Independence of some multiple Poisson stochastic integrals with variable-sign kernels

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Abstract

It is known that the multiple Poisson stochastic integrals $I_n(f_n)$, $I_m(g_m)$ of two symmetric *constant-sign* functions are independent if and only if f_n and g_m satisfy a disjoint support condition. In this paper we present some examples in which this property extends to some variable-sign functions.

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1 Introduction

The independence of two Gaussian random variables can be characterized by the vanishing of their covariance, and this property has been extended to multiple Wiener integrals in [9], [10], by stating that the multiple integrals $I_n(f_n)$, $I_m(g_m)$ with respect to Brownian motion are independent if and only if the contraction

$$\int_0^\infty f_n(x_1,\ldots,x_{n-1},z)g_m(y_1,\ldots,y_{m-1},z)dz = 0,$$

vanishes, $x_1,\ldots,x_{n-1},y_1,\ldots,y_{m-1} \in \mathbb{R}$, where $L^2(\mathbb{R}_+)^{\circ n}$ denotes the subspace of $L^2(\mathbb{R}_+)^{\otimes n} = L^2(\mathbb{R}_+^n)$ made of symmetric functions.

A proof of necessity has been provided in [2] using standard probabilistic arguments, while the original proof of [9], [10] relied on the Malliavin calculus. In addition, $I_n(f_n)$ is independent of $I_m(g_m)$ if and only if

$$E[(I_n(f_n))^2(I_m(g_m))^2] = E[(I_n(f_n))^2] \times E[(I_m(g_m))^2], \tag{1.1}$$

cf. Corollary 5.2 of [7]. Independence criteria for multiple stochastic integrals with respect to other Gaussian processes have been obtained in [1].

Coming back to the case of single integrals $I_1(f)$ with respect to Brownian motion, when $X \simeq \mathcal{N}(0,s)$ and $Y \simeq \mathcal{N}(0,t-s)$ are two independent centered Gaussian random variables it is well known that

$$X - Y \simeq I_1(f), \qquad \text{with } f = \mathbf{1}_{[0,s]} - \mathbf{1}_{[s,t]},$$

is independent of

$$X + Y \simeq I_1(g), \qquad \text{with } g = \mathbf{1}_{[0,s]} + \mathbf{1}_{[s,t]},$$

since $\langle f, g\rangle = \langle \mathbf{1}_{[0,s]} - \mathbf{1}_{[s,t]}, \mathbf{1}_{[0,s]} + \mathbf{1}_{[s,t]}\rangle = 0$.

The situation is completely different when $X \simeq \tilde{\mathcal{P}}(s)$ and $Y \simeq \tilde{\mathcal{P}}(t-s)$ are independent centered Poisson random variables with means s and $t-s$, in which case $X - Y$ is no longer independent of $X + Y$. Although we may also write

$$X - Y \simeq I_1(f) = I_1(\mathbf{1}_{[0,s]} - \mathbf{1}_{[s,t]})$$

and

$$X + Y \simeq I_1(g) = I_1(\mathbf{1}_{[0,s]} + \mathbf{1}_{[s,t]})$$

where $I_1(f)$ is the compensated Poisson integral of f, independence can occur only when $fg = 0$, as follows from Proposition 3.1 below.

When $I_n(f_n)$ and $I_m(g_m)$ are multiple Poisson stochastic integrals on $\mathbb{R}_+$, the condition

$$f_n \otimes_1^0 g_m(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) := f_n(x_1, \ldots, x_n) g_m(y_1, \ldots, y_{n-1}, x_n) = 0, \tag{1.2}$$

$x_1, \ldots, x_n, y_1, \ldots, y_{n-1} \in \mathbb{R}_+$, is sufficient for the independence of $I_n(f_n)$ and $I_m(g_m)$ on the Poisson space, due to the independence of increments of the Poisson process, since by symmetry of these functions, the supports of f_n and g_m can be respectively contained in two Borel sets of the form $A^n \subset \mathbb{R}_+^n$ and $B^m \subset \mathbb{R}_+^m$ with $A \cap B = \emptyset$, cf. [5].

On the other hand, it has been claimed in [4], [5] that (1.2) is also a *necessary* condition for the independence of $I_n(f_n)$ and $I_m(g_m)$ on the Poisson space. A similar result was stated independently in [8].

However that necessity claim was proved only when the functions f_n and g_m have constant signs. More precisely, the necessity condition therein is based on the relation (1.1) which, unlike on the Wiener space, turns out to be weaker than the independence of $I_n(f_n)$ and $I_m(g_m)$ when f_n and g_m have variable signs, as was shown in [7] by a counterexample for $n = 2$ and $m = 1$, cf. also Remark 3.4 below.

In this note we present some examples in which (1.2) holds as a necessary and sufficient condition for the independence of $I_n(f_n)$, $I_m(g_m)$ when f_n and g_m are allowed to have variable signs. Namely we treat the following cases:

- tensor powers: f_n and g_m take the form $f_n = f^{\circ n}$ and $g_m = g^{\circ m}$,
- integrals of first and multiple orders: $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and g_1 has constant sign,
- Integrals of 1st and 2nd orders: $f_2 = f \circ g$, without sign restrictions on f_2 and g_1,

for which we show that $f_n \otimes_1^0 g_m = 0$ is necessary and sufficient for independence. The question whether (1.2) is a necessary condition for independence is still open in full generality.

In Section 2 we recall the necessary conditions for independence obtained in [4], [5]. The examples of multiple integrals with variable-sign kernels for which the disjoint support condition is necessary and sufficient are given in Section 3. Finally in Section 4 we consider the expectation of multiple stochastic integrals.

The results of this paper are stated for a standard Poisson process on the half line $\mathbb{R}_+$, however they can be extended without difficulty to Poisson measures on metric spaces.

2 Necessary condition for independence

We start by recalling some results of [4]. Given $f_n \in L^2(\mathbb{R})^{\circ n}$ a symmetric square-integrable function of n variables on $\mathbb{R}_+^n$, the multiple stochastic integral of f_n with respect to the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is defined as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \ldots, t_n) d(N_{t_1} - t_1) \cdots d(N_{t_n} - t_n). \tag{2.1}$$

We will need the multiplication formula

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n\wedge m)} I_{n+m-s}(h_{n,m,s}), \tag{2.2}$$

cf. e.g. Proposition 4.5.6 of [6], provided

$$h_{n,m,s} := \sum_{s \le 2k \le 2(s\wedge n\wedge m)} k! \binom{n}{k} \binom{m}{k} \binom{k}{s-k} f_n \circ_k^{s-k} g_m$$

is in $L^2(\mathbb{R}_+)^{\circ n+m-s}$, $0 \le s \le 2(n\wedge m)$ where $f_n \circ_k^l g_m$, $0 \le l \le k$, is the symmetrization in $n+m-k-l$ variables of the function

$$f_n \otimes_k^l g_m(x_{l+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) := \int_0^\infty \cdots \int_0^\infty f_n(x_1, \ldots, x_n) g_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m) dx_1 \cdots dx_l.$$

for $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbb{R}_+)^{\circ m}$ such that $f_n \circ_k^l g_m \in L^2(\mathbb{R}_+)^{\circ n+m-k-l}$, $0 \le l \le k \le n \wedge m$.

The following proposition shows that $f_n \circ_1^0 g_m = 0$ is a necessary condition for the independence of $I_n(f_n)$ and $I_m(g_m)$.

Proposition 2.1 *Let* $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ *and* $g_m \in L^2(\mathbb{R}_+)^{\circ m}$ *such that*

$$E[(I_n(f_n))^2(I_m(g_m))^2] = E[(I_n(f_n))^2] \times E[(I_m(g_m))^2].$$

Then we have $f_n \circ_1^0 g_m = 0$.

Proof. ([4]) If $I_n(f_n)$ and $I_m(g_m)$ are independent, then $I_n(f_n)I_m(g_m) \in L^2(\Omega)$ and using the isometry formula

$$E\left[I_n(f_n)I_m(g_m)\right] = n!\langle f_n, g_m\rangle_{L^2(\mathbb{R}_+)^{\otimes n}} \mathbf{1}_{\{n=m\}}, \tag{2.3}$$

for $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, we get

$$\begin{aligned}
(n+m)! \mid f_n \circ g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} &\geq n!m! \mid f_n \otimes g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} \\
&= n!m! \mid f_n \mid^2_{L^2(\mathbb{R}_+)^{\otimes n}} \mid g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes m}} \\
&= E\left[I_n(f_n)^2\right] E\left[I_m(g_m)^2\right] = E\left[(I_n(f_n)I_m(g_m))^2\right] \\
&= \sum_{r=0}^{2(n\wedge m)} (n+m-r)! \mid h_{n,m,r} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-r}} \\
&\geq (n+m)! \mid h_{n,m,0} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} + (n+m-1)! \mid h_{n,m,1} \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-1}} \\
&\geq (n+m)! \mid f_n \circ g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m}} + nm(n+m-1)! \mid f_n \circ_1^0 g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes n+m-1}}
\end{aligned}$$

from (2.2), which implies $f_n \circ_1^0 g_m = 0$. □

It follows from Proposition 2.1 that if $I_n(f_n)$ and $I_m(g_m)$ are independent then we have $f_n \circ_1^0 g_m = 0$. However, $f_n \circ_1^0 g_m = 0$ is in general only a necessary and not sufficient condition for $f_n \otimes_1^0 g_m$ to vanish, since

$$\langle f_n \circ_1^0 g_m, f_n \circ_1^0 g_m\rangle \leq \langle f_n \otimes_1^0 g_m, f_n \otimes_1^0 g_m\rangle,$$

which follows from the fact that $f_n \circ_1^0 g_m$ is the symmetrization of $f_n \otimes_1^0 g_m$ in $n+m-1$ variables.

A counterexample for which we have $f_n \circ_1^0 g_m = f_m \circ_1^1 g_m = 0$ and $f_n \otimes_1^0 g_m \neq 0$ can be found in [7], Example 5.3, as

$$f_2(s,t) = \tag{2.4}$$

$$\mathbf{1}_{\{st<0\}}(\mathbf{1}_{[-1,-1/2]}(s)-\mathbf{1}_{[-1/2,1/2]}(s)+\mathbf{1}_{[1/2,1]}(s))(\mathbf{1}_{[-1,-1/2]}(s)-\mathbf{1}_{[-1/2,1/2]}(s)+\mathbf{1}_{[1/2,1]}(s))$$

and $g_1(t) = -\mathbf{1}_{[-1,0]}(t) + \mathbf{1}_{(0,1]}(t)$. In this example the integrals $I_2(f_2)$ and $I_1(g_1)$ are not independent, however (1.1) holds and therefore (1.1) and (1.2) cannot characterize the independence of $I_n(f_n)$ and $I_m(g_m)$.

In the case $n = 2$ and $m = 1$ we are able to show that (1.1) entails $f_2 \circ_1^1 g_1 = 0$ in addition to $f_2 \circ_1^0 g_1 = 0$.

Proposition 2.2 *Let $f_2 \in L^2(\mathbb{R}_+)^{\circ 2}$ and $g_1 \in L^2(\mathbb{R}_+)$ such that*

$$E[(I_2(f_2))^2(I_1(g_1))^2] = E[(I_2(f_2))^2] \times E[(I_1(g_1))^2].$$

Then we have

$$f_2 \circ_1^0 g_1 = 0, \quad \textit{and} \quad f_2 \circ_1^1 g_1 = 0.$$

Proof. By the same argument as in the proof of Proposition 2.1 we find

$$h_{2,1,1} = f_2 \circ_1^0 g_1 = 0,$$

and

$$h_{2,1,2} := 2 f_2 \circ_1^1 g_1 = 0.$$

□

When $f_2 = f \circ g$ and $g_1 = h$, Proposition 2.2 yields

$$(f \circ g) \circ_1^0 h = \frac{1}{4}(f \otimes (gh) + (gh) \otimes f + g \otimes (fh) + (fh) \otimes g) = 0, \tag{2.5}$$

and

$$(f \circ g) \circ_1^1 h = f\langle g, h\rangle + g\langle f, h\rangle = 0, \tag{2.6}$$

hence

$$\langle g, h\rangle = \langle f, h\rangle = 0, \tag{2.7}$$

provided $f \circ g \neq 0$.

3 Variable-sign kernels

In this section we consider examples of kernels $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $g_m \in L^2(\mathbb{R}_+)^{\circ m}$ for which

$$f_n \circ_1^0 g_m = 0 \quad \Longrightarrow \quad f_n \otimes_1^0 g_m = 0,$$

in which case the condition

$$f_n \otimes_1^0 g_m = f_n(x_1, \ldots, x_{n-1}, z) g_m(y_1, \ldots, y_{m-1}, z) = 0,$$

$x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z \in \mathbb{R}$, becomes both necessary and sufficient for the independence of $I_n(f_n)$ and $I_m(g_m)$, and is equivalent to (1.1), which also becomes necessary and sufficient.

We note that

$$\begin{aligned}
&\langle f_n \circ_1^0 g_m, f_n \circ_1^0 g_m \rangle \qquad (3.1)\\
&= \frac{1}{(n+m-1)!} \sum_{\sigma \in \Sigma_{n+m-1}} \\
&\quad \int_{\mathbb{R}_+^{n+m-1}} (f_n \otimes_1^0 g_m)(x_1, \ldots, x_{n+m-1})(f_n \otimes_1^0 g_m)(x_{\sigma_1}, \ldots, x_{\sigma_{n+m-1}}) dx_1 \cdots dx_{n+m-1} \\
&= \frac{(n-1)!(m-1)!}{(n+m-1)!} \langle f_n \otimes_1^0 g_m, f_n \otimes_1^0 g_m \rangle \\
&\quad + \frac{1}{(n+m-1)!} \sum_{\sigma \in \Theta_{n,m}} \\
&\quad \int_{\mathbb{R}_+^{n+m-1}} f_n \otimes_1^0 g_m(x_1, \ldots, x_{n+m-1}) f_n \otimes_1^0 g_m(x_{\sigma_1}, \ldots, x_{\sigma_{n+m-1}}) dx_1 \cdots dx_{n+m-1},
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{n,m} \quad = \quad & \{\sigma \in \Sigma_{n+m-1} \ : \ \sigma(\{1, \ldots, n-1\}) \neq \{1, \ldots, n-1\} \\
& \text{or } \sigma(\{n+1, \ldots, n+m-1\}) \neq \{n+1, \ldots, n+m-1\}\},
\end{aligned}$$

and our method of proof will rely on this decomposition to show that $f_n \otimes_1^0 g_m = 0$.

Tensor powers

In the next proposition we obtain a necessary and sufficient condition for the independence of integrals of (symmetric) tensor powers.

Proposition 3.1 *Let $f, g \in L^2(\mathbb{R})$. Then $I_n(f^{\otimes n})$ is independent of $I_m(g^{\otimes m})$ if and only if $f^{\otimes n} \otimes_1^0 g^{\otimes m} = 0$, which is equivalent to $fg = 0$.*

Proof. We have

$$f^{\circ n} \circ_1^0 g^{\circ n} = (fg) \circ f^{\circ(n-1)} \circ g^{\circ(m-1)},$$

hence $f^{\circ n} \circ_1^0 g^{\circ n} = 0$ is equivalent to $fg = 0$ and to

$$f^{\otimes n} \otimes_1^0 g^{\otimes n} = f^{\otimes(n-1)} \otimes (fg) \otimes g^{\otimes(m-1)} = 0.$$

□

In particular, $I_1(f)$ is independent of $I_1(g)$ if and only if $fg = 0$, for all $f, g \in L^2(\mathbb{R}_+)$.

Integrals of first and multiple orders

Here we consider the case of $I_n(f_n)$ and $I_1(g_1)$ when only g_1 has constant sign.

Proposition 3.2 *Let $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, and assume that $g_1 \in L^2(\mathbb{R}_+)$ only has constant sign. Then $I_n(f_n)$ is independent of $I_1(g_1)$ if and only if $f_n \otimes_1^0 g_1 = 0$, i.e.*

$$f_n(x_1, \ldots, x_n) g_1(x_1) = 0, \qquad x_1, \ldots, x_n \in \mathbb{R}_+.$$

Proof. We have

$$f_n \circ_1^0 g_1(x_1, \ldots, x_n) = \frac{1}{n} f_n(x_1, \ldots, x_n) \sum_{k=1}^{n} g_1(x_k),$$

and, as in (3.1),

$$\begin{aligned}
\langle f_n \circ_1^0 g_1, f_n \circ_1^0 g_1 \rangle &= \frac{1}{n^2} \int_0^\infty \cdots \int_0^\infty f_n^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \sum_{j=1}^{n} g_1(x_i) g_1(x_j) dx_1 \cdots dx_n \\
&= \frac{1}{n} \int_0^\infty \cdots \int_0^\infty f_n^2(x_1, \ldots, x_n) |g_1(x_i)|^2 dx_1 \cdots dx_n \\
&\quad + \frac{n-1}{n} \int_0^\infty \cdots \int_0^\infty f_n^2(x_1, \ldots, x_n) g_1(x_1) g_1(x_2) dx_1 \cdots dx_n \\
&= \frac{1}{n} \langle f_n \otimes_1^0 g_1, f_n \otimes_1^0 g_1 \rangle + \frac{n-1}{n} \langle f_n \otimes_n^{n-2} f_n, g_1 \otimes g_1 \rangle, \qquad (3.2)
\end{aligned}$$

hence $f_n \circ_1^0 g_1 = 0$ implies $f_n \otimes_1^0 g_1 = 0$ since $f_n \otimes_n^{n-2} f_n \geq 0$. □

Integrals of first and second orders

In the next result the necessary and sufficient condition for independence of multiple stochastic integrals is obtained without any sign assumption on the integrands in the first and second order case. In this section we work in the tensor case $f_2 = f \circ g$, which does not include Example 2.4, see (3.5) below for an example in the tensor case.

Proposition 3.3 *Let $f, g, h \in L^2(\mathbb{R}_+)$. The double Poisson stochastic integral*

$$I_2(f \circ g) = 2 \int_0^\infty f(t) \int_0^{t^-} g(s) d(N_s - s) d(N_t - t)$$

is independent of the single integral $I_1(h) = \int_0^\infty h(t) d(N_t - t)$ if and only if

$$(f \circ g) \otimes_1^0 h = 0,$$

which is equivalent to $fh = gh = 0$.

Proof. We assume for simplicity that $\langle f, f\rangle = \langle g, g\rangle = 1$. If $I_2(f \circ g)$ is independent of $I_1(h)$, Proposition 2.1 shows that $(f \circ g) \circ_1^0 h = 0$ and the conclusion follows from Lemma 3.5 in case $\langle f^2, h\rangle\langle g^2, h\rangle \geq 0$. Next, if

$$\langle f^2, h\rangle\langle g^2, h\rangle < 0,$$

Lemma 3.6 below shows that

$$h = \lambda \mathbf{1}_{\{f \neq 0\}} - \lambda \mathbf{1}_{\{g \neq 0\}}, \tag{3.3}$$

where

$$|\lambda| = \langle f^2, h^2\rangle^{1/2} = \langle g^2, h^2\rangle^{1/2} \neq 0. \tag{3.4}$$

In addition we have $fg = 0$ by Lemma 3.6, and by Proposition 2.2, (2.6) and (3.3), we get

$$\int_0^\infty f(x) dx = \int_0^\infty g(x) dx = 0,$$

hence, letting

$$\alpha_- = \int_0^\infty \mathbf{1}_{\{g(x) \neq 0\}} dx < \infty, \qquad \text{and} \qquad \alpha_+ = \int_0^\infty \mathbf{1}_{\{f(x) \neq 0\}} dx < \infty,$$

which are both finite since $h \in L^2(\mathbb{R}_+)$, we obtain

$$
\begin{aligned}
& E[(I_2(f \circ g))^2 \mathbf{1}_{\{I_1(h)=\lambda n\}}] = E[I_1(f)^2 I_1(g)^2 \mathbf{1}_{\{I_1(h)=\lambda n\}}] \\
&= e^{-\alpha_- - \alpha_+} \sum_{k=0\vee(-n)}^{\infty} \frac{1}{(2k+n)!} \int_{\mathrm{R}_+^k} \left(\sum_{i=1}^{k} f(x_i) \right)^2 dx_1 \cdots dx_k \int_{\mathrm{R}_+^{k+n}} \left(\sum_{i=1}^{n+k} g(y_i) \right)^2 dy_1 \cdots dy_{k+n} \\
&= e^{-\alpha_- - \alpha_+} \sum_{k=0\vee(-n)}^{\infty} \frac{1}{(2k+n)!} \int_{\mathrm{R}_+^k} \sum_{i=1}^{k} f^2(x_i) dx_1 \cdots dx_k \int_{\mathrm{R}_+^{k+n}} \sum_{i=1}^{n+k} g^2(y_i) dy_1 \cdots dy_{k+n} \\
&= e^{-\alpha_- - \alpha_+} \sum_{k=0\vee(-n)}^{\infty} \frac{k(k+n)}{(2k+n)!}.
\end{aligned}
$$

On the other hand, being the difference of two Poisson random variables, $\lambda^{-1} I_1(h)$ has the Skellam distribution.

$$
\begin{aligned}
P(I_1(h) = \lambda n) &= e^{-\alpha_- - \alpha_+} \sum_{k=0\vee(-n)}^{\infty} \frac{\alpha_+^{n+k} \alpha_-^k}{k!(n+k)!} \\
&= e^{-\alpha_- - \alpha_+} \left(\frac{\alpha_+}{\alpha_-} \right)^{n/2} \mathcal{I}_{|n|}(2\sqrt{\alpha_+ \alpha_-}),
\end{aligned}
$$

$n \in \mathbb{Z}$, where

$$
\mathcal{I}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}, \qquad x > 0,
$$

is the modified Bessel function of the first kind with parameter $n \geq 0$. It follows that

$$
E[(I_2(f \circ g))^2 \mid I_1(h) = \lambda n] = \left(\frac{\alpha_+}{\alpha_-} \right)^{-n/2} \frac{\sum_{k=0}^{\infty} \frac{k(k+n)}{(2k+n)!}}{\mathcal{I}_{|n|}(2\sqrt{\alpha_+ \alpha_-})},
$$

which cannot be constant in n because the Bessel function $\mathcal{I}_n(x)$ is not separable in its variables x and n. Therefore the independence of $I_2(f \circ g)$ with $I_1(h)$ imposes $\lambda = 0$ which concludes the proof by contradiction with (3.4). □

From the above proof we note again that, although independence of $I_2(f \circ g)$ and $I_1(h)$ is equivalent to $(f \circ g) \otimes_1^0 h = 0$, in general the statement $(f \circ g) \circ_1^0 h = 0$ does not imply $(f \circ g) \otimes_1^0 h = 0$, as shown by the following example:

$$
f = \mathbf{1}_A, \qquad g = \mathbf{1}_B, \qquad h = \mathbf{1}_A - \mathbf{1}_B, \tag{3.5}
$$

$A, B \subset \mathbb{R}_+$, with $A \cap B = \emptyset$, where we have

$$
(f \circ g) \otimes_1^0 h = (\mathbf{1}_A \circ \mathbf{1}_B) \otimes_1^0 (\mathbf{1}_A - \mathbf{1}_B) = -\mathbf{1}_A \otimes \mathbf{1}_B + \mathbf{1}_B \otimes \mathbf{1}_A \neq 0,
$$

and $(f \circ g) \circ_1^0 h = 0$, while $(f \circ g) \otimes_1^0 h \neq 0$, i.e. (1.2) does not hold. In this case the independence of $I_2(f \circ g)$ and $I_1(h)$ does not hold, as in Example (2.4) above.

Remark 3.4 *Unlike in the Wiener case, Condition* (1.1) *is in general not sufficient for the independence of* $I_n(f_n)$ *and* $I_m(g_m)$*, as shown by the example* (2.4)*. Therefore the search for a necessary and sufficient condition for independence in the general case has to involve more complex criteria than* (1.1)*.*

The next lemma has been used in the proof of Proposition 3.3.

Lemma 3.5 *Let* $f, g, h \in L^2(\mathbb{R}_+)$ *such that* $(f \circ g) \circ_1^0 h = 0$ *and*

$$\langle f^2, h\rangle\langle g^2, h\rangle \geq 0.$$

Then we have $(f \circ g) \otimes_1^0 h = 0$*, and* $fh = gh = 0$*.*

Proof. For simplicity and without loss of generality, we assume that

$$\langle f, f\rangle = \langle g, g\rangle = 1.$$

By Proposition 2.2 we have $(f \circ g) \circ_1^0 h = 0$, and from (3.2) or (2.5) we find

$$\begin{aligned} 0 &= \langle (f \circ g) \circ_1^0 h, (f \circ g) \circ_1^0 h\rangle \\ &= \frac{1}{2}\langle (f \circ g) \otimes_1^0 h, (f \circ g) \otimes_1^0 h\rangle + \frac{1}{2}\langle (f \circ g) \otimes_2^0 (f \circ g), h \otimes h\rangle \\ &= \frac{1}{2}\langle (f \circ g) \otimes_1^0 h, (f \circ g) \otimes_1^0 h\rangle + \frac{1}{4}\langle fg, h\rangle^2 + \frac{1}{4}\langle f^2, h\rangle\langle g^2, h\rangle. \end{aligned}$$

This shows that

$$(f \circ g) \otimes_1^0 h = 0,$$

provided $\langle f^2, h\rangle\langle g^2, h\rangle \geq 0$, and we conclude by Lemma 3.7 below. □

The next lemma has been used in the proof of Proposition 3.3.

Lemma 3.6 *Let* $f, g, h \in L^2(\mathbb{R}_+)$ *such that* $(f \circ g) \circ_1^0 h = 0$ *and*

$$\langle g^2, h\rangle \neq \langle f^2, h\rangle. \tag{3.6}$$

Then we have $fg = 0$ *and*

$$h = \lambda \mathbf{1}_{\{f \neq 0\}} - \lambda \mathbf{1}_{\{g \neq 0\}},$$

for some $\lambda \neq 0$ *such that*

$$\lambda^2 = \frac{\langle f^2, h^2\rangle}{\langle f, f\rangle} = \frac{\langle g^2, h^2\rangle}{\langle g, g\rangle}.$$

In particular it holds that

$$\langle g^2, h\rangle\langle f^2, h\rangle < 0.$$

Proof. For simplicity of exposition we assume that $\langle f, f\rangle = \langle g, g\rangle = 1$. The condition $(f \circ g) \circ_1^0 h = 0$ implies

$$\begin{cases} \langle (f \circ g) \circ_1^0 h, f \otimes f\rangle = \langle f, gh\rangle + \langle f^2, h\rangle\langle f, g\rangle = 0, \\ \\ \langle (f \circ g) \circ_1^0 h, g \otimes g\rangle = \langle f, gh\rangle + \langle g^2, h\rangle\langle f, g\rangle = 0, \end{cases}$$

and since $\langle g^2, h\rangle \neq \langle f^2, h\rangle$ this gives $\langle f, g\rangle = \langle f, gh\rangle = 0$, hence

$$\begin{aligned} \langle (f \circ g) \circ_1^1 h, (f \circ g) \circ_1^1 h\rangle &= \frac{1}{2}\langle f^2, h\rangle\langle g^2, h\rangle + \langle (f \circ g) \otimes_1^0 h, (f \circ g) \otimes_1^0 h\rangle \\ &= \frac{1}{2}\langle f^2, h\rangle\langle g^2, h\rangle + \frac{1}{4}(\langle f, f\rangle\langle g^2, h^2\rangle + \langle g, g\rangle\langle f^2, h^2\rangle) \\ &\geq -\frac{1}{2}\langle f^2, h^2\rangle^{1/2}\langle g^2, h^2\rangle^{1/2} + \frac{1}{4}(\langle g^2, h^2\rangle + \langle f^2, h^2\rangle) \qquad (3.7) \\ &= \frac{1}{4}(\langle f^2, h^2\rangle^{1/2} - \langle g^2, h^2\rangle^{1/2})^2 \\ &\geq 0, \end{aligned}$$

which shows that $\langle (f \circ g) \circ_1^1 h, (f \circ g) \circ_1^1 h\rangle = 0$ implies $\langle f^2, h^2\rangle = \langle g^2, h^2\rangle$, and, by the equality (3.7),

$$fh = \lambda f, \qquad \text{and} \qquad gh = -\lambda g,$$

for some $\lambda \in \mathbb{R}$ such that

$$|\lambda| = \langle f^2, h^2\rangle^{1/2} = \langle g^2, h^2\rangle^{1/2}.$$

Hence we have

$$h = \lambda \mathbf{1}_{\{f \neq 0\}} - \lambda \mathbf{1}_{\{g \neq 0\}}, \qquad (3.8)$$

and

$$\lambda^2 fg = h^2 fg,$$

which imply that $fg = 0$ a.e. on

$$\{h^2 = 0\} = \{f = h = 0\} \cup \{fg \neq 0\},$$

hence $fg = 0$. Finally we note that $\lambda \neq 0$ by (3.6) and (3.8). □

The next lemma has been used in the proof of Lemma 3.5.

Lemma 3.7 *For any $f, g, h \in L^2(\mathbb{R}_+)$, the condition $(f \circ g) \otimes_1^0 h = 0$ is equivalent to $fh = gh = 0$.*

Proof. Assuming again that $\langle f, f\rangle = \langle g, g\rangle = 1$, we note that since

$$(f \circ g) \otimes_1^0 h = \frac{1}{2}(f \otimes (gh) + g \otimes (fh)), \tag{3.9}$$

we have

$$\begin{aligned} \langle (f \circ g) \otimes_1^0 h, (f \circ g) \otimes_1^0 h\rangle &= \frac{1}{4}(\langle g^2, h^2\rangle + \langle f^2, h^2\rangle + 2\langle f, g\rangle\langle fg, h^2\rangle) \\ &\geq \frac{1}{4}(\langle g^2, h^2\rangle + \langle f^2, h^2\rangle - 2|\langle fg, h^2\rangle|) \qquad (3.10) \\ &\geq \frac{1}{4}(\langle g^2, h^2\rangle + \langle f^2, h^2\rangle - 2\langle g^2, h^2\rangle^{1/2}\langle f^2, h^2\rangle^{1/2}) \\ &= \frac{1}{4}(\langle f^2, h^2\rangle^{1/2} - \langle g^2, h^2\rangle^{1/2})^2 \\ &\geq 0. \end{aligned}$$

Assuming that $(f \circ g) \otimes_1^0 h = 0$, if $\langle fg, h^2\rangle \neq 0$ then the equality (3.10) implies $f = -g$ and $fh = gh = 0$ by (3.9). On the other hand, if $\langle fg, h^2\rangle = 0$ we have

$$0 = \langle (f \circ g) \otimes_1^0 h, (f \circ g) \otimes_1^0 h\rangle = \frac{1}{4}(\langle g^2, h^2\rangle + \langle f^2, h^2\rangle),$$

hence $fh = gh = 0$. This shows that $fh = gh = 0$ in all cases, and $(f \circ g) \otimes_1^0 h = 0$ is equivalent to $fh = gh = 0$. □

4 Conditional expectations

In this section we present some complements on conditional expectations. Next is an adaptation of Proposition 3 in [3] to the Poisson case.

Proposition 4.1 *Let $f_n \in L^2(\mathbb{R}_+)$, $n \geq 1$, and let A be a Borel set of $\mathbb{R}_+$ with finite measure. We have*

$$E[I_n(f_n) \mid I_n(\mathbf{1}_A^{\otimes n})] = E[I_n(f_n) \mid I_1(\mathbf{1}_A)] = \frac{\langle f_n, \mathbf{1}_A^{\otimes n}\rangle}{\langle \mathbf{1}_A, \mathbf{1}_A\rangle^n} I_n(\mathbf{1}_A^{\otimes n}),$$

which is a polynomial in $I_1(\mathbf{1}_A)$.

Proof. For any $k \geq 1$, by (2.2) we have

$$I_1(\mathbf{1}_A)^k = \sum_{l=0}^{k} \alpha_l I_l(\mathbf{1}_A^{\otimes l}), \tag{4.1}$$

where $\alpha_0, \ldots, \alpha_k \in (0, \infty)$, hence

$$\begin{aligned} E[I_n(f_n)(I_1(\mathbf{1}_A))^k] &= \alpha_k \mathbf{1}_{\{k \geq n\}} \langle f_n, \mathbf{1}_A^{\otimes n} \rangle \\ &= \frac{\langle f_n, \mathbf{1}_A^{\otimes n} \rangle}{\langle \mathbf{1}_A, \mathbf{1}_A \rangle^n} E[(I_1(\mathbf{1}_A))^k I_n(\mathbf{1}_A^{\otimes n})], \end{aligned}$$

which shows that

$$E[I_n(f_n) \mid I_1(\mathbf{1}_A)] = \frac{\langle f_n, \mathbf{1}_A^{\otimes n} \rangle}{\langle \mathbf{1}_A, \mathbf{1}_A \rangle^n} I_n(\mathbf{1}_A^{\otimes n}).$$

By (2.2) we also have

$$I_n(\mathbf{1}_A^{\otimes n}) = \sum_{l=0}^{n} \beta_l I_l(\mathbf{1}_A^{\otimes l}),$$

where $\beta_0, \ldots, \beta_n \in \mathbb{R}$, which implies $\sigma(I_n(\mathbf{1}_A^{\otimes n})) \subset \sigma(I_1(\mathbf{1}_A))$, and

$$\begin{aligned} E[I_n(f_n) \mid I_n(\mathbf{1}_A^{\otimes n})] &= E[E[I_n(f_n) \mid I_1(\mathbf{1}_A)] \mid I_n(\mathbf{1}_A^{\otimes n})] \\ &= \frac{\langle f_n, \mathbf{1}_A^{\otimes n} \rangle}{\langle \mathbf{1}_A, \mathbf{1}_A \rangle^n} E[E[I_n(\mathbf{1}_A^{\otimes n}) \mid I_1(\mathbf{1}_A)] \mid I_n(\mathbf{1}_A^{\otimes n})] \\ &= \frac{\langle f_n, \mathbf{1}_A^{\otimes n} \rangle}{\langle \mathbf{1}_A, \mathbf{1}_A \rangle^n} I_n(\mathbf{1}_A^{\otimes n}). \end{aligned}$$

□

When $n = 1$ the above result says that

$$E[I_1(f_1) \mid I_1(\mathbf{1}_A)] = \frac{\langle f_1, \mathbf{1}_A \rangle}{\langle \mathbf{1}_A, \mathbf{1}_A \rangle} I_1(\mathbf{1}_A), \tag{4.2}$$

which follows by a direct orthogonal projection argument. In particular when $X \simeq \tilde{\mathcal{P}}(s)$ and $Y \simeq \tilde{\mathcal{P}}(t-s)$ are independent centered Poisson random variables with means s and $t-s$,

$$X \simeq I_1(\mathbf{1}_{[0,s]}) \qquad \text{and} \qquad Y \simeq I_1(\mathbf{1}_{[s,t]})$$

taking $f_1 = \mathbf{1}_{[0,s]}$ and $A = [0, t]$, Relation (4.2) follows from the fact that $X + s$ has a binomial distribution with parameter s/t given $X + Y + t$.

Finally we note that, contrary to the Wiener case and Proposition 2 of [3], the conditional expectation of an odd order integral given an even order integral is not zero in general, indeed,

$$E[I_3(f_3)(I_2(g_2))^2]$$

does not vanish in general, from the multiplication formula (2.2).

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