A characterization of grand canonical Gibbs measures by duality

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Abstract

We introduce Skorohod type integral operators that satisfy an integration by parts formula under Gibbs measures and obtain a characterization of grand canonical Gibbs measures by duality, without use of a differential structure on the underlying configuration space.

Key words: Gibbs measures, Poisson measures, integration by parts.
Mathematics Subject Classification (1991): 60G60, 60K35, 82C22, 60H07, 60H20.

1 Introduction

Integration by parts formulas are important in stochastic analysis, especially when they are formulated with help of gradient and divergence operators. Their applications include criteria for existence and smoothness of densities and extensions of stochastic calculus to anticipating integrands. In this paper we present a duality formula using gradient and divergence operators under Gibbs measures, and obtain in this way a characterization of grand canonical Gibbs measures on a metric space X. Our duality formula uses Skorohod type integral operators, and a gradient that acts by finite differences. No differential structure is needed on X.

Poisson measures have been characterized using Campbell measures in [8], cf. also [6], and the Wiener measure has been characterized by integration by parts, cf. [16] and [17], Th. 1.2. The characterization of Poisson measures has been extended to Gibbs measures in [10], [7]. Gibbs measures on the Wiener pathspace have been considered in [17] in the case of the fixed lattice $\mathbb{Z}^d$ and in [18] in the case of random Poisson lattices. A characterization of Poisson measures has been formulated in terms of duality between finite difference operators and Skorohod integral operators in [15].

On the other hand, a characterization of mixed Poisson measures (or Gibbs canonical free case) by integration by parts has been obtained in [1], and extended to canonical Gibbs measures in [2], cf. also [4]. This result uses true differential operators, defined by infinitesimal shifts of configurations. However, this characterization does not extend to grand canonical Gibbs measures, as mentioned after Remark 4.6 of [2]. This is natural because in the canonical free case, the conditions imposed by this integration by parts formula are characteristic of mixed Poisson measures (which are a particular case of canonical Gibbs measures, cf. [2], [9], [5]), not of Poisson measures.

The main difficulty associated to finite difference operators is that they do not satisfy a chain rule of differentiation, and do not allow to do functional calculus. However, as pointed out in [12], some type of finite difference functional calculus is possible for exponential functionals, i.e. we have $D\exp(F) = \exp(F)(\exp(DF)-1)$ if D is a finite difference operator. This fact allows us to construct an integration by parts formula using finite difference operators since Gibbs measures are constructed via conditional densities which are given as exponential functionals. The use of the duality between gradient and divergence simplifies the proof of the characterization result. More generally, this paper suggests a definition of Skorohod integrals under measures whose densities with respect to the Poisson measure are given by exponential functionals.

In Sect. 2 we introduce some notation on configuration spaces and finite difference operators. Integration by parts under Poisson measures is considered in Sect. 3. A Skorohod type integral operator, which is in duality with the finite difference gradient under grand canonical Gibbs measures, is defined in Sect. 4. Sect. 5 contains a characterization of grand canonical Gibbs measures by duality.

2 Configuration spaces and finite difference operators

We refer to [1], [3], [11], for the analysis on configuration spaces under Poisson measures. Let X be a metric space and let the configuration space Γ^X on X be the set of Radon measures on X of the form $\sum_{i=1}^{i=N} \varepsilon_{x_i}$, with $(x_i)_{i=1}^{i=N} \subset X$, $x_i \neq x_j$, $\forall i \neq j$, and $N \in \mathbb{N} \cup \{\infty\}$, where ε_x denotes the Dirac measure at $x \in X$. The fact that all $\gamma \in \Gamma^X$ is finite on compact subsets of X implies that all sums and products considered in this paper will be finite. We endow Γ^X with the vague topology and associated σ-algebra $\mathcal{B}(X)$, cf. [1], and as a convention we identify $\gamma \in \Gamma^X$ with its support. Let $|\gamma| = \gamma(X)$ denote the cardinal of $\gamma \in \Gamma^X$, let $d(\gamma)$ denote the diameter of $\gamma \in \Gamma^X$, and let $d(x, \Lambda)$ denote the distance from $x \in X$ to $\Lambda \subset X$. Let $\mathcal{B}(X)$, resp. $\mathcal{B}_c(X)$ denote the Borel, resp. compact subsets of X. Following [11], for any

$x \in X$ and any mapping $F : \Gamma^X \longrightarrow \mathbb{R}$ we define the mappings $\varepsilon_x^+ F : \Gamma^X \longrightarrow \mathbb{R}$ and $\varepsilon_x^- F : \Gamma^X \longrightarrow \mathbb{R}$ by

$$\left[\varepsilon_x^- F\right](\gamma) = F(\gamma \backslash \{x\}), \quad \text{and} \quad \left[\varepsilon_x^+ F\right](\gamma) = F(\gamma \cup \{x\}), \quad \gamma \in \Gamma^X,$$

and the difference operator D as $D_x F = \varepsilon_x^+ F - \varepsilon_x^- F$, $x \in X$. We have the relations $\varepsilon_x^- \varepsilon_x^+ = \varepsilon_x^+$ and $\varepsilon_x^+ \varepsilon_x^- = \varepsilon_x^-$, $x \in X$. Let σ be a diffuse Borel Radon measure on X. We have the product rules

$$D_x(FG) = (\varepsilon_x^- F) D_x G + (\varepsilon_x^- G) D_x F + (D_x F)(D_x G), \quad \forall x \in X,$$

and

$$D_x(FG) = (\varepsilon_x^+ F) D_x G + (\varepsilon_x^+ G) D_x F - (D_x F)(D_x G), \quad \forall x \in X,$$

which means, $\sigma(dx)$-a.e.:

$$D_x(FG) = F D_x G + G D_x F + (D_x F)(D_x G), \tag{2.1}$$

and $\gamma(dx)$-a.e.:

$$D_x(FG) = F D_x G + G D_x F - (D_x F)(D_x G). \tag{2.2}$$

These product rules do not lead a priori to a general functional calculus that could express $Df(F)$ in terms of F, DF and f. However, in case f is the exponential function we have $D_x \exp F = \exp \varepsilon_x^+ F - \exp \varepsilon_x^- F$ for $F : \Gamma^X \longrightarrow \mathbb{R}$, i.e.

$$D_x \exp F = (\exp F)(\exp(D_x F) - 1), \quad \sigma(dx) - a.e., \tag{2.3}$$

and

$$D_x \exp F = (\exp F)(1 - \exp(-D_x F)), \quad \gamma(dx) - a.e. \tag{2.4}$$

For $u : \Gamma^X \times X \longrightarrow \mathbb{R}$ measurable, and such that $u(\cdot, \gamma), \varepsilon_\cdot^+ u(\cdot, \gamma) \in L^1(X, d\sigma)$ for all $\gamma \in \Gamma^X$, the negative and positive Skorohod integral operators are defined as

$$\delta^-(u) = \int_X \varepsilon_x^- u(x) \gamma(dx) - \int_X u(x) \sigma(dx), \tag{2.5}$$

and

$$\delta^+(u) = \int_X u(x) \gamma(dx) - \int_X \varepsilon_x^+ u(x) \sigma(dx), \tag{2.6}$$

$\gamma \in \Gamma^X$, cf. [11], [15], [14], with the relations $\delta^-(\varepsilon^+ u) = \delta^+(u)$ and $\delta^+(\varepsilon^- u) = \delta^-(u)$. (In order to simplify the notation we will often omit the variable γ in $u : \Gamma^X \times X \longrightarrow \mathbb{R}$, and write $u(x)$, $x \in X$, instead of $u(x, \gamma)$). For $u : \Gamma^X \times X \longrightarrow \mathbb{R}$ and $G : \Gamma^X \longrightarrow \mathbb{R}$ measurable, the operators δ^-, δ^+ and D are linked by the identities

$$\delta^-(Gu) = G\delta^-(u) - \langle u, DG\rangle_\sigma - \delta^-(uDG), \tag{2.7}$$

and

$$\delta^+(Gu) = G\delta^+(u) - \langle u, DG\rangle_\gamma + \delta^+(uDG). \tag{2.8}$$

provided all integrability and summability conditions are satisfied, see Prop. 2 below for a proof of these identities in a more general context. Let $\mathcal{F}_\Lambda$, $\Lambda \in \mathcal{B}(X)$, denote the σ-algebra $\mathcal{F}_\Lambda = \sigma(\gamma \mapsto \gamma(A) \ : \ A \subset \Lambda)$. For $f_n \in \mathcal{C}_c(X^n)$, the multiple integral of f_n is defined as

$$I_n(f_n) = \int_{\Delta_n} f_n(x_1, \ldots, x_n)(\gamma(dx_1) - \sigma(dx_1)) \cdots (\gamma(dx_n) - \sigma(dx_n))$$

with

$$\Delta_n = \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j, \ \forall i \neq j\}.$$

We denote by $f_n \otimes g_m$ the tensor product of two functions $f_n \in L^2(X, \sigma)^{\otimes n}$, $g_m \in L^2(X, \sigma)^{\otimes m}$, defined as

$$f_n \otimes g_m(x_1, \ldots, x_n, y_1, \ldots, y_m) = f_n(x_1, \ldots, x_n) g_m(y_1, \ldots, y_m),$$

$(x_1, \ldots, x_n, y_1, \ldots, y_m) \in X^{n+m}$. The action of D and δ on multiple integrals is

$$D_x I_n(f_n) = n I_{n-1}(f_n(x, *)), \quad \sigma(dx) - a.e.,$$

$f_n \in \mathcal{C}_c(X^n)$ being symmetric in its n variables. For $g_n \in \mathcal{C}_c(X^{n+1})$, we have

$$\delta^-(I_n(g_n(*, \cdot))) = I_{n+1}(g_n), \tag{2.9}$$

and the Kabanov multiplication formula:

$$I_1(u) I_n(f^{\otimes n}) = I_{n+1}(u \otimes f^{\otimes n}) + n\langle u, f\rangle_\sigma I_{n-1}(f^{\otimes(n-1)}) + n I_n((uf) \otimes f^{\otimes(n-1)}), \tag{2.10}$$

$f, u \in \mathcal{C}_c(X)$, which can be understood as a reformulation of (2.7).

Definition 1 *Given $\Lambda \in \mathcal{B}(X)$, let $\mathcal{S}_b(\Lambda)$ denote the space of bounded $\mathcal{F}_\Lambda$-measurable functionals $F : \Gamma^X \longrightarrow \mathbb{R}$, and let $\mathcal{U}_b(\Lambda)$ denote the set of bounded, measurable mappings $u : \Gamma^X \times X \longrightarrow \mathbb{R}$ whose support is contained in $\Gamma^X \times \Lambda'$, where Λ' is a compact subset of Λ.*

Then $F \in \mathcal{S}_b(\Lambda)$ implies $DF \in \mathcal{U}_b(\Lambda)$ for $\Lambda \in \mathcal{B}(X)$.

3 Poisson measures on configuration spaces

In this section we consider Poisson measures on Γ^X, which can be viewed as a particular case of Gibbs measures. Let π_σ denote the Poisson measure with intensity σ on Γ^X, defined by its characteristic function

$$E_{\pi_\sigma}\left[\exp\left(iz\int_X h d\gamma\right)\right] = \exp\int_X (e^{izh}-1)d\sigma, \quad z\in\mathbb{R},\ h\in\mathcal{C}_c(X).$$

We use the notation $\langle\cdot,\cdot\rangle_\sigma = \langle\cdot,\cdot\rangle_{L^2(X,\sigma)}$ and $\langle\cdot,\cdot\rangle_\gamma = \langle\cdot,\cdot\rangle_{L^2(X,\gamma)}$, $\gamma\in\Gamma^X$. The following statements hold under the Poisson measure π_σ with intensity σ, cf. [11], Cor. 1 and Cor. 5. For all $u\in L^2(\Gamma^X,\pi_\sigma;L^\infty(X,\sigma))$ with support in $\Gamma^X\times\Lambda$, $\Lambda\in\mathcal{B}_c(X)$, we have:

$$\begin{aligned}
&E_{\pi_\sigma}\left[F\delta^-(u)\right] = E_{\pi_\sigma}\left[\langle DF,u\rangle_\sigma\right], \quad F\in\mathcal{S}_b(X),\\
&E_{\pi_\sigma}\left[F\delta^+(u)\right] = E_{\pi_\sigma}\left[\langle DF,u\rangle_\gamma\right], \quad F\in\mathcal{S}_b(X),\\
&E_{\pi_\sigma}\left[\delta^-(u)\right] = 0, \quad F\in\mathcal{S}_b(X),\\
&E_{\pi_\sigma}\left[\delta^+(u)\right] = 0 \quad F\in\mathcal{S}_b(X).
\end{aligned}$$

We recall the following characterization of Poisson measures by duality between δ^-, δ^+ and D, cf. [15].

Theorem 1 *Let π be a finite measure on Γ^X under which $\delta^-(h)$, resp. $\delta^+(h)$, is integrable, $\forall h\in\mathcal{C}_c(X)$. If one of the following conditions holds:*

$$E_\pi\left[F\delta^-(h)\right] = E_\pi\left[\langle DF,h\rangle_\sigma\right], \qquad \forall\ h\in\mathcal{C}_c(X), \quad F\in\mathcal{S}_b(X), \tag{3.1}$$

$$E_\pi\left[F\delta^+(h)\right] = E_\pi\left[\langle DF,h\rangle_\gamma\right], \qquad \forall\ h\in\mathcal{C}_c(X), \quad F\in\mathcal{S}_b(X), \tag{3.2}$$

$$E_\pi\left[\delta^-(u)\right] = 0, \qquad \forall\ u\in\mathcal{U}_b(X), \tag{3.3}$$

$$E_\pi\left[\delta^+(u)\right] = 0, \qquad \forall\ u\in\mathcal{U}_b(X), \tag{3.4}$$

then π is proportional to the Poisson measure π_σ with intensity σ.

Proof. For completeness we recall a short proof of this result, cf. [15]. The facts that (3.3) implies (3.1) and that (3.4) implies (3.2) follow from Relations (2.7) and (2.8). Assume that (3.1) holds, let $h\in\mathcal{C}_c(X)$ and $\psi(z) = E_\pi[\exp(iz\int_X hd\gamma)]$, $z\in\mathbb{R}$. We have

$$\begin{aligned}
\frac{d}{dz}\psi(z) &= iE_\pi\left[\delta^-(h)\exp\left(iz\int_X hd\gamma\right)\right] + iE_\pi\left[\int_X hd\sigma\exp\left(iz\int_X hd\gamma\right)\right]\\
&= iE_\pi\left[\langle h, D\exp\left(iz\int_X hd\gamma\right)\rangle_\sigma\right] + i\psi(z)\int_X hd\sigma\\
&= i\langle h, e^{izh}-1\rangle_\sigma E_\pi\left[\exp\left(iz\int_X hd\gamma\right)\right] + i\psi(z)\int_X hd\sigma = i\psi(z)\langle h, e^{izh}\rangle_\sigma,
\end{aligned}$$

$h \in \mathcal{C}_c(X)$. This gives $\psi(z) = \psi(0) \exp \int_X (e^{izh} - 1) d\sigma$, $z \in \mathbb{R}$, hence $\pi = \psi(0)\pi_\sigma$. $\square$

The integration by parts characterization of mixed Poisson, resp. canonical Gibbs, measures of [1], resp. [2], uses a different operator, denoted as ∇^Γ, which can be defined as

$$(\nabla^\Gamma_\gamma F)(x) = \sum_{i=1}^{i=n} \partial_i f \left(\int_X \psi_1 d\gamma, \ldots, \int_X \psi_n d\gamma \right) \nabla^X \psi_i(x), \quad x \in X,$$

for $F = f\left(\int_X \psi_1 d\gamma, \ldots, \int_X \psi_n d\gamma\right)$, $\psi_1, \ldots, \psi_n \in \mathcal{C}_c^\infty(X)$. This requires a differential structure on X, namely X has to be a Riemannian manifold with gradient ∇^X.

4 Skorohod integration under grand canonical Gibbs measures

Our aim in this section is to define Skorohod type operators in the setting of Gibbs measures. We consider an interaction potential $\Phi : \Gamma^X \longrightarrow \mathbb{R}$. For $\Lambda \in \mathcal{B}_c(X)$, the Hamiltonian potential H_Λ is defined as (see. e.g. [4]):

$$H_\Lambda(\gamma) = \begin{cases} \displaystyle\sum_{\tilde{\gamma} \subset \gamma,\ \tilde{\gamma}(\Lambda) > 0} \Phi(\tilde{\gamma}), & \text{if} \displaystyle\sum_{\tilde{\gamma} \subset \gamma,\ \tilde{\gamma}(\Lambda) > 0} |\Phi(\tilde{\gamma})| < \infty, \\ +\infty \text{ otherwise}, \end{cases}$$

and satisfies $H_\Lambda(\gamma) = H_{\Lambda'}(\gamma)$ if $\gamma \subset \Lambda \subset \Lambda'$. Using the notation $\gamma_\Lambda = \gamma \cap \Lambda$, $\Lambda \in \mathcal{B}(X)$, and $\gamma\tilde{\gamma} = \gamma \cup \tilde{\gamma}$, for $\gamma, \tilde{\gamma} \in \Gamma^X$, we assume that there exists $a > 0$ with

$$H_\Lambda(\gamma) \geq -a|\gamma_\Lambda|, \quad \gamma \in \Gamma^X,\ \Lambda \in \mathcal{B}_c(X). \tag{4.1}$$

This condition ensures in particular the integrability of $\gamma \mapsto \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda))$ under the Poisson measure π_σ, for all $\tilde{\gamma} \in \Gamma^X$ and $\Lambda \in \mathcal{B}_c(X)$. Let

$$Z_\Lambda(\tilde{\gamma}) = \int_{\Gamma^X} \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) d\pi_\sigma(\gamma), \quad \tilde{\gamma} \in \Gamma^X,$$

and

$$\Pi_\Lambda(\tilde{\gamma}, A) = \frac{1}{Z_\Lambda(\tilde{\gamma})} \int_{\Gamma^X} 1_A(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda) \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) d\pi_\sigma(\gamma),$$

$A \in \mathcal{B}(\Gamma^X)$, $\Lambda \in \mathcal{B}_c(X)$. We refer to [2], [4], [13], [18] and references therein for the following definition.

Definition 2 *A probability μ on Γ^X is called a grand canonical Gibbs distribution if*

$$E_\mu[F \mid \mathcal{F}_{\Lambda^c}](\tilde{\gamma}) = \Pi_\Lambda(\cdot, F)(\tilde{\gamma}), \qquad \mu(d\tilde{\gamma}) - a.s., \tag{4.2}$$

for all $F \in \mathcal{S}_b(X)$ and $\Lambda \in \mathcal{B}_c(X)$.

The integral operators under Gibbs measures will be defined on a class of processes which is larger than $\mathcal{U}_b(\Lambda)$.

Definition 3 *Given $\Lambda \in \mathcal{B}(X)$, let $\mathcal{U}_b^\Phi(\Lambda)$ denote the union of $\mathcal{U}_b(\Lambda)$ with the set of measurable mappings $u : \Gamma^X \times X \longrightarrow \mathbb{R}$ whose support is contained in $\Gamma^X \times \Lambda'$, where Λ' is a compact subset of Λ, and such that $u \exp(-H_\Lambda)$ is bounded on Γ^X.*

We now define two families $(\delta^-_{\Phi,\Lambda})_{\Lambda \in \mathcal{B}(X)}$ and $(\delta^+_{\Phi,\Lambda})_{\Lambda \in \mathcal{B}(X)}$ of integral operators indexed by the compact subsets of X.

Definition 4 *For $\Lambda \in \mathcal{B}_c(X)$ and $u \in \mathcal{U}_b^\Phi(\Lambda)$, let*

$$\delta^-_{\Phi,\Lambda}(u) = \exp(H_\Lambda)\delta^-(u \exp(-H_\Lambda)) \qquad \textit{and} \qquad \delta^+_{\Phi,\Lambda}(u) = \exp(H_\Lambda)\delta^+(u \exp(-H_\Lambda)).$$

Since $\varepsilon^-_x D_x F = D_x F$, $\gamma(dx)$-a.e. and $\varepsilon^+_x D_x F = D_x F$, $\sigma(dx)$-a.e., there are multiple expressions for $\delta^-_{\Phi,\Lambda}$ and $\delta^+_{\Phi,\Lambda}$, $\Lambda \in \mathcal{B}_c(X)$. In particular we have

$$\delta^-_{\Phi,\Lambda}(u) = \int_X \varepsilon^-_x u(x) \exp(H_\Lambda - \varepsilon^-_x H_\Lambda)\gamma(dx) - \int_X u(x)\sigma(dx), \tag{4.3}$$

$$\delta^-_{\Phi,\Lambda}(u) = \int_X \varepsilon^-_x u(x) \exp(D_x H_\Lambda)\gamma(dx) - \int_X u(x)\sigma(dx), \tag{4.4}$$

$$\delta^-_{\Phi,\Lambda}(u) = \delta^-(u \exp(DH_\Lambda)) + \int_X u(x)(\exp(D_x H_\Lambda) - 1)\sigma(dx), \tag{4.5}$$

and (4.6)

$$\delta^+_{\Phi,\Lambda}(u) = \int_X u(x)\gamma(dx) - \int_X \varepsilon^+_x u(x) \exp(-\varepsilon^+_x H_\Lambda + H_\Lambda)\sigma(dx), \tag{4.7}$$

$$\delta^+_{\Phi,\Lambda}(u) = \int_X u(x)\gamma(dx) - \int_X \varepsilon^+_x u(x) \exp(-D_x H_\Lambda)\sigma(dx), \tag{4.8}$$

$$\delta^+_{\Phi,\Lambda}(u) = \delta^+(u \exp(-D_x H_\Lambda)) + \int_X u(x)(1 - \exp(-D_x H_\Lambda))\gamma(dx).$$

Also, we have $\delta^-_{\Phi,\emptyset} = \delta^-$, $\delta^+_{\Phi,\emptyset} = \delta^+$ if $\Lambda = \emptyset$, and $\delta^-_{0,\Lambda} = \delta^-$, $\delta^+_{0,\Lambda} = \delta^+$ if $\Phi = 0$, $\Lambda \in \mathcal{B}(X)$.

Proposition 1 *We have the relation*

$$D_x H_\Lambda(\gamma) = \sum_{\tilde{\gamma} \subset \gamma \setminus x} \Phi(\tilde{\gamma} \cup x), \quad x \in \Lambda, \ \gamma \in \Gamma^X, \ \Lambda \in \mathcal{B}(X).$$

Proof. This follows by direct calculation:

$$\begin{aligned} D_x H_\Lambda(\gamma) &= \sum_{\tilde{\gamma} \subset \gamma \cup x, \ \tilde{\gamma}(\Lambda) > 0} \Phi(\tilde{\gamma}) - \sum_{\tilde{\gamma} \subset \gamma \setminus x, \ \tilde{\gamma}(\Lambda) > 0} \Phi(\tilde{\gamma}) \\ &= \sum_{\tilde{\gamma} \subset \gamma \setminus x, \ (\tilde{\gamma} \cup x)(\Lambda) > 0} \Phi(\tilde{\gamma} \cup x). \quad \square \end{aligned}$$

This provides another expression for $\delta^-_{\Phi,\Lambda}(u)$ and $\delta^+_{\Phi,\Lambda}(u)$, $u \in \mathcal{U}_b(\Lambda)$:

$$\delta^-_{\Phi,\Lambda}(u) = \int_X \varepsilon^-_x u(x) \prod_{\tilde\gamma \subset \gamma \backslash x} \exp(\Phi(\tilde\gamma \cup x))\ \gamma(dx) - \int_X u(x)\sigma(dx)$$

and

$$\delta^+_{\Phi,\Lambda}(u) = \int_X u(x)\gamma(dx) - \int_X \varepsilon^+_x u(x) \prod_{\tilde\gamma \subset \gamma \backslash x} \exp(\Phi(\tilde\gamma \cup x))\ \sigma(dx).$$

The following proposition extends Relations (2.7) and (2.8) to $\delta^-_{\Phi,\Lambda}$ and $\delta^+_{\Phi,\Lambda}$, $\Lambda \in \mathcal{B}_c(X)$.

Proposition 2 *Let $\Lambda \in \mathcal{B}_c(X)$. We have for $F \in \mathcal{S}_b(X)$ and $u \in \mathcal{U}^\Phi_b(\Lambda)$:*

$$\delta^-_{\Phi,\Lambda}(Fu) = F\delta^-_{\Phi,\Lambda}(u) - \langle u, DF\rangle_\sigma - \delta^-_{\Phi,\Lambda}(uDF), \tag{4.9}$$

and

$$\delta^+_{\Phi,\Lambda}(Fu) = F\delta^+_{\Phi,\Lambda}(u) - \langle u, DF\rangle_\gamma + \delta^+_{\Phi,\Lambda}(uDF). \tag{4.10}$$

Proof. We have $\varepsilon^-_x D_x F = D_x F$, $x \in X$, $F \in \mathcal{S}_b(X)$, hence with (4.3),

$$\begin{aligned}
\delta^-_{\Phi,\Lambda}(uDF) &= \int_X \varepsilon^-_x u(x) \exp(D_x H_\Lambda) D_x F \gamma(dx) - \int_X u(x) D_x F \sigma(dx) \\
&= F\int_X \varepsilon^-_x u(x) \exp(D_x H_\Lambda)\gamma(dx) \\
&\quad - \int_X \varepsilon^-_x u(x) \varepsilon^-_x F \exp(D_x H_\Lambda)\gamma(dx) - \langle DF, u\rangle_\sigma \\
&= F\int_X \varepsilon^-_x u(x) \exp(D_x H_\Lambda)\gamma(dx) \\
&\quad - F\int_X \varepsilon^-_x u(x)\sigma(dx) - \delta^-_{\Phi,\Lambda}(uF) - \langle DF, u\rangle_\sigma \\
&= F\delta^-_{\Phi,\Lambda}(u) - \delta^-_{\Phi,\Lambda}(uF) - \langle DF, u\rangle_\sigma, \quad u \in \mathcal{U}^\Phi_b(\Lambda).
\end{aligned}$$

Concerning $\delta^+_{\Phi,\Lambda}$ we have $\varepsilon^+_x D_x F = D_x F$, $x \in X$, $F \in \mathcal{S}_b(X)$, and from (4.8),

$$\begin{aligned}
\delta^+_{\Phi,\Lambda}(uDF) &= \int_X u(x) D_x F \gamma(dx) - \int_X \varepsilon^+_x u(x) D_x F \exp(-D_x H_\Lambda)\sigma(dx) \\
&= \langle u, DF\rangle_\gamma - \int_X \varepsilon^+_x u(x) \varepsilon^+_x F \exp(-D_x H_\Lambda)\sigma(dx) \\
&\quad + F\int_X \varepsilon^+_x u(x) \exp(-D_x H_\Lambda)\sigma(dx) \\
&= \langle u, DF\rangle_\gamma + \delta^+_{\Phi,\Lambda}(uF) - F\int_X u(x)\gamma(dx) \\
&\quad + F\int_X \varepsilon^+_x u(x) \exp(-D_x H_\Lambda)\sigma(dx) \\
&= \langle u, DF\rangle_\gamma + \delta^+_{\Phi,\Lambda}(uF) - F\delta^+_{\Phi,\Lambda}(u), \quad u \in \mathcal{U}^\Phi_b(\Lambda). \ \square
\end{aligned}$$

In general there is no reason that $\delta^\pm_{\Phi,\Lambda}$ satisfy the following consistency property:

$$\delta^\pm_{\Phi,\Lambda'}(u) = \delta^\pm_{\Phi,\Lambda}(u), \quad u \in \mathcal{U}^\Phi_b(\Lambda), \quad \Lambda \subset \Lambda', \quad \Lambda, \Lambda' \in \mathcal{B}_c(X), \tag{4.11}$$

hence it is of interest to provide a definition of $\delta^{\pm}_{\Phi,\Lambda}(u)$ that would not depend on the set $\Lambda \in \mathcal{B}_c(X)$. For this we consider the following assumption:

(A) Assume that all closed balls in X are compact, and that long range interactions vanish, i.e. there exists $c > 0$ such that

$$\Phi(\gamma) = 0 \quad \text{if} \;\; d(\gamma) > c, \quad \gamma \in \Gamma^X,$$

where $d(\gamma)$ denotes the diameter of $\gamma \in \Gamma^X$.

As a consequence of Prop. 1 we have:

Lemma 1 *Under assumption (A), let* $\Lambda_c := \{x \in X \; : \; d(x, \Lambda) > c\}$, *for* $\Lambda \in \mathcal{B}_c(X)$. *For any* $\Lambda' \in \mathcal{B}_c(X)$ *such that* $\Lambda' \supset \Lambda_c$ *we have*

$$D_x H_{\Lambda'}(\gamma) = D_x H_{\Lambda_c}(\gamma), \quad x \in \Lambda, \; \gamma \in \Gamma^X.$$

Hence under assumption (A), $(\delta^{\pm}_{\Phi,\Lambda})_{\Lambda \in \mathcal{B}_c(X)}$ satisfies (from (4.3) and (4.8)) the partial consistency property

$$\delta^{\pm}_{\Phi,\Lambda'}(u) = \delta^{\pm}_{\Phi,\Lambda_c}(u), \quad u \in \mathcal{U}_b(\Lambda), \quad \Lambda' \supset \Lambda_c, \quad \Lambda, \Lambda' \in \mathcal{B}_c(X),$$

which is less general than (4.11). This leads to the following definition.

Definition 5 *Under assumption (A) on long range interactions, we let*

$$\delta^{-}_{\Phi}(u) = \delta^{-}_{\Phi,\Lambda_c}(u) \quad \textit{and} \quad \delta^{+}_{\Phi}(u) = \delta^{+}_{\Phi,\Lambda_c}(u), \qquad u \in \mathcal{U}^{\Phi}_b(\Lambda), \;\; \Lambda \in \mathcal{B}_c(X).$$

If X is compact, then $\delta^{\pm}_{\Phi} = \delta^{\pm}_{\Phi,X}$ on $\mathcal{U}^{\Phi}_b(\Lambda)$ under assumption (A), and Relations (4.9) and (4.10) hold also for $\delta^{\pm}_{\Phi}$. The following result extends Th. 1 to the Gibbsian case, showing the duality between $\delta^{-}_{\Phi,\Lambda}$, $\delta^{+}_{\Phi,\Lambda}$ and D.

Theorem 2 *Under the grand canonical Gibbs measure* μ *with Hamiltonian* H_Λ, $\delta^{-}_{\Phi,\Lambda}(u)$ *and* $\delta^{+}_{\Phi,\Lambda}(u)$ *are integrable for all* $u \in \mathcal{U}_b(\Lambda)$ *and we have*

$$E_\mu\left[F\delta^{-}_{\Phi,\Lambda}(h)\right] = E_\mu\left[\langle DF, h\rangle_\sigma\right], \qquad \forall \; h \in \mathcal{U}_b(\Lambda), \quad F \in \mathcal{S}_b(X), \; \Lambda \in \mathcal{B}_c(X), \tag{4.12}$$

$$E_\mu\left[F\delta^{+}_{\Phi,\Lambda}(h)\right] = E_\mu\left[\langle DF, h\rangle_\gamma\right], \qquad \forall \; h \in \mathcal{U}_b(\Lambda), \quad F \in \mathcal{S}_b(X), \; \Lambda \in \mathcal{B}_c(X), \tag{4.13}$$

$$E_\mu\left[\delta^{-}_{\Phi,\Lambda}(u)\right] = 0, \qquad \forall \; u \in \mathcal{U}_b(\Lambda), \; \Lambda \in \mathcal{B}_c(X), \tag{4.14}$$

$$E_\mu\left[\delta^{+}_{\Phi,\Lambda}(u)\right] = 0, \qquad \forall \; u \in \mathcal{U}_b(\Lambda), \; \Lambda \in \mathcal{B}_c(X). \tag{4.15}$$

Proof. We start by proving that (4.12) holds under the Gibbs measure μ. If $x \in \Lambda$, we have $(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \setminus x = (\gamma \setminus x)_\Lambda \tilde{\gamma}_{\Lambda^c}$ and $(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \cup x = (\gamma \cup x)_\Lambda \tilde{\gamma}_{\Lambda^c}$, $\gamma, \tilde{\gamma} \in \Gamma^X$. Hence, given that D and δ^- act on the variable $\gamma \in \Gamma^X$, we have for all $v \in \mathcal{U}_b^\Phi(\Lambda)$:

$$v(x, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) D_x(F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c})) = v(x, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c})(D_x F)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}), \qquad x \in X, \ \gamma \in \Gamma^X,$$

and

$$\delta^-(v(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c})) = \delta^-(v)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}), \qquad \gamma \in \Gamma^X,$$

because the support of v lies in $\Gamma^X \times \Lambda$. Since $\int_X u(x,\gamma)\sigma(dx)$ is uniformly bounded in $\gamma \in \Gamma^X$, we can write for positive $u \in \mathcal{U}_b(\Lambda)$:

$$\begin{aligned}
Z_\Lambda(\tilde{\gamma}) \Pi_\Lambda(\tilde{\gamma}, F\delta^-_{\Phi,\Lambda}(u)) &= \int_{\Gamma^X} F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \delta^-_{\Phi,\Lambda}(u)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \delta^-(u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \exp(-H_\Lambda(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}))) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \langle D(F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c})), u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \rangle_\sigma d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \langle (DF)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}), u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \rangle_\sigma d\pi_\sigma(\gamma) \\
&= Z_\Lambda(\tilde{\gamma}) \Pi_\Lambda(\tilde{\gamma}, \langle DF, u \rangle_\sigma).
\end{aligned}$$

Using the Gibbs property (4.2) we obtain

$$\begin{aligned}
E_\mu[F\delta^-_{\Phi,\Lambda}(u)] &= E_\mu[E_\mu[F\delta^-_{\Phi,\Lambda}(u) \mid \mathcal{F}_{\Lambda^c}]] = E_\mu[\Pi_\Lambda(\tilde{\gamma}, F\delta^-_{\Phi,\Lambda}(u))] \\
&= E_\mu[\Pi_\Lambda(\tilde{\gamma}, \langle u, DF \rangle_\sigma)] = E_\mu[\langle u, DF \rangle_\sigma], \quad F \in \mathcal{S}_b(X), \ u \in \mathcal{U}_b(\Lambda).
\end{aligned}$$

Given the boundedness of $\gamma \longrightarrow \int_X u(x,\gamma)\sigma(dx)$, this also implies the integrability of $\delta^-_{\Phi,\Lambda}(u)$ under μ, $\forall u \in \mathcal{U}_b(\Lambda)$, and (4.14) for $F = 1$. Concerning $\delta^+_{\Phi,\Lambda}$, $\gamma \mapsto \int_X u(x)\gamma(dx)$ is bounded by $\|u\|_\infty \gamma(\Lambda)$ hence it is integrable under μ, and we can write for positive $u \in \mathcal{U}_b(\Lambda)$:

$$\begin{aligned}
Z_\Lambda(\tilde{\gamma}) \Pi_\Lambda(\tilde{\gamma}, F\delta^+_{\Phi,\Lambda}(u)) &= \int_{\Gamma^X} F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \delta^+_{\Phi,\Lambda}(u)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \delta^+(u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \exp(-H_\Lambda(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}))) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \langle D(F(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c})), u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \rangle_\gamma d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)) \langle (DF)(\gamma_\Lambda \tilde{\gamma}_{\Lambda^c}), u(\cdot, \gamma_\Lambda \tilde{\gamma}_{\Lambda^c}) \rangle_\gamma d\pi_\sigma(\gamma) \\
&= Z_\Lambda(\tilde{\gamma}) \Pi_\Lambda(\tilde{\gamma}, \langle DF, u \rangle_\gamma).
\end{aligned}$$

From the Gibbs property (4.2) we have

$$\begin{aligned}
E_\mu[F\delta^+_{\Phi,\Lambda}(u)] &= E_\mu[E_\mu[F\delta^+_{\Phi,\Lambda}(u) \mid \mathcal{F}_{\Lambda^c}]] = E_\mu[\Pi_\Lambda(\tilde{\gamma}, F\delta^+_{\Phi,\Lambda}(u))] \\
&= E_\mu[\Pi_\Lambda(\tilde{\gamma}, \langle u, DF \rangle_\gamma)] = E_\mu[\langle u, DF \rangle_\gamma], \quad F \in \mathcal{S}_b(X), \ u \in \mathcal{U}_b(\Lambda).
\end{aligned}$$

The boundedness of $\gamma \mapsto \int_X u(x,\gamma)\gamma(dx)$ by $\|u\|_\infty \gamma(\Lambda)$ also implies the integrability of $\delta^+_{\Phi,\Lambda}(u)$ under μ, (4.15) for $F = 1$.

$\square$

Under assumption (A) on long range interactions we obtain the following corollary.

Corollary 1 *Under the grand canonical Gibbs measure μ, $\delta^-_\Phi(u)$ and $\delta^+_\Phi(u)$ are integrable for all $u \in \mathcal{U}_b(X)$ and we have*

$$E_\mu\left[F\delta^-_\Phi(h)\right] = E_\mu\left[\langle DF, h\rangle_\sigma\right], \qquad \forall\ h \in \mathcal{U}_b(X), \quad F \in \mathcal{S}_b(X), \tag{4.16}$$

$$E_\mu\left[F\delta^+_\Phi(h)\right] = E_\mu\left[\langle DF, h\rangle_\gamma\right], \qquad \forall\ h \in \mathcal{U}_b(X), \quad F \in \mathcal{S}_b(X), \tag{4.17}$$

$$E_\mu\left[\delta^-_\Phi(u)\right] = 0, \qquad \forall\ u \in \mathcal{U}_b(X), \tag{4.18}$$

$$E_\mu\left[\delta^+_\Phi(u)\right] = 0, \qquad \forall\ u \in \mathcal{U}_b(X). \tag{4.19}$$

We also mention a different proof of (4.12) and (4.13), which is longer but uses (4.3) and (4.8) and exploits the functional calculus for D on exponential functions. From (4.3) we have

$$\begin{aligned}
\exp(-H_\Lambda)\delta^+_{\Phi,\Lambda}(u) &= \int_X \varepsilon^+_x u(x)\exp(-\varepsilon^+_x H_\Lambda)\gamma(dx) - \exp(-H_\Lambda)\int_X u(x)\sigma(dx), \\
&= \delta^+(u\exp(-H_\Lambda)),
\end{aligned}$$

hence from assumption (4.1), $\gamma \mapsto \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^-_{\Phi,\Lambda}(u)(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)$ is integrable under π_σ for all $\tilde\gamma \in \Gamma^X$ and we have, using (2.1):

$$\begin{aligned}
Z_\Lambda(\tilde\gamma)\Pi_\Lambda(\tilde\gamma, F\delta^-_{\Phi,\Lambda}(u)) &= \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^-_{\Phi,\Lambda}(u)(\gamma_\Lambda\tilde\gamma_{\Lambda^c})d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^-(u\exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)))d\pi_\sigma(\gamma) \\
&\quad + \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, \exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)) - 1\rangle_\sigma d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \langle D(F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))), u\exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\sigma d\pi_\sigma(\gamma) \\
&\quad + \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, \exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)) - 1\rangle_\sigma d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle DF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), u\exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\sigma d\pi_\sigma(\gamma) \\
&\quad + \int_{\Gamma^X} F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle \exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)) - 1, u\exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\sigma d\pi_\sigma(\gamma) \\
&\quad + \int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle uDF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), 1 - \exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\sigma d\pi_\sigma(\gamma) \\
&\quad - \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, 1 - \exp(DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\sigma d\pi_\sigma(\gamma) \\
&= \int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle DF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), u\rangle_\sigma d\pi_\sigma(\gamma) \\
&= Z_\Lambda(\tilde\gamma)\Pi_\Lambda(\tilde\gamma, \langle DF, u\rangle_\sigma), \qquad u \in \mathcal{U}_b(\Lambda).
\end{aligned}$$

Using the Gibbs property (4.2) we obtain

$$E_\mu[F\delta^-_{\Phi,\Lambda}(u)] = E_\mu[\langle u, DF\rangle_\sigma], \quad F \in \mathcal{S}_b(X),\ u \in \mathcal{U}_b(\Lambda).$$

The proof of (4.13) for $\delta^+_{\Phi,\Lambda}$ has some similarity with the above. More precisely it requires a careful replacement of σ by γ, combined with the use of (2.2) instead of (2.1), and it is preferable to state it completely. In the following, the operator D acts on the variable $\gamma \in \Gamma^X$ that determines the random scalar product $\langle\cdot,\cdot\rangle_\gamma$, and γ is also the integration variable in the expectation under π_σ. From (4.7) we have

$$\exp(-H_\Lambda)\delta^+_{\Phi,\Lambda}(u) = \exp(-H_\Lambda)\int_X u(x)\gamma(dx) - \int_X \varepsilon^+_x u(x)\exp(-\varepsilon^+_x H_\Lambda)\sigma(dx),$$

hence from (4.1) $\gamma \mapsto \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^+_{\Phi,\Lambda}(u)(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)$ is integrable under π_σ, for all $\tilde\gamma \in \Gamma^X$. Using the relation $\exp(-DF)D\exp(-F) = \exp(-F)(\exp(-DF)-1)$, γ-a.e., cf. (2.4), we have:

$$\begin{aligned}
Z_\Lambda(\tilde\gamma)\Pi_\Lambda(\tilde\gamma, F\delta^+_{\Phi,\Lambda}(u)) &= \int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^+_{\Phi,\Lambda}(u)(\gamma_\Lambda\tilde\gamma_{\Lambda^c})d\pi_\sigma(\gamma)\\
= &\int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\delta^+(u\exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)))d\pi_\sigma(\gamma)\\
&+\int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, 1-\exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\gamma d\pi_\sigma(\gamma)\\
= &\int_{\Gamma^X} \langle D(F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))), u\exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\gamma d\pi_\sigma(\gamma)\\
&+\int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, 1-\exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\gamma d\pi_\sigma(\gamma)\\
= &\int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle uDF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), \exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\gamma d\pi_\sigma(\gamma)\\
&+\int_{\Gamma^X} F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, \exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))-1\rangle_\gamma d\pi_\sigma(\gamma)\\
&-\int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle uDF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), \exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))-1\rangle_\gamma d\pi_\sigma(\gamma)\\
&+\int_{\Gamma^X} F(\gamma_\Lambda\tilde\gamma_{\Lambda^c})\exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle u, 1-\exp(-DH_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\rangle_\gamma d\pi_\sigma(\gamma)\\
= &\int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle DF(\gamma_\Lambda\tilde\gamma_{\Lambda^c}), u\rangle_\gamma d\pi_\sigma(\gamma)\\
= &\ Z_\Lambda(\tilde\gamma)\Pi_\Lambda(\tilde\gamma, \langle DF, u\rangle_\gamma), \quad u \in \mathcal{U}_b(\Lambda),\ F \in \mathcal{S}_b(X).
\end{aligned}$$

Using the Gibbs property we obtain $E_\mu[F\delta^+_{\Phi,\Lambda}(h)] = E_\mu[\langle h, DF\rangle_\gamma]$.

5 Integration by parts characterization of Gibbs measures

In this section we prove that the identities of Th. 2 characterize the grand canonical Gibbs measure μ with interaction potential $\Phi : \Gamma^X \longrightarrow \mathbb{R}$.

Theorem 3 *Let π be a probability measure under which $\delta^-_{\Phi,\Lambda}(u)$, $\delta^+_{\Phi,\Lambda}(u)$ are integrable for all $u \in \mathcal{U}^\Phi_b(\Lambda)$ and $\Lambda \in \mathcal{B}_c(X)$, and assume that one of the following statements holds:*

$$E_\pi\left[F\delta^-_{\Phi,\Lambda}(u)\right] = E_\pi\left[\langle DF, u\rangle_\sigma\right], \qquad \forall\, u \in \mathcal{U}^\Phi_b(\Lambda), \quad F \in \mathcal{S}_b(X),\ \Lambda \in \mathcal{B}_c(X), \tag{5.1}$$

$$E_\pi\left[F\delta^+_{\Phi,\Lambda}(u)\right] = E_\pi\left[\langle DF, u\rangle_\gamma\right], \qquad \forall\, u \in \mathcal{U}^\Phi_b(\Lambda), \quad F \in \mathcal{S}_b(X),\ \Lambda \in \mathcal{B}_c(X), \tag{5.2}$$

$$E_\pi\left[\delta^-_{\Phi,\Lambda}(u)\right] = 0, \qquad \forall\, u \in \mathcal{U}^\Phi_b(\Lambda),\ \Lambda \in \mathcal{B}_c(X), \tag{5.3}$$

$$E_\pi\left[\delta^+_{\Phi,\Lambda}(u)\right] = 0, \qquad \forall\, u \in \mathcal{U}^\Phi_b(\Lambda),\ \Lambda \in \mathcal{B}_c(X). \tag{5.4}$$

Then π is the Gibbs measure μ with interaction potential Φ.

Proof. From Prop. 2, (5.3) implies (5.1), and (5.4) implies (5.2). Writing (5.1) for F of the form $F = F_1F_2$, $F_1 \in \mathcal{S}_b(X)$, $F_2 \in \mathcal{S}_b(\Lambda^c)$, we obtain

$$E_\pi[F_1\delta^-_{\Phi,\Lambda}(u) \mid \mathcal{F}_{\Lambda^c}](\tilde\gamma) \;=\; E_\pi[\langle u, DF_1\rangle_\sigma \mid \mathcal{F}_{\Lambda^c}](\tilde\gamma), \quad \pi(d\tilde\gamma) - a.s., \quad u \in \mathcal{U}^\Phi_b(\Lambda), \tag{5.5}$$

because $uDF_2 = 0$ and $uD(F_1F_2) = uF_2DF_1$. Now we have

$$E_\pi[F \mid \mathcal{F}_{\Lambda^c}](\tilde\gamma) = \int_{\Gamma^X} F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde\gamma), \quad \pi(d\tilde\gamma) - a.s., \quad F \in L^1(\Gamma^X, \pi), \tag{5.6}$$

(this identity can be checked for simple functionals of the form

$$F(\gamma) = 1_{\{\gamma(U)=k\}}1_{\{\gamma(V)=l\}}, \qquad \gamma \in \Gamma^X,$$

$U \in \mathcal{B}(\Lambda)$, $V \in \mathcal{B}(\Lambda^c)$, $k, l \in \mathbb{N}$). Hence (5.5) can be rewritten as

$$\begin{aligned}\int_{\Gamma^X} &\exp(H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))F_1(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\delta^-(h)(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde\gamma)\\ &= \int_{\Gamma^X} \exp(H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))\langle h, DF_1\rangle_\sigma(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde\gamma), \quad \pi(d\tilde\gamma) - a.s.,\end{aligned}$$

$h \in \mathcal{C}_c(\Lambda)$, with $u = h\exp(-H_\Lambda) \in \mathcal{U}^\Phi_b(\Lambda)$. From Th. 1, this implies

$$\int_{\Gamma^X} \exp(H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde\gamma) \;=\; K(\tilde\gamma)\int_{\Gamma^X} F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)d\pi_\sigma(\gamma),$$

$F \in \mathcal{S}_b(X)$, and from (5.6),

$$E_\pi[F \mid \mathcal{F}_{\Lambda^c}](\tilde\gamma) \;=\; K(\tilde\gamma)\int_{\Gamma^X} \exp(-H_\Lambda(\tilde\gamma_{\Lambda^c}\gamma_\Lambda))F(\tilde\gamma_{\Lambda^c}\gamma_\Lambda)d\pi_\sigma(\gamma), \quad \pi(d\tilde\gamma) - a.s., \tag{5.7}$$

$F \in \mathcal{S}_b(X)$, where $\tilde\gamma \mapsto K(\tilde\gamma)$ is a normalization function which is equal to $\tilde\gamma \mapsto 1/Z_\Lambda(\tilde\gamma)$ since $\pi_{|\mathcal{F}_{\Lambda^c}}(\cdot,\tilde\gamma)$ is a probability measure for all $\tilde\gamma \in \Gamma^X$. Hence from Def. 2, π

is the grand canonical Gibbs measure μ with interaction potential Φ. We now prove that (5.2) also characterizes the Gibbs measure μ. We have under the probability π:

$$E_\pi[F_1(\gamma)\delta^+_{\Phi,\Lambda}(u) \mid \mathcal{F}_{\Lambda^c}](\tilde{\gamma}) \quad = \quad E_\pi[\langle u, DF_1\rangle_\gamma \mid \mathcal{F}_{\Lambda^c}](\tilde{\gamma}), \qquad \pi(d\tilde{\gamma}) - a.s.,$$

i.e.

$$E_\pi[\exp(H_\Lambda)F_1\delta^+(u\exp(-DH_\Lambda)) \mid \mathcal{F}_{\Lambda^c}] = E_\pi[\exp(H_\Lambda)\langle uDF_1, \exp(-DH_\Lambda)\rangle_\gamma \mid \mathcal{F}_{\Lambda^c}].$$

Hence

$$\begin{aligned} &\int_{\Gamma^X} \exp(H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda))F_1(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)\delta^+(h)(\tilde{\gamma})\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde{\gamma}) \\ &\qquad = \int_{\Gamma^X} \exp(H_\Lambda(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda))\langle h, DF_1\rangle_\gamma(\tilde{\gamma}_{\Lambda^c}\gamma_\Lambda)\ \pi_{|\mathcal{F}_{\Lambda^c}}(d\gamma;\tilde{\gamma}), \quad \pi(d\tilde{\gamma}) - a.s., \end{aligned}$$

$h \in \mathcal{C}_c(\Lambda)$, with $u = h\exp(-H_\Lambda) \in \mathcal{U}_b^\Phi(\Lambda)$. From Th. 1 we also obtain (5.7), hence μ is the grand canonical Gibbs measure with interaction potential $\Phi : \Gamma^X \longrightarrow \mathbb{R}$. $\square$

Th. 3 does not hold by replacing $\delta^\pm_{\Phi,\Lambda}$ by $\delta^\pm_\Phi$ since even under condition (A) on long range interactions, (5.7) would hold only for Λ_c, $\Lambda \in \mathcal{B}_c(X)$, but not for Λ.

Acknowledgement. I Thank M. Röckner, S. Roelly, H. Zessin and the referee for useful comments.

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