The Fourier-Mehler transform and generalized dilations of Gaussian and Poisson measures

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Abstract

We define a family of random dilations of the Wiener and Poisson measures, and show that they can be represented as generalised Fourier-Mehler transforms. These (not quasi invariant) transformations include transforms given e.g. by time changes on Brownian motion. The generators of one-parameter families of such transformations are computed, and the Poisson case is also considered.

Key words: Wiener measure, Fourier-Mehler transform, Brownian motion, time changes.
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1 Introduction

The Fourier-Mehler transform, cf. [5], [6], [10], is originally a group $(\mathcal{F}_\theta)_{\theta\in\mathbb{R}}$ of transformations of random variables on Gaussian space, with the property that $\mathcal{F}_{-\pi/2}$ coincides with the Fourier transform. The adjoint $\mathcal{G}_\theta$ of the Fourier-Mehler transform $\mathcal{F}_\theta$ also forms a group whose infinitesimal generator is the sum of the Gross Laplacian and a number operator. It has been extended in [2] as a two-parameter family of transformations which includes the group of complex dilations of Gaussian measures, and in [3], [4] to a family of transformations indexed by continuous mappings on $\mathcal{S}(\mathbb{R})$. In this paper we define a general class of transformations of random functionals that includes as particular cases the dilations of Gaussian measures and transformations induced by time changes on Brownian motion. The generators of families of such transformations are computed in a general random setting using a modified Gross Laplacian and a second quantized operator. In the deterministic case, we show that the generalised Fourier-Mehler transform of [3] can be used to express our generalised dilations. Although such transformations are not quasi-invariant with respect to the Gaussian measures they can be defined on a space of smooth functionals. We also examine the counterpart of this construction in the Poisson case.

In Sections 2, 3, 4 and 5 we recall the construction of the Fourier-Mehler transform, the generalised dilations and the Gross Laplacian. In Sect. 6, derivatives of one-parameter families of dilations are computed, and the link with the Fourier-Mehler transform is made in Prop. 2 of Sect. 7. The Poisson case is studied in a similar way in Sect. 8 and Prop. 4.

2 Notation

Let T be a topological space with Borel measure $\sigma(dt)$, and let $H = L^2(T, \sigma)$. Let

$$E = \mathcal{S}(T) \subset H \subset \mathcal{S}'(T) = E^*$$

denote a Gelfand triple satisfying the hypothesis of white noise analysis (see e.g. [2]). We denote by $\langle \cdot, \cdot \rangle$ the canonical product on $\mathcal{S}(T) \times \mathcal{S}'(T)$. In this section, μ is one of the Gaussian or Poisson measures on $\mathcal{S}'(T)$ determined by the characteristic functions

$$\int_{\mathcal{S}'(T)} \exp(i\langle \omega, \xi \rangle)\mu(d\omega) = \exp\left(-\frac{1}{2}\|\xi\|_H^2\right),$$

and

$$\int_{\mathcal{S}'(T)} \exp(i\langle \omega, \xi \rangle)\mu(d\omega) = \exp\left(\int_T (e^{i\xi} - 1)d\sigma\right),$$

cf. [9]. In this case, every $F \in L^2(\mathcal{S}'(T), \mu)$ has a decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in H^{\circ n},$$

where $I_n(f_n)$ is the multiple stochastic integral of the square-integrable symmetric function of n variables $f_n \in H^{\circ n}$ with respect to the chosen random measure (Gaussian or Poisson). Let

$$(E) \subset L^2(\mathcal{S}'(T), \mu) \subset (E)^*$$

denote the standard of white noise triple test and generalised functionals, and let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the canonical product on $(E) \times (E)^*$. We denote respectively by $f_n \otimes g_m$ and $f_n \circ g_m$ the completed symmetric tensor product of $f_n \in L^2(T^n)$, $g_m \in L^2(T^m)$, and its symmetrization. The exponential vector ϕ_ξ associated to $\xi \in L^2(T, \sigma)$ is defined as

$$\phi_\xi = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\xi^{\circ n}).$$

Definition 1 *Let Ξ denote the vector space generated by $\{\phi_\xi \ : \ \xi \in \mathcal{S}(T)\}$.*

The gradient $D : L^2(\mathcal{S}'(T), \mu)$, is densely defined as $D\Phi = (\partial_t \Phi)_{t\in T}$, with

$$\partial_t \Phi = \sum_{n=1}^{\infty} n I_{n-1}(f_n(*, t)), \quad t \in T,$$

if $\Phi \in (E)$ is of the form $\Phi = \sum_{n=0}^{\infty} I_n(f_n)$. We denote by $\mathbb{L}_{1,p}$, $p > 1$, the space of processes defined by the norm

$$\|u\|_{1,p}^p = \|u\|_{L^p(\mathcal{S}'(T), L^2(T))}^p + \|Du\|_{L^p(\mathcal{S}'(T), L^2(T^2))}^p.$$

For $p = 2$ we have

$$\|u\|_{1,2}^2 = \sum_{n=0}^{\infty} (n+1)! \|f_{n+1}\|_{L^2(T^{n+1})}^2,$$

if $u(\cdot) = \sum_{n=0}^{\infty} I_n(f_{n+1}(*, \cdot))$. The unbounded creation operator $\delta : L^2(\mathcal{S}'(T) \times T) \longrightarrow L^2(\mathcal{S}'(T))$ is defined by linearity as

$$\delta(I_n(u_n(*, \cdot))) = I_{n+1}(\tilde{u}_n),$$

where $\tilde{u}_n$ denotes the symmetrization of $u_n \in H^{\circ n} \otimes H$ in n variables. It is continuous from $\mathbb{L}_{1,2}$ into $L^2(\mathcal{S}'(T), \mu)$ with the bound

$$\|\delta(u)\|_{L^2(\mathcal{S}, \mu)} \leq \|u\|_{1,2},$$

and closable on $L^2(\mathcal{S}'(T), \mu)$, its L^2 domain being denoted by $\mathrm{Dom}(\delta)$. The second quantization of the deterministic operator R is defined as

$$\Gamma(R)\phi_u = \phi_{Ru}, \quad u \in \mathcal{S}(T).$$

Given $\Phi \in (E)^*$, the U-transform $U[\Phi] : \mathcal{S}(T) \longrightarrow \mathbb{R}$ of Φ is defined as

$$U[\Phi](\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \xi \in \mathcal{S}(T),$$

cf. [7].

3 The Fourier-Mehler transform

Let μ denote the Gaussian measure on $\mathcal{S}'(T)$, defined by its characteristic function Under the identification between $L^2(\mathcal{S}'(T), \mu)$ and the Fock space on $L^2(T, \sigma)$ we have

$$\phi_\xi = \exp\left(I_1(\xi) - \frac{1}{2}\|\xi\|_H^2 \right), \quad \xi \in L^2(T, \sigma),$$

and the formula

$$\delta(\Phi u) = \Phi\delta(u) - \langle D\Phi, u\rangle, \qquad (1)$$

for $\Phi \in \Xi$ and $u \in \mathrm{Dom}(\delta)$ such that $u\Phi \in \mathrm{Dom}(\delta)$. The Fourier transform $\mathcal{F}\Phi$ of Φ is the unique (cf. [15]) functional in $(E)^*$ satisfying

$$U[\mathcal{F}\Phi](\xi) = \langle\!\langle \Phi, \exp(-i\langle\cdot,\xi\rangle)\rangle\!\rangle, \quad \xi \in \mathcal{S}(T).$$

The Fourier-Mehler transform $\mathcal{F}_\theta\Phi$ of Φ with parameter $\theta \in \mathbb{R}$ is the unique functional in $(E)^*$ satisfying

$$U[\mathcal{F}_\theta\Phi](\xi) = \langle\!\langle \Phi, \exp(e^{i\theta}\langle\cdot,\xi\rangle)\rangle\!\rangle \exp\left(-\frac{1+e^{2i\theta}}{2}\frac{|\xi|^2}{2}\right), \quad \xi \in \mathcal{S}(T),$$

or

$$U[\mathcal{F}_\theta\Phi](\xi) = \exp\left(\frac{ie^{i\theta}\sin\theta}{2}|\xi|^2\right) U[\Phi](e^{i\theta}\xi), \quad \xi \in \mathcal{S}(T).$$

It satisfies the group property

$$\mathcal{F}_\theta\mathcal{F}_{\theta'} = \mathcal{F}_{\theta+\theta'}, \quad \theta,\theta' \in \mathbb{R}.$$

The dual $\mathcal{G}_\theta$ of the Fourier-Mehler transform $\mathcal{G}_\theta$ is given as

$$\mathcal{G}_\theta = \Gamma(e^{i\theta}\mathrm{I})\exp\left(\frac{e^{2i\theta}-1}{4}\Delta_G\right) = \Gamma(e^{i\theta}\mathrm{I})\exp\left(\frac{ie^{i\theta}\sin\theta}{2}\Delta_G\right),$$

where Δ_G is the Gross Laplacian and I denotes the identity operator on H. It has been proved in Cor. 4.4-(v) of [2] that complex dilations of Gaussian measures can be expressed via the adjoint of a generalised Fourier-Mehler transform, as

$$\Lambda(e^{i\theta}\mathrm{I}) \;=\; \Gamma(e^{i\theta}\mathrm{I})\exp\left(ie^{i\theta}(\sin\theta)\Delta_G\right).$$

In the sequel we will extend this type of result to more general families of transformations.

4 Generalised dilations of Gaussian measures

Under the identification between $L^2(\mathcal{S}'(T),\mu)$ and the Fock space on $L^2(T,\sigma)$, the space Ξ is also an algebra, which is dense in $L^2(\mathcal{S}'(T),\mu)$, moreover each element $F \in \Xi$ can be expressed as

$$F = f(I_1(\xi_1),\dots,I_1(\xi_n)), \quad \xi_1,\dots,\xi_n \in \mathcal{S}(T),$$

where $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Definition 2 *Let* $R : \mathcal{S}(T) \longrightarrow \mathrm{Dom}(\delta)$ *be a given mapping. We densely define on* Ξ *the linear operator* $\Lambda(R)$ *as*

$$\Lambda(R)\phi_\xi = \exp\left(-\frac{1}{2}\|\xi\|_H^2\right)\exp(\delta(R\xi)), \quad \xi \in \mathcal{S}(T).$$

The operator $\Lambda(R)$ is well-defined on Ξ since each element $F \in \Xi$ has a unique expression as

$$F = \sum_{i=1}^{i=n} \alpha_i \phi_{\xi_i},$$

since the family $\{\phi_{\xi_1}, \ldots, \phi_{\xi_n}\}$ is linearly independent whenever $\xi_i \neq \xi_j$, $i \neq j$, cf. [1]. It satisfies

$$\Lambda(R)F = f(\delta(R\xi_1), \ldots, \delta(R\xi_n)),$$

if $F = f(I_1(\xi_1), \ldots, I_1(\xi_n))$, $f \in \mathcal{C}^1(\mathbb{R}^n)$, and it is a morphism with respect to the pointwise product:

$$\Lambda(R)f(F_1, \ldots, F_n) = f(\Lambda(R)F_1, \ldots, \Lambda(R)F_n), \quad f \in \mathcal{C}^1(\mathbb{R}^n).$$

We now mention some particular cases generalised dilations, which correspond to deterministic R.

- Dilations of the Gaussian measure, with $R = \alpha \mathrm{I}$, $\alpha \in \mathbb{C}$.
- Time changes on Brownian motion. In this case, $T = \mathbb{R}_+$, $\sigma(dt) = dt$, and R is given by $R\xi = \xi \circ h$ where $h \in L^2(\mathbb{R}_+)$. If $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is bijective, then the action of $\Lambda(R)$ is to evaluate a functional $\Phi \in \Xi$ on a Brownian motion which is time-changed according to h, since a.s. we have

$$\int_0^\infty u(h(t))dB(t) = \int_0^\infty u(t)dB(h^{-1}(t)) = -\int_0^\infty u'(t)B(h^{-1}(t))dt, \quad u \in \mathcal{S}(\mathbb{R}_+).$$

5 Generalized Gross Laplacians

The Gross Laplacian, cf. [7], has been generalized in different directions. In [2], Def. 3.1, Gross Laplacians associated to deterministic continuous operators on $\mathcal{S}(T)$ have been defined. In [16], [17], [18], [19], similar operators have been defined in association with deterministic unbounded derivation operators on $\mathcal{S}(\mathbb{R}_+)$. Let $K : \mathcal{S}(T) \longrightarrow \mathbb{L}_{1,4}$ be a given mapping and let $\tau(K)$ denote the linear trace operator associated to K, and defined as

$$\langle \tau(K), \xi \otimes \eta \rangle = \langle K\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{S}(T), \quad \mu - a.s.$$

Let $\Delta_G(K)$ be defined on Ξ as

$$\Delta_G(K)\Phi = \tau(K)DD\Phi, \quad \Phi \in \Xi,$$

or

$$\Delta_G(K) = \int_{T^2} \tau(K)(s,t)\partial_s\partial_t dsdt.$$

If $K = \mathrm{I}$ is the identity, then $\Delta_G(\mathrm{I}) = \Delta_G$ is the Gross Laplacian. If K is deterministic we have

$$\Delta_G(K)\phi_\xi = \langle K\xi, \xi\rangle\phi_\xi$$

and

$$\exp(\Delta_G(K))\phi_\xi = \exp(\langle K\xi, \xi\rangle)\phi_\xi, \qquad \xi \in \Xi.$$

We also define the differential second quantization $d\Gamma(K)$ as

$$d\Gamma(K)\Phi = \delta(KD\Phi), \quad \Phi \in \Xi,$$

which can also be denoted as

$$d\Gamma(K) = \int_{T^2} \partial_t^* \tau(K)(s,t)\partial_s dsdt,$$

extending the definitions of [4], Sect. 3, to the case of random K.

6 Derivatives of one-parameter families of transformations

In this section we are interested in the derivatives of one-parameter families of transformations of Gaussian measures, without adaptedness restrictions on the transformation R. The following proposition extends analog results proved in [2], [3] in the deterministic case.

Proposition 1 *Let $(R_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of mappings with $R_0 = \mathrm{I}$, such that $(R_\varepsilon\xi)_{\varepsilon\in[0,1]}$ is differentiable in $\mathbb{L}_{1,4}$ at $\varepsilon = 0$, $\forall \xi \in \mathcal{S}(T)$, and let*

$$K\xi = \frac{d}{d\varepsilon} R_\varepsilon \xi_{|\varepsilon=0}, \quad \xi \in \mathcal{S}(T).$$

Then the derivative on Ξ of $(\Lambda(R_\varepsilon))_{\varepsilon\in[0,1]}$ at $\varepsilon = 0$ is

$$d\Lambda(K) := \Delta_G(K) + d\Gamma(K). \tag{2}$$

Proof. By assumption, $\exp(\delta(\xi))K\xi$ belongs to $\mathbb{L}_{1,2}$ and to $\mathrm{Dom}(\delta)$, thus from (1):

$$\begin{aligned}\frac{d}{d\varepsilon}\exp(\delta(R_\varepsilon\xi))_{|\varepsilon=0} &= \delta(K\xi)\exp(\delta(\xi)) \\ &= \langle K\xi,\xi\rangle\exp(\delta(\xi)) + \delta(\exp(\delta(\xi))K\xi) \\ &= (\Delta_G(K) + d\Gamma(K))\exp(\delta(\xi)), \quad \xi\in\mathcal{S}(T),\end{aligned}$$

and by linearity we have

$$\frac{d}{d\varepsilon}\Lambda(R_\varepsilon)\Phi_{|\varepsilon=0} = (\Delta_G(K) + d\Gamma(K))\,\Phi, \quad \Phi\in\Xi,$$

hence (2) holds on Ξ.

□

If K is deterministic then the operator $d\Lambda(K)$ can also be written as

$$d\Lambda(K) = \int_{T^2}\dot{B}(t)\tau(K)(s,t)\partial_s ds dt,$$

which generalises the family of operators studied in Sect. 7 of [4] and Sect. 5.1 of [12], [13], in the particular case $\tau(K)(s,t)dsdt = \xi(t)\delta_s(t)dsdt$, where $\xi:\mathbb{R}\longrightarrow\mathbb{R}$ is a deterministic function.

7 Generalized dilations and Fourier-Mehler transforms

In this section we work in the case where the operators R and K are deterministic. In order to include transformations by time changes, no continuity property is assumed on these operators. If $A,B:\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$ are two continuous linear mappings, the transform $\mathcal{G}(A,B)$ has been defined in [2] as

$$\mathcal{G}(A,B) = \Gamma(B)\exp(\Delta_G(A)).$$

We note that this still defines an operator on Ξ without continuity assumptions on $A,B:\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$. The following result extends Th. 6.4. of [4].

Lemma 1 *Given $B,R:\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$, we have the commutation relation on Ξ:*

$$\exp\left(\Delta_G(B)\right)\Gamma(R) = \Gamma(R)\exp\left(\Delta_G(R^*BR)\right).$$

Proof. This relation is checked on exponential vectors:

$$\begin{aligned}\Gamma(R)\exp\left(\Delta_G(R^*BR)\right)\phi_\xi &= \Gamma(R)\exp\left(\langle R^*BR\xi,\xi\rangle\right)\phi_\xi\\ &= \exp\left(\langle BR\xi,R\xi\rangle\right)\phi_{R\xi}=\exp\left(\Delta_G(B)\right)\phi_{R\xi}\\ &= \exp\left(\Delta_G(B)\right)\Gamma(R)\phi_\xi. \quad\square\end{aligned}$$

The following proposition extends the formula

$$\Lambda(e^{i\theta}\mathrm{I}) = \Gamma(e^{i\theta}\mathrm{I})\exp\left(ie^{i\theta}(\sin\theta)\Delta_G\right)=\mathcal{G}_\theta$$

of Cor. 4.4-(v) of [2] which concerns complex dilations, to generalised random dilations.

Proposition 2 *Let $R:\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$ be deterministic and admitting an adjoint $R^*:\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$. Then*

$$\Lambda(R)=\Gamma(R)\exp\left(\frac{1}{2}\Delta_G(R^*R-\mathrm{I})\right)=\mathcal{G}\left(\frac{1}{2}(R^*R-\mathrm{I}),R\right).$$

If moreover R is invertible, then

$$\Lambda(R)=\exp\left(\frac{1}{2}\Delta_G(\mathrm{I}-(R^*R)^{-1})\right)\Gamma(R). \tag{3}$$

Proof. We have

$$\begin{aligned}\Lambda(R)\phi_\xi &= \Lambda(R)\exp\left(\delta(\xi)-\frac{1}{2}\mid\xi\mid^2\right)=\exp\left(\delta(R\xi)-\frac{1}{2}\mid\xi\mid^2\right)\\ &= \exp\left(\frac{1}{2}(\mid R\xi\mid^2-\mid\xi\mid^2)\right)\exp\left(\delta(R\xi)-\frac{1}{2}\mid R\xi\mid^2\right)\\ &= \exp\left(\frac{1}{2}(\mid R\xi\mid^2-\mid\xi\mid^2)\right)\phi_{R\xi}=\exp\left(\frac{1}{2}(\mid R\xi\mid^2-\mid\xi\mid^2)\right)\Gamma(R)\phi_\xi\\ &= \Gamma(R)\exp\left(\frac{1}{2}\Delta_G\left(R^*R-\mathrm{I}\right)\right)\phi_\xi.\end{aligned}$$

From Prop. 2 and Lemma 1 we obtain (3).

□

By differentiation of this identity we also obtain Prop. 2 since the derivative of $(\Gamma(R_\varepsilon))_{\varepsilon\in[0,1]}$ is $d\Gamma(K)$, and the derivative of $\frac{1}{2}(R_\varepsilon R_\varepsilon^*-\mathrm{I})$ at $\varepsilon=0$ is $\frac{1}{2}(K+K^*)$. In the particular case $R=r\mathrm{I}$, $r\in\mathbb{C}$,

$$\Lambda(r\mathrm{I}) = \Gamma(r\mathrm{I})\exp\left(\frac{1}{2}(r^2-1)\Delta_G\right)=\exp\left(\frac{1}{2}(1-r^{-2})\Delta_G\right)\Gamma(r\mathrm{I}),$$

$$\Lambda(e^t\mathrm{I}) = \Gamma(e^t\mathrm{I})\exp\left(\frac{1}{2}(e^{2t}-1)\Delta_G\right)=\exp\left(\frac{1}{2}(1-e^{-2t})\Delta_G\right)\Gamma(e^t\mathrm{I}),\quad t\in\mathbb{C},$$

and

$$\Lambda(e^{i\theta}\mathrm{I}) \;=\; \Gamma(e^{i\theta}\mathrm{I})\exp\left(ie^{i\theta}\sin\theta\Delta_G\right), \quad \theta\in\mathbb{R},$$

which is Cor. 4.4-v of [2]. With $R_\varepsilon = e^{\alpha\varepsilon}\mathrm{I}$, $\alpha\in\mathbb{C}$, we obtain

$$d\Lambda(\mathrm{I}) = \alpha(\Delta_G + d\Gamma(\mathrm{I})).$$

Remark 1 *From the expression of characteristic functions, $\Lambda(R)$ preserves Gaussian measures if and only if R is an isometry. The above Lemma shows that this condition is equivalent to $\Lambda(R)=\Gamma(R)$.*

We now turn to the particular case of time changes on Brownian motion i.e. R_ε is given by $R_\varepsilon\xi = \xi\circ\nu$ where $\nu\in\mathcal{S}(\mathbb{R}_+)$ with $\nu(0)=0$. Let K_h, $h\in\mathcal{S}(\mathbb{R})$, denote the operator $K_h : \mathcal{S}(\mathbb{R}_+)\longrightarrow\mathcal{S}(\mathbb{R}_+)$ defined as

$$K_h\xi(t) = h(t)\xi'(t), \quad t\in\mathbb{R},\ \xi\in\mathcal{S}(\mathbb{R}_+).$$

Assume that ν is bijective with

$$\nu(t) = t + h(t) \ \text{ and } \ \nu^{-1}(t) = t+\tilde{h}(t), \quad t\in\mathbb{R}_+.$$

The following corollary is an application of Prop. 2 in the case of time changes.

Corollary 1 *We have $\Lambda(R_\nu) = \mathcal{G}\left(-K_{\tilde{h}}, R_\nu\right)$, i.e. on Ξ:*

$$\Lambda(R_\nu) = \Gamma(R_\nu)\exp\left(-\Delta_G(K_{\tilde{h}})\right) = \exp\left(\Delta_G(K_h)\right)\Gamma(R_\nu). \tag{4}$$

Proof. We have

$$\begin{aligned}
\langle R\xi, R\xi\rangle &= \int_0^\infty \xi(\nu(t))\xi(\nu(t))dt \\
&= -2\int_0^\infty \xi(\nu(t))\xi'(\nu(t))\nu'(t)tdt = -2\int_0^\infty \xi(t)\xi'(t)\nu^{-1}(t)dt \\
&= -2\int_0^\infty \xi(t)\xi'(t)\tilde{h}(t)dt - 2\int_0^\infty \xi(t)\xi'(t)\nu^{-1}(t)dt \\
&= -2\int_0^\infty \xi(t)\xi'(t)\tilde{h}(t)dt + \int_0^\infty \xi(t)\xi(t)dt \\
&= -2\langle\xi, K_{\tilde{h}}\xi\rangle + \langle\xi,\xi\rangle, \quad \xi\in\mathcal{S}(\mathbb{R}_+),
\end{aligned}$$

hence

$$\Delta_G(R^*R) = -2\Delta_G(K_{\tilde{h}}) + \Delta_G.$$

On the other hand,

$$\begin{aligned}\langle R^{-1}\xi, R^{-1}\xi\rangle &= \int_0^\infty \xi(\nu^{-1}(t))\xi(\nu^{-1}(t))dt \\ &= \int_0^\infty \xi(t)\xi(t)\nu'(t)dt = -2\int_0^\infty \xi(t)\xi'(t)\nu(t)dt \\ &= -2\int_0^\infty \xi(t)\xi'(t)h(t)dt - 2\int_0^\infty \xi(t)\xi'(t)tdt \\ &= -2\int_0^\infty \xi(t)\xi'(t)h(t)dt + \int_0^\infty \xi(t)\xi(t)dt \\ &= -2\langle \xi, K_h\xi\rangle + \langle \xi,\xi\rangle, \quad \xi \in \mathcal{S}(\mathbb{R}_+),\end{aligned}$$

hence

$$\Delta_G(R^{-1}(R^*)^{-1}) = -2\Delta_G(K_h) + \Delta_G.$$

It remains to apply Prop. 2.

□

In [8], related calculations are performed in the case of changes of variances for Brownian motion. We note that if $(R_\varepsilon)_{\varepsilon\in[0,1]}$ is a family of (random) transformations such that $\Lambda(R_\varepsilon)$, $\varepsilon \in [0,1]$ preserves the Gaussian measure, then $\Delta_G(K)$ is conservative, i.e. $E[\Delta_G(K)\Phi] = 0$, $\Phi \in \Xi$, since if R_ε is measure preserving, then $E[\Lambda(R_\varepsilon)\Phi] = E[\Phi]$, $\Phi \in \Xi$, hence by differentiation $E[\Delta_G(K)\Phi] = 0$ since $E[d\Gamma(K)\Phi] = 0$.

8 The Poisson case

Let π_σ denote the Poisson measure on $\mathcal{S}'(T)$ with finite diffuse intensity measure σ on T, defined by

$$\int_{\mathcal{S}'(T)} \exp(i\langle\omega,\xi\rangle)\mu(d\omega) = \exp\left(\int_T (e^{i\xi}-1)d\sigma\right).$$

We assume that for all $\xi \in \mathcal{S}(T)$ such that $\|\xi\|_\infty < 1$ we have $e^\xi - 1 \in \mathcal{S}(T)$ and $\log(1+\xi) \in \mathcal{S}(T)$. Moreover, in this section Ξ denotes the vector space generated by

$$\{\phi_\xi \ : \ \xi \in \mathcal{S}(T), \quad \|\xi\|_\infty < 1\}.$$

The exponential vector $\phi_\xi \in \Xi$ satisfies

$$\begin{aligned}\phi_\xi &= \exp\left(-\int_T \xi d\sigma\right)\prod_{t\in T}(1+\xi(t)\omega(\{t\})) \\ &= \exp\left(-\int_T \xi d\sigma + \int_T \log(1+\xi(t))\omega(dt)\right), \quad \mu(d\omega) - a.s., \quad \xi \in \mathcal{S}(T).\end{aligned}$$

We let

$$\omega\backslash t = \omega - \epsilon_t 1_{\{\omega(\{t\})=1\}} = \begin{cases} \omega & \text{if } \omega(\{t\}) = 1 \\ \omega - \epsilon_t & \text{if } \omega(\{t\}) = 0, \quad \omega \in \mathcal{S}'(T),\ t \in T, \end{cases}$$

and

$$\omega \cup t = \omega + \epsilon_t 1_{\{\omega(\{t\})=0\}} = \begin{cases} \omega + \epsilon_t & \text{if } \omega(\{t\}) = 0 \\ \omega & \text{if } \omega(\{t\}) = 1, \quad \omega \in \mathcal{S}'(T),\ t \in T, \end{cases}$$

where ϵ_t denotes the Dirac measure at $t \in T$. We have the identity

$$\Phi\delta(u) = \delta(u\Phi) + \langle u, D\Phi\rangle + \delta(uD\Phi), \tag{5}$$

for $\Phi \in \Xi$ and $u \in \mathbb{L}_{1,4}$. If $u : \mathcal{S}'(T) \times T \longrightarrow \mathbb{R}$ is square-integrable with respect to $\mu \otimes \sigma$ and belongs to $\mathrm{Dom}(\delta)$, then from Prop. 2 of [11] we have

$$\delta(u) = \int_T u(t, \omega \setminus t)\omega(dt) - \int_T u(t,\omega)\sigma(dt), \quad a.s., \tag{6}$$

cf. also [14], and ∂_t is a finite difference operator:

$$\partial_t \Phi(\omega) = \Phi(\omega \setminus t) - \Phi(\omega), \quad \mu(d\omega) \otimes \sigma(dt) - a.e., \quad \Phi \in \Xi.$$

Definition 3 *Let $R : \mathcal{S}(T) \longrightarrow L^2(\mathcal{S}'(T) \times T)$ be a mapping such that $R\log(1+\xi) \in \mathrm{Dom}(\delta)$, $\xi \in \mathcal{S}(T)$, $\|\xi\|_\infty < 1$. The transformation $\Lambda(R)$ is defined on Ξ as*

$$\Lambda(R)\phi_\xi = \exp\left(-\int_T \xi d\sigma + \int_T R(\log(1+\xi))(t, \omega \setminus t)\omega(dt)\right).$$

From (6) we can also write

$$\Lambda(R)\phi_\xi = \exp\left(-\int_T \xi d\sigma + \delta(R(\log(1+\xi)) + \int_T R(\log(1+\xi))d\sigma\right), \quad a.s.,$$

and

$$\Lambda(R)\Phi = f\left(\int_T (R\xi_1)(t,\omega \setminus t)\omega(dt), \ldots, \int_T (R\xi_n)(t,\omega \setminus t)\omega(dt)\right), \quad \xi_1, \ldots, \xi_n \in \mathcal{S}(T),$$

for $\Phi = f\left(\int_T \xi_1 d\omega, \ldots, \int_T \xi_n d\omega\right) \in \Xi$, and the transformation $\Lambda(R)$ is well-defined on Ξ. We now mention some particular cases generalised dilations.

- Dilations of Poisson measures, with $R = \alpha \mathrm{I}$, $\alpha \in \mathbb{C}$.
- Shifts of configurations. In this case, R is given by $R\xi = \xi \circ h$ where $h : T \longrightarrow T$ is measurable. If $h : T \longrightarrow T$ is bijective, then the action of $\Lambda(R)$ is to evaluate a functional Φ on configurations ω^h that are shifted according to h, since

  $$\int_T \xi \circ h \; d\omega = \int_T \xi d\omega^h,$$

  where ω^h denotes the image measure of ω by h.

Given a mapping $A : \mathcal{S}(T) \longrightarrow \mathcal{S}(T)$, we denote by $\nabla(A)$ the operator defined on Ξ as

$$\nabla(A)\Phi = \int_T (A \otimes \mathrm{I}) D\Phi d\sigma, \quad \Phi \in \Xi.$$

Proposition 3 *Let $(R_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of operators from $\mathcal{S}(T)$ into $\mathbb{L}_{1,4}$ such that $R_\varepsilon\xi$ is differentiable in $\mathbb{L}_{1,4}$, $\forall\xi \in \mathcal{S}(T)$. We let*

$$K\xi = \frac{d}{d\varepsilon} R_\varepsilon \xi_{|\varepsilon=0}, \quad \xi \in \mathcal{S}(T).$$

If K has the derivation property, then the derivative of $(\Lambda(R_\varepsilon))_{\varepsilon\in[0,1]}$ is

$$d\Lambda(K) = d\Gamma(K) + \nabla(K).$$

Proof. Since K is a derivation, we have from (5) and the identity $D\exp(\delta(\xi)) = (e^\xi - 1)\exp(\delta(\xi))$, $\xi \in \mathcal{S}(T)$:

$$\begin{aligned}
(d\Gamma(K) + \nabla(L))\exp(\delta(\xi)) &= \delta(KD\exp(\delta(\xi))) + \int_T KD\exp(\delta(\xi))d\sigma \\
&= \delta(\exp(\delta(\xi))K(e^\xi - 1)) + \exp(\delta(\xi))\int_T K(e^\xi - 1)d\sigma \\
&= \delta(e^\xi(K\xi)\exp(\delta(\xi))) + \exp(\delta(\xi))\int_T e^\xi(K\xi)d\sigma \\
&= \delta((K\xi)D\exp(\delta(\xi))) + \delta((K\xi)\exp(\delta(\xi))) \\
&\quad + \int_T (K\xi)D\exp(\delta(\xi))d\sigma + \exp(\delta(\xi))\int_T K\xi d\sigma \\
&= \exp(\delta(\xi))\delta(K\xi) + \exp(\delta(\xi))\int_T K\xi d\sigma \\
&= \exp(\delta(\xi))(d\Gamma(K) + \nabla(K))\delta(\xi).
\end{aligned}$$

(we used the fact that $\exp(\delta(\xi))K(e^\xi - 1) \in \mathbb{L}_{1,2}$), hence

$$\begin{aligned}
\frac{d}{d\varepsilon}\left[\Lambda(R_\varepsilon)\exp(\delta(\xi))\right]_{|\varepsilon=0} &= \frac{d}{d\varepsilon}\exp\left(\delta(R_\varepsilon\xi) + \int_T R_\varepsilon\xi d\sigma - \int_T \xi d\sigma\right)_{|\varepsilon=0} \\
&= \left(\delta(K\xi) + \int_T K\xi d\sigma\right)\exp(\delta(\xi)) \\
&= (\nabla(K) + d\Gamma(K))\exp(\delta(\xi)). \quad \square
\end{aligned}$$

From (5) and (6) we also have the expression

$$d\Lambda(K) = \int_T \left[\int_T \tau(K)(t,s)\partial_s\Phi ds\right](\omega \setminus t)\omega(dt),$$

and K has the derivation property if R_ε is multiplicative, $\varepsilon \in [0,1]$. This suggests the definition of a generalised Fourier-Mehler type operator $\mathcal{G}(A, B)$ in the Poisson case as

$$\mathcal{G}(A, B) = \Gamma(B)\exp(\nabla(A)).$$

Lemma 2 *Let $A, B : \mathcal{S}(T) \longrightarrow \mathcal{S}(T)$. We have the commutation relation*

$$\Gamma(B) \exp(\nabla(A)) = \exp(\nabla(A \circ B)) \Gamma(A).$$

Proof. We have

$$\begin{aligned}\Gamma(B) \exp(\nabla(A)) \phi_\xi &= \Gamma(B) \exp\left(\int_T A\xi d\sigma\right) \phi_\xi = \exp\left(\int_T A\xi d\sigma\right) \phi_{B\xi} \\ &= \exp(\nabla(A \circ B)) \phi_{B\xi} = \exp(\nabla(A \circ B)) \Gamma(B) \phi_\xi. \quad \square\end{aligned}$$

The following is the Poisson analog of Prop. 2.

Proposition 4 *Let $R : \mathcal{S}(T) \longrightarrow \mathcal{S}(T)$ be a deterministic and bounded operator. Then*

$$\Lambda(R) = \Gamma(R) \exp(\nabla(R - \mathrm{I})) = \mathcal{G}(R - \mathrm{I}, R). \tag{7}$$

If $R : \mathcal{S}(T) \longrightarrow \mathcal{S}(T)$ is invertible, then

$$\Lambda(R) = \exp\left(\nabla(\mathrm{I} - R^{-1})\right) \Gamma(R). \tag{8}$$

Proof. We have

$$\begin{aligned}\Lambda(R)\phi_\xi &= \exp\left(\int_T (R\xi - \xi) d\sigma\right) \phi_{R\xi} \\ &= \Gamma(R) \exp\left(\int_T (R\xi - \xi) d\sigma\right) \phi_\xi = \Gamma(R) \exp(\nabla(R - \mathrm{I})) \phi_\xi.\end{aligned}$$

Relation (8) follows from (7) and Lemma. 2.

□

If R is deterministic, then $\Lambda(R)$ preserves the Poisson measure if and only if R preserves the measure σ, from the characteristic function of the Poisson measure. From the above Lemma we check that this is equivalent to saying that $\Lambda(R) = \Gamma(R)$.
We note that if $(R_\varepsilon)_{\varepsilon \in [0,1]}$ is a family of transformations preserving the Poisson measure, then $d\Lambda(K)$ is conservative, since if $\Lambda(R_\varepsilon)$ is measure preserving then $E[\Lambda(R_\varepsilon)F] = E[F]$, $F \in \mathcal{S}$, hence by differentiation $E[\nabla(K)F] = 0$, since $E[d\Gamma(K)F] = 0$.

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