Computation of coverage probabilities in a spherical germ-grain model

Ian Flint and Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
637371 Singapore

September 10, 2019

Abstract

We consider a spherical germ-grain model on $\mathbb{R}^d$ in which the centers of the spheres are driven by a possibly non-Poissonian point process. We show that various covering probabilities can be expressed using the cumulative distribution function of the random radii on one hand, and distances to certain subsets of $\mathbb{R}^d$ on the other hand. This result allows us to compute the spherical and linear contact distribution functions, and to derive expressions which are suitable for numerical computation. Determinantal point processes are an important class of examples for which the relevant quantities take the form of Fredholm determinants.

Keywords: Boolean model; germ-grain model; capacity functional; multipoint probability function; determinantal point process.
Mathematics Subject Classification: 60G60; 60D05; 60G55.

1 Introduction

We consider a point process Φ on $\mathbb{R}^d$, i.e. a random locally finite subset of points in $\mathbb{R}^d$, and to each point $x_i \in \Phi$ we associate a random radius R_i which is independent of x_i, and forms a sequence of i.i.d. random variables distributed according to a given probability distribution μ. The spherical germ-grain model Ξ is the union of the Euclidean balls centered around the points $x_i \in \Phi$, with radii R_i. When Φ is a Poisson point process, Ξ is a Boolean model.

The study of random sets can be traced back to the 1930s (see [2, Section 6.1]) and the Boolean model has been thoroughly studied since its introduction in the 1970s (see Section 3 in [2] for a summary, and in particular Section 3.1.2 for a wide range of applications).

Lifting the Poissonian assumption introduces some technical difficulties, and indeed many formulas become more complicated without this assumption. In [2, Section 6.5.2], the authors present a formula for the capacity functional (see Section 3.1.5 therein) of general germ-grain models, and derive their two-point probability function

$$(z_1, z_2) \mapsto \mathbb{P}(z_1 \in \Xi,\ z_2 \in \Xi), \qquad z_1, z_2 \in \mathbb{R}^d,$$

assuming that their grains are non-overlapping. Other authors have studied germ-grain models, see e.g. [9] for their first rigorous construction, and [12] for several formulas for the computation of their capacity functionals on open balls of $\mathbb{R}^d$. The model has also been studied under additional assumptions on the distribution of Φ, see [1].

Our main motivation is to compute multipoint probability functions defined by

$$\mathbb{P}\big(\forall i \in \Theta_1,\ z_i \in \Xi,\ \forall j \in \Theta_2,\ z_j \notin \Xi\big), \qquad z_1, \dots, z_n \in \mathbb{R}^d, \tag{1}$$

where Θ_1 and Θ_2 are two disjoint subsets of $\{1, \dots, n\}$, see Theorem 1. Another aim of this work is to derive an expression for the capacity functional which can be implemented numerically, see Proposition 1 and its consequences.

The above quantity appears naturally when the spherical germ-grain model is used as a model for wireless networks, as it represents the probability that the nodes in Θ_1 are covered by a network of Radio Frequency (RF) sources while the ones in Θ_2 are not. For example, one of the main objectives of [7] is the explicit computation of such quantities in the case of the Boolean model in order to estimate the average throughput in wireless energy harvesting sensor networks, see for example § V therein.

An important class of point processes is given by the α-determinantal point processes introduced in [15], which are widely used in the modeling of wireless networks [6, 8, 17] due to their ability to model both repulsion and clustering among mobile users and base stations, see e.g. [5] and references therein. Therefore we are particularly interested in obtaining concise formulas when Φ is distributed as an α-determinantal point process.

In Theorem 1 below we show that the multipoint probability functions (1) can be expressed using the cumulative distribution function of μ and the moment generating functional of the point process Φ. From this result we deduce a formula for the capacity functional of the spherical germ-grain model at a given compact set Λ in terms of the Euclidean distance to Λ, see Proposition 1. Our results are written in a form which is suitable for numerical computation.

This result is then specialized in Proposition 1 as the expression (12) providing the capacity functional of the spherical germ-grain model. This leads to formulas for spherical and linear contact distribution functions in Corollary 1, again in terms of the moment generating functional of Φ. Our examples of application include the Bernoulli point process, Poisson point processes, and α-determinantal point processes treated in Section 4. In particular, when Φ is distributed as an α-determinantal point process, we exploit the fact that the moment generating functional is a Fredholm determinant and derive additional bounds on the increments of the contact distribution functions, see Section 4.3. More generally, our results can be applied to any point process with known moment generating functional, encompassing doubly stochastic Poisson point processes [3] and α-stable point processes [4].

Our main results are presented in Section 3, and specialized in Section 4 to some specific examples for which the moment generating functional is known.

2 Preliminaries on the spherical germ-grain model

We give in this section a short review of the spherical germ-grain model. Let $\|\cdot\|$ be the Euclidean distance, and denote by $\mathrm{B}(x, r) := \{y \in \mathbb{R}^d : \|x - y\| < r\}$ the open Euclidean ball of $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r \in [0, \infty)$. For a point $x \in \mathbb{R}^d$, we denote by $(x^{(1)}, \ldots, x^{(d)})$ the coordinates of x.

We consider a simple locally finite point process Φ on $\mathbb{R}^d$, and we denote the corresponding expectation by $\mathbb{E}$. For any Borel function $f : \mathbb{R}^d \to [0, 1]$ we define the moment generating functional of Φ at f as

$$\mathcal{G}_\Phi(f) := \mathbb{E}\left[\prod_{x\in\Phi} f(x)\right].$$

In the spherical germ-grain model driven by Φ, each point in Φ is the center of a Euclidean ball with random radius distributed according to a probability measure $\mu(\mathrm{d}x)$ on $[0, \infty)$ with cumulative distribution function $F_\mu(r) := \mu([0, r])$, independently of the other radii and of Φ. We let $(R_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables, independent of Φ, constructed on the same probability space and with same distribution μ, and we let $\{Y_i\}$ denote the points of Φ. We define the marked point process Ψ on $\mathbb{R}^d \times [0, \infty)$ as $\Psi := \{(Y_i, R_i)\}$. Each point $(x, r) \in \Psi$ models the location $x \in \mathbb{R}^d$ along with the radius $r \in [0, \infty)$ corresponding to the radius of the ball centered around it.

We consider the subset of $\mathbb{R}^d$ covered by the Euclidean balls centered around the points of the point process Φ, i.e.,

$$\Xi = \bigcup_{(x,r)\in\Psi} \mathrm{B}\,(x, r),$$

which consists of all points covered by at least one ball. We call the set Ξ the spherical germ-grain model, with Ψ as its driving point process.

Note that unlike in e.g. [2] and [14], the spherical germ-grain model that we consider is a random open set, however our results also hold when replacing the open Euclidean balls with the closed Euclidean balls, except for Proposition 1 and its consequences, which require the openness of Ξ. In addition, our setting is not restricted to Euclidean balls as it extends to non-Euclidean distances on $\mathbb{R}^d$ with minor modifications.

3 Main result

Our main result relies on the following well-known Lemmas 1 and 2 pertaining to the spherical germ-grain model. We start with the following result for the computation of the void probabilities of Ψ, see e.g. Table 1 page 16 of [10].

Lemma 1. *The void probabilities of Ψ are given for any Borel set B in $\mathbb{R}^d \times [0,\infty)$ by*

$$\mathbb{P}\big(\Psi \cap B = \emptyset\big) = \mathcal{G}_\Phi\bigg(\int_0^\infty \mathbb{1}_{B^c}(\cdot, r)\, \mu(\mathrm{d}r)\bigg), \tag{2}$$

where $\mathbb{1}_A$ denotes the indicator function of a set A, and B^c is the complement of the set B in $\mathbb{R}^d \times [0,\infty)$.

Next, the probability that a fixed point in $\mathbb{R}^d$ is covered by the spherical germ-grain model is computed in the following lemma.

Lemma 2. *The probability that a point $z \in \mathbb{R}^d$ belongs to Ξ is computed as*

$$\mathbb{P}\big(z \in \Xi\big) = 1 - \mathcal{G}_\Phi\big(F_\mu(\| \cdot - z\|)\big). \tag{3}$$

Proof. Given a point $z \in \mathbb{R}^d$ we consider the set

$$\mathcal{C}_z = \big\{(x,r) \in \mathbb{R}^d \times [0,\infty) : x \in \mathrm{B}\,(z,r)\big\}\,.$$

Then, we have

$$z \notin \Xi \iff \forall (x,r) \in \Psi,\ z \notin \mathrm{B}\,(x,r) \iff \forall (x,r) \in \Psi,\ x \notin \mathrm{B}\,(z,r) \iff \Psi \cap \mathcal{C}_z = \emptyset, \tag{4}$$

as well as

$$\int_0^\infty \mathbb{1}_{(\mathcal{C}_z)^c}(x,r)\, \mu(\mathrm{d}r) = \int_0^\infty \mathbb{1}_{\{\|x-z\| \geq r\}}\, \mu(\mathrm{d}r) = F_\mu\big(\|x-z\|\big), \quad x \in \mathbb{R}^d, \tag{5}$$

where we recall that F_μ is the cumulative distribution function of μ. We obtain

$$\mathbb{P}\big(z \in \Xi\big) = 1 - \mathbb{P}\big(\Psi \cap \mathcal{C}_z = \emptyset\big) = 1 - \mathcal{G}_\Phi\bigg(\int_0^\infty \mathbb{1}_{(\mathcal{C}_z)^c}(\cdot, r)\, \mu(\mathrm{d}r)\bigg) = 1 - \mathcal{G}_\Phi\big(F_\mu\big(\| \cdot - z\|\big)\big),$$

where the first equality follows from (2), the second one from (4) and the third one from (5). □

We note that the above coverage probability (3) generalizes the known result in the Poisson setting. Indeed, when Φ is the Poisson point process on $\mathbb{R}^d$ with the intensity measure $\lambda(\mathrm{d}x) = c\ell(\mathrm{d}x)$, for $c > 0$ a constant and $\ell(\mathrm{d}x)$ the Lebesgue measure, its moment generating functional is given by (15) for Borel $[0,1]$-valued functions f, and so (3) yields

$$\begin{aligned}
\mathbb{P}\big(z \notin \Xi\big) = \mathcal{G}_\Phi\big(F_\mu(\|\cdot - z\|)\big) &= \exp\left(-c\int_{\mathbb{R}^d}\big(1 - F_\mu(\|x - z\|)\big)\ell(\mathrm{d}x)\right) \\
&= \exp\left(-c\int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{\{x\in \mathrm{B}(z,r)\}}\,\ell(\mathrm{d}x)\mu(\mathrm{d}r)\right) \\
&= \exp\left(-cv_d\int_0^\infty r^d\,\mu(\mathrm{d}r)\right), \qquad (6)
\end{aligned}$$

where v_d denotes the volume of the d-dimensional unit ball.

Our main result is the following generalization of (3) to an arbitrary number of points $z_1, \ldots, z_n \in \mathbb{R}^d$. For all $x \in \mathbb{R}^d$ we take $\min_{k\in\emptyset} \|x - z_k\| = +\infty$ by convention. Note that this implies that when $\Theta_2 = \emptyset$, the summand corresponding to $\theta = \emptyset$ in (7) below is equal to one.

Theorem 1. *Let $n \geq 1$ be fixed and let $z_1, \ldots, z_n \in \mathbb{R}^d$ denote the (fixed) locations of n points. We consider two subsets of points $(z_i)_{i\in\Theta_1}$ and $(z_j)_{j\in\Theta_2}$, for $\Theta_1, \Theta_2 \subseteq \{1, \ldots, n\}$ such that $\Theta_1 \cap \Theta_2 = \emptyset$. We have*

$$\mathbb{P}\big(\forall i \in \Theta_1,\ z_i \in \Xi,\ \forall j \in \Theta_2,\ z_j \notin \Xi\big) = \sum_{\theta\,\subseteq\,\Theta_1} (-1)^{|\theta|}\mathcal{G}_\Phi\Big(F_\mu\Big(\min_{k\in\theta\cup\Theta_2} \|\cdot - z_k\|\Big)\Big), \qquad (7)$$

where $|\theta|$ denotes the cardinal of the set θ, and F_μ is the cumulative distribution function of μ.

Proof. We have

$$\begin{aligned}
&\mathbb{P}\big(\forall i \in \Theta_1,\ z_i \in \Xi,\ \forall j \in \Theta_2,\ z_j \notin \Xi\big) \\
&\quad = \mathbb{P}\big(\forall i \in \Theta_1,\ z_i \in \Xi \mid \forall j \in \Theta_2,\ z_j \notin \Xi\big)\cdot \mathbb{P}\big(\forall j \in \Theta_2,\ z_j \notin \Xi\big) \\
&\quad = \left(1 - \mathbb{P}\left(\bigcup_{i\in\Theta_1}\{z_i \notin \Xi\} \mid \forall j \in \Theta_2,\ z_j \notin \Xi\right)\right)\cdot \mathbb{P}\big(\forall j \in \Theta_2,\ z_j \notin \Xi\big) \\
&\quad = \left(1 + \sum_{\theta\,\subseteq\,\Theta_1,\ \theta\neq\emptyset} (-1)^{|\theta|}\mathbb{P}\big(\forall i \in \theta,\ z_i \notin \Xi \mid \forall j \in \Theta_2,\ z_j \notin \Xi\big)\right)\cdot \mathbb{P}\big(\forall j \in \Theta_2,\ z_j \notin \Xi\big) \qquad (8)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\theta \subseteq \Theta_1} (-1)^{|\theta|} \mathbb{P}\big(\forall i \in \theta,\ z_i \notin \Xi,\ \forall j \in \Theta_2,\ z_j \notin \Xi\big) \\
&= \sum_{\theta \subseteq \Theta_1} (-1)^{|\theta|} \mathbb{P}\big(\forall i \in \theta \cup \Theta_2,\ z_i \notin \Xi\big), \qquad (9)
\end{aligned}
$$

where we have used the inclusion-exclusion formula in (8). At this point we note that when $\Theta_2 = \emptyset$, it is clear that the summand corresponding to $\theta = \emptyset$ in (9) is equal to one. Letting $\theta \subseteq \Theta_1$ be fixed, each of the summands in the above equation is computed as

$$
\begin{aligned}
\mathbb{P}\big(\forall i \in \theta \cup \Theta_2,\ z_i \notin \Xi\big) &= \mathbb{P}\big(\big\{\forall (x,r) \in \Psi,\ \forall i \in \theta \cup \Theta_2,\ z_i \notin \mathrm{B}\,(x,r)\big\}\big) \\
&= \mathbb{P}\big(\big\{\forall (x,r) \in \Psi,\ \forall i \in \theta \cup \Theta_2,\ x \notin \mathrm{B}\,(z_i,r)\big\}\big) \\
&= \mathbb{P}\big(\big\{\forall (x,r) \in \Psi,\ x \notin \cup_{i \in \theta \cup \Theta_2} \mathrm{B}\,(z_i,r)\big\}\big) \\
&= \mathbb{P}\big(\Psi \cap A = \emptyset\big), \qquad (10)
\end{aligned}
$$

where

$$
A := \Big\{(x,r) \in \mathbb{R}^d \times [0,\infty) : x \in \bigcup_{i \in \theta \cup \Theta_2} \mathrm{B}\,(z_i,r)\Big\}.
$$

Additionally, we have

$$
\int_0^\infty \mathbb{1}_{A^c}(\cdot, r)\,\mu(\mathrm{d}r) = \int_0^\infty \mathbb{1}_{\{\forall i \in \theta \cup \Theta_2,\ \|\cdot - z_i\| \geq r\}}\,\mu(\mathrm{d}r) = F_\mu\Big(\min_{i \in \theta \cup \Theta_2} \|\cdot - z_i\|\Big). \qquad (11)
$$

By applying (11) and Lemma 1 to compute (10), we find

$$
\mathbb{P}\big(\forall i \in \theta \cup \Theta_2,\ z_i \notin \Xi\big) = \mathcal{G}_\Phi\Big(F_\mu\Big(\min_{i \in \theta \cup \Theta_2} \|\cdot - z_i\|\Big)\Big).
$$

The proof is concluded by plugging the above equation into (9). □

In Figure 1 below we plot the two-point probability functions $\mathbb{P}\big(z_1 \in \Xi, z_2 \in \Xi\big)$ for a stationary Poisson point process and a standard Ginibre point process [6], both restricted to $\mathrm{B}\,(0,10)$ with the same intensity $1/\pi$, where $\mu(\mathrm{d}r) = \exp(-r)\ \mathrm{d}r$. Since both point processes are stationary and isotropic, we have fixed z_1 at the origin, and plotted the resulting two-point probability function as a function of $\|z_1 - z_2\|$.

As expected from the repulsive nature of the determinantal point process, the coverage probability of a single point (when $\|z_1 - z_2\| = 0$ on the graph) is higher when Φ is a Ginibre point

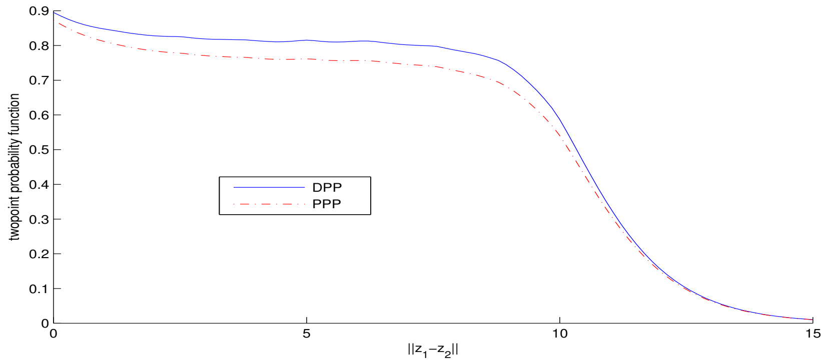

Figure 1: Comparison of the two-point probability functions $\mathbb{P}(z_1 \in \Xi, z_2 \in \Xi)$ of a Poisson point process (red dot-dashed line) and a Ginibre Determinantal Point Process (blue solid line).

process. In addition, this covering property is preserved when $\|z_1 - z_2\|$ increases, i.e. two given points have a larger probability of being simultaneously covered when Φ is a Ginibre point process than when Φ is a Poisson point process.

Interpreting the points $z_1, \ldots, z_n \in \mathbb{R}^d$ appearing in Theorem 1 as n nodes in a wireless network, the theorem provides in computable form the probability that the nodes with indices in Θ_1 are covered by the spherical germ-grain model while the nodes in Θ_2 are not. Such quantities have been shown to be of interest in the study of sensor networks, see [7].

For example, when Φ is the Poisson point process on $\mathbb{R}^d$ with the intensity measure $\lambda(\mathrm{d}x)$, each term in (7) considerably simplifies since

$$\begin{aligned}
\mathcal{G}_\Phi\Big(F_\mu\Big(\min_{i\in\theta\cup\Theta_2} \|x - z_i\|\Big)\Big) &= \exp\Big(-\int_{\mathbb{R}^d}\Big(1 - F_\mu\Big(\min_{i\in\theta\cup\Theta_2}\|x - z_i\|\Big)\Big)\lambda(\mathrm{d}x)\Big) \\
&= \exp\Big(-\int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{\{r > \min_{i\in\theta\cup\Theta_2}\|x - z_i\|\}}\lambda(\mathrm{d}x)\mu(\mathrm{d}r)\Big) \\
&= \exp\Big(-\int_0^\infty \lambda\big(\cup_{i\in\theta\cup\Theta_2}\mathrm{B}\left(z_i, r\right)\big)\,\mu(\mathrm{d}r)\Big),
\end{aligned}$$

where in the above derivation we have used arguments similar to those in (6) above. The formula (7) thus extends the results proved in [7, Theorem 2] to the case where the nodes $(z_i)_{1\le i\le n}$ are not necessarily aligned.

In particular when $n = 2$, $\Theta_1 = \{1,2\}$ and $\lambda(\mathrm{d}x) = c\,\ell(\mathrm{d}x)$ for $c > 0$ a constant and $\ell(\mathrm{d}x)$ the Lebesgue measure, Theorem 1 yields the two-point probability function (see [2, Section 3.1.6])

$$\begin{aligned}
&\mathbb{P}\big(z_1 \in \Xi, z_2 \in \Xi\big) \\
&\qquad = 1 - 2\exp\left(-cv_d \int_0^\infty r^d\,\mu(\mathrm{d}r)\right) + \exp\left(-c\int_0^\infty \ell\big(\mathrm{B}\,(z_1,r) \cup \mathrm{B}\,(z_2,r)\big)\,\mu(\mathrm{d}r)\right) \\
&\qquad = 1 - 2\exp\left(-cv_d \int_0^\infty r^d\,\mu(\mathrm{d}r)\right) \\
&\qquad\quad + \exp\left(-2cv_d \int_0^\infty r^d\,\mu(\mathrm{d}r) + c\int_{\|z_1-z_2\|/2}^\infty \ell\big(\mathrm{B}\,(z_1,r) \cap \mathrm{B}\,(z_2,r)\big)\,\mu(\mathrm{d}r)\right) \\
&\qquad = 1 - 2\exp\left(-cv_d \int_0^\infty r^d\,\mu(\mathrm{d}r)\right) \\
&\qquad\quad + \exp\left(-2cv_d \int_0^\infty r^d\,\mu(\mathrm{d}r) + cv_d\int_{\|z_1-z_2\|/2}^\infty I_{1-\|z_1-z_2\|^2/(4r^2)}\Big(\frac{d+1}{2},\frac{1}{2}\Big) r^d\,\mu(\mathrm{d}r)\right),
\end{aligned}$$

where I_x denotes the incomplete beta function. This also recovers Theorem 1 in [7] for the Boolean model.

In the next Proposition 1 we derive a general expression for the capacity functional of the spherical germ-grain model as a corollary of Theorem 1. This expression is simple to implement numerically, especially when the moment generating functional is easily computable, see Section 4 for examples. Other related expressions for the capacity functional have been obtained for germ-grain models, see e.g. [9, eq. (2.5)] and [2, eq. (6.96)], however, contrary to Proposition 1 these assume that Ξ is a closed set.

Proposition 1. *For any compact set $\Lambda \subset \mathbb{R}^d$ we have*

$$\mathbb{P}\big(\Xi \cap \Lambda \neq \emptyset\big) = 1 - \mathcal{G}_\Phi\big(F_\mu\big(\mathrm{d}(\cdot,\Lambda)\big)\big), \tag{12}$$

where

$$\mathrm{d}(x,\Lambda) := \inf_{y\in\Lambda} \|x-y\|, \quad x \in \mathbb{R}^d.$$

Proof. Let $\Lambda \subset \mathbb{R}^d$ be a compact set, and denote by $(z_n)_{n>0}$ a countable set which is dense in Λ. Since Ξ is a (random) open set, we have equality between the event $\{\forall x \in \Lambda,\ x \notin \Xi\}$ and

$\{\forall n > 0,\ z_n \notin \Xi\}$. Thus, by Theorem 1 we have

$$\mathbb{P}\big(\Xi \cap \Lambda = \emptyset\big) = \mathbb{P}\big(\forall x \in \Lambda,\ x \notin \Xi\big) = \mathbb{P}\Bigg(\bigcap_{n>0} \big\{z_1 \notin \Xi, \ldots, z_n \notin \Xi\big\}\Bigg)$$
$$= \lim_{n\to\infty} \mathbb{P}\big(z_1 \notin \Xi, \ldots, z_n \notin \Xi\big)$$
$$= \lim_{n\to\infty} \mathcal{G}_\Phi\Big(F_\mu\Big(\min_{k=1,\ldots,n} \|\cdot - z_k\|\Big)\Big). \tag{13}$$

By the right-continuity of F_μ and the density of $(z_n)_{n>0}$ in Λ, for any fixed $x \in \mathbb{R}^d$ we have

$$F_\mu\Big(\min_{k=1,\ldots,n} \|x - z_k\|\Big) \xrightarrow[n\to\infty]{} F_\mu\Big(\min_{k\in\mathbb{N}} \|x - z_k\|\Big) = F_\mu\big(\mathrm{d}(x, \Lambda)\big),$$

and so by the monotone convergence theorem, we obtain

$$\prod_{x\in\Phi} F_\mu\Big(\min_{k=1,\ldots,n} \|x - z_k\|\Big) = \exp\Bigg(-\sum_{x\in\Phi} -\log\Big(F_\mu\Big(\min_{k=1,\ldots,n} \|x - z_k\|\Big)\Big)\Bigg)$$
$$\xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} \exp\Bigg(-\sum_{x\in\Phi} -\log\big(F_\mu\big(\mathrm{d}(x, \Lambda)\big)\big)\Bigg) = \prod_{x\in\Phi} F_\mu\big(\mathrm{d}(x, \Lambda)\big).$$

Combining the above with the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathcal{G}_\Phi\Big(F_\mu\Big(\min_{k=1,\ldots,n} \|\cdot - z_k\|\Big)\Big) = \mathcal{G}_\Phi\big(F_\mu\big(\mathrm{d}(x, \Lambda)\big)\big),$$

and plugging this into (13) concludes the proof. □

In the next corollary we compute the capacity functional of Proposition 1 for a class of compact sets of the form

$$\mathcal{A}_p(r) := \Big\{\big(x^{(1)}, \ldots, x^{(p)}, 0, \ldots, 0\big) \in \mathbb{R}^d : \sqrt{\big(x^{(1)}\big)^2 + \cdots + \big(x^{(p)}\big)^2} \le r\Big\}, \quad 1 \le p \le d,$$

made of p-dimensional balls in $\mathbb{R}^d$.

Corollary 1. *For any $p \in \{1, \ldots, d\}$, we have*

$$\mathbb{P}\big(\Xi \cap A_p(r) \neq \emptyset\big) = 1 - \mathcal{G}_\Phi\big(\, g_{r,p}\,\big), \quad r \ge 0,$$

where for $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$, we let

$$g_{r,p}\,(x) := F_\mu\Bigg(\sqrt{\max\Big\{\sqrt{\big(x^{(1)}\big)^2 + \cdots + \big(x^{(p)}\big)^2} - r, 0\Big\}^2 + \big(x^{(p+1)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\Bigg).$$

The above corollary yields an expression for the spherical and linear contact distribution functions defined in [2, Section 3.1.7]. Indeed, the spherical contact distribution function $H_{\mathrm{s}}(r)$ defined by

$$H_{\mathrm{s}}(r) := \mathbb{P}\big(\Xi \cap \overline{\mathrm{B}\,(0,r)} \neq \emptyset\big), \quad r \geq 0$$

is obtained with $p = d$ as

$$H_{\mathrm{s}}(r) = 1 - \mathcal{G}_{\Phi}\big(F_{\mu}\big(\max\ \{\ \|\cdot\| - r, 0\ \}\ \big)\big), \quad r \geq 0.$$

Similarly, the linear contact distribution function $H_{\mathrm{l}}(r)$ is given for $p = 1$ by

$$H_{\mathrm{l}}(r) := \mathbb{P}\big(\Xi \cap \mathcal{L}(r) \neq \emptyset\big) = 1 - \mathcal{G}_{\Phi}\big(h_r\big), \quad r \geq 0,$$

with

$$\mathcal{L}(r) = \big\{(t, 0, \ldots, 0) \in \mathbb{R}^d, \ \text{ where } \ t \in [-r, r]\big\}, \tag{14}$$

and for $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$, we define

$$h_r(x) = F_{\mu}\left(\sqrt{\max\ \{\ \big|x^{(1)}\big| - r, 0\ \}^{\ 2} + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\right).$$

4 Examples

4.1 Bernoulli point process

By a Bernoulli point process we mean a binomial point process with one point, i.e. a point process which has no points with probability $p \in [0,1]$ and one point with probability $1-p$, distributed according to a given probability measure $\nu(\mathrm{d}x)$ on $\mathbb{R}^d$, see e.g. [11], pages 27-28. When Φ is a Bernoulli point process, its moment generating functional is given for measurable non-negative functions f by

$$\mathcal{G}_{\Phi}(f) = p + (1-p)\int_{\mathbb{R}^d} f(x)\,\nu(\mathrm{d}x).$$

In the next proposition, we specialize Proposition 1 to this point process.

Proposition 2. *Assume that Φ is a Bernoulli point process characterized by $p \in [0,1]$ and the probability measure $\nu(\mathrm{d}x)$ on $\mathbb{R}^d$. Then, for any compact set $\Lambda \subset \mathbb{R}^d$, the capacity functional is given by*

$$\mathbb{P}\big(\Xi \cap \Lambda \neq \emptyset\big) = (1-p)\int_{\mathbb{R}^d}\big(1 - F_{\mu}\big(\mathrm{d}(x,\Lambda)\big)\big)\,\nu(\mathrm{d}x).$$

The above proposition can obviously be recovered by using the definition of the point process to compute $\mathbb{P}(\Xi \cap \Lambda \neq \emptyset)$ directly. We also remark that Proposition 2 can readily be generalized to a binomial point process with n points, i.e. a point process with at most n independent points distributed according to ν, each point appearing with the probability p_i, $i \in \{1, \dots, n\}$.

4.2 Poisson point process

When Φ is distributed as the Poisson point process on $\mathbb{R}^d$ with the intensity measure $\lambda(\mathrm{d}x)$, we recall that its moment generating functional is given for Borel $[0,1]$-valued functions f by

$$\mathcal{G}_\Phi(f) = \exp\Big(-\int_{\mathbb{R}^d}\big(1 - f(x)\big)\lambda(\mathrm{d}x)\Big). \tag{15}$$

In the next proposition, we specialize Proposition 1 to the Poissonian setting.

Proposition 3. *Assume that Φ is the Poisson point process on $\mathbb{R}^d$ with the intensity measure $\lambda(\mathrm{d}x)$. Then, for any compact set $\Lambda \subset \mathbb{R}^d$, the capacity functional is given by*

$$\mathbb{P}\big(\Xi \cap \Lambda \neq \emptyset\big) = 1 - \exp\Big(-\int_{\mathbb{R}^d}\big(1 - F_\mu\big(\mathrm{d}(x,\Lambda)\big)\big)\lambda(\mathrm{d}x)\Big). \tag{16}$$

Note that (16) is more explicit than its counterpart equation (3.7) on p. 72 of [2].

4.3 α-determinantal point processes

The class of α-determinantal point processes [15] models a wide range of phenomena [6, 8, 13] while having tractable statistics. We recall here some simple facts required for our purposes and refer the reader to [15] for the details. Let $\alpha \in \{2/m : m \in \mathbb{N}\} \cup \{-1/m : m \in \mathbb{N}\}$ and let K be a bounded symmetric integral operator on the space $L^2(\mathbb{R}^d)$ of square-integrable functions on $\mathbb{R}^d$, which is assumed to be locally of trace class. In the following, we identify K with its kernel which is a function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{C}$. The moment generating functional of the α-determinantal point process on $\mathbb{R}^d$ with the kernel K with respect to a reference measure $\lambda(\mathrm{d}x)$ is given for $[0,1]$-valued Borel functions f with compact support by $\mathrm{Det}\,(\mathrm{Id} + \alpha K_f)^{-1/\alpha}$, where Det stands for the Fredholm determinant, see e.g. Chapter 3 of [16], and the kernel K_f is defined as

$$K_f(x,y) := \sqrt{1 - f(x)}\; K(x,y)\; \sqrt{1 - f(y)}, \quad x, y \in \mathbb{R}^d.$$

In the next proposition, we specialize Proposition 1 to the α-determinantal setting.

Proposition 4. *Assume that Φ is an α-determinantal point process on $\mathbb{R}^d$ with a kernel K with respect to a reference measure $\lambda(\mathrm{d}x)$. Assume further that either*

(i) the radii of the balls in Ξ are bounded, so that $F_\mu(r) = 1$ for a sufficiently large r; or

(ii) the α-determinantal point processes Φ is restricted to a compact set.

Then, for any compact set $\Lambda \subset \mathbb{R}^d$, the capacity functional is given by

$$\mathbb{P}\big(\Xi \cap \Lambda \neq \emptyset\big) = 1 - \mathrm{Det}\,(\mathrm{Id} + \alpha K_\Lambda)^{-1/\alpha}, \tag{17}$$

where K_Λ is the kernel defined as

$$K_\Lambda(x,y) := \sqrt{1 - F_\mu\big(\mathrm{d}(x,\Lambda)\big)}\; K(x,y)\; \sqrt{1 - F_\mu\big(\mathrm{d}(y,\Lambda)\big)}, \quad x, y \in \mathbb{R}^d.$$

When Φ is a determinantal point process (i.e. $\alpha = -1$) and the eigenvalues $(\lambda_n)_{n\geq 1}$ of K_Λ are known, the relation (17) allows us to write the capacity functional as $1 - \prod_{n\geq 1}(1 - \lambda_n)$. In case the eigenvalues of K_Λ are unknown, the capacity functional (17) still yields bounds of interest in specific cases, using Fredholm determinants.

Proposition 5. *Let the setting and assumptions of Proposition 4 prevail. Assume further that $\alpha = -1$, i.e. that the point process is a determinantal point process, that the kernel K is a continuous function on $\mathbb{R}^d \times \mathbb{R}^d$, and that the distribution function F_μ is continuous on $\mathbb{R}^d$. Then, for any compact sets $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ such that $\Lambda_1 \subseteq \Lambda_2$, we have*

$$\begin{aligned}\big|\mathbb{P}\big(\Xi \cap \Lambda_1 \neq \emptyset\big) - \mathbb{P}\big(\Xi \cap \Lambda_2 \neq \emptyset\big)\big| \leq \int_{\mathbb{R}^d} \big(F_\mu\big(\mathrm{d}(x,\Lambda_1)\big) - F_\mu\big(\mathrm{d}(x,\Lambda_2)\big)\big) K(x,x)\,\lambda(\mathrm{d}x) \\ \times \exp\left(1 + \int_{\mathbb{R}^d} \big(1 - F_\mu\big(\mathrm{d}(x,\Lambda_2)\big)\big) K(y,y)\,\lambda(\mathrm{d}y)\right).\end{aligned} \tag{18}$$

Proof. By [16, p. 45] we have

$$\big|\mathrm{Det}\,(\mathrm{Id} - K_{\Lambda_1}) - \mathrm{Det}\,(\mathrm{Id} - K_{\Lambda_2})\big| \leq \big\|K_{\Lambda_2} - K_{\Lambda_1}\big\|_1 \exp\big(1 + \max\,\{\ \|K_{\Lambda_1}\|_1, \|K_{\Lambda_2}\|_1\ \}\,\big), \tag{19}$$

where $\|\cdot\|_1$ is the trace-norm. Since $\Lambda_1 \subseteq \Lambda_2$, we see that $K_{\Lambda_2} - K_{\Lambda_1}$ is again a non-negative trace-class operator with continuous kernel $K_{\Lambda_2}(x,y) - K_{\Lambda_1}(x,y)$. Thus, we get

$$\big\|K_{\Lambda_2} - K_{\Lambda_1}\big\|_1 = \int_{\mathbb{R}^d} \big(F_\mu\big(\mathrm{d}(x,\Lambda_1)\big) - F_\mu\big(\mathrm{d}(x,\Lambda_2)\big)\big) K(x,x)\,\lambda(\mathrm{d}x). \tag{20}$$

Additionally, we have $\mathrm{d}(x, \Lambda_1) \geq \mathrm{d}(x, \Lambda_2)$ and thus

$$\max \{ \|K_{\Lambda_1}\|_1, \|K_{\Lambda_2}\|_1 \} = \int_{\mathbb{R}^d} \big(1 - F_\mu\big(\mathrm{d}(x, \Lambda_2)\big)\big) K(x,x)\, \lambda(\mathrm{d}x). \tag{21}$$

Combining equations (19), (20) and (21) with the result of Proposition 4, we obtain the upper bound (18). □

In the next corollary we note that the condition (22) holds e.g. if μ has a bounded density.

Corollary 2. *Let the setting and assumptions of Proposition 5 prevail and assume additionally that there exists a number $C > 0$ such that*

$$\big|F_\mu(s) - F_\mu(t)\big| \leq C|s - t|, \quad s, t \geq 0. \tag{22}$$

We have the following bound on the increments of the spherical contact distribution function:

$$\begin{aligned}\big|H_\mathrm{s}(r_1) - H_\mathrm{s}(r_2)\big| \leq C\big|r_2 - r_1\big| &\int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_1)} K(x,x)\,\lambda(\mathrm{d}x) \\ &\times \exp\bigg(1 + \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} \big(1 - F_\mu\big(\|y\| - r_2\big)\big) K(y,y)\,\lambda(\mathrm{d}y)\bigg), \quad r_1 < r_2, \end{aligned} \tag{23}$$

where $M \in [0, \infty]$ is a constant which upper bounds the radii, i.e. $F_\mu(r) = 1$ for all $r \geq M$.

Proof. Applying Proposition 5 with $\Lambda_1 = \mathrm{B}(0, r_1)$ and $\Lambda_2 = \mathrm{B}(0, r_2)$ for some $r_2 > r_1$, we obtain

$$\begin{aligned}
&\big|H_\mathrm{s}(r_1) - H_\mathrm{s}(r_2)\big| \\
&\leq \int_{\mathbb{R}^d} \big(F_\mu\big(\max\{ \|x\| - r_1, 0 \}\big) - F_\mu\big(\max\{ \|x\| - r_2, 0 \}\big)\big) K(x,x)\,\lambda(\mathrm{d}x) \\
&\qquad \times \exp\bigg(1 + \int_{\mathbb{R}^d} \big(1 - F_\mu\big(\max\{ \|y\| - r_2, 0 \}\big)\big) K(y,y)\,\lambda(\mathrm{d}y)\bigg) \\
&\leq \int_{\mathrm{B}(0,r_2)\setminus \mathrm{B}(0,r_1)} F_\mu\big(\|x\| - r_1\big) K(x,x)\,\lambda(\mathrm{d}x) + C\big(r_2 - r_1\big) \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} K(x,x)\,\lambda(\mathrm{d}x) \\
&\qquad \times \exp\bigg(1 + \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} \big(1 - F_\mu\big(\|x\| - r_2\big)\big) K(y,y)\,\lambda(\mathrm{d}y)\bigg) \\
&\leq \int_{\mathrm{B}(0,r_2)\setminus \mathrm{B}(0,r_1)} C\big(\|x\| - r_1\big) K(x,x)\,\lambda(\mathrm{d}x) + C\big(r_2 - r_1\big) \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} K(x,x)\,\lambda(\mathrm{d}x) \\
&\qquad \times \exp\bigg(1 + \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} \big(1 - F_\mu\big(\|x\| - r_2\big)\big) K(x,x)\,\lambda(\mathrm{d}x)\bigg)
\end{aligned}$$

$$\le C\big(r_2 - r_1\big) \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_1)} K(x,x)\,\lambda(\mathrm{d}x)$$
$$\times \exp\left(1 + \int_{\mathrm{B}(0,M+r_2)\setminus \mathrm{B}(0,r_2)} \big(1 - F_\mu\big(\|x\| - r_2\big)\big) K(x,x)\,\lambda(\mathrm{d}x)\right),$$

which concludes the proof of (23). □

Corollary 3. *Let the setting and assumptions of Corollary 2 prevail. We have the following bound on the increments of the linear contact function:*

$$\big|H_l(r_1) - H_l(r_2)\big| \le C\big|r_2 - r_1\big| \int_{\mathrm{B}(0,M+r_2)} K(x,x)\,\lambda(\mathrm{d}x)$$
$$\times \exp\left(1 + \int_{\mathrm{B}(0,M+r_2)} \big(1 - F_\mu\big(\|y\| - r_2\big)\big) K(y,y)\,\lambda(\mathrm{d}y)\right), \quad r_1 < r_2, \tag{24}$$

where we recall that C is defined by (22) and $M \in [0,\infty]$ is a constant which upper bounds the radii.

Proof. Applying Proposition 5 with $\Lambda_1 = \mathcal{L}(r_1)$ and $\Lambda_2 = \mathcal{L}(r_2)$, where $\mathcal{L}$ is defined in (14), for any $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$ we have

$$\begin{aligned}
F_\mu\big(\mathrm{d}(x,\Lambda_1)\big) - F_\mu\big(\mathrm{d}(x,\Lambda_2)\big) &= F_\mu\left(\sqrt{\max\ \left\{\ \big|x^{(1)}\big| - r_1, 0\ \right\}^2 + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\right) \\
&\quad - F_\mu\left(\sqrt{\max\ \left\{\ \big|x^{(1)}\big| - r_2, 0\ \right\}^2 + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\right) \\
&\le C\left(\sqrt{\max\ \left\{\ \big|x^{(1)}\big| - r_1, 0\ \right\}^2 + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\right. \\
&\quad \left. - \sqrt{\max\ \left\{\ \big|x^{(1)}\big| - r_2, 0\ \right\}^2 + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}\right) \\
&\le C\big(r_2 - r_1\big), \quad r_1 < r_2,
\end{aligned}$$

where the last inequality follows since $g(r) := \sqrt{\max\ \left\{\ \big|x^{(1)}\big| - r, 0\ \right\}^2 + \big(x^{(2)}\big)^2 + \cdots + \big(x^{(d)}\big)^2}$ is continuously differentiable with $g'(r) \in [-1, 0]$. Along with the fact that $F_\mu(\mathrm{d}(y,\Lambda_2)) = 1$ for $\|y\| \ge M + r_2$, this implies (24). □

Acknowledgments

The authors would like to thank the two anonymous referees for their suggestions which have greatly improved an earlier draft of this paper.

References

[1] Jean-Christophe Breton, Adrien Clarenne, and Renan Gobard. Macroscopic analysis of determinantal random balls. *Bernoulli*, 25:1568–1601, 2019.

[2] Sung Nok Chiu, Dietrich Stoyan, Wilfrid S. Kendall, and Joseph Mecke. *Stochastic Geometry and its Applications.* Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester, third edition, 2013.

[3] Daryl J. Daley and David Vere-Jones. *An Introduction to the Theory of Point Processes. Vol. I.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2003.

[4] Youri Davydov, Ilya Molchanov, and Sergei Zuyev. Strictly stable distributions on convex cones. *Electron. J. Probab.*, 13:259–32, 2008.

[5] Laurent Decreusefond, Ian Flint, Nicolas Privault and Giovanni Torrisi. Determinantal point processes. In G. Peccati and M. Reitzner, editors, *Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Itô Chaos Expansions and Stochastic Geometry*, volume 7 of *Bocconi & Springer Series*, pages 311–342, Berlin, 2016. Springer.

[6] Na Deng, Wuyang Zhou, and Martin Haenggi. The Ginibre point process as a model for wireless networks with repulsion. *IEEE Trans. on Wireless Comm.*, 14(1):107–121, 2014.

[7] Ian Flint, Han-Bae Kong, Nicolas Privault, Wang Ping, and Niyato Dusit. Wireless energy harvesting sensor networks: Boolean-Poisson modeling and analysis. *IEEE Trans. on Wireless Comm.*, 16(11):7108–7122, 2017.

[8] Ian Flint, Xiao Lu, Nicolas Privault, Dusit Niyato, and Ping Wang. Performance analysis of ambient RF energy harvesting: A stochastic geometry approach. In *IEEE Global Communications Conference (GLOBECOM 2014)*, 1448-1453, 2014.

[9] Lothar Heinrich. On existence and mixing properties of germ-grain models. *Statistics*, 23(3):271–286, 1992.

[10] Olav Kallenberg. *Random Measures.* Akademie-Verlag, Berlin, fourth edition, 1986.

[11] Alan F. Karr. *Point processes and their statistical inference.* M. Dekker, New York, Fourth Edition, 1986.

[12] Günter Last and Markus Holtmann. On the empty space function of some germ-grain models. *Pattern Recognition*, 32(9):1587–1600, 1999.

[13] Frédéric Lavancier, Jesper Møller, and Ege Rubak. Determinantal point process models and statistical inference. *Journal of the Royal Statistical Society, Series B*, 77:853–877, 2015.

[14] Georges Matheron. *Random Sets and Integral Geometry.* Wiley, 1975.

[15] Tomoyuki Shirai and Yoichiro Takahashi. Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J. Funct. Anal.*, 205(2):414–463, 2003.

[16] Barry Simon. *Trace Ideals and their Applications*, volume 120 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2005.

[17] Anaïs Vergne, Ian Flint, Laurent Decreusefond, and Philippe Martins. Disaster recovery in wireless networks: A homology-based algorithm. In *ICT 2014*, pages 226–230, Lisbon, Portugal, 2014.