Applied Probability Trust (July 8, 2008)

HYPOTHESIS TESTING AND SKOROKHOD STOCHASTIC INTEGRATION

NICOLAS PRIVAULT,* *Université de La Rochelle*

Abstract

We define a class of anticipating flows on Poisson space and compute its Radon-Nikodym derivative. This result is applied to statistical testing in an anticipating queuing problem.

HYPOTHESIS TESTING; QUEUING THEORY; SKOROKHOD INTEGRAL; POINT PROCESSES

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 62F03; 60H07
SECONDARY 60K25; 60J75

1. Introduction

In the Itô construction of stochastic integration, adaptedness conditions are imposed on the integrand. In the anticipative case there exists several extensions of the stochastic integral. Among these extensions the ones that seems to be the most closely related to concrete situations are the pathwise Stratonovich and forward integral, cf. e.g. [17] and the references therein. However these integrals do not retain certain natural properties of the Itô integral, for example they do not have expectation zero in the anticipative case. On the other hand, the Skorokhod integral, cf. [18] is an extension of the stochastic integral that possesses the latter property, and acts on stochastic processes without adaptedness requirement. It can be defined as the dual of a gradient operator, which makes it useful in the analysis on Wiener space and the Malliavin calculus, cf. [11]. See for example [6] for a discussion on the connection of the Skorokhod integral on the Wiener space to engineering problems. On the Poisson space, as in most non-gaussian settings, cf. [4], [12], [14], there exists two different Skorokhod integral operators defined as the adjoints of different gradient operators. Their common property is to coincide with the Itô integral on adapted integrands.

The aim of this paper is to show that on the Poisson space one of the constructions of the Skorokhod integral can be connected via hypothesis testing to an engineering problem.

We proceed as follows. Sect. 2 consists in a description of a queuing problem in which jobs are processed by a server. At time zero a prediction of expected completion times is made, and has a Poisson distribution over $\mathbf{R}_+$. The processing speed of the server changes over time, and its increase or decrease at time t is governed by a function that may depend on all of the processing times, including predicted completion times. In

* Postal address: Département de Mathématiques, Université de La Rochelle, Avenue Marillac, 17042 La Rochelle Cedex 1, France.

this way a flow of transformations of Poisson trajectories is constructed, and this flow is naturally anticipating with respect to the Poisson filtration since it acts on whole Poisson trajectories. The construction of this flow is formalised in Sect. 3. In Sect. 4 we devise statistical procedures for testing and estimation using the Radon-Nikodym density function of the flow as a likelihood ratio, see Th. 1. Given a sample trajectory made of completion times that are predicted or have been measured at time t, we test the hypothesis "the sample is Poisson distributed" at time t. The main tool is an anticipative Girsanov theorem on Poisson space, which is presented in Sect. 5. In Sect. 6 the queuing problem is formulated in a more abstract way, and a Girsanov theorem for anticipating flows on Poisson space is proved. We refer to [2], [3], for the analog of this result on the Wiener space, to [9] for the anticipative Girsanov theorem on the Wiener space and to [19] for its extension to non-invertible shifts. Whereas in the adapted case the equation satisfied by the process of Radon-Nikodym densities is a well-known linear stochastic differential equation, in our case the equation remains formally the same except that the Itô integral has to be replaced by the Skorokhod integral. This shows the relation between anticipative stochastic integration in the Skorokhod sense and the queuing problem considered above. Anticipating stochastic differential equations on the Poisson space have been studied in [10], [13] using the Skorokhod integral of [12] and in [16] using the integral of [4].

2. An anticipating queuing problem

The aim of this section is to state the considered queuing problem. For simplicity of exposition we adopt an intuitive approach that will be formalized in the next section. Let B be the vector space of sequences

$$B = \{(\omega_k)_{k\geq 1} \ : \ \omega_k \in \mathbf{R}, \ k \geq 1\},$$

with the norm

$$\|w\|_B = \sup_{k\geq 1} \frac{|\omega_k|}{k}.$$

Let $H = l^2(\mathbf{N}^*)$, and let $(e_k)_{k\geq 1}$ denote the canonical basis of H. Let P be the probability measure on the Borel σ-algebra of B under which the coordinate functionals

$$\begin{aligned} \tau_k : B \quad & \longrightarrow \mathbf{R}, \quad k \geq 1, \\ \omega \quad & \mapsto \tau_k(\omega) = \omega_k \end{aligned}$$

are independent identically distributed exponential random variables, cf. [14]. We let $T_0 = 0$ and define the family $(T_k)_{k\geq 1}$ as $T_k = \sum_{i=1}^{i=k} \tau_i$, $k \geq 1$, i.e. $(T_k)_{k\geq 1}$ represents the jump times of the standard Poisson process

$$N_t = \sum_{k=1}^{\infty} 1_{[T_k,\infty[}(t), \ t \in [0,1].$$

We consider that the sequence $(\tau_k)_{k\geq 1}$ represents an estimation $(\tau_k^{0,0})_{k\geq 1}$ made at time $t = 0$ of processing times of a given countable sequence of jobs. While the k-th job is being processed, the server is able to modify its speed by taking into account the processing times of all jobs in the sequence, whether they are completed or not, i.e. predicted completion times may also be taken into account. For $t \in [0,1]$ we denote by $T_k^{0,t}$ the estimation or measure at time $t \geq 0$ of the completion time of job n°k, where $T_k^0 = T_k$, $k \geq 1$, and

- $0 \leq t < T_{k-1}^{0,t}$ means job k is not yet processed,
- $T_{k-1}^{0,t} \leq t < T_k^{0,t}$ means job k is being processed,
- $T_k^{0,t} \leq t$ means job k is already completed.

We also let $\tau_k^{0,t} = T_k^{0,t} - T_{k-1}^{0,t}$, $k \geq 1$, $t \in [0,1]$. The processing speed is controlled by a function $\sigma : [0,1] \times B \longrightarrow \mathbf{R}$ which depends on time as well as on all processing times (measured or predicted), and satisfies the following hypothesis.

Hypothesis (H) *We assume that* ess sup $\sigma < 1$ *and that for any* $k \geq 1$ *there is a random variable* G_k *which is* $\sigma(\tau_i \ : \ i \neq k)$ *measurable with*

$$|\sigma_t(\omega) - \sigma_t(\omega + xe_k)| \leq |x| G_k(\omega), \ x \in \mathbf{R}, \ t \in [0,1], \ \omega \in B, \ k \geq 1.$$

With this notation, the evolution of $t \mapsto T_k^{0,t}$ is described inductively on $k \geq 1$ as follows. Let $T_0^{0,\infty} = 0$ and let $t \mapsto \tau_{k+1}^{0,t}$ be the solution, for $t > T_k^{0,\infty}$, of the ordinary differential equation

$$\frac{d}{dt}\tau_{k+1}^{0,t} = \sigma_t(\tau_1^{0,\infty}, \ldots, \tau_k^{0,\infty}, \tau_{k+1}^{0,t}, \tau_{k+2}^0, \ldots), \quad t \geq T_k^{0,\infty}, \quad k \geq 0. \tag{2.1}$$

Hypothesis (H) ensures the existence and uniqueness of a solution to (2.1).
One lets

$$T_{k+1}^{0,\infty} = \inf\{t \in \mathbf{R}_+ \ : \ T_k^{0,\infty} + \tau_{k+1}^{0,t} = t\}, \quad \tau_{k+1}^{0,\infty} = T_{k+1}^{0,\infty} - T_k^{0,\infty},$$

and

$$T_{k+1}^{0,t} = \begin{cases} T_k^{0,t} + \tau_{k+1}^0 & \text{for } 0 \leq t \leq T_k^{0,\infty}, \\ T_k^{0,t} + \tau_{k+1}^{0,t} & \text{for } T_k^{0,\infty} \leq t \leq T_{k+1}^{0,\infty}, \\ T_{k+1}^{0,\infty} & \text{for } t > T_{k+1}^{0,\infty}, \end{cases}$$

The expected remaining time until completion of job k is $T_k^{0,t} - t$. This definition of the flow can be summarized as

$$\frac{d}{dt}\tau_k^{0,t} = 1_{[T_{k-1}^{0,t}, T_k^{0,t}[}(t)\sigma_t(\tau_1^{0,\infty}, \ldots, \tau_{k-1}^{0,\infty}, \tau_k^{0,t}, \tau_{k+1}^0, \ldots), \quad k \geq 1, \ t \in [0,1],$$

with the initial condition $\tau_k^{0,0} = \tau_k$, $k \geq 1$. Hypothesis (H) also implies that $\lim_{k\to\infty} T_k^{0,\infty} = +\infty$. The condition ess sup $\sigma < 1$, ensures that all jobs can terminate in finite time. Fig. 1 gives a typical graphic representation of $t \mapsto (T_k^{0,t})_{k\geq 1}$.

Remark 1 The statements $T_{k-1}^{0,s} \leq s < T_k^{0,s}$ *and* $T_{k-1}^{0,t} \leq s < T_k^{0,t}$ *are equivalent,* $0 \leq s \leq t \leq 1$, $k \geq 1$.

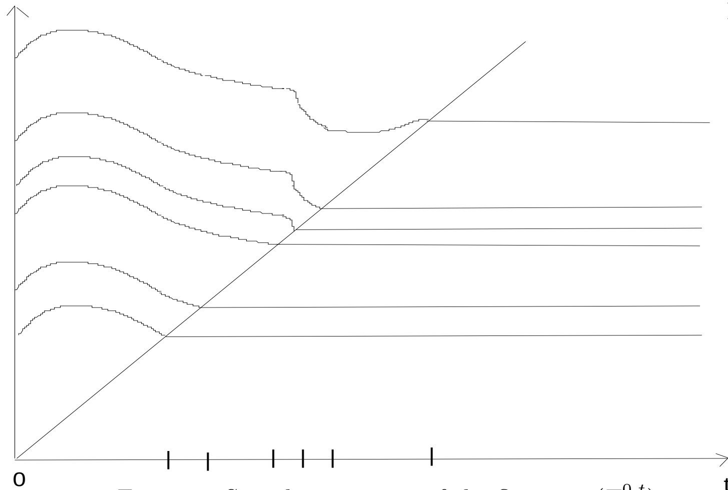

Figure 1. Sample trajectory of the flow $t \mapsto (T_k^{0,t})_{k\geq 1}$.

3. Construction of the flow

In this section we formalise the definition of the flow as a differential equation in the Banach space B. Let $i : H \to L^2([0,1])$ be the random mapping defined as

$$i_t(f) = f(N_t + 1) = \sum_{k=1}^{\infty} f(k) 1_{[T_{k-1},T_k[}(t), \quad t \in [0,1]. \tag{3.1}$$

Let $(j_s)_{s\in[0,1]}$ denote the H-valued process defined as

$$j_s = (i_s(e_k))_{k\geq 1}, \quad s \in [0,1],$$

i.e. $j_s = e_k \in H$ if and only if $s \in [T_{k-1}, T_k[$, $k \geq 1$.

Proposition 3.1 Let $\sigma : [0,1] \times B \longrightarrow \mathbf{R}$ satisfy **(H)**.

- *The equation in B*

$$\phi_{s,t}\omega = \omega + \int_s^t (j_r\sigma_r)(\phi_{s,r}\omega)dr, \quad \omega \in B, \quad 0 \leq s,t \leq 1, \tag{3.2}$$

  has a unique solution that defines $\phi_{s,t} : B \longrightarrow B$.

- *We have $j_s = j_s \circ \phi_{t,s}$, $0 \leq s < t$, and (3.2) is equivalent to*

$$\phi_{t,s}\omega = \omega - \int_s^t j_r\sigma_r(\phi_{t,r}\omega)dr, \quad \omega \in B, \quad 0 \leq s \leq t \leq 1. \tag{3.3}$$

- *The family $(\phi_{s,t} : B \longrightarrow B)_{0\leq s\leq t}$ satisfies the flow property*

$$\phi_{s,t} \circ \phi_{u,s} = \phi_{u,t}, \quad u,s,t \geq 0, \tag{3.4}$$

and $\phi_{s,t} : B \longrightarrow B$, is invertible with inverse $\phi_{t,s}$, $0 \leq s,t \leq 1$.

Proof. Existence and uniqueness of the solution of (3.2) follow from the Lipschitz hypothesis (H) on σ. The flow property (3.4) follows from

$$\begin{aligned}\phi_{s,t}\circ\phi_{u,s}\omega &= \phi_{u,s}\omega + \int_s^t (j_r\sigma_r)(\phi_{s,r}\circ\phi_{u,s}\omega)dr \\ &= \omega + \int_u^s (j_r\sigma_r)(\phi_{u,r}\omega)dr + \int_s^t (j_r\sigma_r)(\phi_{s,r}\circ\phi_{u,t}\omega)dr.\end{aligned}$$

From Remark 1 we have $j_r\circ\phi_{0,r} = j_r\circ\phi_{0,t}$ hence $j_r\circ\phi_{t,r} = j_r$, $r\leq t$, by composition with $\phi_{t,0}$, and (3.2) is equivalent to (3.3). Note that (3.3) is wrong if $s>t$, this point will be important in the calculations of Sect. 6, Lemma 5. The notation

$$T_k^{s,t} = \sum_{i=1}^{i=k}\phi_{s,t}(i), \quad 0\leq s,t\leq 1,$$

is consistent with that of the preceding section.

4. Hypothesis testing

Statistical testing for point processes, cf. e.g. [8], often aims to test an hypothesis on the intensity of a Poisson process. The central tool of this approach is the computation of the Radon-Nikodym derivative $L_{0,t} = d\tilde{P}^{0,t}/dP$ where $\tilde{P}^{0,t}$ is a probability under which $(T_k^{0,t})_{k\geq 1}$ is Poisson distributed. Let $t\in[0,1]$. We will test the hypothesis

$$H_0\text{: } (\tau_k^{0,t})_{k\geq 1} \text{ is not exponentially i.i.d.}$$

against the hypothesis

$$H_1\text{: } (\tau_k^{0,t})_{k\geq 1} \text{ is exponentially i.i.d.,}$$

i.e. the sample $(T_k^{0,t})_{k\geq 1}$ is Poisson distributed, e.g. $(\tau_k^{0,t})_{k\geq 1}$ does not result of a perturbation of $(\tau_k)_{k\geq 1}$ driven by the function σ. The following decision rule is justified from the fact that if E is an event such that $P(E)\leq\beta$ then $\tilde{P}^{0,t}(L_{0,t}\geq\alpha)\geq\tilde{P}^{0,t}(E)$, cf. [1], [8].

Decision rule *Let $\alpha\in\mathbf{R}$ and $\beta>0$ such that $P(L_{0,t}\geq\alpha)=\beta$. Then the hypothesis H_1 is accepted at the level β whenever $L_{0,t}\geq\alpha$.*

The Likelihood ratio $L_{0,t}$ is usually computed via the Girsanov theorem for point processes, cf. [1], [7]. However this theorem relies on the adaptedness assumptions of the Itô stochastic calculus, hence it is not applicable to our problem. For this reason we use an anticipative Girsanov theorem on Poisson space, cf. [15], in order to find a probability $\tilde{P}^{0,t}$ under which $(\tau_k^{0,t})_{k\geq 1}$ is exponentially i.i.d. and to compute $d\tilde{P}^{0,t}/dP$. We define a space of smooth random variables

$$\mathcal{S} = \left\{ f_n(\tau_1,...,\tau_n) \ : \ f_n\in\mathcal{C}_c^\infty(\mathbf{R}_+^n),\ n\geq 1\right\},$$

and an operator $\tilde{D}:\mathcal{S}\longrightarrow L^2(B\times[0,1])$ by

$$\tilde{D}F = -\sum_{k=1}^{\infty} 1_{[T_{k-1},T_k[}\partial_k f_n(\tau_1,\dots,\tau_n), \tag{4.1}$$

where $F \in \mathcal{S}$ is of the form $f_n(\tau_1, \ldots, \tau_n)$. A discrete-time gradient $D : \mathcal{S} \longrightarrow L^2(B \times \mathbb{N}^*)$ is also defined by

$$DF = (D_k F)_{k \geq 1} = (\partial_k f_n(\tau_1, \ldots, \tau_n)), \quad F \in \mathcal{S}. \tag{4.2}$$

The operators $D : L^2(B) \to L^2(B) \otimes H$, and $\tilde{D} : L^2(B) \to L^2(B \times [0,1])$ are closable and $\mathbb{D}_{1,2}$ denotes the domain of the closed extension of $\tilde{D}$. They are linked by the relation $\tilde{D} = -i \circ D$, where $i : H \to L^2([0,1])$ is the random mapping defined in (3.1). The closable adjoint $\tilde{\delta} : L^2(B \times [0,1]) \longrightarrow L^2(B)$ of $\tilde{D}$ corresponds to one of two notions of Skorokhod integral on the Poisson space, cf. [4] and [14]. Its interpretation as an extension of the stochastic integral with respect to the compensated Poisson process comes from the fact that $\tilde{\delta}(u)$ coincides with the Itô stochastic integral of u if u is adapted and square-integrable.

Definition 1 We call $\mathbb{D}_{1,\infty}$ the subspace of $\mathbb{D}_{1,2}$ made of the random variables F such that

$$\|F\|_{\mathbb{D}_{1,\infty}} = \|F\|_\infty + \||DF|_H\|_\infty < \infty.$$

We also let $\mathbb{L}_{1,\infty} = L^\infty([0,1], \mathbb{D}_{1,\infty})$, and $\mathbb{L}_{1,2} = L^2([0,1], \mathbb{D}_{1,2})$.

If $\mathcal{T} : B \longrightarrow B$ is measurable we denote by $\mathcal{T}P$ the image measure of P by $\mathcal{T}$. We say that $\mathcal{T}$ is absolutely continuous if $\mathcal{T}P$ is absolutely continuous with respect to P. The following is the main result of this paper, and will be proved in Sect. 6. For clarity we may denote $\sigma_t(\omega)$ by $\sigma(t, \omega)$. In particular, $\sigma(T_k^{0,t}, \phi_{0,T_k^{0,t}})$ denotes $\left(\sigma_r \circ \phi_{0,T_k^{0,t}}\right)_{|r=T_k^{0,t}}$.

Theorem 1 Let $\sigma \in \mathbb{L}_{1,\infty}$. We assume that σ has a version with continuous trajectories and $\sup \sigma < 1$. Then the equation in B

$$\phi_{s,t}\omega = \omega + \int_s^t (j_r \sigma_r)(\phi_{s,r}\omega) dr, \quad \omega \in B, \quad 0 \leq s,t \leq 1,$$

has a unique solution. Moreover, $\phi_{s,t}P$ is absolutely continuous, $0 \leq s,t \leq 1$, and for $0 \leq s \leq t \leq 1$ we have

$$\frac{d\phi_{t,s}P}{dP} = \exp\left(-\int_s^t \left[\tilde{D}_r \sigma_r\right] \circ \phi_{s,r} dr - \int_s^t \sigma_r \circ \phi_{s,r} dr \right) \prod_{k=1}^{k=N_t} (1 - \sigma(T_k^{s,t}, \phi_{s,T_k^{s,t}}))^{-1}, \tag{4.3}$$

and

$$\frac{d\phi_{s,t}P}{dP} = \exp\left(\int_s^t \left[\tilde{D}_r \sigma_r\right] \circ \phi_{t,r} dr + \int_s^t \sigma_r \circ \phi_{t,r} dr \right) \prod_{k=1}^{k=N_t} (1 - \sigma(T_k, \phi_{t,T_k})). \tag{4.4}$$

Note that hypothesis (H) is not assumed here in order to obtain the existence and uniqueness of $\phi_{s,t}$. Also, the process $(\tilde{D}_r \sigma_r)_{r \in [0,1]}$ is well-defined since $t \mapsto \tilde{D}_t \sigma_r$ is constant on each interval $]T_{k-1}, T_k[$, $k \geq 1$, a.s. In fact, from Remark 1 we have $j_r \circ \phi_{s,r} = j_r \circ \phi_{s,t}$, $0 \leq s \leq r \leq t \leq 1$, and (4.3) can be rewritten using D as

$$\frac{d\phi_{t,s}P}{dP} = \exp\left(-\int_s^t \sigma_r \circ \phi_{s,r} dr + \sum_{k=1}^{\infty} \int_{t \wedge T_{k-1}^{s,t}}^{t \wedge T_k^{s,t}} [D_k \sigma_r] \circ \phi_{s,r} dr \right) \prod_{k=1}^{k=N_t} \frac{1}{1 - \sigma(T_k^{s,t}, \phi_{s,T_k^{s,t}}))},$$

$0 \leq s \leq t \leq 1$. In the adapted case, $\tilde{D}_r \sigma_r = 0$, cf. [14], hence the terms $\int_0^t \left[\tilde{D}_r \sigma_r\right] \circ \phi_{s,r} dr$ and $\int_0^t \left[\tilde{D}_r \sigma_r\right] \circ \phi_{t,r} dr$ vanish in (4.3) and (4.4) and we obtain the classical expression of the Radon-Nikodym density function. Denoting by $(N_r^{0,t})_{r \in [0,1]}$ the point process whose jump times are given by $(T_k^{0,t})_{k \geq 1}$:

$$N_s^{0,t} = \sum_{k \geq 1} 1_{[T_k^{0,t}, \infty[}(s), \quad s \in [0,1].$$

We have $L_{0,t} = \frac{d\phi_{t,0}P}{dP}$ and the log-likelihood ratio becomes

$$l_{0,t} = -\int_0^t \left[\tilde{D}_r \sigma_r\right] \circ \phi_{0,r} dr - \int_0^t \sigma_r \circ \phi_{0,r} dr - \int_0^t \log(1 - \sigma_r \circ \phi_{0,r}) dN_r^{0,t}.$$

From Remark 1 we have explicitly

$$l_{0,t} = -\int_0^t \sigma_r \circ \phi_{0,r} dr - \int_0^t \log(1 - \sigma_r \circ \phi_{0,r}) dN_r^{0,t} + \sum_{k=1}^{\infty} \int_{t \wedge T_{k-1}^{0,t}}^{t \wedge T_k^{0,t}} [D_k \sigma_r] \circ \phi_{0,r} dr.$$

The evaluation of $l_{0,t}$ is made according to measures or estimations between time 0 and time t of completion times.

5. Anticipating Girsanov theorem

We now introduce a formalism which is helpful for the proof of the anticipating Girsanov theorem Th. 3. Given a real separable Hilbert space X with orthonormal basis $(h_i)_{i \geq 1}$, let

$$\mathcal{S}(X) = \left\{ \sum_{i=1}^{i=n} F_i h_i \ : \ F_1, \ldots, F_n \in \mathcal{S}, \quad n \geq 1 \right\},$$

with $\mathcal{S} = \mathcal{S}(\mathbf{R})$. Let $H \otimes X$ denote the completed Hilbert-Schmidt tensor product of H with X. Any $u \in \mathcal{S}(H \otimes X)$, is written as

$$u = \sum_{k=1}^{\infty} u_k e_k, \quad u_k \in \mathcal{S}(X), \ k \geq 1.$$

Let

$$\mathcal{U}(X) = \left\{ \sum_{k=1}^{\infty} \tau_k u_k e_k \ : \ u \in \mathcal{S}(H \otimes X) \right\}.$$

It is known that $\mathcal{S}(X)$ is dense in $L^2(B,P;X)$, and that $\mathcal{U}(X)$ is dense in $L^2(B\times \mathbb{N}^*;X)$, cf. [14]. We extend the definition of D to $\mathcal{S}(X)$ as

$$(DF,h)_H = \lim_{\varepsilon\to 0} \frac{F(\omega+\varepsilon h)-F(\omega)}{\varepsilon}, \quad F\in\mathcal{S}(X).$$

We call $\mathbb{D}_{1,2}(X)$ the completion of $\mathcal{S}(X)$ with respect to the norm

$$\|F\|_{\mathbb{D}_{1,2}(X)} = \||F|_X\|_2 + \||DF|_{H\otimes X}\|_2,$$

We define $\delta : \mathcal{S}(H\otimes X) \longrightarrow L^2(B;X)$ by

$$\delta(u) = \sum_{k=1}^{\infty} u_k - D_k u_k, \quad u\in\mathcal{S}(H\otimes X). \tag{5.1}$$

The operators $D:\mathcal{S}(X)\to L^2(B\times\mathbb{N}^*;X)$ and $\delta:\mathcal{U}(X)\to L^2(B;X)$ are closable and mutually adjoint:

$$E\left[(DF,u)_{H\otimes X}\right] = E\left[(\delta(u),F)_X\right], \quad u\in\mathcal{U}(X), F\in\mathcal{S}(X).$$

With this notation the anticipating Girsanov theorem (Th. 1 of [15]) can be formulated as follows.

Theorem 2 Let $F:B\to H$ be a measurable mapping such that

- *$h\mapsto F(\omega+h)$ is continuously differentiable on H in the completed tensor product $H\otimes H$, for any $\omega\in B$,*
- *$F_k = 0$ on $\{\tau_k = 0\}$, $k\geq 1$,*
- *$I_B + F$ leaves invariant the cone $\{(\omega_k)_{k\geq 1}\in B : \omega_k > 0,\ k\geq 1\}$ of strictly positive sequences,*
- *$\det_2(I_H + DF) \neq 0$, a.s., and*
- *$I_B + F$ is a.s. bijective.*

Then

$$E[f] = E\left[f\circ(I_B+F)|\Lambda_F|\right], \quad f\in\mathcal{C}_b^+(B).$$

The functional Λ_F is the density $d(I_B+F)^{-1}P/dP$, with

$$\Lambda_F = \det_2(I_H+DF)\exp(-\delta(F)), \quad F\in\mathbb{D}_{1,2}(H)\cap\mathrm{Dom}(\delta), \tag{5.2}$$

where $\det_2(I_H+K)$ is the Carleman-Fredholm determinant of I_H+K:

$$\det_2(I_H+K) = \prod_{i=1}^{\infty}(1+\lambda_i)\exp(-\lambda_i),$$

$(\lambda_k)_{k\geq 1}$ being the eigenvalues of the Hilbert-Schmidt operator K, counted with their multiplicities, cf. [5], Th. 26. If $F\in Dom(\delta)\cap\mathbb{D}_{1,2}(H)$ and DF is a trace class operator, then $\sum_{k=1}^{\infty} F_k$ is summable, a.s., and Λ_F admits from (5.1) the simpler factorization

$$\Lambda_F = \det(I_H + DF) \exp\left(- \sum_{k=1}^{\infty} F_k \right), \tag{5.3}$$

where det is the limit of finite-dimensional classical determinants. We note that Th. 2 is not directly applicable in the present situation where the transformation $I_B + F = \phi_{0,t}$ is given by a flow which is solution of a differential equation. In particular the differentiability hypothesis of Th. 2 are not directly verified and the Carleman-Fredholm determinant $\det_2(I_H + DF)$ remains to be explicitly computed in terms of σ. For this reason we use Th. 1.

6. Proofs

We now prove Th. 1, using the formalism of Sect. 5.

Definition 2 Let $\mathcal{V}$ denote the class of processes of the form $v(\cdot, \tau_1, \ldots, \tau_n)$, where $v \in \mathcal{C}_c^1([0,1] \times \mathbf{R}^n)$ satisfies $v(t, x_1, \ldots, x_n) = 0$ if $t \geq x_1 + \cdots + x_n$, $n \geq 1$.

We will need the following Lemmas, which are adapted from [3]. They do not rely on the nature of the underlying (Poisson or Wiener) measure, hence their proofs are similar to that of [3], see also [16].

Lemma 1 Let $F \in \mathbb{D}_{1,2}$. For any $\varepsilon > 0$, there is a sequence $(F_n)_{n \in \mathbf{N}} \subset \mathcal{S}$ that converges to F in $\mathbb{D}_{1,2}$ and such that

- ess inf $F < F_n <$ ess sup F, $n \in \mathbf{N}$.
- $\| \| DF_n |_H \|_\infty \leq \| \| DF |_H \|_\infty + \varepsilon$, $n \in \mathbf{N}$.

Lemma 2 Let $\sigma \in \mathbb{L}_{1,\infty}$ with ess sup $\sigma < 1$.

- *There is a sequence $(\sigma^n)_{n \in \mathbf{N}} \subset \mathcal{V}$, uniformly bounded in $\mathbb{L}_{1,\infty}$, that converges to σ in $\mathbb{L}_{1,2}$, with $\sup_{n,t,\omega} \sigma_t^n(\omega) < 1$.*
- *If σ has a version (also denoted by σ) with continuous trajectories, then the sequence $(\sigma^n)_{n \in \mathbf{N}}$ can be chosen such that $(\sigma^n_{T_k})_{n \in \mathbf{N}}$ converges in $L^2(B)$ to σ_{T_k}, $k \geq 1$.*

Lemma 3 Let $\mathcal{T}, \mathcal{R}$ be two absolutely continuous transformations, respectively defined by

$$\mathcal{T}(\omega) = \omega + \int_0^1 j_s(\omega) U_s(\omega) ds,$$

and

$$\mathcal{R}(\omega) = \omega + \int_0^1 j_s(\omega) V_s(\omega) ds,$$

$\omega \in B$, with $U, V \in L^2(B) \otimes L^2([0,1])$. Let $F \in I\!D_{1,\infty}$. We have

$$|F \circ \mathcal{T} - F \circ \mathcal{R}| \leq \| |DF|_H \|_\infty |U - V|_{L^2([0,1])}, \quad a.s.,$$

and

$$|F(\omega) - F(\omega + h)| \leq \| |DF|_H \|_\infty \|h\|_H, \quad h \in H, \ a.s. \tag{6.1}$$

Lemma 4 Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous transformations with

$$\mathcal{T}_n\omega = \omega + \int_0^1 U_s^n(\omega)ds,$$

defined by a sequence $(U^n)_{n\in\mathbb{N}} \subset L^2(B) \otimes L^2([0,1]; H)$ of processes that converges to $U \in L^2(B) \otimes L^2([0,1]; H)$, such that the sequence of densities $(L^n)_{n\in\mathbb{N}} = (d\mathcal{T}_n P/dP)_{n\in\mathbb{N}}$ is uniformly integrable. If $(F_n)_{n\in\mathbb{N}}$ converges to F in probability, then $(F_n \circ \mathcal{T}_n)_{n\in\mathbb{N}}$ converges to $F \circ \mathcal{T}$ in probability, where $\mathcal{T} : B \longrightarrow B$ is defined by

$$\mathcal{T}\omega = \omega + \int_0^1 U_s(\omega)ds.$$

Moreover, $\mathcal{T} : B \longrightarrow B$ is absolutely continuous.

The proofs of Lemma 1, 2, 3 and 4 are postponed to the end of this section.

Lemma 5 Let $\sigma \in \mathcal{V}$ and $0 \leq s \leq t \leq 1$. We have for $F = \phi_{t,s} - I_B$:

$$\Lambda_F = \exp\left(-\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{s,r}dr - \int_s^t \sigma_r \circ \phi_{s,r}dr\right) \prod_{k=1}^{k=N_t} (1 - \sigma(T_k^{s,t}, \phi_{s,T_k^{s,t}}))^{-1},$$

and for $F = \phi_{s,t} - I_B$:

$$\Lambda_F = \exp\left(\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{s,r}dr + \int_s^t \sigma_r \circ \phi_{t,r}dr\right) \prod_{k=1}^{k=N_t} (1 - \sigma(T_k, \phi_{t,T_k})).$$

Proof. (of Lemma 5). We use the expression (5.2) of Λ_F. We have for $0 \leq s \leq t \leq 1$, using (3.3):

$$D_k\phi_{t,s}(l) = \begin{cases} -\sigma(T_l, \phi_{t,T_l})1_{\{s<T_l<t\}} + \sigma(T_{l-1}, \phi_{t,T_{l-1}})1_{\{s<T_{l-1}<t\}} \\ -\displaystyle\int_s^t i_r(e_l) \sum_{i=1}^{i=n} D_k\phi_{t,r}(i) \left[D_i\sigma_r\right] \circ \phi_{t,r}dr, & 1 \leq k < l \leq n, \\ \\ 1 - \sigma(T_l, \phi_{t,T_l})1_{\{k\leq l\}}1_{\{s<T_l<t\}} \\ -\displaystyle\int_s^t i_r(e_l) \sum_{i=1}^{i=n} D_k\phi_{t,r}(i) \left[D_i\sigma_r\right] \circ \phi_{t,r}dr, & 1 \leq k = l \leq n, \\ \\ -\displaystyle\int_s^t i_r(e_l) \sum_{i=1}^{i=n} D_k\phi_{t,r}(i) \left[D_i\sigma_r\right] \circ \phi_{t,r}dr, & 1 \leq l < k \leq n, \\ \\ 1_{\{k=l\}}, & k > n. \end{cases}$$

Let $M_{t,s} \in \mathcal{M}_n(\mathbf{R})$ be the $n \times n$ matrix

$$M_{t,s} = (D_k \phi_{t,s}(l))_{1 \le k,l \le n}, \quad 0 \le s \le t \le 1.$$

The above computation can be rewritten as

$$M_{t,s} = A_{t,s} + \int_s^t M_{t,r} Q_{t,r} dr, \quad 0 \le s \le t \le 1, \tag{6.2}$$

where $A_{t,s} \in \mathcal{M}_n(\mathbf{R})$ is the $n \times n$ matrix

$$A_{t,s}(i,j) = \begin{cases} -\sigma(T_j, \phi_{t,T_j}) 1_{\{s<T_j<t\}} + \sigma(T_{j-1}, \phi_{t,T_{j-1}}) 1_{\{s<T_{j-1}<t\}}, & 1 \le i < j \le n, \\ 1 - \sigma(T_j, \phi_{t,T_j}) 1_{\{s<T_j<t\}}, & 1 \le i = j \le n, \\ 0, & 1 \le j < i \le n, \end{cases}$$

and $Q_{t,r} \in \mathcal{M}_n(\mathbf{R})$ is the $n \times n$ matrix $Q_{t,r} = (i_r(e_l)\,[D_k \sigma_r] \circ \phi_{t,r})_{1 \le k,l \le n}$. We have from (3.4):

$$D\phi_{t,u} = D(\phi_{s,u} \circ \phi_{t,s}) = [D\phi_{t,s}]\,[D\phi_{s,u}] \circ \phi_{t,s}, \quad 0 \le u \le s \le t \le 1,$$

hence

$$M_{t,u} = [M_{t,s}]\,[M_{s,u} \circ \phi_{t,s}], \qquad 0 \le u \le s \le t \le 1.$$

We will show that for fixed $\omega \in B$,

$$\det M_{t,s} = \exp\left(-\int_s^t \text{trace } Q_{t,r} dr\right), \quad T_{k-1} < s \le t < T_k, \ k \ge 1. \tag{6.3}$$

Let $\varepsilon > 0$. We have

$$\begin{aligned} \det M_{t,s} - \det M_{t,s-\varepsilon} &= \det M_{t,s} - \det(M_{t,s} M_{s,s-\varepsilon} \circ \phi_{t,s}) \\ &= (1 - \det M_{s,s-\varepsilon} \circ \phi_{t,s}) \det M_{t,s} \\ &= (\det M_{t,s}) \left(1 - det\left(A_{s,s-\varepsilon} \circ \phi_{t,s} - \int_{s-\varepsilon}^s M_{t,r} \circ \phi_{t,s} Q_{t,r} \circ \phi_{t,s} dr\right)\right). \end{aligned}$$

Moreover,

$$\det\left(A_{s,s-\varepsilon} \circ \phi_{t,s} - \int_{s-\varepsilon}^s M_{t,r} \circ \phi_{t,s} Q_{t,r} \circ \phi_{t,s} dr\right)$$

is equivalent to $1 - \varepsilon \text{trace } Q_{s,s} \circ \phi_{t,s}$ as ε goes to zero since for $T_{k-1} < s < s + \varepsilon < t < T_k$, we have $A_{s,s-\varepsilon} = I_{\mathbf{R}^n}$ and $M_{t,r} \longrightarrow I_{\mathbf{R}^n}$ as $r \downarrow s$. Moreover, from Remark 1 we have $Q_{s,s} \circ \phi_{t,s} = Q_{t,s}$, hence for $T_{k-1} < s \le t < T_k$,

$$\frac{d}{ds}(\det M_{t,s}) = \text{trace } (Q_{t,s}) \det M_{t,s}, \tag{6.4}$$

which proves (6.3). Since $Q_{t,s}(i,j)$ may be nonzero only if $T_{j-1} \leq s < T_j$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
\text{trace } (Q_{t,s}) &= \sum_{k=1}^{\infty} 1_{[T_{k-1},T_k[}(s) \, [D_k\sigma_s] \circ \phi_{t,s} = \sum_{k=1}^{\infty} i_s(e_k) \, [D_k\sigma_s] \circ \phi_{t,s} \\
&= \sum_{k=1}^{\infty} i_s(e_k) \circ \phi_{t,s} \, [D_k\sigma_s] \circ \phi_{t,s} = -\left[\tilde{D}_s\sigma_s\right] \circ \phi_{t,s},
\end{aligned}$$

hence for $T_{l-1} < s \leq t < T_l$:

$$\det(M_{t,s}) = \exp\left(\int_s^{T_l \wedge t} \left[\tilde{D}_r\sigma_r\right] \circ \phi_{t,r} dr\right) \det M_{t,t\wedge T_l}, \quad l \in \mathbf{N},$$

and

$$\det(M_{t,T_{l-1}}) = (1 - \sigma(T_{l-1}, \phi_{t,T_{l-1}})) \exp\left(\int_s^{t\wedge T_l} \left[\tilde{D}_r\sigma_r\right] \circ \phi_{t,r} dr\right) \det M_{t,t\wedge T_l}.$$

Hence for $0 \leq s \leq t \leq 1$,

$$\det(D\phi_{t,s}) = \exp\left(\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{t,r} dr\right) \prod_{s<T_k<t} (1 - \sigma(T_k, \phi_{t,T_k})) \,. \tag{6.5}$$

Hence from (5.3), (6.5) and $\displaystyle\sum_{k=1}^{\infty} F_k = \sum_{k=1}^{\infty} \int_{t\wedge T_{k-1}}^{t\wedge T_k} \sigma_r \circ \phi_{t,r} dr$, we obtain

$$\begin{aligned}
\frac{d\phi_{s,t}P}{dP} &= \exp\left(\sum_{k=1}^{\infty} \int_{t\wedge T_{k-1}}^{t\wedge T_k} \sigma_r \circ \phi_{t,r} dr\right) \det D\phi_{t,s} \\
&= \exp\left(\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{t,r} dr + \int_s^t \sigma_r \circ \phi_{t,r} dr\right) \prod_{k=1}^{k=N_t} (1 - \sigma(T_k, \phi_{t,T_k})).
\end{aligned}$$

Moreover, we have $\phi_{t,T_k} \circ \phi_{s,t} = \phi_{s,T_k^{s,t}}$, $T_k < t$, since

$$\begin{aligned}
\phi_{t,T_k} \circ \phi_{s,t}\omega &= \phi_{s,t}\omega - \left(\int_{T_k}^t \sigma_r \circ \phi_{t,r} dr\right) \circ \phi_{s,t}\omega \\
&= \phi_{s,t}\omega - \int_{T_k^{s,t}}^t \sigma_r(\phi_{t,r} \circ \phi_{s,t}\omega) dr \\
&= \omega + \int_s^t \sigma_r(\phi_{s,r}\omega) dr - \int_{T_k^{s,t}}^t \sigma_r(\phi_{t,r} \circ \phi_{s,t}\omega) dr \\
&= \omega + \int_s^{T_k^{s,t}} \sigma_r(\phi_{s,r}\omega) dr = \phi_{s,T_k^{s,t}}\omega, \quad \omega \in B.
\end{aligned}$$

Hence for $0 \le s \le t \le 1$,

$$\begin{aligned}
\frac{d\phi_{t,s}P}{dP} &= \left(\frac{d\phi_{s,t}P}{dP}\right)^{-1} \circ \phi_{s,t} \\
&= \exp\left(-\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{s,r} dr - \int_s^t \sigma_r \circ \phi_{s,r} dr\right) \prod_{k=1}^{k=N_t} (1-\sigma(T_k, \phi_{t,T_k}) \circ \phi_{s,t})^{-1} \\
&= \exp\left(-\int_s^t \left[\tilde{D}_r\sigma_r\right] \circ \phi_{s,r} dr - \int_s^t \sigma_r \circ \phi_{s,r} dr\right) \prod_{k=1}^{k=N_t} (1-\sigma(T_k^{s,t}, \phi_{s,T_k^{s,t}}))^{-1}.
\end{aligned}$$

Proof. (of Th. 1). We start by assuming that $\sigma \in \mathcal{V}$. In this case the assumptions of Th. 2 are satisfied by $F = \phi_{s,t} - I_B$:

$$E\left[f \circ \phi_{t,s}\right] = E\left[f \Lambda_F\right], \qquad E\left[f\right] = E\left[f \circ \phi_{s,t}\Lambda_F\right], \quad f \in \mathcal{C}_b^+(B),$$

where the expression of $\Lambda_F = d\phi_{t,s}/dP$ is given by Lemma 5, $0 \le s \le t \le 1$, this statement is in fact finite dimensional. It remains to extend this result to $\sigma \in \mathbb{L}_{1,\infty}$. For this we follow [16] which uses ideas applied in the Wiener case by [3]. From (6.1), $\sigma \in \mathbb{L}_{1,\infty}$ satisfies **(H)**. From Lemma 2 we choose a sequence $(\sigma^n)_{n\in\mathbf{N}} \subset \mathcal{V}$ that converges to σ in $\mathbb{L}_{1,2}$, with $\|\log(1-\sigma^n)\|_\infty < C$ and $\|\sigma^n\|_\infty + \||D\sigma^n|_H\|_\infty \le C$, $n \in \mathbf{N}$, for some $C > 0$. The sequence $(\sigma^n)_{n\in\mathbf{N}}$ defines a sequence of transformations $\left(\phi_{s,t}^n\right)_{n\in\mathbf{N}}$, and a sequence of density functions $(L_{s,t}^n)_{n\in\mathbf{N}} = d\phi_{s,t}^n P/dP$, $0 \le s,t \le 1$. We have

$$\begin{aligned}
E\left[L_{s,t}^n |\log L_{s,t}^n|\right] &= E\left[|\log L_{s,t}^n \circ \phi_{s,t}^n|\right] \\
&\le E\left[\int_0^1 \|\log(1-\sigma_r^n)\|_\infty dN_r\right] + \int_0^t \|\tilde{D}_r\sigma_r^n\|_\infty dr + \int_0^t \|\sigma_r^n\|_\infty dr \\
&\le \int_0^1 \|\log(1-\sigma_r^n)\|_\infty dr + \int_0^1 \||D\sigma_r^n|_H\|_\infty dr + \int_0^t \|\sigma_r^n\|_\infty dr \le 3C,
\end{aligned}$$

hence the sequence $(L_{s,t}^n)_{n\in\mathbf{N}}$ is uniformly integrable. We now have that $(\phi_{s,t}^n - I_B)_{n\in\mathbf{N}}$ converges in $L^2(B) \otimes H$ since

$$\begin{aligned}
E\left[|\phi_{s,t}^n - \phi_{s,t}^m|_H^2\right] &\le E\left[\int_s^t |\sigma_r^n(\phi_{s,r}^n) - \sigma_r^m(\phi_{s,r}^m)|^2 dr\right] \\
&\le 2E\left[\int_s^t |\sigma_r^n - \sigma_r^m|^2 L_{s,r}^n dr + \int_s^t |\sigma_r^m(\phi_{s,r}^n) - \sigma_r^m(\phi_{s,r}^m)|^2 dr\right] \\
&\le 2E\left[\int_s^t |\sigma_r^n - \sigma_r^m|^2 L_{s,r}^n dr + C^2 \int_s^t \int_s^r |\sigma_u^n(\phi_{s,u}^n) - \sigma_u^m(\phi_{s,u}^m)|^2 du dr\right] \\
&\le 2E\left[\int_s^t |\sigma_r^n - \sigma_r^m|^2 L_{s,r}^n dr\right] \exp\left(|t-s|C^2\right)
\end{aligned}$$

$n,m \in \mathbf{N}$, $0 \le s,t \le 1$, by the Gronwall lemma and Lemma 3. As n and m go to infinity, $E\left[|\phi_{s,t}^n - \phi_{s,t}^m|_H^2\right]$ converges to 0 by uniform integrability of $(L_{s,t}^n)_{n\in\mathbf{N}}$, and from Lemma 4, the sequence $(\sigma_r^n(\phi_{s,r}^n))_{n\in\mathbf{N}}$ converges to $\sigma_r(\phi_{s,r})$ in $L^2(B)$, for $r\in[0,1]$, hence by boundedness of σ the limit of $(\phi_{s,t}^n)_{n\in\mathbf{N}}$ solves (3.2) and coincides with $\phi_{s,t}$. The uniqueness of $\phi_{s,t}$ follows from Lemma 3 and its absolute continuity from Lemma 4. The above argument also shows that $\left(\phi_{t,T_k}^n - I_B\right)_{n\in\mathbf{N}}$ converges in $L^2(B)\otimes L^2([0,1],H)$ to $\phi_{t,T_k}-I_B$ which is absolutely continuous, $k\geq 1$ (we let $\sigma=0$ outside of $[0,1]\times B$). Consequently, $(\sigma^n(T_k,\phi_{t,T_k}^n))$ and $(\sigma^n(T_k,\phi_{t,T_k}^n))\circ\phi_{s,t}^n$ converge respectively to $\sigma(T_k,\phi_{t,T_k})$ and $\sigma(T_k,\phi_{t,T_k})\circ\phi_{s,t} = \sigma(T_k^{s,t},\phi_{s,T_k^{s,t}})$ in probability as $n\to\infty$, from Lemma 2 and Lemma 4, $k\geq 1$. Moreover, $\left(\left[\tilde{D}.\sigma_\cdot^n\right]\circ\phi_{s,\cdot}^n\right)_{n\in\mathbf{N}}$ converges to $\left[\tilde{D}.\sigma.\right]\circ\phi_{s,\cdot}$ in $L^2(B\times[0,1])$:

$$E\left[\int_0^1 \left|\left[\tilde{D}_r\sigma_r^n\right]\circ\phi_{s,r}^n - \left[\tilde{D}_r\sigma_r\right]\circ\phi_{s,r}\right|^2 dr\right]$$
$$\leq\ 2E\left[\int_0^1 |\,[D\sigma_r^n]\circ\phi_{s,r}^n - [D\sigma_r]\circ\phi_{s,r}^n|_H^2 dr + \int_0^1 |\,[D\sigma_r]\circ\phi_{s,r}^n - [D\sigma_r]\circ\phi_{s,r}|_H^2 dr\right]$$
$$\leq\ 2E\left[\int_0^1 |D(\sigma_r^n-\sigma_r)|_H^2 L_r^n dr + \int_0^1 |\,[D\sigma_r]\circ\phi_{s,r}^n - [D\sigma_r]\circ\phi_{s,r}|_H^2 dr\right],$$

which converges to 0 as n goes to infinity since $|D\sigma_r(\phi_{s,r}^n)|_H \leq \||D\sigma_r|_H\|_\infty$, $r\in[0,1]$. Finally, a subsequence of $(L_{s,t}^n)_{n\in\mathbf{N}}$, resp. $(L_{t,s}^n)_{n\in\mathbf{N}}$, also denoted by $(L_{s,t}^n)_{n\in\mathbf{N}}$, resp. $(L_{t,s}^n)_{n\in\mathbf{N}}$, converges almost surely to (4.3), resp. (4.4), $0\leq s\leq t\leq 1$. By uniform integrability of $\left\{L_{s,t}^n\ :\ n\in\mathbf{N}\right\}$ we have for $f\in\mathcal{C}_b^+(B)$:

$$E[f] \quad = \quad \lim_{n\to\infty} E\left[f\circ\phi_{s,t}^n L_{s,t}^n\right] = E\left[f\circ\phi_{s,t}L_{s,t}\right]. \quad 0\leq s,t\leq 1.$$

For completeness we state the proofs of Lemmas 1, 2, 3 and 4. They are the respective analogs of Props. 2.5, 2.6, 2.7 and 2.10 in [3], see also Props. 2,3,4 and 5 in [16].

Proof. (of Lemma 1). Let $\mathcal{F}_n$ denote the σ-algebra generated by $\tau_1,\ldots,\tau_n$ and let $F_n = (1-\frac{1}{n})E[F|\mathcal{F}_n]$, $n\geq 1$. We have $\operatorname{ess\,inf} F < F_n < \operatorname{ess\,sup} F$, $n\geq 1$. We have $\||DF_n|_H\|_\infty \leq \||DF|_H\|_\infty$, and $(F_n)_{n\geq 1}$ converges to F in $I\!D_{1,2}$. Hence it suffices to prove the result for $F=f(\tau_1,\ldots,\tau_n)\in I\!D_{1,2}$. Assume first that f has a compact support in $\mathbf{R}_+^n$, let $\Psi\in\mathcal{C}_c^\infty(\mathbf{R}_+^n)$ with $\int_{\mathbf{R}_+^n}\Psi(x)dx=1$, $\Psi\geq 0$, and $f_k(y)=\frac{1}{k^n}\int_{\mathbf{R}_+^n}\Psi(kx)f(y+x)dx$, $k>0$, $y\in\mathbf{R}_+^n$. With $F_k=f_k(\tau_1,\ldots,\tau_n)$, we have $\operatorname{ess\,inf} F\leq F_k\leq \operatorname{ess\,sup} F$, $k\geq 1$, and $\||DF_k|_H\|_\infty\leq\||DF|_H\|_\infty$. If f does not have a compact support, let $\Phi\in\mathcal{C}_c^\infty(\mathbf{R}^n,[0,1])$ such that $\Phi(x)=1$ for $|x|<1$. Let $F_k=E[F|\mathcal{F}_n]\Phi(\tau_1/k,\ldots,\tau_m/k)$. Then $(F_k)_{k\geq 1}$ converges to F in $I\!D_{1,2}$ and

$$\||DF_k|_H\|_\infty \leq \||DF|_H\|_\infty + \frac{1}{k}\|F\|_\infty \sup\sum_{i=0}^{i=n}(\partial_i\Phi)^2 \leq \||DF|_H\|_\infty+\varepsilon$$

for k great enough.

Proof. (of Lemma 2). For $\pi=\{\Delta_1,\ldots,\Delta_n\}$ a partition of $[0,1]$, let

$$\sigma^{\pi} = \sum_{i=1}^{i=n} \frac{1}{|\Delta_i|} 1_{\Delta_i} \int_{\Delta_i} \sigma_r dr.$$

Let $(\pi_n)_{n\in\mathbf{N}}$ be a sequence of partitions of $[0,1]$, mutually increasing with $\max_{1\leq i\leq n} |\Delta_i^n|$ converging to 0 as n goes to infinity. We have that $(\sigma^{\pi_n})_{n\in\mathbf{N}}$ converges to σ in $L_{1,2}$ with $|\sigma_s^{\pi_n}| \leq \|\sigma\|_\infty$ and ess sup $\sigma^{\pi_n} \leq$ ess sup $\sigma < 1$. We apply Lemma 1 to construct a sequence $(\sigma^{\pi_n,m})_{m\in\mathbf{N}} \subset \mathcal{V}$, bounded in $\mathbb{L}_{1,\infty}$, such that $\sigma_t^{\pi_n,m}$ converges a.s. to $\sigma_t^{\pi_n}$, $t \in [0,1]$, as $m \to \infty$. If σ has a version with continuous trajectories, then $\sigma_{T_k}^{\pi_n}$ converges a.s. to σ_{T_k}, and a subsequence of $(\sigma_{T_k}^{\pi_n,m})_{m\in\mathbf{N}}$ converges a.s. to $\sigma_{T_k}^{\pi_n}$, $t \in [0,1]$, $k \geq 1$, as $m \to \infty$.

Proof. (of Lemma 3). Let $\delta > 0$ and $\varepsilon > 0$. If $F \in \mathbb{D}_{1,\infty}$, then from Lemma 1 there is a sequence $(F_n)_{n\in\mathbf{N}} \subset \mathcal{S}$ that converges to F in $\mathbb{D}_{1,2}$ and

$$\begin{aligned} |F_n \circ \mathcal{T} - F_n \circ \mathcal{R}| &\leq \||DF_n|_H\|_\infty |U - V|_{L^2([0,1])} \\ &\leq (\||DF|_H\|_\infty + \varepsilon)|U - V|_{L^2([0,1])}, \quad a.s. \end{aligned}$$

Since $\mathcal{T}$ and $\mathcal{R}$ are absolutely continuous, we have that $P(|F_n \circ \mathcal{T} - F \circ \mathcal{T}| \geq \delta)$ and $P(|F_n \circ \mathcal{R} - F \circ \mathcal{R}| \geq \delta)$ converge to 0 as n goes to ∞. Hence

$$|F \circ \mathcal{T} - F \circ \mathcal{R}| \leq (\||DF|_H\|_\infty + \varepsilon)\, |U - V|_{L^2([0,1])}, \quad a.s.$$

Proof. (of Lemma 4). Let $\varepsilon > 0$. By uniform integrability there is $M_\varepsilon > 0$ such that

$$\sup_{n\in\mathbf{N}} E\left[L^n 1_{\{L^n > M_\varepsilon\}}\right] \leq \varepsilon/2.$$

There is $n_0 \in \mathbf{N}$ such that

$$\begin{aligned} P(|F(\mathcal{T}_n) - F_n(\mathcal{T}_n)| \geq \delta) &= E\left[1_{\{|F-F_n|\geq\delta\}} L^n\right] \\ &\leq E\left[1_{\{L^2 > M_\varepsilon\}} L^n\right] + M_\varepsilon P(|F - F_n| \geq \delta) \\ &\leq \varepsilon/2 + M_\varepsilon P(|F - F_n| \geq \delta) \leq \varepsilon, \quad n \geq n_0. \end{aligned}$$

Let $(G_n)_{n\in\mathbf{N}} \subset \mathcal{S}$ be a sequence that converges to F in $L^2(B)$. There is $k_0 \in \mathbf{N}$ such that

$$\begin{aligned} &P(|F \circ \mathcal{T}_n - G_{k_0} \circ \mathcal{T}_n| \geq \delta) + P(|F \circ \mathcal{T} - G_{k_0} \circ \mathcal{T}| \geq \delta) \\ &\qquad \leq \quad E\left[1_{\left\{|F-G_{k_0}|\geq\delta\right\}}(L^n + L)\right] \leq 2\varepsilon, \quad n \geq k_0. \end{aligned}$$

From Lemma 3, there is $n_1 \in \mathbf{N}$ such that

$$P(|G_{k_0} \circ \mathcal{T} - G_{k_0} \circ \mathcal{T}_n| \geq \delta) \leq \frac{1}{\delta} \||DG_{k_0}|_H\|_\infty |\sigma - \sigma^n|_{L^2([0,1])} \leq \varepsilon, \quad n \geq n_1.$$

Hence

$$P(|F \circ \mathcal{T} - F \circ \mathcal{T}_n|3 \geq \delta) \leq 3\varepsilon, \quad n \geq \max(n_0, k_0, n_1).$$

The transformation $\mathcal{T}$ is absolutely continuous because its density is obtained as the weak limit of $(L^n)_{n\in\mathbb{N}}$ in the weak topology $\sigma(L^1(B), L^\infty(B))$. Finally we mention

a result that shows the link between the queuing problem exposed in Sect. 2 and the anticipative Skorokhod integral $\tilde{\delta}$, cf. [16].

Theorem 3 Let $\sigma \in \mathbb{L}_{1,\infty}$ *have a version with continuous trajectories, such that* ess inf $\sigma < 1$. *Let* $b \in L^2([0,1], L^\infty(B))$ *and* $\eta \in L^\infty(B)$. *The anticipating stochastic differential equation*

$$X_t = \eta - \tilde{\delta}\left(1_{[0,t]}\sigma X\right) + \int_0^t b_s X_s ds \quad t \in [0,1] \tag{6.6}$$

has for solution

$$\begin{aligned} X_t &= \eta \circ \phi_{t,0} \exp\left(\int_0^t b_s \circ \phi_{t,s} ds\right) \frac{d\phi_{0,t}P}{dP} \\ &= \eta \circ \phi_{t,0} \exp\left(\int_0^t \left[\tilde{D}_s \sigma_s\right] \circ \phi_{t,s} ds + \int_0^t \sigma_s \circ \phi_{t,s} ds + \int_0^t b_s \circ \phi_{t,s} ds\right) \\ &\quad \times \prod_{k=1}^{k=N_t} (1 - \sigma(T_k, \phi_{t,T_k})), \quad t \in [0,1]. \end{aligned}$$

Acknowledgement

I thank Prof. E. Çınlar for his help in the construction of the queueing model.

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