \mathbb{R}^d, \tag{3.1}$$

for all sufficiently large t, where c_D does not depend on x and t.

Proof. By Theorem 2.1 we have

$$p_t(x,y) \leq c_2 G_t(c_3(x-y)) \exp\left(-c_1\left(\left(\frac{t}{1+|x|^b}\right)^{c_4} + \left(\frac{t}{1+|y|^b}\right)^{c_4}\right)\right) \tag{3.2}$$

for certain constants $c_1, c_2, c_3, c_4 > 0$. Condition (3.1) is obviously fulfilled for any positive γ if $b = 0$, hence let us assume that $0 < b < 2$. For any bounded measurable $D \subset \mathbb{R}^d$ we have, provided $t > \|D\|^2 := \sup_{y\in D}\|y\|^2$,

$$\begin{aligned}
S_t\mathbf{1}_D(x) &\leq c_2 \int_D G_t(c_3(x-y)) e^{-c\left(\frac{t}{1+|y|^b}\right)^{c_4}}\, dy \\
&\leq \frac{c_2}{(4\pi t)^{d/2}} \int_D dy \\
&\leq c_D t^{-(1+\gamma)},
\end{aligned}$$

with $\gamma = (d-2)/2 > 0$. □

4 Explosion in subcritical dimensions

Recall that if u_t, v_t respectively solve

$$\frac{\partial u_t}{\partial t}(y) = \Delta u_t(y) + \zeta_t(y)u_t(y), \qquad \frac{\partial v_t}{\partial t}(y) = \Delta v_t(y) + \xi_t(y)v_t(y),$$

with $u_0 \geq v_0$ and $\zeta_t \geq \xi_t$ for all $t \geq 0$, then $u_t \geq v_t$, $t \geq 0$. In particular, if $\varphi \geq 0$ is bounded and measurable, and if u_t is a subsolution of

$$\frac{\partial w_t}{\partial t}(y) = \Delta w_t(y) + \kappa w_t^{1+\beta}(y), \qquad w_0 = \varphi, \tag{4.1}$$

where $\kappa, \beta > 0$, then any solution of

$$\frac{\partial v_t}{\partial t}(y) = \Delta v_t(y) + \kappa u_t^{\beta}(y)v_t(y), \qquad v_0 = \varphi,$$

remains a subsolution of (4.1).

Theorem 4.1 *Let $d \geq 3$, $b \geq 2$, $\beta > 0$ and $a > 0$, and assume that*

$$0 \leq V(x) \leq \frac{a}{1 + |x|^b}, \qquad x \in \mathbb{R}^d.$$

Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be such that

$$\frac{G(z)}{z} \geq \kappa z^{\beta}, \qquad z > 0, \tag{4.2}$$

for some $\kappa > 0$. Let $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function satisfying

$$v_t(x) \geq t^{\zeta} \mathbf{1}_{B_{t^{1/2}}}(x) \tag{4.3}$$

for some $\zeta > 0$. Consider the semilinear equation

$$\frac{\partial u_t(x)}{\partial t} = \Delta u_t(x) - V(x)u_t(x) + v_t(x)G(u_t(x)), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{4.4}$$

where $\varphi \geq 0$ is bounded and measurable.

a) If $b > 2$ and

$$0 < \beta < \frac{2(1+\zeta)}{d},$$

then any nontrivial positive solution of (4.4) blows up in finite time.

b) If $b = 2$ *and*

$$0 < \beta < \beta_*(a) := \frac{1+\zeta-2ac}{ac+d/2} < \frac{2(1+\zeta)}{d},$$

where $2ac < 1+\zeta$ *and* $c > 0$ *is given in Remark 2.3, then any nontrivial positive solution of* (4.4) *blows up in finite time.*

Proof. Let g_t denote the mild solution of

$$\frac{\partial g_t}{\partial t}(x) = \Delta g_t(x) - V(x)g_t(x) + v_t(x)\frac{G(f_t(x))}{f_t(x)}g_t(x), \quad g_0(x) = \varphi(x),$$

where $f_t = S_t\varphi$ satisfies

$$\frac{\partial f_t}{\partial t}(x) = \Delta f_t(x) - V(x)f_t(x), \quad f_0(x) = \varphi(x).$$

By the Feynman-Kac formula (1.7) we have

$$g_t(y) = \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)E_x\left[\exp\int_0^t v_s(X_s)\frac{G(f_s(X_s))}{f_s(X_s)}\,ds\,\middle|\,X_t = y\right]dx.$$

Let $\alpha_2(a) = 0$ if $b > 2$, and $\alpha_2(a) = ca$ if $b = 2$. Then, for $y \in B_{t^{1/2}}$, and for certain positive constants K_1, K_2, K_3, we have by Lemma 3.2 that

$$\begin{aligned}
g_t(y) &\geq \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)E_x\left[\exp K_1\int_0^t v_s(X_s)(f_s(X_s))^\beta ds\,\middle|\,X_t = y\right]dx \\
&\geq \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)E_x\left[\exp\int_1^{t/2} K_2 s^{\zeta-d\beta/2-\beta\alpha_2(a)}\mathbf{1}_{B_{s^{1/2}}}(X_s)\,ds\,\middle|\,X_t = y\right]dx \\
&\geq \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)\exp\left(K_2\int_1^{t/2} s^{\zeta-d\beta/2-\beta\alpha_2(a)}\mathbb{P}_x\left(X_s \in B_{s^{1/2}}|X_t = y\right)\,ds\right)dx \\
&\geq \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)\exp\left(K_3 t^{-2\alpha_2(a)}\int_1^{t/2} s^{\zeta-d\beta/2-\beta\alpha_2(a)}\,ds\right)dx \\
&\geq \int_{\mathbb{R}^d} \varphi(x)p_t(x,y)dx\exp\left(K_4 t^{\zeta-d\beta/2-(\beta+2)\alpha_2(a)+1}\right),
\end{aligned}$$

where we used Lemma 3.3 to obtain the fourth inequality. The above argument shows that g eventually grows to $+\infty$ uniformly on the unit ball B_1 provided

$$\zeta - d\beta/2 - (\beta+2)\alpha_2(a) > -1.$$

This condition is satisfied for all $0 < \beta < 2(1+\zeta)/d$ if $b > 2$, and for all $0 < \beta < \beta_*(a)$ if $b = 2$. Since g is subsolution of (4.4), the comparison result recalled at the beginning of this section shows that the solution u_t of (4.4) also grows to $+\infty$ uniformly on B_1.

A well-known argument [6] involving Condition (4.2) then shows blow-up of (4.4). For the sake of completeness we include this argument here. Given $t_0 \geq 1$, let $\tilde{u}_t = u_{t+t_0}$ and $K(t_0) = \inf_{x\in B_1} u_{t_0}(x)$. The mild solution of (4.4) is given by

$$\tilde{u}_t(x) = \int_{\mathbb{R}^d} p_t(x,y)\tilde{u}_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x,y)v_{s+t_0}(y)G(\tilde{u}_s(y))\,dy\,ds.$$

Thus, for all $t \in (1,2]$ and $x \in B_1$ we get from Theorem 2.2:

$$\begin{aligned}\tilde{u}_t(x) &\geq \int_{B_1} p_t(x,y)\tilde{u}_0(y)\,dy + \kappa \int_0^t s^\zeta \int_{B_1} p_{t-s}(x,y)\tilde{u}_s^{1+\beta}(y)\,dy\,ds \\ &\geq c_6 K(t_0)\int_{B_1} G_t(c_4(x-y))\,dy + \kappa c_6 \int_0^t s^\zeta \int_{B_1} G_{t-s}(c_4(x-y))\tilde{u}_s^{1+\beta}(y)\,dy\,ds.\end{aligned}$$

Since $\xi := c_4^{-d} \min_{x\in B_1}\min_{s\in[1,2]} \mathbb{P}_x(W_s \in B_{c_4}) > 0$, we have

$$\min_{x\in B_1} \tilde{u}_t(x) \geq \xi c_6 K(t_0) + \kappa\xi c_6 \int_0^t s^\zeta (\min_{x\in B_1} \tilde{u}_s(x))^{1+\beta}\,ds.$$

It remains to choose $t_0 > 0$ sufficiently large so that the blow-up time of the equation

$$v(t) = \xi c_6 K(t_0) + \kappa\xi c_6 \int_0^t s^\zeta v^{1+\beta}(s)\,ds$$

is smaller than 2. $\square$

The following result gives an explosion criterion which is actually valid for any $\alpha \in (1,2]$ and $d = 1$; its proof uses Theorem 2.4 instead of Theorem 2.2 and Lemma 3.3. Here the potential V need not be bounded.

Theorem 4.2 *Let $\alpha \in (1,2]$, $\beta > 0$ and assume that $V : \mathbb{R} \to \mathbb{R}_+$ is integrable. Then the solution of*

$$\frac{\partial u_t}{\partial t}(x) = -(-\Delta)^{\alpha/2}u_t(x) - V(x)u_t(x) + \kappa t^\zeta u_t^{1+\beta}(x), \quad u_0(x) = \varphi(x), \quad x \in \mathbb{R},$$

blows up in finite time whenever $0 < \beta < 1 + \alpha\zeta$. If $\beta = 1 + \alpha\zeta$ the same happens provided $\int_{\mathbb{R}} V(z)\,dz$ is sufficiently small.

Proof. Let g_t denote the mild solution of

$$\frac{\partial g_t}{\partial t}(x) = -(-\Delta)^{\alpha/2}g_t(x) - V(x)g_t(x) + \kappa t^\zeta f_t^\beta(x)g_t(x), \quad g_0(x) = \varphi(x), \quad x \in \mathbb{R},$$

where $f_t = P_t\varphi$ satisfies

$$\frac{\partial f_t}{\partial t}(x) = -(-\Delta)^{\alpha/2}f_t(x), \quad f_0(x) = \varphi(x),$$

and $(P_t)_{t\in\mathbb{R}_+}$ is the α-stable semigroup. The Feynman-Kac formula and Jensen's inequality yield

$$g_t(y) \geq \int_{\mathbb{R}} \varphi(x) G_t^\alpha(x-y) \exp\left(E_x \left[\int_0^t \left(-V(W_s^\alpha) + s^\zeta \left(P_s\varphi(W_s^\alpha)\right)^\beta \right) ds \Big| W_t^\alpha = y \right] \right) dx,$$

where, for any $t \geq 1$,

$$\begin{aligned} E_x \left[\int_0^t s^\zeta \left(P_s\varphi(W_s^\alpha)\right)^\beta ds \Big| W_t^\alpha = y \right] &\geq c_2 E_x \left[\int_1^t s^{-\beta/\alpha+\zeta} \mathbf{1}_{\{B_{s^{1/\alpha}}\}}(W_s^\alpha) \Big| W_t^\alpha = y \right] \\ &\geq c_2 \int_1^t \mathbb{P}_x(W_s^\alpha \in B_{s^{1/\alpha}} \mid W_t^\alpha = y) s^{-\beta/\alpha+\zeta} ds \\ &\geq c_5 \int_1^t s^{\zeta-\beta/\alpha} ds \\ &= \frac{c_5}{1+\zeta-\beta/\alpha}(t^{1-\beta/\alpha+\zeta} - 1); \end{aligned}$$

here we applied Lemma 2.2 of [2]. The last inequality together with (2.3) renders

$$g_t(y) \geq e^{-C_\alpha t^{1-1/\alpha} \int_{\mathbb{R}} V(z)\, dz + \frac{c_5}{1-\beta/\alpha+\zeta}(t^{1+\zeta-\beta/\alpha}-1)},$$

hence by the same steps as in the proof of Theorem 4.1 (comparison result for PDEs and blow-up argument of [6]), finite time explosion occurs if $\beta < 1+\alpha\zeta$, or if $\beta = 1+\alpha\zeta$ and $\int_{\mathbb{R}} V(z)\, dz$ is sufficiently small. □

Since $0 \leq V(x) \leq (1+|x|^b)^{-1}$, $x \in \mathbb{R}$, and $1 < b \leq 2$ imply integrability of $V(x)$ on $\mathbb{R}$, Theorem 4.2 yields a partial extension of Theorem 4.1 to the case $0 < \alpha \leq 2$.

5 Existence of global solutions

We have the following non-explosion result, which is a generalization of Theorem 4.1 in [9].

Theorem 5.1 *Consider the semilinear equation*

$$\frac{\partial w_t}{\partial t}(x) = \Delta w_t(x) - V(x) w_t(x) + t^\zeta G(w_t(x)), \qquad w_0(x) = \varphi(x), \quad x \in \mathbb{R}^d, \quad (5.1)$$

where $\zeta \in \mathbb{R}$, φ is bounded and measurable, and $G : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function satisfying

$$0 \leq \frac{G(z)}{z} \leq \lambda z^\beta, \quad z \in (0, c), \quad (5.2)$$

for some $\lambda, \beta, c > 0$. Assume that $\varphi \geq 0$ is such that

$$\lambda\beta \int_0^\infty r^\zeta \|S_r\varphi\|_\infty^\beta \, dr < 1$$

and

$$\|\varphi\|_\infty \leq c \left(1 - \lambda\beta \int_0^\infty r^\zeta \|S_r\varphi\|_\infty^\beta dr \right)^{1/\beta}. \tag{5.3}$$

Then Equation (5.1) admits a global solution $u_t(x)$ that satisfies

$$0 \leq u_t(x) \leq \frac{S_t\varphi(x)}{\left(1 - \lambda\beta \int_0^t r^\zeta \|S_r\varphi\|_\infty^\beta dr\right)^{1/\beta}}, \qquad x \in \mathbb{R}^d, \quad t \geq 0.$$

Proof. This is an adaptation of the proof of Theorem 3 in [16], see also [9]. Recall that the mild solution of (5.1) is given by

$$u_t(x) = S_t\varphi(x) + \int_0^t r^\zeta S_{t-r} G(u_r(x)) \, dr. \tag{5.4}$$

Setting

$$B(t) = \left(1 - \lambda\beta \int_0^t r^\zeta \|S_r\varphi\|_\infty^\beta dr \right)^{-1/\beta}, \qquad t \geq 0,$$

it follows that $B(0) = 1$ and

$$\frac{d}{dt} B(t) = \lambda t^\zeta \|S_t\varphi\|_\infty^\beta \left(1 - \lambda\beta \int_0^t r^\zeta \|S_r\varphi\|_\infty^\beta dr \right)^{-1-1/\beta} = \lambda t^\zeta \|S_t\varphi\|_\infty^\beta B^{1+\beta}(t),$$

hence

$$B(t) = 1 + \lambda \int_0^t r^\zeta \|S_r\varphi\|_\infty^\beta B^{1+\beta}(r) \, dr.$$

Let $(t, x) \mapsto v_t(x)$ be a continuous function such that $v_t(\cdot) \in C_0(\mathbb{R}^d)$, $t \geq 0$, and

$$S_t\varphi(x) \leq v_t(x) \leq B(t) S_t\varphi(x), \qquad t \geq 0, \ x \in \mathbb{R}^d. \tag{5.5}$$

Let now

$$R(v)(t, x) = S_t\varphi(x) + \int_0^t r^\zeta S_{t-r} G(v_r(x)) \, dr.$$

Since $v_r(x) \leq B(r)\|S_r\varphi\|_\infty$, $r \geq 0$, we have from (5.5), (5.3) and (5.2) that

$$\begin{aligned} R(v)(t, x) &= S_t\varphi(x) + \int_0^t r^\zeta S_{t-r} \left(\frac{G(v_r)}{v_r} v_r \right)(x) \, dr \\ &\leq S_t\varphi(x) + \lambda \int_0^t r^\zeta (B(r))^\beta \|S_r\varphi\|_\infty^\beta S_{t-r} v_r(x) \, dr \\ &\leq S_t\varphi(x) + \lambda \int_0^t r^\zeta B^{1+\beta}(r) \|S_r\varphi\|_\infty^\beta S_{t-r} \left(S_r\varphi(x) \right) \, dr \end{aligned}$$

$$= \quad S_t\varphi(x)\left(1+\lambda\int_0^t r^{\zeta}\|S_r\varphi\|_\infty^{\beta}B^{1+\beta}(r)\,dr\right),$$

where the last inequality follows from (5.5). Hence

$$S_t\varphi(x) \le R(v)(t,x) \le B(t)S_t\varphi(x), \qquad t\ge 0,\ x\in\mathbb{R}^d.$$

Let

$$u_t^0(x) = S_t\varphi(x), \quad \text{and} \quad u_t^{n+1}(x) = R(u^n)(t,x), \quad n\in\mathbb{N}.$$

Then $u_t^0(x) \le u_t^1(x)$, $t\ge 0$, $x\in\mathbb{R}^d$. Since S_t is non-negative, using induction we obtain

$$0 \le u_t^n(x) \le u_t^{n+1}(x), \qquad n\ge 0.$$

Letting $n\to\infty$ yields, for $t\ge 0$ and $x\in\mathbb{R}^d$,

$$0 \le u_t(x) = \lim_{n\to\infty} u_t^n(x) \le B(t)S_t\varphi(x) \le \frac{S_t\varphi(x)}{\left(1-\lambda\beta\int_0^t r^{\zeta}\|S_r\varphi\|_\infty^{\beta}dr\right)^{1/\beta}} < \infty.$$

Thus, u_t is a global solution of (5.4) due to the monotone convergence theorem. □

As a consequence of Theorem 5.1, an existence result can be obtained under an integrability condition on φ.

Theorem 5.2 *Let $G:\mathbb{R}_+\to\mathbb{R}_+$ and $v:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}_+$ be measurable functions such that $G(z)\le \kappa_1 z^{1+\beta}$, $z>0$, and $v_t(x)\le\kappa_2 t^{\zeta}$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, where $\beta,\zeta,\kappa_1,\kappa_2>0$. Let $0\le b\le 2$, $a>0$, and assume that*

$$V(x) \ge \frac{a}{1+|x|^b}, \qquad x\in\mathbb{R}^d.$$

i) If $b<2$, then the equation

$$\frac{\partial u_t}{\partial t}(x) = \Delta u_t(x) - V(x)u_t(x) + v_t(x)G(u_t(x)), \qquad w_0=\varphi, \tag{5.6}$$

admits a global solution for all $\beta>0$.

ii) If $b=2$ and

$$\beta > \beta^*(a) := \frac{2(1+\zeta)}{d+\alpha_1(a)}$$

then (5.6) admits a global solution.

Proof. Clearly, it suffices to consider the semilinear equation

$$\frac{\partial u_t}{\partial t}(x) = \Delta u_t(x) - V(x)u_t(x) + \kappa t^\zeta u_t^{1+\beta}(x), \qquad u_0(x) = \varphi(x), \tag{5.7}$$

for a suitable constant $\kappa > 0$. Suppose that for some $\sigma > 0$,

$$0 \leq \varphi(x) \leq \frac{C}{1+|x|^\sigma}, \qquad x \in \mathbb{R}^d.$$

i) Assume that $\sigma > b(1+\zeta)/\beta$, and let $\varepsilon \in (0,1)$ be such that $(1-\varepsilon)\beta\sigma/b > 1+\zeta$. From Proposition 3.1.i we get

$$\int_1^\infty t^\zeta \|S_t\varphi\|_\infty^\beta dt < 1,$$

provided C is sufficiently small.

ii) If $b = 2$ and $\beta > 2(1+\zeta)/(d+\alpha_1(a))$, let $\varepsilon \in (0,1)$ be such that $\beta(d/2 + (1-\varepsilon)\alpha_1(a)) > 1+\zeta$. From Proposition 3.1.ii there exists $\sigma > d$ such that

$$\int_1^\infty t^\zeta \|S_t\varphi\|_\infty^\beta dt < 1$$

provided C is sufficiently small. □

Remark 5.3 An alternative proof of Theorem 5.2-i) consists in letting the initial value φ in (5.7) be such that

$$\varphi(x) \leq \tau S_1 \mathbf{1}_D(x),$$

for a sufficiently small constant $\tau > 0$, where $D \subset \mathbb{R}^d$ is bounded and Borel measurable. By Lemma 3.4,

$$S_t\varphi(x) \leq \tau S_{t+1}\mathbf{1}_D(x) \leq \tau c_D(1+t)^{-(1+\gamma)},$$

thus showing that $\int_1^\infty t^\zeta \|S_t\varphi\|^\beta \, dt$ can be made arbitrarily close to 0 by choosing τ sufficiently small. By Theorem 5.1 we conclude that (5.7) admits positive global solutions.

Remark 5.4 In the same way as in the above remark we can deal with the semilinear system

$$\begin{cases} \dfrac{\partial u_t}{\partial u}(x) = \Delta u_t(x) - V_1(x)u_t(x) + u_t(x)v_t(x), \quad u_0(x) = \varphi(x), \\ \\ \dfrac{\partial v_t}{\partial t}(x) = \Delta v_t(x) - V_2(x)v_t(x) + u_t(x)v_t(x), \quad v_0(x) = \psi(x), \end{cases} \tag{5.8}$$

where $x \in \mathbb{R}^d$, $d \geq 2$, $\varphi, \psi \geq 0$, and

$$V_1(x) \sim \frac{a_1}{1+|x|^{b_1}}, \qquad V_2(x) \sim \frac{a_2}{1+|x|^{b_2}}, \qquad x \in \mathbb{R}^d, \tag{5.9}$$

with $a_i > 0$ and $b_i \geq 0$, $i = 1, 2$.

Theorem 5.5 *If* $\max(b_1, b_2) < 2$*, then* (5.8) *admits nontrivial positive global solutions.*

Proof. Without loss of generality let us assume that $b := b_1 < 2$. Let $(S_t^1)_{t\geq 0}$ denote the semigroup with generator $L = \Delta - V_1$. By Lemma 3.4, there exists $\gamma > 0$ such that

$$S_t^1 \mathbf{1}_D(x) \leq c_D t^{-(1+\gamma)}, \quad x \in \mathbb{R}^d,$$

for all sufficiently large $t > 0$, where c_D does not depend on x and t. The proof is finished by an application of Theorem 1.1 in [10]. □

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