Computation of Fredholm determinants for quadratic Ornstein-Uhlenbeck functionals

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Abstract

We derive closed form expressions for the Laplace transform of certain quadratic Brownian functionals based on the Ornstein-Uhlenbeck process, using both Fredholm determinants and PDE arguments. Classical and new bond pricing formulas in quadratic Brownian models are obtained as particular cases.

Keywords: Ornstein-Uhlenbeck process; Quadratic Brownian functionals; Fredholm expansions and equations; Volterra operators; Cox-Ingersoll-Ross model; Bond pricing.
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1 Introduction

The Laplace transform of quadratic functional of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of the form

$$F = I_1(\psi) + \frac{1}{2} I_2(\varphi) = \int_0^T \psi(t) dB_t + \int_0^T \int_0^t \varphi(s,t) dB_s dB_t, \tag{1.1}$$

$\psi \in L^2([0,T])$ and where $\varphi \in L^2([0,T]^2)$ is symmetric in two variables, has been computed on abstract Wiener spaces in [3] and [4], Theorem 2.1, as

$$E\left[e^{-F}\right] = (\det{}_2(I + A^\varphi))^{-1/2} \exp\left(\frac{1}{2}\int_0^T \psi(s)(I + A^\varphi)^{-1}\psi(s)\, \mathrm{d}s\right), \tag{1.2}$$

cf. also Proposition 4.1 in [7], where $\det_2(I + A^\varphi)$ is the Carleman-Fredholm determinant

$$\det_2(I + A^\varphi) = e^{-\operatorname{tr} A^\varphi} \det(I + A^\varphi), \tag{1.3}$$

extended to the (symmetric Hilbert-Schmidt) Volterra operator A^φ defined by

$$(A^\varphi f)(t) := \int_0^T \varphi(s,t) f(s)\, \mathrm{d}s, \qquad f \in L^2([0,T]), \tag{1.4}$$

such that $I + A^\varphi$ has positive spectrum, where

$$\det(I + A^\varphi) = \prod_{i=0}^\infty (1+\lambda_i), \quad \operatorname{trace} A^\varphi = \sum_{i=0}^\infty \lambda_i,$$

and $(\lambda_i)_{i\geq 0}$ are the eigenvalues of A^φ, counted with their multiplicities.

The Laplace transform of quadratic Brownian random variable is relevant to the computation of Feynman path integrals in quantum field theory, cf. e.g. [6] pages 211-212, and it is also used for bond pricing. From a probabilistic point of view, quadratic Brownian functionals are infinitely divisible random variables, and closed form expressions for their Lévy measures have been given in [9], based on (1.2).

In this paper, we use PDE arguments and Fredholm expansions to provide closed form expression for the determinant (1.3), with application to the Laplace transform of functionals of the form (1.1). We show in particular in Corollary 2.2 that when

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T], \tag{1.5}$$

$\alpha \in \mathbb{R}$, $\beta \geq -\alpha$, $b \geq \min(0, -2\alpha)$, the determinant of $I + A^\varphi$ is given by

$$\det(I + A^\varphi) = e^{-bT} \left(\cosh\left(T\sqrt{b^2+2\alpha b}\right) + (b+\alpha+\beta) T \operatorname{sinhc}\left(T\sqrt{b^2+2\alpha b}\right)\right), \tag{1.6}$$

where $\operatorname{sinhc} x = (\sinh x)/x$, by comparison of the PDE solution (2.1) below with (1.2).

In general the spectrum of A^φ is unknown, nevertheless the determinant $\det(I + A^\varphi)$ can also be computed by the Fredholm expansion

$$\det(I + A^\varphi) = \sum_{n=0}^\infty \frac{1}{n!} \int_{(0,1)^n} \det(\varphi(t_p, t_q))_{p,q=1}^n \, \mathrm{d}t_1 \cdots \mathrm{d}t_n, \tag{1.7}$$

cf. Theorem 3.10 of [10], showing that

$$\det(I + A^\varphi) = 1 + Te^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} (\alpha\, {}_1F_1(n+1; 2n, 2bT) + \beta\, {}_1F_1(n; 2n, 2bT)),$$

cf. Proposition 4.1 below, where ${}_1F_1(a; b, z)$ is the hypergeometric function

$${}_1F_1(a; b, z) := \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}$$

and $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial. In the limiting case $b \to 0$ in (1.5) with $\alpha = -\beta = \sigma^2/b$, the eigenvalues of A^φ can be computed explicitly as

$$\lambda_k = 8\sigma^2 T^2 \pi^{-2} (2k+1)^{-2}, \qquad k \geq 1,$$

which yields $\det(I + A^\varphi) = \cosh(\sigma T \sqrt{2})$, cf. (3.12) below, and recovers (1.6).

The above results apply in particular to the computation of the bond price

$$P(t, T) = F(t, r_t) = E\left[e^{-\int_t^T r_s \, \mathrm{d}s} \,\middle|\, r_t \right]$$

of a zero-coupon bond with maturity T, where the short term interest rate process r_t is given in the Cox-Ingersoll-Ross (CIR) model by

$$\mathrm{d}r_t = (\gamma - 2br_t)\,\mathrm{d}t + 2\sigma\sqrt{r_t}\,\mathrm{d}B_t, \tag{1.8}$$

with $\gamma = \sigma^2$ and $b, \sigma > 0$, based on the Laplace transform of quadratic Brownian functionals. The link between (1.8) and quadratic functionals of Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ has been pointed out in [7] using squared Gaussian processes, in the chaos expansion framework of [8].

Indeed, under the condition $\gamma = \sigma^2$, the process r_t solution of (1.8) can be written as $r_t = X_t^2$ where X_t is the Ornstein-Uhlenbeck process

$$X_t = xe^{-bt} + \sigma \int_0^t e^{-b(t-s)}\,\mathrm{d}B_s, \tag{1.9}$$

solution of the equation

$$\mathrm{d}X_t = -bX_t\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \qquad X_0 = x, \tag{1.10}$$

where $b, \sigma > 0$.

We also apply (1.5) and (1.7) to the computation of the joint moment generating functions of quadratic functionals such as

$$\int_0^T X_s^2 ds, \qquad \int_0^T X_s dB_s, \qquad \int_0^T X_s dX_s,$$

using both the Fredholm determinant expansions and PDE expressions. This allows us in particular to recover the Laplace transform of $\int_0^T r_t dt$ where r_t is the CIR process solution of (1.8) under the condition $\gamma = \sigma^2$.

We proceed as follows. Section 2 contains the main results of the paper, which are based on the comparison of determinant expansions and PDE solutions. In Section (4) we compute the determinant of Volterra operators of the form (1.5), using the Fredholm expansion (1.7). Section 3 contains several lemmas, including the computation of trace terms appearing in the exponential component of (1.2). The finite dimensional determinants needed for Fredholm expansions are evaluated in Section 4.

2 Main results

We start by computing the bivariate Laplace transform of $\left(\int_0^T X_s \, \mathrm{d}B_s, \int_0^T X_s^2 ds\right)$ using standard PDE arguments and stochastic calculus applied to the Ornstein-Uhlenbeck process X_t defined in (1.9).

Proposition 2.1 *For all $\rho \geq 0$ and $\mu \in \mathbb{R}$ such that $b^2 + 2\rho b\sigma + 2\mu\sigma^2 \geq 0$ we have*

$$E\left[e^{-\rho \int_0^T X_s \, \mathrm{d}B_s - \mu \int_0^T X_s^2 \, \mathrm{d}s} \Big| X_0 = x\right] \tag{2.1}$$
$$= \left(\cosh(hT) + \frac{b + \sigma\rho}{h} \sinh(hT)\right)^{-1/2} \exp\left(\frac{b + \sigma\rho}{2} T - \frac{x^2(\mu - \rho^2/2)}{b + \rho\sigma + h \coth(hT)}\right),$$

where $h = \sqrt{b^2 + 2\rho b\sigma + 2\mu\sigma^2}$.

Proof. By standard stochastic calculus and martingale arguments it can be shown that the function

$$H(t,x) := E\left[\exp\left(-\rho \int_t^T X_s \, \mathrm{d}B_s - \mu \int_t^T X_s^2 \, \mathrm{d}s\right) \Big| X_t = x\right], \qquad t \in [0,T],$$

solves the PDE

$$\begin{cases} \dfrac{\sigma^2}{2}\dfrac{\partial^2 H}{\partial x^2}(t,x) - (\sigma\rho + b)\, x \dfrac{\partial H}{\partial x}(t,x) + \dfrac{\partial H}{\partial t}(t,x) - x^2(\mu - \rho^2/2) H(t,x) = 0, \\ \\ H(T,x) = 1, \end{cases} \tag{2.2}$$

whose solution $H(t,x)$ is given by (2.1), cf. Lemma 3.1 below. □

In order to compare the result of Proposition 2.1 to the determinant identity (1.2) we rewrite

$$F = \rho \int_t^T X_s \, \mathrm{d}B_s + \mu \int_t^T X_s^2 \, \mathrm{d}s$$

in the form of (1.1) as

$$F = E[F] + x\, I_1(\psi) + \frac{1}{2} I_2(\varphi), \tag{2.3}$$

where ψ and φ are given in (2.6) and (2.7), cf. Lemma 3.2 below. As a consequence of Proposition 2.1, Lemma 3.4, and Relations (1.2) and (2.3) we obtain the following proposition.

Corollary 2.2 *Assume that $\alpha + \beta \geq 0$ and $b \geq -2\alpha$, and let*

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T]. \tag{2.4}$$

Then $I + A^\varphi$ has positive spectrum and we have

$$\det(I + A^\varphi) = e^{-bT} \left(\cosh(hT) + \frac{b + \alpha + \beta}{h} \sinh(hT) \right), \tag{2.5}$$

where $h = \sqrt{b^2 + 2\alpha b}$.

Proof. By Lemma 3.2 the functions in (2.3) are given by

$$\psi(t) = \left(\rho + \frac{\mu\sigma}{b} \right) e^{-bt} - \frac{\mu\sigma}{b} e^{-b(2T-t)}, \tag{2.6}$$

and

$$\varphi(s,t) = \left(\rho\sigma + \frac{\mu\sigma^2}{b} \right) e^{-b|t-s|} - \frac{\mu\sigma^2}{b} e^{-b(2T-s-t)}, \tag{2.7}$$

and

$$E[F] = \frac{\mu x^2}{2b}(1 - e^{-2bT}) + \frac{1}{2}\mathrm{trace}(A^\varphi) - \frac{1}{2}\rho\sigma T, \tag{2.8}$$

where

$$\mathrm{trace}(A^\varphi) = \rho\sigma T + \frac{\mu\sigma^2}{2b^2}\left(e^{-2bT} + 2bT - 1\right).$$

By applying (3.13) below with

$$y = -\mu\sigma e^{-2bT}/b, \quad z = \rho + \mu\sigma/b, \quad \alpha = \rho\sigma + \sigma^2\mu/b, \text{ and } \beta = -\mu\sigma^2/b,$$

cf. (2.6), (2.7) and (2.8) we have

$$\frac{1}{2}\int_0^T \psi(t)(I + A^\varphi)^{-1}\psi(t)\,\mathrm{d}t = \frac{\mu}{2b}(1 - e^{-2bT}) - \frac{\mu - \rho^2/2}{b + \rho\sigma + h\coth(hT)}, \tag{2.9}$$

and (2.5) is obtained by comparison of (2.1) in Proposition 2.1 and (2.9) with (1.2) under the change of variable $\mu = -\beta b/\sigma^2$ and $\rho = (\alpha + \beta)/\sigma$. Here, (1.2) holds since $I + A^\varphi$ has positive spectrum by Lemma 3.3. □

On the other hand, by the Fredholm expansion (1.7) and Proposition 4.1 below we obtain the following alternative to Proposition 2.1, which provides a series expansion in σ.

Corollary 2.3 *For all $\rho \geq 0$ and $\mu \in \mathbb{R}$ such that $b^2 + 2\rho b\sigma + 2\mu\sigma^2 \geq 0$ we have*

$$E\left[e^{-\rho\int_0^T X_t\,\mathrm{d}B_t - \mu\int_0^T X_t^2\,\mathrm{d}t}\Big| X_0 = x\right]$$
$$= \left|1 + e^{-2bT}\sum_{n=1}^\infty (2\sigma T^2)^n \frac{(\rho b + \mu\sigma)^{n-1}}{(2n)!}\left(\frac{n\rho}{T}\,{}_1F_1(n+1; 2n, 2bT) + \mu\sigma\,{}_1F_1(n+1; 2n+1, 2bT)\right)\right|^{-1/2}$$
$$\times \exp\left(\frac{\sigma\rho T}{2} - \frac{x^2(\mu - \rho^2/2)}{b + \rho\sigma + h\coth(hT)}\right). \tag{2.10}$$

Proof. By Proposition 4.1 we have

$$\det(I + A^\varphi) = 1 + Te^{-2bT}\sum_{n=1}^\infty \frac{(2\alpha bT^2)^{n-1}}{(2n-1)!}(\alpha\,{}_1F_1(n+1; 2n, 2bT) + \beta\,{}_1F_1(n; 2n, 2bT)),$$

where φ is given by (2.4) with $\alpha = \rho\sigma + \mu\sigma^2/b$ and $\beta = -\mu\sigma^2/b$ by Lemma 3.2, hence (2.10) follows from (1.2) by the relation

$${}_1F_1(n+1; 2n, 2bT) - {}_1F_1(n; 2n, 2bT) = 2bT\,{}_1F_1(n+1; 2n+1, 2bT).$$

□

Examples

Next we consider some particular cases of quadratic Brownian functionals built from the Ornstein-Uhlenbeck process.

Laplace transform of $\int_0^T X_s^2 \, \mathrm{d}s$

By taking $\mu = 1$ and $\rho = 0$ in (2.1) we recover the Laplace transform

$$\begin{aligned} E\left[e^{-\int_0^T X_s^2 ds} \middle| X_0 = x \right] &= \left(\cosh(hT) + \frac{b}{h} \sinh(hT) \right)^{-1/2} \exp\left(\frac{bT}{2} - \frac{x^2}{b + h\coth(hT)} \right) \\ &= \left(\frac{he^{(b+h)T}}{h + (b+h)(e^{2hT}-1)/2} \right)^{1/2} \exp\left(-\frac{x^2(e^{2hT}-1)}{2h + (b+h)(e^{2hT}-1)} \right), \quad (2.11) \end{aligned}$$

where $h = \sqrt{b^2 + 2\sigma^2}$, cf. [5] and [2]. In addition it follows from (2.10) that

$$\begin{aligned} &E\left[e^{-\int_0^T X_s^2 \, \mathrm{d}s} \middle| X_0 = x \right] \\ &= \left(1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\sigma^2 T^2)^n}{(2n)!} \, {}_1F_1(n+1; 2n+1, 2bT) \right)^{-1/2} \exp\left(-\frac{x^2(e^{2hT}-1)}{2h + (b+h)(e^{2hT}-1)} \right). \end{aligned}$$

In this case we have

$$\varphi(s,t) = \alpha(e^{-b|s-t|} - e^{-b(2T-s-t)}), \qquad s, t \in [0, T],$$

and

$$\det(I + A^\varphi) = e^{-bT} \left(\cosh(hT) + \frac{b}{h} \sinh(hT) \right),$$

or

$$\det(I + A^\varphi) = 1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^n}{(2n)!} \, {}_1F_1(n+1; 2n+1, 2bT),$$

where $h = \sqrt{b^2 + 2\alpha b}$.

Laplace transform of $\int_0^T X_s \, \mathrm{d}B_s$

By taking $\mu = 0$ and $\rho = 1$ in (2.1) we find

$$\begin{aligned} &E\left[e^{-\int_0^T X_t \, \mathrm{d}B_t} \middle| X_0 = x \right] \\ &= \left(\cosh(hT) + \frac{b+\sigma}{h} \sinh(hT) \right)^{-1/2} \exp\left(\frac{(b+\sigma)T}{2} + \frac{x^2/2}{b + \sigma + h\coth(hT)} \right) \end{aligned}$$

$$= \left(1+\sigma T e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\sigma b T^2)^{n-1}}{(2n-1)!}\,{}_1F_1(n+1;2n,2bT)\right)^{-1/2}\exp\left(\frac{\sigma T}{2}+\frac{x^2/2}{b+\sigma+h\coth(hT)}\right),$$

where $h=\sqrt{b^2+2\sigma b}$. In this case we have

$$\varphi(s,t)=\sigma e^{-b|t-s|}, \qquad s,t\in[0,T],$$

and

$$\det(I+A^{\varphi})=e^{-bT}\left(\cosh(hT)+\frac{b+\sigma}{h}\sinh(hT)\right),$$

and

$$\det(I+A^{\varphi})=1+\sigma T e^{-2bT}\sum_{n=1}^{\infty}\frac{(2\sigma b T^2)^{n-1}}{(2n-1)!}\,{}_1F_1(n+1;2n,2bT).$$

Laplace transform of $\int_0^T X_t\,\mathrm{d}X_t$

Similarly by taking $\rho=\sigma$ and $\mu=-b$, i.e. $h=b$, we can show that

$$\begin{aligned} &E\left[e^{-\int_0^T X_s\,\mathrm{d}X_s}\middle|X_0=x\right] \\ &= \left(\cosh(bT)+\frac{(b+\sigma^2)}{b}\sinh(bT)\right)^{-1/2}\exp\left(\frac{b+\sigma^2}{2}T+\frac{x^2(\sigma^2+2b)/2}{b+\sigma^2+b\coth(bT)}\right) \\ &= \left(1+\frac{\sigma^2}{2b}(1-e^{-2bT})\right)^{-1/2}\exp\left(\frac{\sigma^2}{2}T+\frac{x^2(2b+\sigma^2)/2}{b+\sigma^2+b\coth(bT)}\right), \end{aligned}$$

by (1.4). In addition we have

$$\varphi(s,t)=\beta e^{-b(2T-s-t)}, \qquad s,t\in[0,T],$$

and

$$\det(I+A^{\varphi})=1+\frac{\beta}{2b}(1-e^{-2bT})=1+\frac{\beta}{b}e^{-bT}\sinh(bT).$$

3 Main lemmas

PDE solution

First we derive the PDE solution which has been used in the proof of Proposition (2.1).

Lemma 3.1 *Let $b, \sigma > 0$ and $\mu \in \mathbb{R}$ such that $b^2 + 2\rho b\sigma + 2\mu\sigma^2 \geq 0$. The PDE* (2.2) *has solution*

$$\begin{aligned} H(t,x) &= \left(\cosh(h(T-t)) + \frac{b+\sigma\rho}{h}\sinh(h(T-T))\right)^{-1/2} \\ &\quad \times \exp\left(\frac{b+\sigma\rho}{2}(T-t) - \frac{x^2(\mu-\rho^2/2)}{b+\rho\sigma+h\coth(h(T-t))}\right), \end{aligned} \tag{3.1}$$

where $h = \sqrt{b^2 + 2\rho b\sigma + 2\mu\sigma^2}$.

Proof. In order to solve (2.2), we search for a solution of the form

$$H(t,x) = \exp\left(C(T-t) + E(T-t)x^2\right),$$

with $C(0) = E(0) = 0$, which implies

$$\begin{cases} \dfrac{\partial E}{\partial t}(T-t) = -2(b+\sigma\rho)E(T-t) + 2\sigma^2 E^2(T-t) - \mu + \dfrac{\rho^2}{2} & \text{(3.2a)} \\[2ex] \dfrac{\partial C}{\partial t}(T-t) = \sigma^2 E(T-t), \\[2ex] C(0) = E(0) = 0. \end{cases}$$

The Riccati equation (3.2a) is solved as

$$E(T-t) = -\frac{(\mu-\rho^2/2)(1-e^{2h(t-T)})}{b+\sigma\rho+h-(b+\sigma\rho-h)e^{2h(t-T)}} = -\frac{\mu-\rho^2/2}{b+\rho\sigma+h\coth(h(T-t))}, \tag{3.3}$$

where $h = \sqrt{b^2 + 2\rho b\sigma + 2\mu\sigma^2}$, which also yields

$$\begin{aligned} C(T-t) &= \sigma^2 \int_t^T E(T-s)\,\mathrm{d}s \\ &= \frac{b+\sigma\rho}{2}(T-t) - \frac{1}{2}\ln\left(\cosh(h(T-t)) + \frac{b+\sigma\rho}{h}\sinh(h(T-t))\right). \end{aligned} \tag{3.4}$$

□

Quadratic Ornstein-Uhlenbeck functionals

In the next lemma we derive the representation of quadratic functionals of the Ornstein-Uhlenbeck process (1.9) solution of (1.10) with $b, \sigma > 0$.

Lemma 3.2 *(i) The integral* $\int_0^T X_t \, \mathrm{d}B_t$ *has the representation*

$$\int_0^T X_t \, \mathrm{d}B_t = X_0 \, I_1(\psi) + \frac{1}{2} I_2(\varphi),$$

where

$$\psi(t) = e^{-bt} \quad \text{and} \quad \varphi(s,t) = \sigma e^{-b|t-s|}, \qquad s,t \in [0,T].$$

(ii) The integral $\int_0^T X_t^2 dt$ *has the representation*

$$\int_0^T X_t^2 dt = \frac{X_0^2}{2b}(1 - e^{-2bT}) + \frac{\sigma^2}{4b^2}\left(e^{-2bT} + 2bT - 1\right) + X_0 \, I_1(\psi) + \frac{1}{2} I_2(\varphi), \qquad (3.5)$$

where

$$\psi(t) = \frac{\sigma}{b}\left(e^{-bt} - e^{-b(2T-t)}\right) \quad \text{and} \quad \varphi(s,t) = \frac{\sigma^2}{b}(e^{-b|s-t|} - e^{-b(2T-s-t)}),$$

$s, t \in [0,T]$.

Proof. (*i*) We have

$$\int_0^T X_t \, \mathrm{d}B_t = X_0 \int_0^T e^{-bt} \, \mathrm{d}B_t + \sigma \int_0^T \int_0^t e^{-b|t-s|} \, \mathrm{d}B_s \mathrm{d}B_t.$$

(*ii*) We have

$$\int_0^T X_t^2 \, \mathrm{d}t = \int_0^T \left(X_0 e^{-bt} + \sigma e^{-bt} \int_0^t e^{bs} \, \mathrm{d}B_s \right)^2 \mathrm{d}t$$
$$= \sigma^2 \int_0^T e^{-2bt} \left(\int_0^t e^{bs} \, \mathrm{d}B_s \right)^2 \mathrm{d}t + 2\sigma X_0 \int_0^T \int_0^t e^{-2bt} e^{bs} \, \mathrm{d}B_s \mathrm{d}t + \int_0^T X_0^2 e^{-2bt} \, \mathrm{d}t.$$

By the Itô formula we have

$$\left(\int_0^t e^{bs} \, \mathrm{d}B_s \right)^2 = 2 \int_0^t e^{bs} \int_0^s e^{bu} \, \mathrm{d}B_u \mathrm{d}B_s + \frac{e^{2bt} - 1}{2b},$$

hence

$$\int_0^T e^{-2bt} \int_0^t e^{bs} \int_0^s e^{bu} \, \mathrm{d}B_u \mathrm{d}B_s \mathrm{d}t = \frac{1}{2b} \int_0^T \int_0^s (e^{-b|s-u|} - e^{-b(2T-s-u)}) \, \mathrm{d}B_u \mathrm{d}B_s,$$

and

$$\int_0^T \int_0^t e^{-2bt} e^{bs} \, \mathrm{d}B_s \mathrm{d}t = \frac{e^{-bT}}{2b} \int_0^T (e^{b(T-s)} - e^{-b(T-s)}) \, \mathrm{d}B_s = \frac{e^{-bT}}{b} \int_0^T \sinh(b(T-s)) \, \mathrm{d}B_s.$$

□

Spectrum of Volterra operators

Next we compute the spectrum of the Volterra operator A^φ with φ given in (3.6) below.

Lemma 3.3 *Assume that $\alpha+\beta\geq 0$ and $b\geq -2\alpha$, and let φ be given by (4.1), i.e.*

$$\varphi(s,t)=\alpha e^{-b|s-t|}+\beta e^{-b(2T-s-t)}, \qquad s,t\in[0,T]. \tag{3.6}$$

Then $I+A^\varphi$ has positive spectrum and if h is an eigenvector of A^φ with nonzero eigenvalue $\lambda>-1$ then
(i) If $b>2\alpha/\lambda$, the eigenvector h is given by

$$h(t)=\left(\sqrt{b^2-2\alpha b/\lambda}+b\right)e^{t\sqrt{b^2-2\alpha b/\lambda}}+\left(\sqrt{b^2-2\alpha b/\lambda}-b\right)e^{-t\sqrt{b^2-2\alpha b/\lambda}}$$

and the corresponding eigenvalue λ satisfies the equation

$$\left(\sqrt{b^2-2\alpha b/\lambda}+b-(\alpha+\beta)/\lambda\right)e^{2T\sqrt{b^2-2\alpha b/\lambda}}+\sqrt{b^2-2\alpha b/\lambda}-b+(\alpha+\beta)/\lambda=0. \tag{3.7}$$

(ii) If $b<2\alpha/\lambda$, the eigenvector h is given by

$$h(t)=\sqrt{2\alpha b/\lambda-b^2}\cos\left(t\sqrt{2\alpha b/\lambda-b^2}\right)+b\sin\left(t\sqrt{2\alpha b/\lambda-b^2}\right),$$

while the corresponding eigenvalue λ satisfies

$$(\alpha+\beta-\lambda b)\sin\left(T\sqrt{2\alpha b/\lambda-b^2}\right)-\lambda\sqrt{2\alpha b/\lambda-b^2}\cos\left(T\sqrt{2\alpha b/\lambda-b^2}\right)=0.$$

Proof. By twice differentiation of the relation $A^\varphi h(t)=\lambda h(t)$, i.e.

$$\int_0^T\left(\alpha e^{-b|s-t|}+\beta e^{-b(2T-s-t)}\right)h(s)\,\mathrm{d}s=\lambda h(t), \tag{3.8}$$

we obtain

$$b\left(-\alpha\int_0^t e^{b(s-t)}h(s)\,\mathrm{d}s+\alpha\int_t^T e^{b(t-s)}h(s)\,\mathrm{d}s+\beta\int_0^T e^{-b(2T-s-t)}h(s)\,\mathrm{d}s\right)=\lambda h'(t), \tag{3.9}$$

and

$$b^2\left(\alpha\int_0^T e^{-b|t-s|}h(s)\,\mathrm{d}s+\beta\int_0^T e^{-b(2T-s-t)}h(s)\,\mathrm{d}s\right)-2\alpha b h(t)=\lambda h''(t). \tag{3.10}$$

Applying (3.8) in (3.10), we obtain

$$h''(t) = \left(b^2 - 2b\alpha/\lambda\right) h(t)$$

and $h'(0) = bh(0)$ by taking $t = 0$ in (3.8) and (3.9). If $b > 2\alpha/\lambda$, the eigenvector h satisfies

$$h(t) = \left(\sqrt{b^2 - 2\alpha b/\lambda} + b\right) e^{t\sqrt{b^2-2\alpha b/\lambda}} + \left(\sqrt{b^2 - 2\alpha b/\lambda} - b\right) e^{-t\sqrt{b^2-2\alpha b/\lambda}}, \quad t \in [0, T].$$

Taking $t = 0$ in (3.8) shows that the eigenvalue λ satisfies

$$\left(b + \sqrt{b^2 - 2\alpha b/\lambda} - (\alpha + \beta)/\lambda\right) e^{2T\sqrt{b^2-2\alpha b/\lambda}} + \sqrt{b^2 - 2\alpha b/\lambda} - b + (\alpha + \beta)/\lambda = 0.$$

In case $b < 2\alpha/\lambda$, the eigenvector is given by

$$h(t) = \sqrt{2\alpha b/\lambda - b^2} \cos\left(t\sqrt{2\alpha b/\lambda - b^2}\right) + b \sin\left(t\sqrt{2\alpha b/\lambda - b^2}\right),$$

and taking $t = 0$ in (3.8) shows that the eigenvalue λ satisfies

$$(\alpha + \beta - \lambda b) \sin\left(T\sqrt{2\alpha b/\lambda - b^2}\right) - \lambda\sqrt{2\alpha b/\lambda - b^2} \cos\left(T\sqrt{2\alpha b/\lambda - b^2}\right) = 0.$$

Finally, if $\lambda < -1$ is an eigenvalue of A^φ then under the condition $b \geq -2\alpha$ we get $b > 2\alpha/\lambda$ and we check that (3.7) has no solution since $-(\alpha+\beta)/\lambda > 0$, which shows that $I + A^\varphi$ has positive spectrum. □

When $\alpha = -\beta = \sigma^2/b$ Lemma 3.3 shows that any eigenvalue λ of A^φ should satisfy

$$\sqrt{2\sigma^2/\lambda - b^2} \cos\left(T\sqrt{2\sigma^2/\lambda - b^2}\right) = 0. \tag{3.11}$$

As b tends to 0 we get

$$\varphi(s, t) = 2\sigma^2 \left(T - \frac{s + t - |s - t|}{2}\right) = 2\sigma^2(T - (s \vee t)),$$

and in this case the spectrum

$$\lambda_k = \frac{8\sigma^2 T^2}{\pi^2(2k + 1)^2} \qquad \text{and} \qquad h_k(t) = \cos\left(\frac{2\sigma}{\sqrt{\lambda}} t\right), \qquad k \geq 1,$$

can be explicitly computed from (3.11).

As a consequence we have trace$A^\varphi = \sigma^2 T^2$, and by e.g. § 4.5.69 page 85 of [1] we obtain

$$\det(I + A^\varphi) = \prod_{k=0}^{\infty} \left(1 + \frac{8\sigma^2 T^2}{(2k+1)^2\pi^2}\right) = \cosh\left(\sigma T\sqrt{2}\right), \tag{3.12}$$

which recovers (2.5), i.e.

$$\det{}_2(I + A^\varphi) = e^{-\sigma^2 T^2} \cosh\left(\sigma T\sqrt{2}\right),$$

as b tends to 0.

Exponential term

We close this section with a computation of the term $\int_0^T \psi(s)(I + A^\varphi)^{-1}\psi(s)\,\mathrm{d}s$ appearing in (1.2).

Lemma 3.4 *Let $y, z, \alpha, \beta \in \mathbb{R}$, $b \geq -2\alpha$, and*

$$\psi(t) = ye^{bt} + ze^{-bt}, \qquad t \in [0, T],$$

and

$$\varphi(s,t) = \alpha e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

We have

$$\begin{aligned} &\int_0^T \psi(t)(I + A^\varphi)^{-1}\psi(t)\,\mathrm{d}t \qquad (3.13)\\ = \;& \frac{zy}{2\alpha}\frac{(b+\alpha+\beta)e^{hT} - (b+\alpha-h+\beta)e^{-hT} - 2he^{bT}}{(b+\alpha+\beta)\sinh(hT) + h\cosh(hT)} \\ &+ \frac{z^2}{2\alpha^2}\frac{(\alpha^2 + \beta(b+\alpha-h))e^{hT} - (\alpha^2 + \beta(b+\alpha+h))e^{-hT} + 2h\beta e^{-bT}}{(b+\alpha+\beta)\sinh(hT) + h\cosh(hT)}, \end{aligned}$$

with $h = \sqrt{b^2 + 2\alpha b}$.

Proof. The function $u(t)$ solves the Fredholm equation of the second kind

$$A^\varphi u(t) = \psi(t) - u(t),$$

i.e.

$$\int_0^T \left(\alpha e^{-b|s-t|} + \beta e^{-b(2T-s-t)}\right) u(s)\,\mathrm{d}s = \psi(t) - u(t). \tag{3.14}$$

Differentiating (3.14) on both sides with respect to t we get

$$b\left(-\alpha\int_0^t e^{b(s-t)}u(s)\,\mathrm{d}s+\alpha\int_t^T e^{b(t-s)}u(s)\,\mathrm{d}s+\beta\int_0^T e^{-b(2T-s-t)}u(s)\,\mathrm{d}s\right) \quad (3.15)$$
$$= \psi'(t)-u'(t),$$

and

$$b^2\left(\alpha\int_0^T e^{-b(t-s)/2}u(s)\,\mathrm{d}s+\beta\int_0^T e^{-b(2T-s-t)}u(s)\,\mathrm{d}s\right)-2\alpha bu(t)=\psi''(t)-u''(t). \quad (3.16)$$

From (3.14) we can simplify (3.16) to

$$b^2(\psi-u)-2\alpha bu=\psi''-u'', \quad \text{i.e.} \quad u''-(b^2+2\alpha b)u=\psi''-b^2\psi. \quad (3.17)$$

By twice differentiation of $\psi(t)$ we find

$$\psi''=b^2\psi. \quad (3.18)$$

Substituting the right hand side of (3.17) with (3.18) this yields

$$u''-(b^2+2\alpha b)u=0, \quad \text{hence} \quad u(t)=c_1e^{ht}+c_2e^{-ht}, \quad (3.19)$$

where $h=\sqrt{b^2+2\alpha b}$ and c_1, c_2 are constants. Letting $t=0$ both in (3.14) and (3.15), we get

$$\psi(0)-u(0)=\int_0^T\left(\alpha e^{-bs}+\beta e^{-b(2T-s)}\right)u(s)\,\mathrm{d}s$$

and

$$\psi'(0)-u'(0)=b\left(\psi(0)-u(0)\right).$$

Next, substitute $u(t)$ with (3.19) and plug in $\psi(0)$, $\psi'(0)$ in the above relations, therefore c_1 and c_2 are the solutions of following two equations

$$(h-b)c_1-(h+b)c_2=-2bz,$$

and

$$c_1(e^{(h-b)T}(\alpha(h+b)+\beta(h-b))-(h-b)\beta e^{-2bT}-\alpha(h-b))$$
$$+c_2(e^{-(h+b)T}(-\alpha(h-b)-\beta(h+b))+(h+b)\beta e^{-2bT}+\alpha(h+b))=2\alpha b(y+z)$$

hence

$$c_1 = \frac{z\left(e^{-hT}((\alpha+\beta)h + (\beta-\alpha)b) - \beta(h+b)e^{-bT}\right) + (h+b)\alpha y e^{bT}}{2\alpha\left((b+\alpha+\beta)\sinh(hT) + h\cosh(hT)\right)}, \tag{3.20}$$

and

$$c_2 = \frac{z\left(e^{hT}((\alpha+\beta)h + (-\beta+\alpha)b) - \beta(h-b)e^{-bT}\right) + (h-b)\alpha y e^{bT}}{2\alpha\left((b+\alpha+\beta)\sinh(hT) + h\cosh(hT)\right)}, \tag{3.21}$$

which shows that

$$\begin{aligned}(I+A^\varphi)^{-1}\psi(t) &= \frac{z}{\alpha}\frac{h(\alpha+\beta)\cosh(h(T-t)) + b(\alpha-\beta)\sinh(h(T-t))}{(b+\alpha+\beta)\sinh(hT) + h\cosh(hT)} \\ &\quad + \frac{1}{\alpha}\frac{(h\cosh(ht) + b\sinh(ht))(aye^{bT} - \beta z e^{-bT})}{(b+\alpha+\beta)\sinh(hT) + h\cosh(hT)}, \qquad t \in [0,T],\end{aligned}$$

and (3.13) follows. □

4 Fredholm expansions

In this section we show that the determinant $\det(I + A^\varphi)$ appearing in (1.4) can be computed using Fredholm expansions and hypergeometric series.

Proposition 4.1 *Given $b > 0$, $\alpha, \beta \in \mathbb{R}$, and*

$$\varphi(s,t) = \alpha e^{-b|s-t|} + \beta e^{-b(2-s-t)}, \qquad s,t \in [0,1], \tag{4.1}$$

we have

$$\det(I + A^\varphi) = 1 + e^{-2b}\sum_{n=1}^\infty \frac{(2\alpha b)^{n-1}}{(2n-1)!}\left(\alpha\, {}_1F_1(n+1; 2n, 2b) + \beta\, {}_1F_1(n; 2n, 2b)\right).$$

Proof. The determinant $\det(I + A^\varphi)$ is computed by the Fredholm expansion (1.7), where

$$\det(\varphi(t_p,t_q))_{p,q=1}^n = \alpha^n\left(1 + \frac{\beta}{\alpha}e^{-2b(1-t_n)}\right)\prod_{i=1}^{n-1}\left(1 - e^{-2b(t_{i+1}-t_i)}\right),$$

$0 < t_1 < \cdots < t_n < 1$, cf. (4.6). By the change of variable

$$x_i = 2b(t_{i+1} - t_i), \quad 1 \le i < n, \quad x_n = 2b(1 - t_n),$$

we get

$$
\begin{aligned}
\det(I+A^\varphi) &= 1+\sum_{n=1}^\infty \frac{\alpha^n}{(2b)^n}\int_{\left\{ {x_1+\cdots+x_n\le 2b, \atop x_1,\ldots,x_n>0} \right\}} \left(1+\frac{\beta}{\alpha}e^{-x_n}\right)\prod_{i=1}^{n-1}(1-e^{-x_i})\,\mathrm{d}x_1\cdots \mathrm{d}x_n \\
&= 1+\sum_{n=1}^\infty \frac{\alpha^n}{(2b)^n}\int_{\left\{ {x_1+\cdots+x_n\le 2b, \atop x_1,\ldots,x_n>0} \right\}} \left(-\frac{\beta}{\alpha}\prod_{i=1}^{n}(1-e^{-x_i})+(1+\frac{\beta}{\alpha})\prod_{i=1}^{n-1}(1-e^{-x_i})\right)\mathrm{d}x_1\cdots \mathrm{d}x_n \\
&= 1+\sum_{n=1}^\infty \frac{\alpha^n}{(2b)^n}\left(-\frac{\beta}{\alpha}\int_{\left\{ {x_1+\cdots+x_n\le 2b, \atop x_1,\ldots,x_n>0} \right\}}\prod_{i=1}^{n}(1-e^{-x_i})\,\mathrm{d}x_1\cdots \mathrm{d}x_n\right. \\
&\qquad \left.+\left(1+\frac{\beta}{\alpha}\right)\int_{\left\{ {x_1+\cdots+x_n\le 2b, \atop x_1,\ldots,x_n>0} \right\}}\prod_{i=1}^{n-1}(1-e^{-x_i})\,\mathrm{d}x_1\cdots \mathrm{d}x_n\right) \\
&= 1+e^{-2b}\sum_{n=1}^\infty \alpha^n\left(-\frac{\beta}{\alpha}\sum_{m=n}^\infty\frac{(2b)^m}{(m+n)!}\binom{m}{n}+\left(1+\frac{\beta}{\alpha}\right)\sum_{m=n-1}^\infty\frac{(2b)^m}{(n+m)!}\binom{m+1}{n}\right) \\
&= 1+e^{-2b}\sum_{n=1}^\infty \alpha^n\left(\sum_{m=n-1}^\infty\frac{(2b)^m}{(n+m)!}\binom{m+1}{n}\right. \\
&\qquad \left.+\frac{\beta}{\alpha}\frac{(2b)^{n-1}}{(2n-1)!}+\frac{\beta}{\alpha}\sum_{m=n}^\infty\frac{(2b)^m}{(m+n)!}\frac{n}{m+1-n}\binom{m}{n}\right) \\
&= 1+e^{-2b}\sum_{n=1}^\infty \alpha^n\sum_{m=n-1}^\infty\frac{(2b)^m}{(n+m)!}\binom{m+1}{n} \\
&\qquad +\beta e^{-2b}\sum_{n=1}^\infty\frac{(2\alpha b)^{n-1}}{(2n-1)!}\sum_{m=0}^\infty\frac{(2b)^m(2n-1)!}{(m+2n-1)!}\frac{(n+m-1)!}{(n-1)!m!} \\
&= 1+e^{-2b}\sum_{n=1}^\infty \alpha^n\sum_{m=0}^\infty\frac{(2b)^{n-1+m}}{(2n-1+m)!}\binom{m+n}{n}+\beta e^{-2b}\sum_{n=1}^\infty\frac{(2\alpha b)^{n-1}}{(2n-1)!}\,{}_1F_1(n;2n,2b) \\
&= 1+\alpha e^{-2b}\sum_{n=1}^\infty\frac{(2\alpha b)^{n-1}}{(2n-1)!}\,{}_1F_1(n+1;2n,2b)+\beta e^{-2b}\sum_{n=1}^\infty\frac{(2\alpha b)^{n-1}}{(2n-1)!}\,{}_1F_1(n;2n,2b).
\end{aligned}
$$

□

In the particular case where $\beta=0$ and

$$\varphi=\sigma e^{-b|s-t|},\qquad s,t\in[0,T],$$

Proposition 4.1 shows that

$$\det(I+A^\varphi)=1+\sigma T e^{-2bT}\sum_{n=1}^\infty\frac{(2\sigma bT^2)^{n-1}}{(2n-1)!}\,{}_1F_1(n+1;2n,2bT). \tag{4.2}$$

In the particular case where $\alpha = -\beta$ we get the following corollary.

Corollary 4.2 *Let $b > 0$, $\alpha \in \mathbb{R}$ and*

$$\varphi(s,t) = \alpha(e^{-b|s-t|} - e^{-b(2T-s-t)}), \qquad s,t \in [0,T].$$

We have

$$\det(I + A^{\varphi}) = 1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^n}{(2n)!} \, {}_1F_1(n+1; 2n+1, 2bT). \tag{4.3}$$

Proof. By Proposition 4.1 we have

$$\begin{aligned} \det(I + A^{\varphi}) &= 1 + \alpha T e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^{n-1}}{(2n-1)!} \left({}_1F_1(n+1; 2n, 2bT) - {}_1F_1(n; 2n, 2bT) \right) \\ &= 1 + e^{-2bT} \sum_{n=1}^{\infty} \frac{(2\alpha b T^2)^n}{(2n)!} \, {}_1F_1(n+1; 2n+1, 2bT). \end{aligned}$$

□

We also have

$$\text{trace}(A^{\varphi}) = \int_0^T \varphi(s,s) \, \mathrm{d}s = \frac{\alpha}{2b} \left(e^{-2bT} + 2bT - 1 \right) = \frac{2\alpha}{\sigma^2} \left(E\left[\int_0^T X_t^2 dt \right] - \frac{x^2}{2b}(1 - e^{-2bT}) \right).$$

Finite-dimensional determinants

Finally we compute the finite-dimensional determinants needed in the Fredholm expansion (1.7).

Lemma 4.3 *Let $n \geq 2$ and $c_1, \ldots, c_{n-1} \in \mathbb{R}$. Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a symmetric $n \times n$ matrix such that*
(i) $a_{i,i} = 1 + (a_{i+1,i+1} - 1)c_i^2, \qquad 1 \leq i < n,$
and
(ii) $a_{i,j} = c_j a_{i,j+1}, \qquad 1 \leq j < i \leq n.$
Then we have

$$\det(A) = a_{n,n} \prod_{i=1}^{n-1} (1 - c_i^2). \tag{4.4}$$

Proof. According to Condition (ii) we have $a_{2,1} = c_1 a_{2,2}$, and since A is symmetric we have $a_{1,2} = c_1 a_{2,2}$ as well. In addition the minors of A are given by

$$M_{1,j} = 0, \quad 3 \le j \le n, \quad \text{and} \quad M_{1,2} = c_1 M_{1,1}.$$

We will prove (4.4) by induction on $n \ge 1$. For $n = 1$ we have

$$\det A = \begin{vmatrix} a_{1,1} & c_1 a_{2,2} \\ c_1 a_{2,2} & a_{2,2} \end{vmatrix} = |a_{1,1}||a_{2,2}| - |c_1|^2 |a_{2,2}|^2 = a_{2,2}(1 - |c_1|^2)$$

by Condition (i), hence (4.4) holds. Next, assuming that (4.4) holds at the rank $n \ge 1$, we have

$$M_{1,1} = a_{n,n} \prod_{i=2}^{n-1} (1 - |c_i|^2), \tag{4.5}$$

and

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} (-1)^{1+j} a_{1,j} M_{1,j} \\ &= a_{1,1} M_{1,1} - a_{1,2} M_{1,2} \\ &= (a_{1,1} - c_1 a_{1,2}) M_{1,1} \\ &= (a_{1,1} - |c_1|^2 a_{2,2}) M_{1,1} \\ &= (1 - |c_1|^2) M_{1,1} \\ &= (1 - |c_1|^2) a_{n,n} \prod_{i=2}^{n-1} (1 - |c_i|^2) \\ &= a_{n,n} \prod_{i=1}^{n-1} (1 - |c_i|^2), \end{aligned}$$

where we used (4.5), thereby showing that (4.4) holds for any $n \ge 1$. □

Proposition 4.4 *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$, and*

$$\varphi(s,t) = \alpha e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

We have

$$\det(\varphi(t_p, t_q))_{p,q=1}^{n} = \alpha^n \left(1 + \frac{\beta}{\alpha} e^{-2b(T-t_n)}\right) \prod_{i=1}^{n-1} (1 - e^{-2b(t_{i+1}-t_i)}), \tag{4.6}$$

where $0 \le t_1 \le \cdots \le t_n \le T$.

Proof. For simplification we take $\alpha = 1$ here and we prove (4.6) by applying Lemma 4.3 with

$$\varphi(s,t) = e^{-b|t-s|} + \beta e^{-b(2T-s-t)}, \qquad s,t \in [0,T].$$

Let $A = (a_{i,j})_{1\le i,j\le n}$, where

$$a_{i,j} = \varphi(t_i,t_j) = e^{-b|t_i-t_j|} + \beta e^{-b(2T-t_i-t_j))}, \quad 1 \le i,j \le n,$$

so that $c_j = e^{-b(t_{j+1}-t_j)}$, $1 \le j < n$, and $a_{j,j} = 1 + \beta e^{-2b(T-t_j)}$, $1 \le j \le n$, which shows that

$$a_{j,j} = 1 + \beta e^{-2b(T-t_1)} = (a_{j+1,j+1} - 1)c_j^2 + 1, \qquad 1 \le j < n.$$

Hence the assumptions of Lemma 4.3 are satisfied and we get

$$\det(A) = (1 + \beta e^{-2b(T-t_n)}) \prod_{i=1}^{n-1} (1 - e^{-2b(t_{i+1}-t_i)}),$$

with $t_{n+1} = T$, which proves (4.6). □

In particular when

$$\varphi(s,t) = e^{-b|t-s|} - e^{-b(2T-s-t)}, \qquad s,t \in [0,T],$$

we get

$$\det(\varphi(t_p,t_q))_{p,q=1}^n = 2^n e^{b(t_1-T)} \prod_{i=1}^{n} \sinh(b(t_{i+1}-t_i)),$$

$0 \le t_1 \le \cdots \le t_n \le T$, and if

$$\varphi(s,t) = e^{-b|s-t|}, \qquad s,t \in [0,T],$$

we find

$$\det(\varphi(t_p,t_q))_{p,q=1}^n = 2^{n-1} e^{b(t_1-t_n)} \prod_{i=1}^{n-1} \sinh(b(t_{i+1}-t_i)),$$

$0 \le t_1 \le \cdots \le t_n \le T$.

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