Computations of Greeks in a market with jumps via the Malliavin calculus

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Abstract. Using the Malliavin calculus on Poisson space we compute Greeks in a market driven by a discontinuous process with Poisson jump times and random jump sizes, following a method initiated on the Wiener space in [5]. European options do not satisfy the regularity conditions required in our approach, however we show that Asian options can be considered due to a smoothing effect of the integral over time. Numerical simulations are presented for the Delta and Gamma of Asian options, and confirm the efficiency of this approach over classical finite difference Monte-Carlo approximations of derivatives.

Key words: Greeks, markets with jumps, Asian options, Poisson process, Malliavin calculus

JEL Classification: C15, G12

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1 Introduction

The Malliavin calculus has been recently applied to numerical computations of price sensitivities in continuous financial markets, cf. [4], [5]. In this paper we deal with Asian options in a market model with jumps, and present formulas for the computation of Greeks using a particular version of the Malliavin calculus on Poisson space. The family of jump processes we consider includes sums of independent Poisson processes with arbitrary jump sizes. In the jump case there exist two main approaches to the Malliavin calculus, relying either on finite difference gradients [6], [8], or on differential operators [1], [2]. Finite difference gradients are not appropriate in our context which requires a chain rule of derivation. We choose to use a version of the operator introduced in [2], [3] because it has the derivation property and its adjoint coincides with the Poisson stochastic integral, which provides a natural way to make explicit computations of weights. We will essentially consider an asset price with dynamics given under the risk-neutral probability by

$$dS_t = r_t(N_t)S_t dt + \sigma_t(N_{t^-})S_{t^-}(\beta_{N_{t^-}} dN_t - \nu dt), \tag{1.1}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with constant intensity λ, $(\beta_k)_{k\in\mathbb{N}}$ is a discrete-time stochastic process independent of $(N_t)_{t\in\mathbb{R}_+}$, and $r_t(N_t)$ denotes the interest rate. For example $(\beta_k)_{k\in\mathbb{N}}$ can be a Markov chain taking values in a finite set $\{b_1,\dots,b_d\}$. If $(\beta_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables with distribution $P(\beta_k = b_i) = p_i$, $i = 1,\dots,d$, $k \in \mathbb{N}$, it is well known that we have the identity in law

$$\beta_{N_{t^-}} dN_t = b_1 dN_t^1 + \cdots + b_d dN_t^d,$$

where $N^1,\dots,N^d$ are independent Poisson processes with intensities

$$(\lambda_i)_{i=1,\dots,d} = (p_i\lambda)_{i=1,\dots,d},$$

and $\nu = \lambda\sum_{i=1}^d b_i p_i$. Hence $\beta_{N_{t^-}} dN_t$ can be used to model a finite sum of Poisson processes with arbitrary jump sizes and intensities.

The gradient used in this paper acts only on the Poisson component $(N_t)_{t\in\mathbb{R}_+}$ of this process, described by its jump times $(T_k)_{k\geq 1}$. Given an element w of the Cameron-Martin space H and a smooth functional $F = f(T_1,\dots,T_n)$ of the Poisson process, let

$$D_w F = -\sum_{k=1}^{k=n} w_{T_k}\partial_k f(T_1,\dots,T_n),$$

cf. [9]. The interest in the operator D is that it admits a closable adjoint δ which coincides with the compensated Poisson stochastic integral on adapted processes. The L^2 domain of D_w does not contain the value N_T at time T of the Poisson process (cf. [10] for an extension of D in distribution sense to such functionals), and this excludes in particular European claims of the form $f(N_T)$ from this analysis. Nevertheless, functionals of the form

$$\int_0^T F(t,N_t)dt \tag{1.2}$$

do belong to the domain of D provided that $F(t,k) \in \mathrm{Dom}\,(D)$, $k \in \mathbb{N}$, due to the smoothing effect of the integral. In particular it turns out that when $S_t^\zeta = F^\zeta(t,N_t)$ is the solution of (1.1) and ζ is the value of a parameter (initial condition x, interest rate r, or volatility σ), D_w can be applied to differentiate the value

$$f\left(\int_0^T S_u^\zeta du\right)$$

of an Asian option.
Using an integration by parts formula for the gradient D we will compute the

following Greeks for Asians options in discontinuous markets governed by a Poisson process:

$$\text{Delta} = \frac{\partial C}{\partial x}, \qquad \text{Gamma} = \frac{\partial^2 C}{\partial x^2}, \qquad \text{Rho} = \frac{\partial C}{\partial r}, \qquad \text{Vega} = \frac{\partial C}{\partial \sigma},$$

where

$$C(\zeta) = E\left[f\left(\int_0^T S_u^\zeta du \right) \right]$$

i.e. $C(\zeta)$ is the value of an Asian option with price process $(S_t^\zeta)_{t\in\mathbb{R}_+}$, with respectively $\zeta = x, r, \sigma$. When f is not differentiable, no analytic expression is in general available for such derivatives.

We proceed as follows. Section 2 contains preliminaries on the Malliavin calculus on Poisson space and on the differentiability of functionals of the form (1.2). In Section 3 we present the integration by parts formula which is the main tool to compute the Greeks (i.e. derivatives with respect to ζ) using a random variable called a weight. The market models are presented in Section 4 and explicit computations are carried out for price processes of the form (1.1). In Section 5 we consider the Delta of a binary Asian option, i.e. $f = 1_{[K,\infty[}$, and the Gamma of a standard Asian option, with numerical simulations. These simulations show that the Malliavin approach applied to Asian options in the case of a market driven by a Poisson process is more efficient than the finite difference method. In Section 7 we consider several settings to which our method can be extended.

2 Malliavin Calculus on Poisson space

Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with intensity λ on a probability space $(\Omega, \mathcal{F}, P)$ and let $\tilde{N}_t = N_t - \lambda t$ denote the associated compensated process. Let H denote the Cameron-Martin space

$$H = \left\{ \int_0^{\cdot} \dot{w}_t dt \ : \ \dot{w} \in L^2(\mathbb{R}_+) \right\}.$$

Let $\mathcal{S}$ denote the set of smooth functionals of the form

$$F = f(T_1, \ldots, T_n), \quad f \in \mathcal{C}_b^1(\mathbb{R}^n), \quad n \geq 1,$$

and let

$$\mathcal{U} = \left\{ \sum_{i=1}^{i=n} G_i u_i, \quad G_1, \ldots, G_n \in \mathcal{S}, \ u_1, \ldots, u_n \in H \right\}.$$

Given $w \in H$, let D denote the gradient operator

$$D_w f(T_1, \ldots, T_d) = -\sum_{k=1}^{k=d} w_{T_k} \partial_k f(T_1, \ldots, T_d).$$

Given $u \in \mathcal{U}$ a process of the form

$$u = \sum_{i=1}^{i=n} G_i u_i, \quad G_i \in \mathcal{S}, \ u_i \in H,$$

we also define

$$D_u F = \sum_{i=1}^{i=n} G_i D_{u_i} F.$$

This definition extends to $u \in L^2(\Omega, H)$ with the bound

$$|D_u F| \leq C_F \|\dot{u}\|_{L^2(\mathbb{R}_+)}, \quad a.s.,$$

where $\dot{u}$ denotes the time derivative of $u(t,\omega)$ and C_F is a random variable depending on $F \in \mathcal{S}$. The following proposition is well-known, cf. e.g. [2], [9], [10].

Proposition 1 *a) The operator D is closable and admits an adjoint δ such that*

$$E[D_u F] = E[F\delta(u)], \quad u \in \mathcal{U}, \ F \in \mathcal{S}.$$

b) We have for $F \in \mathrm{Dom}\ (D)$ and $u \in \mathrm{Dom}\ (\delta)$ such that $uF \in \mathrm{Dom}\ (\delta)$:

$$\delta(uF) = F \int_0^T \dot{u}_t d\tilde{N}_t - D_u F. \tag{2.1}$$

c) Moreover, δ coincides with the compensated Poisson stochastic integral on the adapted processes in $L^2(\Omega; H)$:

$$\delta(u) = \int_0^\infty \dot{u}_t d\tilde{N}_t.$$

The domain of the closed extension of D is denoted by Dom (D). Given

$$F : \mathbb{R}_+ \times \mathbb{N} \times \Omega \to \mathbb{R}$$

we define the partial finite difference operator ∇_k as

$$\nabla_k F(t,k) = F(t,k) - F(t,k-1).$$

The following propositions provide general derivation rules for the quantities $\int_0^T F(t,N_t)dt$ and $\int_0^T F(t,N_t)dN_t$, which appear in the solutions of stochastic differential equations such as (1.1).

Proposition 2 *Let $w \in H$ and assume that $F(t,k) \in \mathrm{Dom}\ (D)$, $t \in \mathbb{R}_+$, $k \in \mathbb{N}$. We have*

$$D_w \int_0^T F(t,N_t)dt = \int_0^T w_t \nabla_k F(t,N_t) dN_t + \int_0^T [D_w F](t,N_t)dt.$$

Proof. We have

$$\begin{aligned}
D_w \int_0^T F(t, N_t) dt &= D_w \sum_{k\geq 0} \int_{T_k\wedge T}^{T_{k+1}\wedge T} F(t,k)dt \\
&= -\sum_{l\geq 1} w_{T_l} 1_{[0,T]}(T_l)(F(T_l, l-1) - F(T_l, l)) + \sum_{k\geq 0} \int_{T_k\wedge T}^{T_{k+1}\wedge T} [D_w F](t,k)dt \\
&= \int_0^T w_t \nabla_k F(t, N_t) dN_t + \int_0^T [D_w F](t, N_t) dt.
\end{aligned}$$

□

Proposition 3 *Let $w \in H$ and assume that $F(t,k) \in \mathrm{Dom}\ (D)$, $t \in \mathbb{R}_+$, and $F(\cdot, k) \in \mathcal{C}_c^1([0,T])$ a.s., $k \in \mathbb{N}$. Then*

$$D_w \int_0^T F(t, N_t) dN_t = -\int_0^T w_t \partial_1 F(t, N_t) dN_t + \int_0^T [D_w F](t, N_t) dN_t,$$

where ∂_1 denotes the derivative of $F(t,k)$ with respect to its first variable t.

Proof. We have

$$\begin{aligned}
D_w \int_0^T F(t, N_t) dN_t &= D_w \sum_{k=1}^{\infty} 1_{[0,T]}(T_k) F(T_k, k) \\
&= D_w \sum_{k=1}^{\infty} F(T_k, k) = \lim_{n\to\infty} D_w \sum_{k=1}^{k=n} F(T_k, k) \\
&= -\sum_{k=1}^{\infty} w_{T_k} \partial_1 F(T_k, k) + \sum_{k=1}^{\infty} [D_w F](T_k, k) \\
&= -\int_0^T w_t \partial_1 F(t, N_t) dN_t + \int_0^T [D_w F](t, N_t) dN_t.
\end{aligned}$$

□

The following corollary is a consequence of Prop. 2 and Prop. 3.

Corollary 1 *Let $w, v \in H$ and assume that $F(t,k) \in \mathrm{Dom}\ (D)$, $t \in \mathbb{R}_+$, and $F(\cdot, k) \in \mathcal{C}_c^1([0,T])$, $k \in \mathbb{N}$. Then*

$$\begin{aligned}
D_v D_w \int_0^T F(t, N_t) dt &= -\int_0^T v_t(\dot{w}_t \nabla_k F(t, N_t) + w_t \partial_1 \nabla_k F(t, N_t)) dN_t \\
&\quad + \int_0^T w_t [D_v \nabla_k F](t, N_t) dN_t \\
&\quad + \int_0^T v_t \nabla_k [D_w F](t, N_t) dN_t \\
&\quad + \int_0^T [D_v D_w \nabla_k F](t, N_t) dt.
\end{aligned}$$

Proof. From Prop. 2 we have

$$D_v D_w \int_0^T F(t, N_t) dt \quad = \quad D_v \int_0^T w_t \nabla_k F(t, N_t) dN_t + D_v \int_0^T [D_w F](t, N_t) dt,$$

and the terms in the above summand are computed from Prop. 2 and Prop. 3 respectively. □

The next corollary is stated for deterministic $F(t,k)$ only for the sake of simplicity. The case of a random $F(t,k)$ can also be treated using Prop. 2 and Prop. 3 although with longer calculations.

Corollary 2 *Assume that $F(t,k)$ does not depend on Poisson jump times, i.e.*

$$[D_w F](t,k) = 0, \quad t \in \mathbb{R}_+, \quad k \in \mathbb{N}, \quad w \in H,$$

and that $F(\cdot, k) \in \mathcal{C}_c^2([0,T])$ a.s., $k \in \mathbb{N}$. We have for all $w \in \mathcal{C}_c^2([0,T])$ and $u, v \in H$:

$$D_u D_v D_w \int_0^T F(t, N_t) dt = \int_0^T u_t (\dot{v}_t \dot{w}_t + v_t \ddot{w}_t) \nabla_k F(t, N_t) dN_t$$
$$+ \int_0^T u_t (2 v_t \dot{w}_t + w_t \dot{v}_t) \partial_1 \nabla_k F(t, N_t) dN_t + \int_0^T u_t v_t w_t \partial_1^2 \nabla_k F(t, N_t) dN_t,$$

where $\ddot{w}_t$ denotes the second derivative of ω_t with respect to t.

Proof. We use the expression

$$D_v D_w \int_0^T F(t, N_t) dt = - \int_0^T v_t (\dot{w}_t \nabla_k F(t, N_t) + w_t \partial_1 \nabla_k F(t, N_t)) dN_t$$

obtained from Cor. 1, and apply Prop. 3. □

3 Computations of Greeks

We present the integration by parts formula which follows from a classical Malliavin calculus argument applied to the derivation operator D, and is essential to the computation of Greeks. Let (a,b) be an open interval of $\mathbb{R}$.

Proposition 4 *Let $(F^\zeta)_{\zeta \in (a,b)}$ and $(G^\zeta)_{\zeta \in (a,b)}$, be two families of random functionals, continuously differentiable in* Dom (D) *in the parameter $\zeta \in (a,b)$. Let $(w_t)_{t \in [0,T]}$ be a process satisfying*

$$D_w F^\zeta \neq 0, \quad a.s. \text{ on } \{\partial_\zeta F^\zeta \neq 0\}, \quad \zeta \in (a,b),$$

and such that $w G^\zeta \partial_\zeta F^\zeta / D_w F^\zeta$ is continuous in ζ in Dom (δ). *We have*

$$\frac{\partial}{\partial \zeta} E\left[G^\zeta f(F^\zeta)\right] \quad = \quad E\left[f(F^\zeta) \delta\left(G^\zeta w \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right] + E\left[\partial_\zeta G^\zeta f(F^\zeta)\right], \quad (3.1)$$

for any function f such that $f(F^\zeta) \in L^2(\Omega)$, $\zeta \in (a,b)$.

Proof. Assuming that $f \in \mathcal{C}_b^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\frac{\partial}{\partial\zeta} E\left[G^\zeta f(F^\zeta)\right] &= E\left[G^\zeta f'\left(F^\zeta\right)\partial_\zeta F^\zeta\right] + E\left[\partial_\zeta G^\zeta f(F^\zeta)\right] \\
&= E\left[G^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta} D_w f(F^\zeta)\right] + E\left[\partial_\zeta G^\zeta f(F^\zeta)\right] \\
&= E\left[f(F^\zeta)\delta\left(wG^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right] + E\left[\partial_\zeta G^\zeta f(F^\zeta)\right].
\end{aligned}$$

The extension to square-integrable f can be obtained from the same argument as in p. 400 of [5], using the bound

$$\begin{aligned}
&\left|\frac{\partial}{\partial\zeta} E\left[G^\zeta f_n(F^\zeta)\right] - E\left[f(F^\zeta)\left(\delta\left(G^\zeta w \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) + \partial_\zeta G^\zeta\right)\right]\right| \\
&\quad \leq \|f(F^\zeta) - f_n(F^\zeta)\|_{L^2(\Omega)} \left\|\delta\left(G^\zeta w \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) + \partial_\zeta G^\zeta\right\|_{L^2(\Omega)},
\end{aligned}$$

and an approximating sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions. □

Using (2.1), the weight $\delta\left(wG^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)$ can be computed using Poisson stochastic integrals:

$$\begin{aligned}
\delta\left(wG^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) &= G^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\int_0^T \dot{w}_t d\tilde{N}_t - D_w\left(G^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) \\
&= G^\zeta \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\int_0^T \dot{w}_t dN_t - G^\zeta \frac{D_w \partial_\zeta F^\zeta}{D_w F^\zeta} \\
&\quad + G^\zeta \frac{\partial_\zeta F^\zeta}{\left(D_w F^\zeta\right)^2} D_w D_w F^\zeta - \frac{\partial_\zeta F^\zeta}{D_w F^\zeta} D_w G^\zeta.
\end{aligned}$$

First derivatives

In particular, first derivatives such as the Delta, Rho and Vega can be computed from

$$\frac{\partial}{\partial\zeta} E\left[f(F^\zeta)\right] = E\left[f(F^\zeta)\delta\left(w\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right],$$

with, from (2.1):

$$\delta\left(w\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right) = \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\int_0^T \dot{w}_t dN_t - \frac{D_w \partial_\zeta F^\zeta}{D_w F^\zeta} + \frac{\partial_\zeta F^\zeta}{\left(D_w F^\zeta\right)^2} D_w D_w F^\zeta. \quad (3.2)$$

Second derivatives

Assume that $w \in \mathcal{C}_c^2([0,T])$. Concerning second derivatives we have

$$\frac{\partial^2}{\partial\zeta^2} E\left[f(F^\zeta)\right] = \frac{\partial}{\partial\zeta} E\left[f(F^\zeta)\delta(G^\zeta w)\right] \quad (3.3)$$

$$= \quad E\left[f(F^\zeta)\frac{\partial}{\partial\zeta}\delta(G^\zeta w)\right] + E\left[f(F^\zeta)\delta\left(\delta(G^\zeta w)G^\zeta w\right)\right],$$

with $G^\zeta = \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}$, and from (2.1):

$$\begin{aligned}
\delta\left(\delta(G^\zeta w)G^\zeta w\right) &= G^\zeta\delta(G^\zeta w)\int_0^T \dot{w}_t d\tilde{N}_t - D_w(G^\zeta\delta(G^\zeta w)) \\
&= G^\zeta\delta(G^\zeta w)\int_0^T \dot{w}_t d\tilde{N}_t - \delta(G^\zeta w)D_w G^\zeta \\
&\quad - G^\zeta D_w\left(G^\zeta\int_0^T \dot{w}_t d\tilde{N}_t - D_w G^\zeta\right) \\
&= \left(G^\zeta\int_0^T \dot{w}_t d\tilde{N}_t - D_w G^\zeta\right)^2 \\
&\quad - G^\zeta\left(D_w G^\zeta\int_0^T \dot{w}_t d\tilde{N}_t + G^\zeta\int_0^T w_t\ddot{w}_t dN_t - D_w D_w G^\zeta\right) \\
&= \left(\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\int_0^T \dot{w}_t d\tilde{N}_t + \frac{\partial_\zeta F^\zeta}{(D_w F^\zeta)^2}D_w D_w F^\zeta - \frac{D_w\partial_\zeta F^\zeta}{D_w F^\zeta}\right)^2 \\
&\quad - \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\left(\left(-\frac{\partial_\zeta F^\zeta}{(D_w F^\zeta)^2}D_w D_w F^\zeta + \frac{D_w\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\int_0^T \dot{w}_t d\tilde{N}_t\right. \\
&\quad + \frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\int_0^T w_t\ddot{w}_t dN_t - \frac{D_w D_w\partial_\zeta F^\zeta}{D_w F^\zeta} + 2D_w\partial_\zeta F^\zeta\frac{D_w D_w F^\zeta}{(D_w F^\zeta)^2} \\
&\quad \left. + \partial_\zeta F^\zeta\frac{D_w D_w D_w F^\zeta}{(D_w F^\zeta)^2} - 2\partial_\zeta F^\zeta\frac{(D_w D_w F^\zeta)^2}{(D_w F^\zeta)^3}\right).
\end{aligned}$$

Delta in the linear case

This is a first derivative with $F^x = xF$. Then $\partial_x F^x = F$ and the weight for the Delta is

$$\delta\left(w\frac{\partial_x F^x}{D_w F^x}\right) = \frac{1}{x}\left(\frac{F}{D_w F}\int_0^T \dot{w}_t d\tilde{N}_t - 1 + \frac{F}{(D_w F)^2}D_w D_w F\right). \tag{3.4}$$

Gamma in the linear case

This is a second derivative, with $F^x = xF$. The weight associated to the Gamma is computed via (3.3) with

$$G^x = \frac{\partial_x F^x}{D_w F^x} = \frac{F}{xD_w F} \quad \text{and} \quad \frac{\partial}{\partial x}G^x = -\frac{1}{x^2}\frac{F}{D_w F},$$

i.e.

$$\text{Gamma} = \frac{-\text{Delta}}{x} + E[f(F^\zeta)\delta\left(\delta(G^x w)G^x w\right)], \tag{3.5}$$

with

$$\delta\left(\delta(G^x w) G^x w\right) = \frac{1}{x^2}\left(\frac{F}{D_w F}\int_0^T \dot{w}_t d\tilde{N}_t - 1 + \frac{F}{(D_w F)^2} D_w D_w F\right)^2 \tag{3.6}$$
$$-\frac{F}{x^2 D_w F}\left(\left(\left(1 - \frac{F}{(D_w F)^2} D_w D_w F\right)\int_0^T \dot{w}_t d\tilde{N}_t + \frac{F}{D_w F}\int_0^T w_t \ddot{w}_t dN_t\right.\right.$$
$$\left.\left. + F\left(\frac{D_w D_w D_w F}{(D_w F)^2} - 2\frac{(D_w D_w F)^2}{(D_w F)^3}\right) + \frac{D_w D_w F}{D_w F}\right),$$

with $w \in \mathcal{C}_c^2([0,T])$. In the next section, these general formulas are specialized to the model described by (1.1).

4 Market model

In this section we make explicit computations for an underlying asset price given under the risk-neutral probability by the linear equation

$$dS_t = r_t(N_t) S_t dt + \sigma_t(N_{t^-}) S_{t^-}(\beta_{N_{t^-}} dN_t - \nu dt), \tag{4.1}$$

whose solution can be written under the form $F(t, N_t)$. For simplicity the random dependence on β_k will not be mentioned as it plays no role in the integration by parts since β_k is independent of $(N_t)_{t\in\mathbb{R}_+}$. As noted in the introduction we may consider as a particular case d independent Poisson processes $N^1, \dots, N^d$ with intensities $\lambda_1, \dots, \lambda_d$, $\lambda = \lambda_1 + \cdots + \lambda_d$, and a sequence $(\beta_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with values in $b_1, \dots, b_d$, and distribution

$$P(\beta_k = b_i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_d}, \quad i = 1, \dots, d, \quad k \in \mathbb{N}.$$

In this case we have the identity in law:

$$b_1 dN_t^1 + \cdots + b_d dN_t^d = \beta_{N_{t^-}} dN_t,$$

and (4.1) can be written as

$$dS_t = r_t(N_t) S_t dt + S_{t^-}\sigma_t(N_{t^-}) \sum_{i=1}^{d} b_i (dN_t^i - \lambda_i dt), \tag{4.2}$$

with $\nu = \sum_{i=1}^{d} b_i \lambda_i$, i.e. we are in a market driven by a sum of independent Poisson processes with arbitrary jump sizes. Coming back to the general case we write (4.1) as

$$dS_t = \alpha_t(N_t) S_t dt + \sigma_t(N_{t^-}) S_{t^-} \beta_{N_{t^-}} dN_t, \quad S_0 = x,$$

where

$$\alpha_t(k) = r_t(k) - \nu\sigma_t(k), \quad k \in \mathbb{N}.$$

The next result is an application of Prop. 2 to the solution of (4.1) which can be written as

$$S_t = F(t, N_t),$$

with

$$F(t,k) = xe^{\int_0^t \alpha_s(N_s)ds} \prod_{i=1}^{i=k} (1 + \beta_{i-1}\sigma_{T_i}(i-1)).$$

A differentiability hypothesis is required on σ.

Proposition 5 *Assume that $\sigma_\cdot(k) \in \mathcal{C}_b^1(\mathbb{R}_+)$ and $1+\beta_k\sigma_\cdot(k) > 0$, for all $k \in \mathbb{N}$. We have*

$$\begin{aligned} D_w \int_0^T S_u du &= \int_0^T w_t \sigma_t(N_{t^-}) S_{t^-} \beta_{N_{t^-}} dN_t + \int_0^T S_t \int_0^t w_s \nabla_k \alpha_s(N_s) ds dt \\ &\quad - \int_0^T S_t \int_0^t \frac{\dot{\sigma}_s(N_{s^-})}{1+\beta_{N_{s^-}}\sigma_s(N_{s^-})} \beta_{N_{s^-}} dN_s dt. \end{aligned} \tag{4.3}$$

Proof. We have

$$\nabla_k F(t,k) = \beta_{k-1}\sigma_{T_k}(k-1) F(t,k-1),$$

moreover $\partial_1 F(t,k) = \alpha_t(k) F(t,k)$, hence

$$\begin{aligned} D_w F(t,k) &= F(t,k) D_w \int_0^t \alpha_s(N_s) ds + F(t,k) \sum_{i=1}^{i=k} \frac{\beta_{i-1}\dot{\sigma}_{T_i}(i-1)}{1+\beta_{i-1}\sigma_{T_i}(i-1)} \\ &= F(t,k) \int_0^t w_s \nabla_k \alpha_s(N_s) ds + F(t,k) \sum_{i=1}^{i=k} \frac{\beta_{i-1}\dot{\sigma}_{T_i}(i-1)}{1+\beta_{i-1}\sigma_{T_i}(i-1)}, \end{aligned}$$

and

$$\begin{aligned} [D_w F](t, N_t) &= F(t,N_t) \int_0^t w_s \nabla_k \alpha_s(N_s) ds \\ &\quad + F(t,N_t) \int_0^t \frac{\dot{\sigma}_s(N_{s^-})}{1+\beta_{N_{s^-}}\sigma_s(N_{s^-})} \beta_{N_{s^-}} dN_s. \end{aligned}$$

We conclude using Prop. 2. □

The second and third derivatives are obtained as applications of Cor. 1 and Cor. 2 in the following proposition.

Proposition 6 *Let $w \in \mathcal{C}_c^1([0,T])$. Assume that α_t does not depend on k and that σ is constant. We have*

$$D_w D_w \int_0^T S_u du = -\int_0^T w_t (\dot{w}_t \sigma S_{t^-} + w_t \sigma \alpha_t S_{t^-}) \beta_{N_{t^-}} dN_t. \tag{4.4}$$

Assuming further that α does not depend on t and $w \in \mathcal{C}_c^2([0,T])$, we have:

$$D_w D_w D_w \int_0^T S_u du = \int_0^T w_t \sigma \left(\dot{w}_t^2 + 3\alpha w_t \dot{w}_t + w_t \ddot{w}_t + \alpha^2 w_t^2 \right) S_{t^-} \beta_{N_{t^-}} dN_t. \tag{4.5}$$

Again, the hypothesis of the above proposition are stated only to simplify the calculations of the Greeks:

Delta in the linear case

The corresponding weight is obtained from (3.4) and (4.3), (4.4) and is equal to:

$$\frac{1}{x\sigma}\left(\frac{\int_0^T S_t dt \int_0^T \dot{w}_t d\tilde{N}_t}{\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t} - 1 - \frac{\int_0^T S_t dt \int_0^T w_t\left(\dot{w}_t + \alpha w_t\right) S_{t^-}\beta_{N_{t^-}} dN_t}{\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\right).$$

Note that unlike in the Brownian case ([4]), the weight is not a function of $\left(S_T, \int_0^T S_u du\right)$.

Gamma in the linear case

The corresponding weight is given by (3.5) and (3.6), with from (4.3)-(4.5):

$$\delta\left(\delta(G^x w)G^x w\right) = \tag{4.6}$$

$$\begin{aligned}
&\frac{1}{x^2\sigma^2}\left(\frac{\int_0^T S_t dt \int_0^T \dot{w}_t d\tilde{N}_t}{\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t} - 1 - \frac{\int_0^T S_t dt \int_0^T w_t\left(\dot{w}_t + \alpha w_t\right) S_{t^-}\beta_{N_{t^-}} dN_t}{\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\right)^2 \\
&-\frac{\int_0^T S_t dt \int_0^T \dot{w}_t d\tilde{N}_t}{x^2\sigma \int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t}\left(1 + \frac{\int_0^T S_t dt \int_0^T w_t\left(\dot{w}_t + \alpha w_t\right) S_{t^-}\beta_{N_{t^-}} dN_t}{\sigma\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\right) \\
&+\frac{\left(\int_0^T S_t dt\right)^2}{x^2\sigma^2\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\int_0^T w_t \ddot{w}_t dN_t \\
&+\frac{\left(\int_0^T S_t dt\right)^2}{x^2\sigma \int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t}\left(\frac{2(\int_0^T w_t\left(\dot{w}_t + \alpha w_t\right) S_{t^-}\beta_{N_{t^-}} dN_t)^2}{\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\right. \\
&\left.-\frac{\int_0^T w_t\left(\dot{w}_t^2 + 3\alpha w_t \dot{w}_t + w_t \ddot{w}_t + \alpha^2 w_t^2\right) S_{t^-}\beta_{N_{t^-}} dN_t}{\sigma\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}\right) \\
&-\int_0^T S_t dt \frac{\int_0^T w_t\left(\dot{w}_t + \alpha w_t\right) S_{t^-}\beta_{N_{t^-}} dN_t}{\sigma\left(\int_0^T w_t S_{t^-}\beta_{N_{t^-}} dN_t\right)^2}.
\end{aligned} \tag{4.7}$$

Vega

The Vega of an Asian option with payoff $f(F^\sigma) = f\left(\int_0^T S_u du\right)$ is given by (3.2) and

$$\partial_\sigma F^\sigma = \int_0^T S_t \left(\sum_{k=1}^{N_t} \frac{\beta_{k-1}}{1+\sigma\beta_{k-1}} - \nu t\right) dt,$$

hence from Prop. 2:

$$D_w \partial_\sigma F^\sigma = \int_0^T w_u S_{u^-} \left(\beta_{N_{u^-}}(1+\sigma) \sum_{k=1}^{N_{u^-}} \frac{\beta_{k-1}}{1+\sigma\beta_{k-1}} - \nu u\right) dN_u,$$

with $D_w F^\sigma$, $D_w D_w F^\sigma$ given by (4.3), (4.4).

Rho

The Rho of an Asian option with payoff $f\left(\int_0^T S_u du\right)$ is given by (3.2) and

$$\partial_r F^r = \int_0^T t S_t dt, \qquad D_w \partial_r F^r = \sigma \int_0^T t w_t S_{t^-} dN_t,$$

with $D_w F^\sigma$, $D_w D_w F^\sigma$ given by (4.3), (4.4).

5 Numerical simulations

We present simulations for the Delta and the Gamma of Asian options, successively the Delta of a binary Asian option with strike price K:

$$C(x) = e^{-rT} E\left[1_{[K,\infty[}\left(\frac{1}{T}\int_0^T S_t^x dt\right)\right],$$

and the Gamma of a standard Asian option:

$$C(x) = e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_t^x dt - K\right)^+\right].$$

We consider a simplified model with constant parameters σ and r, first with a fixed jump size, and then with multiple random jump sizes independent from $(N_t)_{t\in\mathbb{R}_+}$. In the case of constant interest rate and volatility, the price of the underlying asset is given by

$$S_t = x e^{\alpha t} \prod_{i=1}^{i=N_t} (1+\sigma\beta_{i-1}) = f(x, t, N_t), \quad t \in [0, T],$$

with $f(x,t,k) = xe^{\alpha t} \prod_{i=1}^{i=k}(1+\sigma\beta_{i-1})$. Prop. 4 can be applied to $F^x = \int_0^T S_t dt$, with

$$\begin{aligned} D_w \int_0^T S_t dt &= \sigma \int_0^T w_t S_{t^-} \beta_{N_{t^-}} dN_t \\ &= x\sigma \sum_{k=1}^{k=N_T} w_{T_k} \beta_{k-1} \prod_{i=1}^{i=k-1} (1+\sigma\beta_{i-1}) e^{\alpha T_k}, \qquad (5.1) \end{aligned}$$

$$\begin{aligned} D_w D_w \int_0^T S_t dt &= -\sigma \int_0^T w_t \left(\dot{w}_t + \alpha w_t\right) S_{t^-} \beta_{N_{t^-}} dN_t \qquad (5.2) \\ &= -x\sigma \sum_{k=1}^{k=N_T} w_{T_k} \beta_{k-1} e^{\alpha T_k} (\dot{w}_{T_k} + \alpha w_{T_k}) \prod_{i=1}^{i=k-1} (1+\sigma\beta_{i-1}), \end{aligned}$$

and

$$\begin{aligned} D_w D_w D_w \int_0^T S_t dt &= \sigma \int_0^T w_t \left(\dot{w}_t^2 + 3\alpha w_t \dot{w}_t + w_t \ddot{w}_t + \alpha^2 w_t^2\right) S_{t^-} \beta_{N_{t^-}} dN_t \\ = \; & x\sigma \sum_{k=1}^{k=N_T} w_{T_k} \beta_{k-1} e^{\alpha T_k} \left(\dot{w}_{T_k}^2 + 3\alpha w_{T_k} \dot{w}_{T_k} + w_{T_k} \ddot{w}_{T_k} + \alpha^2 w_{T_k}^2\right) \\ & \times \prod_{i=1}^{i=k-1} (1+\sigma\beta_{i-1}), \qquad (5.3) \end{aligned}$$

if $w \in \mathcal{C}_c^2([0,T])$. The finite difference method gives Delta as

$$\text{Delta} = \frac{C(x+\epsilon) - C(x-\epsilon)}{2\epsilon}.$$

For the Malliavin approach we take $w_t = \sin(\pi t/T)$ (so that $\int_0^T \dot{w}_t dt = 0$), and $T = 500$, $x = 10$, $K = 15000$, $\alpha = 0.009$, $\sigma = 0.01$, $\lambda = 1$, and $\epsilon = 0.001$.

The following graphs allow to compare both methods on several sample sizes. We start with the case of a fixed jump size $\beta = 1$.

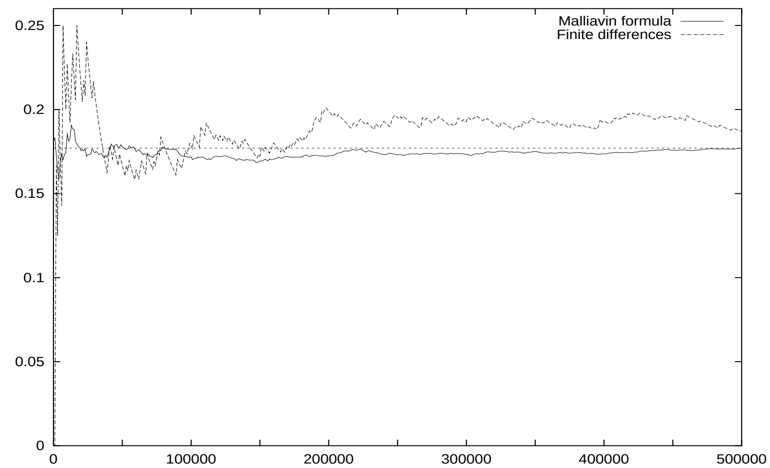

Figure 1 - 500000 simulations for Delta with $K = 15000$ and $\beta = 1$

The same simulation is presented with a larger sample size:

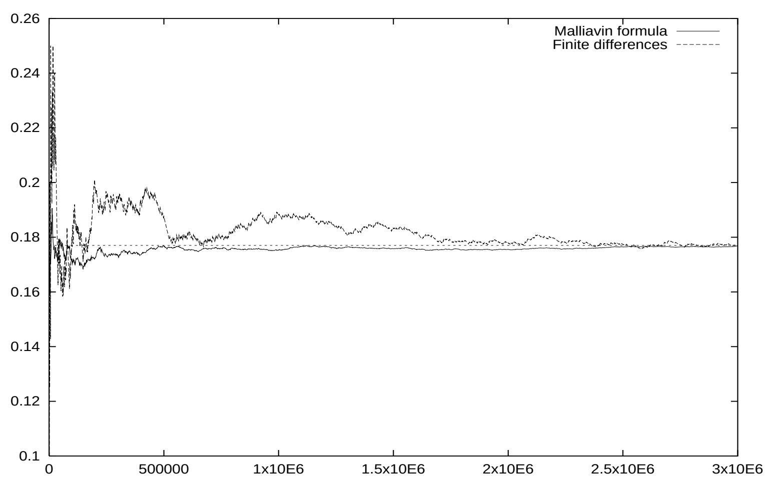

Figure 2 - 3×10^6 simulations for Delta with $K = 15000$ and $\beta = 1$

In the next simulation we increase the value of K to $K = 28000$.

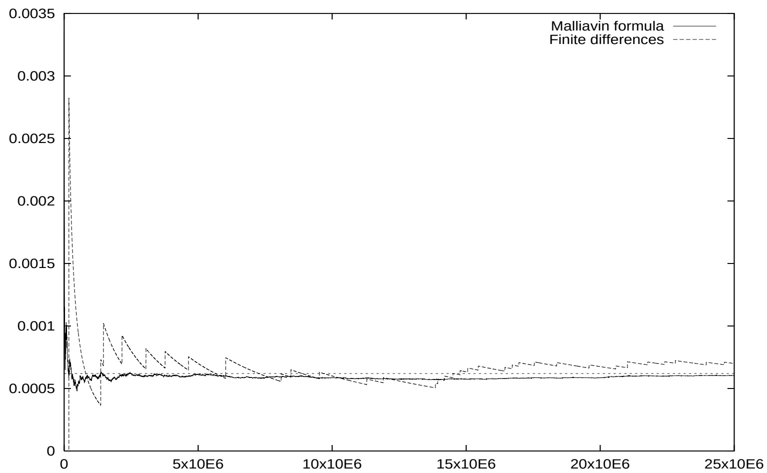

Figure 3 - 25×10^6 simulations for Delta with $K = 28000$ and $\beta = 1$

Next we present two simulations of Delta in models with multiple random jump sizes, for K=2500.

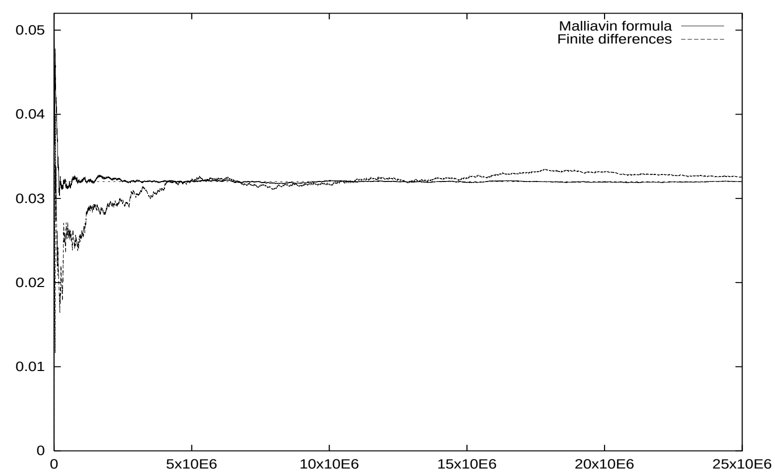

Figure 4 - 25×10^6 simulations for Delta with jump sizes in $\{-1, 2\}$

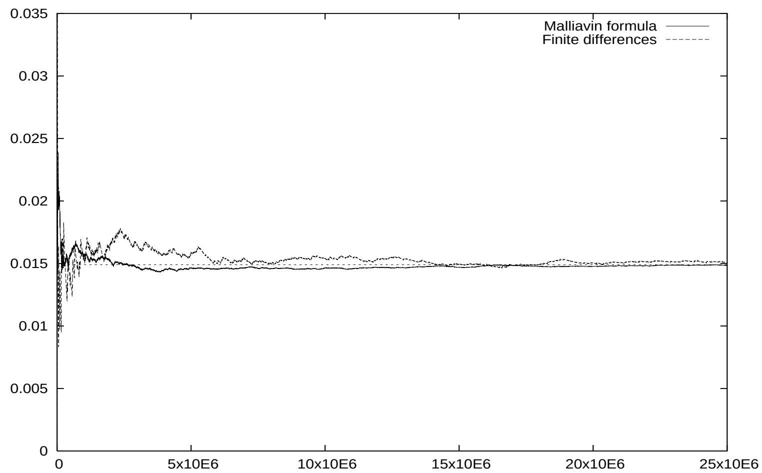

Figure 5 - 25×10^6 simulations for Delta with jump sizes in $\{-2.5, -1.5, 1, 2.2, 3\}$

For the Gamma, the finite difference are computed via

$$\text{Gamma} = \frac{C(x+\epsilon) - 2C(x) + C(x-\epsilon)}{\epsilon^2}.$$

The Malliavin method uses (3.3) and (4.6). We take $w_t = \sin(\pi t/T)$, and the values $T = 100$, $x = 10$, $K = 30$, $r = 0.009$, $\sigma = 0.01$, $\epsilon = 0.001$, and a fixed jump size $\beta = 1$.

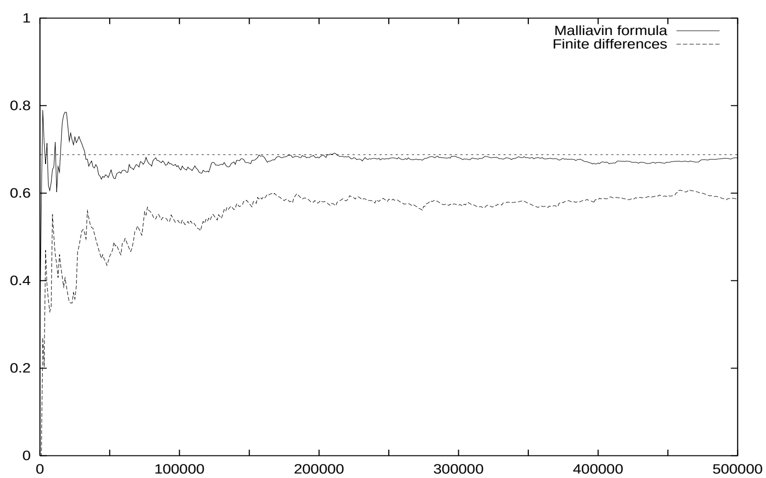

Figure 6 - 500000 simulations for Gamma

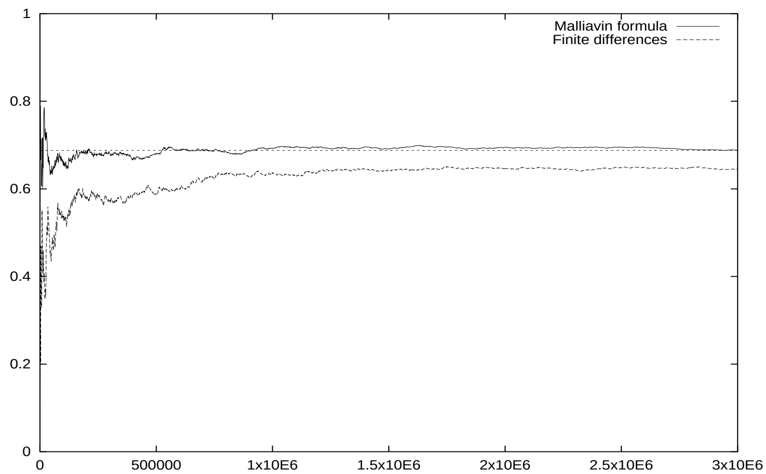

Figure 7 - 3×10^6 simulations for Gamma

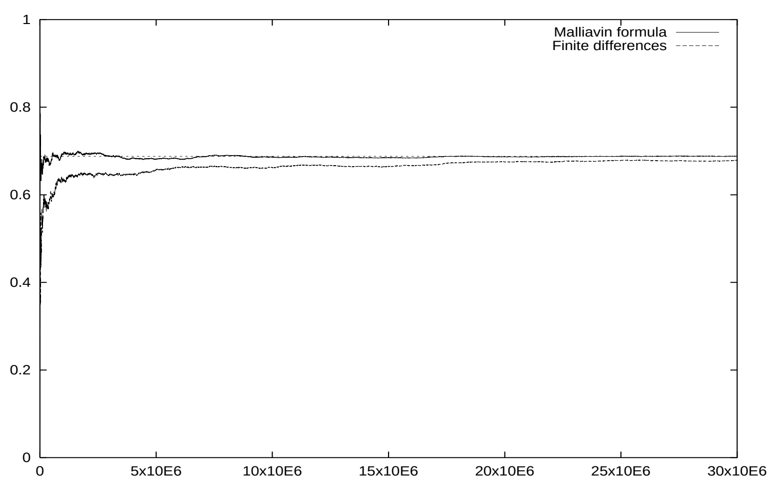

Figure 8 - 30×10^6 simulations for Gamma

6 Conclusion

The simulation graphs show a faster and better convergence of the Greeks obtained from the Malliavin method on Poisson space for Asian options in a market with jumps, when compared to the finite difference approximations. When performing simulations, the Malliavin method turned out to be more efficient for out-of-the-money options.

7 Extensions

In this section we consider two more general settings which can be treated by the above method. We first consider a model with state-dependent coefficients given by a nonlinear equation of the form

$$dS_t = \alpha_t(S_t)dt + \sigma_t(S_{t^-})\beta_{N_{t^-}} dN_t, \quad S_0 = x, \tag{7.1}$$

since S_t does have an expression in terms of the jump times and the flow associated to $dx_t = \alpha_t(x_t)dt$. In this model and the following, the computations of $\int_0^T S_t dt$ and its derivatives are still possible recursively (although more complicated) using the general results of Section 3. More precisely we have on $\{N_t = k\}$:

$$S_t = \Phi_{T_k,t}(S_{T_k}),$$

and

$$S_{T_k} = (1 + \beta_{k-1}\sigma_{T_k}(S_{T_{k-1}})\Phi_{T_{k-1},T_k}(S_{T_{k-1}}))\Phi_{T_{k-1},T_k}(S_{T_{k-1}}),$$

where $\Phi_{s,t}$ is the flow defined by

$$dx_t = \alpha_t(x_t)dt,$$

i.e.

$$\Phi_{s,t}(x) = x + \int_s^t \alpha_u(x_u)du, \quad x_s = x.$$

Secondly, although this paper focuses on the Poisson case an independent diffusion term can be introduced in the driving stochastic differential equation as in the complete market model of [7]:

$$dS_t = r_t S_t dt + \sigma_t S_{t^-}\left(1_{\{\phi_t=0\}} dB_t + \phi_t(\beta_{N_{t^-}} dN_t - \nu_t dt)\right), \quad t \in \mathbb{R}_+,$$

where $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a deterministic bounded functions satisfying $1 + \sigma_t\beta_{N_{t^-}}\phi_t > 0$, $t \in \mathbb{R}_+$, and $(B_t)_{t\in\mathbb{R}_+}$ is a Brownian motion independent of $(N_t)_{t\in\mathbb{R}_+}$. In this case S_t still has an explicit form in terms of jump times:

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \sigma_s 1_{\{\phi_s=0\}} dB_s + \int_0^t (r_s - \phi_s\nu_s\sigma_s)ds - \frac{\sigma_s^2}{2}\int_0^t 1_{\{\phi_s=0\}} ds\right) \\ &\quad \times \prod_{k=1}^{k=N_t} (1 + \sigma_{T_k}\beta_{k-1}\phi_{T_k}), \quad t \in \mathbb{R}_+. \end{aligned}$$

In this way one can use either the method of [5] to perturb the Brownian component, or our method to deal with the Poisson part. Note however that the Poisson and Brownian components should mutually exclude each other (as a result of the presence of $(\phi_t)_{t\in\mathbb{R}}$), otherwise $D_w \int_0^T S_t dt$ will contain Brownian indefinite stochastic integrals evaluated at Poisson jump times, which will not belong to the domain of D_w.

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