

# Variational calculus for a Lévy process based on a Lie group

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## Abstract

The tools of the stochastic calculus of variations are constructed for Poisson processes on Lie group, and the corresponding analysis on the Lie-Wiener space is recovered as the limiting case of this construction.

**Key words:** Stochastic analysis, Lie groups, Lévy processes.

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## 1 Introduction

In this paper we develop analytic tools for the stochastic analysis on Lie groups in the jump case, in the framework of [11] and using the construction of Lévy processes of [2], [7]. This can be considered as a first step toward the construction of a variational calculus for the Lévy processes on manifolds of [1]. We refer to [4] for regularity results for the law of Lévy processes (stable semigroups) on Lie groups in terms of their Lévy measure, which are not covered in this paper. The main consequence of non-commutativity in the Poisson case is the introduction of left and right finite difference gradient operators. On the Lie-Wiener space, (cf. [9]), the Left gradient is linked to the classical gradient by the adjoint representation. Similarly, the left Poisson gradient is related to its classical counterpart via the inner automorphisms of the Lie group  $G$ .

In Sect. 3 we consider a Poisson random measure on a  $d$ -dimensional manifold  $G$  and a Wiener process taking values in the tangent space  $\mathcal{G}$  to  $G$  at a given point  $e$ . We define a chaotic decomposition for functionals of this process, considering simultaneously its the Wiener and Poisson components. In Sect. 4,  $G$  is assumed to be a Lie group and we define left and right finite difference operators for functionals of a Lévy process on  $G$ . The (left) divergence operator is defined as the dual of the left gradient and is linked to stochastic integration, chaos expansions and the Itô-Clark representation theorem. In addition to the invariance of the inner product

of the Lie algebra under inner automorphisms, we assume that the intensity of the Poisson random measure is left and right invariant, e.g. it is the Haar measure on a unimodular Lie group. In our framework, the Lie-Wiener gradient is obtained by differentiation of the left finite difference operator.

## 2 Notation

Let  $\mathbf{G}$  be a  $d$ -dimensional manifold with tangent space  $\mathcal{G}$  at some point  $e \in \mathbf{G}$ . Let  $(X_1, \dots, X_d)$  denote a basis of  $\mathcal{G}$ , with inner product  $(\cdot, \cdot)_{\mathcal{G}}$  and norm  $\|\cdot\|_{\mathcal{G}}$ , and let  $\nabla$  denote the gradient on  $\mathbf{G}$ . We define

$$\Omega = \mathcal{C}_0(\mathbb{R}_+, \mathcal{G}) \times \left\{ \sum_{i=1}^{i=N} \delta_{x_i} : x_i \in \mathbf{G} \times \mathbb{R}_+, i = 1, \dots, N, x_i \neq x_j, i \neq j, N \in \mathbb{N} \cup \{\infty\} \right\},$$

where  $\mathcal{C}_0(\mathbb{R}_+, \mathcal{G})$  is the space of continuous  $\mathcal{G}$ -valued functions starting at 0, and  $\delta_x$  is the Dirac measure at  $x \in \mathbf{G} \times \mathbb{R}_+$ . Let  $(B(t))_{t \in \mathbb{R}_+}$  and  $(N(A))_{A \in \mathcal{B}(\mathbf{G} \times \mathbb{R}_+)}$  be the applications defined on  $\Omega$  as

$$B(t)(\omega_1, \omega_2) = \omega_1(t), \quad N(A)(\omega_1, \omega_2) = \omega_2(A),$$

$t \in \mathbb{R}_+$ ,  $A \in \mathcal{B}(\mathbf{G} \times \mathbb{R}_+)$ ,  $(\omega_1, \omega_2) \in \Omega$ . We let

$$\mathcal{F}_t = \sigma(\{B(s), N(E \times [0, u]), E \in \mathcal{B}(\mathbf{G}), 0 \leq u, s \leq t\}), \quad t \in \mathbb{R}_+ \cup \{\infty\}.$$

Let  $P$  denote the probability measure on  $(\Omega, \mathcal{F}_\infty)$  such that  $B$  and  $N$  are independent standard  $\mathcal{G}$ -valued Brownian motion and Poisson random measure with intensity  $d\mu dt$  on  $\mathbf{G} \times \mathbb{R}_+$ , where  $\mu$  is a finite diffuse measure on  $\mathbf{G}$ . The couple  $(N, B)$  will be denoted by  $M$ , and we let  $\tilde{N}(d\sigma, dt) = N(d\sigma, dt) - \mu(d\sigma)dt$ ,  $L^2(\Omega) = L^2(\Omega, P)$ , and  $L^2(\mathbf{G} \times \mathbb{R}_+) = L^2(\mathbf{G} \times \mathbb{R}_+, d\mu dt)$ .

**Convention 1** *In this paper, for all normed vector spaces  $H$  and  $K$ , the tensor product  $H \otimes K$  and the direct sum  $H \oplus K$  are completed is and only if  $H$  and  $K$  are closed. Otherwise,  $H \otimes K$  and  $H \oplus K$  are simply algebraic.*

We construct a chaotic decomposition for the functionals of  $B$  and  $N$ , using the differential structure of  $\mathbf{G}$ . The notation “ $\circ$ ” denotes the symmetric tensor product. The Fock space  $\Gamma(H)$  on a normed vector space  $H$ , is defined as the direct sum

$$\Gamma(H) = \bigoplus_{n \geq 0} H^{\circ n},$$

where the symmetric tensor product  $H^{\circ n}$  is endowed with the norm

$$\|\cdot\|_{H^{\circ n}}^2 = n! \|\cdot\|_{H^{\otimes n}}^2, \quad n \in \mathbf{N}.$$

**Definition 1** Let  $\mathcal{S}$  be the vector space generated by

$$\{I_n(f_1 \circ \cdots \circ f_n) \ : \ f_1, \dots, f_n \in H\},$$

and let

$$\mathcal{U} = \left\{ \sum_{i=1}^{i=n} F_i u_i \ : \ u_1, \dots, u_n \in H, F_1, \dots, F_n \in \mathcal{S}, n \geq 1 \right\}.$$

The gradient and divergence operators  $D : \Gamma(H) \rightarrow \Gamma(H) \otimes H$  and  $\delta : \Gamma(H) \otimes H \rightarrow \Gamma(H)$  are densely defined on  $\mathcal{S}$  by linearity and polarization as

$$Dh^{\circ n} = nh^{\circ n-1} \otimes h, \quad \text{and} \quad \delta(h^{\circ n} \otimes g) = h^{\circ n} \circ g, \quad n \in \mathbf{N}. \quad (1)$$

The composition  $\delta D$  is the number operator, and  $\delta$  is adjoint of  $D$  in the following sense:

$$E[(DF, u)_H] = E[F\delta(u)], \quad F \in \mathcal{S}, \quad u \in \mathcal{U}.$$

Given  $(\gamma, t) \in \mathbf{G} \times \mathbf{R}_+$ , the operator  $\varepsilon_{\gamma, t}^+$  is defined on measurable functions  $F : \Omega \rightarrow \mathbf{R}$  as

$$\varepsilon_{\gamma, t}^+ F(\omega_1, \omega_2) = F(\omega_1, \omega_2 + (1 - \omega_2(\{(\gamma, t)\}))\delta_{\gamma, t}), \quad (\omega_1, \omega_2) \in \Omega,$$

i.e.  $\varepsilon_{\gamma, t}^+$  evaluates  $F$  at the configuration  $\omega_2$  modified by addition of a point  $(\gamma, t)$ , except if this point already belongs to  $\omega_2$ , cf. [8].

### 3 Chaos expansion

We introduce a chaos expansion for a Poisson random measure on  $\mathbf{G}$  and a Brownian motion in the  $d$ -dimensional space  $\mathcal{G}$ . Although it is related to Brownian motion on  $\mathbf{G}$ , our chaos expansion refers to a Brownian motion in a linear space, hence it is not the decomposition of [5]. The classical Wiener-Poisson decomposition identifies  $L^2(\Omega)$  to the Fock space over  $L^2(\mathbf{R}_+, L^2(\mathbf{G}) \oplus \mathcal{G})$ , whereas our decomposition is only based on  $L^2(\mathbf{R}_+, L^2(\mathbf{G}))$  but takes into account the fact that  $\mathcal{G}$  is the tangent space of  $\mathbf{G}$  at  $e$ . It is only identified to a dense subspace of  $L^2(\Omega)$ , nevertheless this is sufficient to define the closable operators of stochastic analysis.

Let  $\mathcal{H}$  denote the space of functions  $f \in \mathcal{C}_c^1(\mathbf{G})$  such that  $f(e) = 0$ , equipped with the norm

$$\|f\|_{\mathcal{H}}^2 = \|f\|_{L^2(\mathbf{G})}^2 + \|\nabla f(e)\|_{\mathcal{G}}^2, \quad f \in \mathcal{C}_c^1(\mathbf{G}),$$

and scalar product

$$(f, g)_{\mathcal{H}} = (f, g)_{L^2(\mathbf{G})} + (\nabla f(e), \nabla g(e))_{\mathcal{G}}, \quad f, g \in \mathcal{C}_c^1(\mathbf{G}).$$

The space  $\mathcal{S}$  are still defined as in Def. 1, taking  $H = L^2(\mathbf{R}_+, \mathcal{H})$ .

**Definition 2** We define the compensated stochastic integral  $\int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) d\tilde{M}_{\gamma, t}$  of a square-integrable  $(\mathcal{F}_t)_{t \in \mathbf{R}_+}$ -adapted process  $u \in L^2(\Omega) \otimes L^2(\mathbf{R}_+, \mathcal{H})$  with respect to the Lévy process  $M$  as

$$\int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) d\tilde{M}_{\gamma, t} = \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) \tilde{N}(d\gamma, dt) + \int_0^\infty dB_t u(e, t), \quad (2)$$

where the last integral is understood as  $\int_0^\infty (\nabla u(e, t), dB_t)_{\mathcal{G}}$ .

Since  $\mu$  is finite on  $\mathbf{G}$ , the non-compensated integral is similarly defined as

$$\int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) dM_{\gamma, t} = \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) N(d\gamma, dt) + \int_0^\infty dB_t u(e, t).$$

We have the isometry property

$$E \left[ \left( \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) d\tilde{M}_{\gamma, t} \right)^2 \right] = E \left[ \|u\|_{L^2(\mathbf{R}_+, \mathcal{H})}^2 \right]. \quad (3)$$

Let  $\pi_t, t \in \mathbf{R}_+$ , denote the projection operator on  $L^2(\mathbf{G} \times \mathbf{R}_+)$  defined as  $\pi_t f(\gamma, s) = f(\gamma, s) 1_{[0, t]}(s)$ ,  $\gamma \in \mathbf{G}$ ,  $s \in \mathbf{R}_+$ ,  $f \in L^2(\mathbf{G} \times \mathbf{R}_+)$ . The multiple stochastic integral  $I_n(h_n)$  of  $h_n \in L^2(\mathbf{R}_+, \mathcal{H})^{\otimes n}$  is defined as

$$I_n(h_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} h_n(\gamma_1, t_1, \dots, \gamma_n, t_n) d\tilde{M}_{\gamma_1, t_1} \cdots d\tilde{M}_{\gamma_n, t_n},$$

with  $I_0(h_0) = h_0$ ,  $h_0 \in \mathbf{R}$ . From (3) we have

$$E \left[ I_n(h_n)^2 \right] = (n!)^2 E \left[ \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} \|h_n(t_1, \dots, t_n)\|_{\mathcal{H}^{\otimes n}}^2 dt_1 \cdots dt_n \right].$$

(This relation is first checked on elements of the form  $h_n = g_1 \circ \cdots \circ g_n$  and then extended to  $L^2(\mathbf{R}_+, \mathcal{H})^{\otimes n}$  by bilinearity). Hence

$$E \left[ I_n(h_n)^2 \right] = n! \|h_n\|_{L^2(\mathbf{R}_+, \mathcal{H})^{\otimes n}}^2.$$

Elements of  $\Gamma(L^2(\mathbf{R}_+, \mathcal{H}))$  are identified to random variables in  $L^2(\Omega)$ , by associating  $h_n \in L^2(\mathbf{R}_+, \mathcal{H})^{\otimes n}$  to its multiple stochastic integral  $I_n(h_n)$ , building a linear isometry  $\mathcal{I} : \Gamma(L^2(\mathbf{R}_+, \mathcal{H})) \longrightarrow L^2(\Omega)$ . The image of  $\Gamma(L^2(\mathbf{R}_+, \mathcal{H}))$  under this injection is dense in  $L^2(\Omega)$ , but this suffices in order to define closable gradient operators. In

the sequel the operators  $D$  and  $\delta$  will act on random variables under the above identification. The spaces  $\mathcal{S}$  and  $\mathcal{U}$  of Def. 1 are also identified to spaces of smooth random variables and processes. The multiplication formula for multiple stochastic integrals with respect to  $d\tilde{M}$  and  $d\tilde{N}$  (see Sect. 5) has the same form:

$$I_n(f^{\circ n})I_1(g) = I_{n+1}(g \circ f^{\circ n}) + n(f, g)_{L^2(\mathbb{R}_+, \mathcal{H})} I_{n-1}(f^{\circ(n-1)}) + nI_n((fg) \circ f^{\circ(n-1)}), \quad (4)$$

$f, g \in L^2(\mathbb{R}_+, \mathcal{H})$ . From (4),  $\mathcal{S}$  is an algebra contained in  $L^p(\Omega)$ ,  $p \geq 2$ . In the following proposition we use the fact that the elements of  $\mathcal{S}$  are defined for every trajectory of  $N$  since they are polynomials in stochastic integrals of smooth functions with finite measure supports.

**Proposition 1** *The annihilation operator  $D$  is interpreted as a finite difference operator:*

(i) for  $F \in \mathcal{S}$  we have

$$D_{\gamma, t}F = \varepsilon_{\gamma, t}^+ F - F, \quad (\gamma, t) \in \mathbf{G} \times \mathbb{R}_+, \quad (5)$$

where  $\dot{B}$  and  $\dot{N}$  denote the Brownian and Poisson noises,

(ii)  $D$  satisfies the product rule

$$D_{\gamma, t}(FG) = FD_{\gamma, t}G + GD_{\gamma, t}F + D_{\gamma, t}FD_{\gamma, t}G, \quad (\gamma, t) \in \mathbf{G} \times \mathbb{R}_+, \quad (6)$$

(iii) by duality we have

$$F\delta(h) = \delta(hF) + (DF, h)_{L^2(\mathbb{R}_+, \mathcal{H})} + \delta(hDF), \quad F \in \mathcal{S}, \quad h \in \mathcal{U}. \quad (7)$$

*Proof.* Since (4) has the same form as on the Poisson space, we have

$$\begin{aligned} D_{\gamma, t}(I_n(f^{\circ n})I_1(g)) &= g(\gamma, t)I_n(f^{\circ n}) + nf(\gamma, t)I_n((fg) \circ f^{\circ(n-1)}) \\ &\quad + n(n-1)(f, g)_{L^2(\mathbb{R}_+, \mathcal{H})} f(\gamma, t)I_{n-2}(f^{\circ(n-2)}) \\ &\quad + nf(\gamma, t)g(\gamma, t)I_{n-1}(f^{\circ(n-1)}) \\ &\quad + n(n-1)f(\gamma, t)I_{n-1}((fg) \circ f^{\circ(n-2)}) \\ &= I_1(g)D_{\gamma, t}I_n(f^{\circ n}) + I_n(f^{\circ n})D_{\gamma, t}I_1(g) + D_{\gamma, t}I_n(f^{\circ n})D_{\gamma, t}I_1(g). \end{aligned}$$

By induction on  $k \in \mathbb{N}$  we also obtain

$$D_{\gamma, t}(I_n(f^{\circ n})I_1(g)^k) = I_1(g)^k D_{\gamma, t}I_n(f^{\circ n}) + I_n(f^{\circ n})D_{\gamma, t}I_1(g)^k + D_{\gamma, t}I_n(f^{\circ n})^k D_{\gamma, t}I_1(g)^k,$$

hence  $D$  satisfies the product rule (6) on  $\mathcal{S}$ , i.e.  $D_{\gamma,t}$  is a finite difference operator on  $\mathcal{S}$ . It is then sufficient to check that (5) holds for  $F = I_1(f) \in \mathcal{S}$ , by density of  $\mathcal{S}$  in  $L^2(\Omega)$ . By duality we obtain from (6)

$$\begin{aligned} E[(D(GF), h)_{L^2(\mathbb{R}_+, \mathcal{H})}] &= E[F(DG, h)_{L^2(\mathbb{R}_+, \mathcal{H})}] + E[G(DF, h)_{L^2(\mathbb{R}_+, \mathcal{H})}] \\ &\quad + E[(hDG, DF)_{L^2(\mathbb{R}_+, \mathcal{H})}], \end{aligned}$$

and

$$E[GF\delta(h)] = E[G\delta(hF)] + E[G(DF, h)_{L^2(\mathbb{R}_+, \mathcal{H})}] + E[G\delta(hDF)],$$

hence (7) holds. □

Let  $\mathcal{D} : L^2(\Omega) \rightarrow L^2(\Omega) \otimes L^2(\mathbb{R}_+, \mathcal{G})$  denote the gradient of the Malliavin calculus on Wiener space, defined on  $\mathcal{S}$  as

$$\int_0^\infty u(t) \mathcal{D}_t F dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( F \left( N, B. + \varepsilon \int_0^\cdot u(s) ds \right) - F \right), \quad u \in L^2(\mathbb{R}_+, \mathcal{G}),$$

$F \in \mathcal{S}$ , which satisfies from (2):

$$\mathcal{D}_t f(I_1(h_1), \dots, I_1(h_n)) = \sum_{i=1}^{i=n} \nabla h_i(e, t) \partial_i f(I_1(h_1), \dots, I_1(h_n)), \quad t \geq 0. \quad (8)$$

The Wiener-Skorohod integral operator  $\delta^W : L^2(\mathcal{D}_{\mathbb{G}}) \otimes L^2(\mathbb{R}_+, \mathcal{G}) \rightarrow L^2(\mathcal{D}_{\mathbb{G}})$  is the adjoint of  $\mathcal{D} : L^2(\Omega) \rightarrow L^2(\Omega) \otimes L^2(\mathbb{R}_+, \mathcal{G})$ .

**Proposition 2** *The operator  $\mathcal{D}$  is obtained as the derivative at the identity of the finite difference operator  $D_{\sigma,t}$ , i.e. we have*

$$\mathcal{D}_t F = \nabla D_{e,t} F, \quad t \in \mathbb{R}_+, F \in \mathcal{S}.$$

*Proof.* This is a consequence of Relations (5) and (8) which imply

$$\nabla D_{e,t} f(I_1(h_1), \dots, I_1(h_n)) = \sum_{i=1}^{i=n} (\nabla h_i(e, t)) \partial_i f(I_1(h_1), \dots, I_1(h_n)), \quad (9)$$

$t \in \mathbb{R}_+$ ,  $f \in \mathcal{P}(\mathbb{R}^n)$ ,  $h_1, \dots, h_n \in L^2(\mathbb{R}_+, \mathcal{H})$ . □

We also have

$$(\mathcal{D}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})} = \int_0^\infty (\mathcal{D}F, h(t))_{\mathcal{G}} dt = \int_0^\infty h(t) D_{e,t} F dt, \quad h \in L^2(\mathbb{R}_+, \mathcal{G}),$$

and

$$\|\mathcal{D}F\|_{L^2(\mathbb{R}_+, \mathcal{H})}^2 = \|\mathcal{D}F\|_{L^2(\mathbb{G} \times \mathbb{R}_+)}^2 + \|\mathcal{D}F\|_{L^2(\mathbb{R}_+, \mathcal{G})}^2.$$

**Proposition 3** Let  $u \in \mathcal{U} \subset L^2(\Omega, P) \otimes L^2(\mathbf{G} \times \mathbb{R}_+)$  be a simple process written as  $u = \sum_{i=1}^{i=n} F_i u_i$ .

(i) The (anticipating) stochastic integral  $\int_{\mathbf{G} \times \mathbb{R}_+} u(\sigma, t) d\tilde{M}_{\sigma, t}$  is defined as

$$\int_{\mathbf{G} \times \mathbb{R}_+} u(\sigma, t) d\tilde{M}_{\sigma, t} = \sum_{i=1}^{i=n} F_i \int_{\mathbf{G} \times \mathbb{R}_+} u_i(\sigma, t) d\tilde{M}_{\sigma, t},$$

(ii) we have

$$\int_{\mathbf{G} \times \mathbb{R}_+} u(\sigma, t) d\tilde{M}_{\sigma, t} = \delta(u) + \int_{\mathbf{G} \times \mathbb{R}_+} D_{\sigma, t} u(\sigma, t) \mu(d\sigma) dt + \delta(D.u(\cdot)). \quad (10)$$

(iii) if  $u \in L^2(\Omega) \otimes L^2(\mathbb{R}_+, \mathcal{H})$  is  $(\mathcal{F}_t)$ -adapted, then

$$\delta(u) = \int_{\mathbf{G} \times \mathbb{R}_+} u(\sigma, t) d\tilde{M}_{\sigma, t}. \quad (11)$$

*Proof.* Relation (7) implies (10) by linearity. If  $u$  is adapted, then the last two terms of (10) vanish and we obtain (11) which is extended by density from the Itô isometry (3). □

## 4 Left and right difference operators

From now on,  $\mathbf{G}$  is a connected Lie group of dimension  $d$  with Lie algebra  $\mathcal{G}$  of left-invariant vector fields. For  $\sigma \in \mathbf{G}$ , let  $\text{ad}_\sigma : \mathbf{G} \rightarrow \mathbf{G}$  denote the inner automorphism of  $\mathbf{G}$  defined by

$$\text{ad}_\sigma \gamma = \sigma \gamma \sigma^{-1}, \quad \gamma \in \mathbf{G},$$

let  $r_\sigma, l_\sigma$  be the right and left multiplications by  $\sigma \in \mathbf{G}$ , and let  $r_\sigma f = f \circ r_\sigma$ ,  $l_\sigma f = f \circ l_\sigma$ ,  $\text{ad}_\sigma f = f \circ \text{ad}_\sigma$ ,  $f \in \mathcal{C}(G)$ . By left invariance, given  $\sigma \in \mathbf{G}$  and  $h \in \mathcal{G}$  we have  $hl_\sigma f = l_\sigma hf$ . Let  $\text{Ad}_\sigma$  denote the adjoint representation, which satisfies  $\text{Ad}_\sigma hf = h \text{ad}_\sigma f$ , and  $(\text{Ad}_\sigma h)r_\sigma f = l_\sigma hf = hl_\sigma f$ ,  $\sigma \in \mathbf{G}$ ,  $h \in \mathcal{G}$ . From Th. 3.5 of [2],  $N$  and  $B$  define a process  $(\phi_t)_{t \in \mathbb{R}_+}$  with values in  $\mathbf{G}$  via the (uncompensated) stochastic differential equation written as

$$f(\phi_t) = f(e) + \int_{\mathbf{G}} \int_0^t (f(\phi_{s-\gamma}) - f(\phi_{s-})) dN_{\gamma, s} + \int_0^t (\nabla f(\phi_s), \circ dB_s)_{\mathcal{G}},$$

$t \in \mathbb{R}_+$ ,  $f \in \mathcal{C}^2(\mathbf{G})$ , using the Stratanovich differential  $\circ dB_s$ , or in Itô form as

$$f(\phi_t) = f(e) + \int_{\mathbf{G}} \int_0^t (f(\phi_{s-\gamma}) - f(\phi_{s-})) d\tilde{N}_{\gamma, s} + \int_0^t (\nabla f(\phi_s), dB_s)_{\mathcal{G}} + \int_0^t \mathcal{A}f(\phi_s) ds,$$

$t \in \mathbb{R}_+$ ,  $f \in \mathcal{C}^2(\mathbf{G})$ , where  $\mathcal{A}$  is the generator of  $(\phi_t)_{t \in \mathbb{R}_+}$ , defined as

$$\mathcal{A}f(\gamma) = \frac{1}{2} \text{trace} \nabla \nabla f(\gamma) + \int_{\mathbf{G}} (f(\gamma\sigma) - f(\gamma)) \mu(d\sigma). \quad (12)$$

This equation can also be rewritten in our framework as

$$f(\phi_t) = f(e) + \int_{\mathbf{G}} \int_0^t (f(\phi_{s-\gamma}) - f(\phi_{s-})) d\tilde{M}_{\gamma,s} + \int_0^t \mathcal{A}f(\phi_s) ds, \quad t \in \mathbb{R}_+, \quad (13)$$

Denoting by  $\nu_t$  the law of  $\phi_t$ ,  $t \in \mathbb{R}_+$ ,  $(\nu_t)_{t \in \mathbb{R}_+}$  is a semi-group of measures whose generator is  $\mathcal{A}$ , cf. [7], and  $(\phi_t)_{t \in \mathbb{R}_+}$  is called a Lévy process on  $\mathbf{G}$ , cf. [2]. The interest in this type of construction is that its driving Poisson measure is directly based on the Lie group  $\mathbf{G}$ . We refer to [3] for a different approach to the construction of jump processes on Lie groups from  $\mathcal{G}$ -valued point processes.

**Definition 3** *Let  $\mathcal{P}$  denote the set of functionals of the form*

$$f(\phi_{t_1}, \dots, \phi_{t_n}), \quad f \in \mathcal{C}_b^1(\mathbf{G}^n), \quad t_1, \dots, t_n, \quad n \in \mathbb{N}, \quad 0 \leq t_1 < \dots < t_n.$$

The mapping  $\phi : \Omega \rightarrow \mathcal{D}_{\mathbf{G}}$  defines an image measure  $\nu$  on the set  $\mathcal{D}_{\mathbf{G}}$  of cadlag functions from  $\mathbb{R}_+$  to  $\mathbf{G}$ , and  $D$  is extended to  $\mathcal{P}$  by (5).

**Definition 4** ([10]) *We define on the domain  $\mathcal{P}$  the left and right finite difference operators*

$$L : L^2(\mathcal{D}_{\mathbf{G}}, \nu) \rightarrow L^2(\mathcal{D}_{\mathbf{G}}, \nu) \otimes L^2(\mathbf{G} \times \mathbb{R}_+),$$

and

$$R : L^2(\mathcal{D}_{\mathbf{G}}, \nu) \rightarrow L^2(\mathcal{D}_{\mathbf{G}}, \nu) \otimes L^2(\mathbf{G} \times \mathbb{R}_+)$$

by

$$L_{\sigma,t}F = \sum_{k=1}^{k=n} 1_{]t_{k-1}, t_k]}(t) \left( f(\phi_{t_1}, \dots, \phi_{t_{k-1}}, \sigma\phi_{t_k}, \dots, \sigma\phi_{t_n}) - f(\phi_{t_1}, \dots, \phi_{t_n}) \right),$$

$$R_{\sigma,t}F = \sum_{k=1}^{k=n} 1_{]t_{k-1}, t_k]}(t) \left( f(\phi_{t_1}, \dots, \phi_{t_{k-1}}, \phi_{t_k}\sigma, \dots, \phi_{t_n}\sigma) - f(\phi_{t_1}, \dots, \phi_{t_n}) \right),$$

$\sigma \in \mathbf{G}$ ,  $t \in \mathbb{R}_+$ , for  $F \in \mathcal{P}$  of the form  $F = f(\phi_{t_1}, \dots, \phi_{t_n})$ , with  $t_0 = 0$ .

With the notation

$$\tilde{\phi}^{\sigma,t}(s) = \begin{cases} \phi_s, & 0 \leq s < t \\ \sigma\phi_s, & t \leq s \end{cases} \quad \hat{\phi}^{\sigma,t}(s) = \begin{cases} \phi_s, & 0 \leq s < t \\ \phi_s\sigma, & t \leq s \end{cases} \quad \sigma \in \mathbf{G}, \quad s \in \mathbb{R}_+,$$



we can also write

$$L_{\sigma,t}F(\phi) = F(\tilde{\phi}^{\sigma,t}) - F(\phi), \quad R_{\sigma,t}F(\phi) = F(\hat{\phi}^{\sigma,t}) - F(\phi), \quad F \in \mathcal{P}.$$

We will check in Prop. 4 that the definition of  $L$  is independent of the particular representation of  $F \in \mathcal{P}$  as  $F = f(\phi_{t_1}, \dots, \phi_{t_n})$ , although  $\phi \mapsto \tilde{\phi}^{\sigma,t}$  is not absolutely continuous with respect to the Poisson measure. We have the finite difference product identities

$$L_{\sigma,t}(FG) = FL_{\sigma,t}G + GL_{\sigma,t}F + (L_{\sigma,t}F)(L_{\sigma,t}G),$$

$$R_{\sigma,t}(FG) = FR_{\sigma,t}G + GR_{\sigma,t}F + (R_{\sigma,t}F)(R_{\sigma,t}G),$$

$(\sigma, t) \in \mathbf{G} \times \mathbb{R}_+$ ,  $F, G \in \mathcal{P}$ . Until the end of this paper we assume that  $\mathbf{G}$  is unimodular, and that the intensity of the Poisson random measure  $N$  on  $G \times \mathbb{R}_+$  is  $d\mu \times dt$ , where  $\mu$  is the Haar measure on  $\mathbf{G}$ . Hence  $\mu$  is left and right invariant under the inner automorphisms  $\text{ad}_\sigma : \mathbf{G} \rightarrow \mathbf{G}$ ,  $\sigma \in \mathbf{G}$ . We also assume that the inner product of  $\mathcal{G}$  is invariant under the inner automorphisms of  $\mathcal{G}$ . We define the operator  $\theta$  as

$$\begin{aligned} \theta : L^2(\mathbf{G} \times \mathbb{R}_+) &\rightarrow L^2(\mathbf{G} \times \mathbb{R}_+) \\ u(\sigma, t) &\mapsto u(\phi_t \sigma \phi_t^{-1}, t) = \text{ad}_{\phi_t} u(\sigma, t). \end{aligned}$$

From the invariance assumptions on  $d\mu$  and  $(\cdot, \cdot)_{\mathcal{G}}$ , the operator  $\theta$  is an isometry on  $L^2(\mathbb{R}_+, \mathcal{H})$ , a.s.:

$$\|\theta u\|_{L^2(\mathbb{R}_+, \mathcal{H})}^2 = \|\theta u\|_{L^2(\mathbb{R}_+, L^2(G))}^2 + \int_0^\infty |\text{Ad}_{\phi_t} \nabla u(e, t)|_{\mathcal{G}}^2 dt = \|u\|_{L^2(\mathbb{R}_+, \mathcal{H})}^2,$$

$u \in L^2(\mathbb{R}_+, \mathcal{H})$ .

**Proposition 4** *The operator  $L$  is closable, due to the relation*

$$(LF) \circ \phi = \theta^{-1} \circ D(F \circ \phi), \quad F \in \mathcal{P}, \quad (14)$$

which implies

$$\|LF\|_{L^2(\mathbb{R}_+, \mathcal{H})} \circ \phi = \|D(F \circ \phi)\|_{L^2(\mathbb{R}_+, \mathcal{H})}.$$

*Proof.* Given  $t \in \mathbb{R}_+$ , Relation (14) follows from the construction of  $(\tilde{\phi}_s^{\sigma,t})_{s \in \mathbb{R}_+}$  as a solution of the stochastic differential equation (13). Let

$$\psi_s^{\sigma,t} = \varepsilon_{\phi_t^{-1} \sigma \phi_t}^+(\phi_s), \quad s \in \mathbb{R}_+.$$

We have  $\psi_s^{\sigma,t} = \phi_s$ ,  $0 \leq s < t$ , and for  $s \geq t$ ,  $\psi_s^{\sigma,t}$  satisfies the stochastic differential equation

$$\begin{aligned} f(\psi_s^{\sigma,t}) &= f(e) + \int_{\mathbb{G}} \int_0^{t^-} (f(\psi_{u^-}^{\sigma,t} \gamma) - f(\psi_{u^-}^{\sigma,t})) dM_{\gamma,u} + \int_0^t \mathcal{A}f(\psi_u^{\sigma,t}) du \\ &\quad + f(\phi_t \phi_{t^-}^{-1} \sigma \phi_{t^-}) - f(\phi_{t^-}) + \int_{\mathbb{G}} \int_t^s (f(\psi_{u^-}^{\sigma,t} \gamma) - f(\psi_{u^-}^{\sigma,t})) d\tilde{M}_{\gamma,u} + \int_t^s \mathcal{A}f(\sigma \psi_u^{\sigma,t}) du \\ &= f(\sigma \phi_{t^-}) + \int_{\mathbb{G}} \int_t^s (f(\sigma \psi_{u^-}^{\sigma,t} \gamma) - f(\sigma \psi_{u^-}^{\sigma,t})) d\tilde{M}_{\gamma,u} + \int_t^s \mathcal{A}f(\sigma \psi_u^{\sigma,t}) du, \end{aligned} \quad (15)$$

$f \in \mathcal{C}_b^2(\mathbb{G})$ . On the other hand, the stochastic differential equation satisfied by  $(\phi_s)_{s \in \mathbb{R}_+}$  shows that for  $t \geq s$ ,

$$\begin{aligned} f(\sigma \phi_t) &= (l_\sigma f)(\phi_s) \\ &= l_\sigma f(\phi_{t^-}) + \int_{\mathbb{G}} \int_t^s (l_\sigma f(\phi_{u^-} \gamma) - l_\sigma f(\phi_{u^-})) d\tilde{M}_{\gamma,u} + \int_t^s \mathcal{A}l_\sigma f(\phi_u) du, \end{aligned} \quad (16)$$

i.e.

$$f(\sigma \phi_s) = f(\sigma \phi_{t^-}) + \int_{\mathbb{G}} \int_t^s (f(\sigma \phi_{u^-} \gamma) - f(\sigma \phi_{u^-})) \tilde{M}_{\gamma,u} + \int_t^s \mathcal{A}f(\sigma \phi_u) du, \quad (17)$$

$f \in \mathcal{C}_b^2(\mathbb{G})$ , since by left invariance we have  $\nabla l_\sigma f = l_\sigma \nabla f$  and  $\mathcal{A}l_\sigma f = l_\sigma \mathcal{A}f$ . Consequently,  $(\sigma \phi_s)_{s \geq t}$  and  $(\psi_s^{\sigma,t})_{s \geq t}$  satisfy the same equations (15) and (17),  $s \geq t$ , hence they coincide. In other terms, for  $0 \leq t \leq s$ , multiplication of  $\phi_t$  by  $\sigma$  on the left is equivalent to the addition of a point at  $(\phi_t^{-1} \sigma \phi_{t^-}, t)$  to the Poisson sample driving the stochastic differential equation that defines  $(\phi_s)_{s \geq 0}$  (provided that such a point does not already belong to the Poisson sample). This implies  $L_{\sigma,t} F = D_{\phi_t^{-1} \sigma \phi_{t^-}, t} F$ ,  $F \in \mathcal{P}$ , from (5).  $\square$

We can write for  $F = f(\phi_{t_1}, \dots, \phi_{t_n})$ :

$$D_{\sigma,t} F = \sum_{i=1}^n 1_{]t_{k-1}, t_k]}(t) \left( f(\phi_{t_1}, \dots, \phi_{t_{k-1}}, \phi_{t_k} \text{ad}_{\phi_{t^-} \phi_{t_k}^{-1} \sigma}, \dots, \phi_{t_n} \text{ad}_{\phi_{t^-} \phi_{t_n}^{-1} \sigma}) - F \right),$$

hence the right finite difference operator has no global expression in terms of the finite difference gradient  $D$  on Poisson space.

**Proposition 5** *We define the left divergence operator  $L^*$  on  $\mathcal{U}$  as*

$$(L^* u) \circ \phi = \delta \circ \theta(u \circ \phi), \quad u \in \mathcal{U}.$$

*We have the duality relation*

$$E_\nu[(LF, u)_{L^2(\mathbb{R}_+, \mathcal{H})}] = E_\nu[FL^*(u)], \quad F \in \mathcal{P}, \quad u \in \mathcal{U},$$

*and  $L, L^*$  are closable.*

*Proof.* Since  $\theta$  is unitary on  $L^2(\mathbb{R}_+, \mathcal{H})$ ,

$$\begin{aligned} E_\nu[(LF, u)_{L^2(\mathbb{R}_+, \mathcal{H})}] &= E[(LF) \circ \phi, u \circ \phi]_{L^2(\mathbb{R}_+, \mathcal{H})} = E[(\theta^{-1} \circ D(F \circ \phi), u \circ \phi)_{L^2(\mathbb{R}_+, \mathcal{H})}] \\ &= E[(D(F \circ \phi), \theta u \circ \phi)_{L^2(\mathbb{R}_+, \mathcal{H})}] = E[(F \circ \phi) \delta(\theta(u \circ \phi))] \\ &= E[(F \circ \phi)(L^*u) \circ \phi] = E_\nu[FL^*u], \quad u \in \mathcal{U}. \end{aligned}$$

The closability of  $L$  and  $L^*$  follows from this duality relation and the fact that they have dense domains. □

We will also identify to  $\pi_t$  the linear operator that acts on random variables as  $\pi_t I_n(h_n) = I_n(\pi_t^{\otimes n} h_n)$ ,  $n \in \mathbb{N}$ . Let  $\tilde{\pi}$  denote the continuous adapted projection operator defined on  $L^2(\Omega) \otimes L^2(\mathbb{G} \times \mathbb{R}_+)$  as  $((\tilde{\pi}u)(t))_{t \in \mathbb{R}_+} = (\pi_t u(t))_{t \in \mathbb{R}_+}$ .

**Proposition 6** *The operator  $L^*L = \delta D$  is the number operator on  $\Gamma(L^2(\mathbb{R}_+, \mathcal{H}))$ , and the Itô-Clark representation formula holds on  $\mathcal{P}$  as*

$$F = E[F] + L^*(\tilde{\pi}LF), \quad F \in \mathcal{P}. \quad (18)$$

*Proof.* This result follows from the fact that  $\tilde{\pi}$  and  $\theta$  commute:

$$\tilde{\pi}(\theta u)(\sigma, t) = \pi_t(u(\phi_t^{-1} \sigma \phi_t, t)) = (\pi_t u)(\phi_t^{-1} \sigma \phi_t, t) = \theta \tilde{\pi}u(\sigma, t),$$

$t \in \mathbb{R}_+$ ,  $u = LF$ ,  $F \in \mathcal{P}$ , and from the representation formula

$$F \circ \phi = E[F \circ \phi] + \delta(\pi D(F \circ \phi)), \quad F \in \mathcal{P}. \quad \square \quad (19)$$

For  $h \in L^2(\mathbb{R}_+, \mathcal{G})$ , let  $(e_t(h))_{t \in \mathbb{R}_+}$  denote the solution of the equation

$$f(e_t(h)) = f(e) + \int_0^t h(s) f(e_s(h)) ds, \quad f \in \mathcal{C}^2(\mathbb{G}).$$

The partial Lie-Wiener left and right derivatives  $(\mathcal{L}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})}$  and  $(\mathcal{R}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})}$  of  $F \in \mathcal{P}$  in the direction  $h \in L^2(\mathbb{R}_+, \mathcal{G})$ , cf. [9], are defined as

$$(\mathcal{L}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})} = \lim_{\varepsilon \rightarrow 0} \frac{F(e(\varepsilon h)\phi) - F}{\varepsilon},$$

and

$$(\mathcal{R}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})} = \lim_{\varepsilon \rightarrow 0} \frac{F(\phi e(\varepsilon h)) - F}{\varepsilon}.$$

We refer to [6] for the stochastic Campbell-Hausdorff formula.

**Lemma 1** *We have the Campbell-Hausdorff formula*

$$e(\text{Ad}_\phi h)\phi = \phi(N, dB_t + hdt).$$

*Proof.* For any  $f \in \mathcal{C}_b^2(\mathbf{G})$ , by left invariance we have:

$$\begin{aligned} f(e_t(\text{Ad}_\phi h)\phi_t) &= f(e) + \int_{\mathbf{G}} \int_0^t (f(e_s(\text{Ad}_\phi h)\phi_s\sigma) - f(e_s(\text{Ad}_\phi h)\phi_s)) d\tilde{M}_{\sigma,s} \\ &\quad + \int_0^t \mathcal{A}f(e_s(\text{Ad}_\phi h)\phi_s) ds + \int_0^t \text{Ad}_{\phi_s} h(s)(r_{\phi_s} f)(e_s(\text{Ad}_\phi h)) ds \\ &= f(e) + \int_{\mathbf{G}} \int_0^t (f(e_s(\text{Ad}_\phi h)\phi_s\sigma) - f(e_s(\text{Ad}_\phi h)\phi_s)) d\tilde{M}_{\sigma,s} \\ &\quad + \int_0^t \mathcal{A}f(e_s(\text{Ad}_\phi h)\phi_s) ds + \int_0^t h(s)f(e_s(\text{Ad}_\phi h)\phi_s) ds, \quad t \in \mathbb{R}_+, \end{aligned}$$

hence  $(e_t(\text{Ad}_\phi h)\phi_t)_{t \in \mathbb{R}_+}$  satisfies the stochastic differential equation defining  $(\phi_s)_{s \in \mathbb{R}_+}$ , driven by  $(dN, dB_s + h(s)ds)$  instead of  $(dN, dB)$ .  $\square$

The Campbell-Hausdorff formula is needed in the proof of the following proposition.

**Proposition 7** *The operator  $\mathcal{L}$  is obtained by differentiation at  $e$  of the left finite difference operator  $L$ :*

$$(\mathcal{L}F, h)_{L^2(\mathbb{R}_+, \mathcal{G})} = \int_0^\infty h(s)L_{e,s}F ds, \quad F \in \mathcal{P}, \quad h \in L^2(\mathbb{R}_+, \mathcal{G}),$$

*i.e.*  $\mathcal{L} = \nabla L$ .

*Proof.* First we remark that from the Campbell-Hausdorff formula,

$$\mathcal{L}_t(F \circ \phi) = (\text{Ad}_{\phi_{t^-}^{-1}} h(t)) D_{e,t}(F \circ \phi), \quad t \in \mathbb{R}_+.$$

Differentiating Relation (14):

$$(L_{\sigma,t}F) \circ \phi = D_{\phi_{t^-}^{-1}\sigma\phi_{t^-}, t}(F \circ \phi) = \text{ad}_{\phi_{t^-}^{-1}} D_{\sigma,t}(F \circ \phi)$$

at  $\sigma = e$  gives

$$(h(t)L_{e,t}F) \circ \phi = (\text{Ad}_{\phi_{t^-}^{-1}} h(t)) D_{e,t}(F \circ \phi) = \mathcal{L}_t(F \circ \phi), \quad t \in \mathbb{R}_+. \quad \square$$

By differentiation of (14) at  $\sigma = e$  we also obtain the relation between the Lie-Wiener right derivative  $\mathcal{L}$  and the right gradient  $\mathcal{R}$ , cf. [11]:

$$\begin{aligned} \nabla D_{e,s}f(\phi_{t_1}, \dots, \phi_{t_n}) &= \sum_{i=1}^{i=n} 1_{]t_{k-1}, t_k]}(s) \sum_{i=k}^{i=n} \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_{t_k}^{-1}} \nabla_i f(\phi_{t_1}, \dots, \phi_{t_n}) \\ &= \sum_{i=1}^{i=n} \sum_{j=1}^{j=i} 1_{]t_{j-1}, t_j]}(s) \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_{t_i}^{-1}} \nabla_i f(\phi_{t_1}, \dots, \phi_{t_n}) \\ &= \sum_{i=1}^{i=n} 1_{[0, t_i]}(s) \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_{t_i}^{-1}} \nabla_i f(\phi_{t_1}, \dots, \phi_{t_n}), \end{aligned}$$

where  $\nabla_i$  is applied to the  $i$ -th variable of  $f$ . Hence

$$h(s)D_{e,s}f(\phi_{t_1}, \dots, \phi_{t_n}) = \sum_{i=1}^{i=n} 1_{[0,t_i]}(s) \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_{t_i}^{-1}} h(s) f(\phi_{t_1}, \dots, \phi_{t_n}),$$

and

$$\int_0^\infty h(s)D_{e,s}f(\phi_{t_1}, \dots, \phi_{t_n})ds = \sum_{i=1}^{i=n} \int_0^{t_i} \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_{t_i}^{-1}} h(s) ds f(\phi_{t_1}, \dots, \phi_{t_n}),$$

i.e.

$$\int_0^\infty h(s)D_{e,s}F ds = \mathcal{R}_{\tilde{h}}F,$$

or  $(\mathcal{L}F, h)_{L^2(\mathbb{R}_+), \mathcal{G}} = \mathcal{R}_{\tilde{h}}F$ , with  $\tilde{h}(t) = \int_0^t \text{Ad}_{\phi_{s^-}} \text{Ad}_{\phi_t^{-1}} h(s) ds$ ,  $t \in \mathbb{R}_+$ . Similarly to the operator  $R$ , the right Lie-Wiener gradient has no expression in terms of  $\mathcal{D}$ .

## 5 Appendix

In this section we prove the Itô formula for the differential  $d\tilde{M}$  and the multiplication formula for multiple stochastic integrals with respect to  $d\tilde{M}$ .

**Proposition 8** *The Itô formula for the stochastic differential  $dM$  can be written as*

$$\begin{aligned} \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) d\tilde{M}_{\gamma, t} \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) d\tilde{M}_{\gamma, t} &= \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) \left( \int_{\mathbb{G}} \int_0^t u(\sigma, s) d\tilde{M}_{\sigma, s} \right) d\tilde{M}_{\gamma, t} \\ &+ \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) \left( \int_{\mathbb{G}} \int_0^t v(\sigma, s) d\tilde{M}_{\sigma, s} \right) d\tilde{M}_{\gamma, t} + \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) v(\gamma, t) d\tilde{M}_{\gamma, t} + (u, v)_{L^2(\mathbb{R}_+, \mathcal{H})}, \end{aligned}$$

for adapted processes  $u, v \in L^2(\Omega) \otimes L^2(\mathbb{R}_+, \mathcal{H})$ .

*Proof.* We have

$$\begin{aligned} \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) d\tilde{M}_{\gamma, t} \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) d\tilde{M}_{\gamma, t} &= \left( \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) d\tilde{N}_{\gamma, t} + \int_0^\infty (\nabla u(e, t), dB_t)_{\mathcal{G}} \right) \\ &\times \left( \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) d\tilde{N}_{\gamma, t} + \int_0^\infty (\nabla v(e, t), dB_t)_{\mathcal{G}} \right) \\ &= \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) d\tilde{N}_{\gamma, t} \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) d\tilde{N}_{\gamma, t} + \int_0^\infty (\nabla u(e, t), dB_t)_{\mathcal{G}} \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) d\tilde{N}_{\gamma, t} \\ &+ \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) d\tilde{N}_{\gamma, t} \int_0^\infty (\nabla v(e, t), dB_t)_{\mathcal{G}} \\ &+ \int_0^\infty (\nabla u(e, t), dB_t)_{\mathcal{G}} \int_0^\infty (\nabla v(e, t), dB_t)_{\mathcal{G}} \\ &= \int_{\mathbb{G} \times \mathbb{R}_+} u(\gamma, t) \int_{\mathbb{G}} \int_0^t v(\sigma, s) d\tilde{N}_{\sigma, s} d\tilde{N}_{\gamma, t} + \int_{\mathbb{G} \times \mathbb{R}_+} v(\gamma, t) \int_{\mathbb{G}} \int_0^t u(\sigma, s) d\tilde{N}_{\sigma, s} d\tilde{N}_{\gamma, t} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbf{G} \times \mathbf{R}_+} u(\sigma, t)v(\sigma, t)dN_{\sigma, t} + \int_{\mathbf{G} \times \mathbf{R}_+} \int_0^t v(\gamma, s)d\tilde{N}_{\gamma, s}(\nabla u(e, t), dB_t)_{\mathcal{G}} \\
& + \int_0^\infty v(\gamma, t) \int_{\mathbf{G}} \int_0^t (\nabla u(e, s), dB_s)_{\mathcal{G}} d\tilde{N}_{\gamma, t} + \int_{\mathbf{G} \times \mathbf{R}_+} \int_0^t u(\gamma, s)d\tilde{N}_{\gamma, s}(\nabla v(e, t), dB_t)_{\mathcal{G}} \\
& + \int_0^\infty u(\gamma, t) \int_{\mathbf{G}} \int_0^t (\nabla v(e, s), dB_s)_{\mathcal{G}} d\tilde{N}_{\gamma, t} + \int_0^\infty \int_0^t (\nabla v(e, s), dB_s)_{\mathcal{G}}(\nabla u(e, t), dB_t)_{\mathcal{G}} \\
& + \int_0^\infty \int_0^t (\nabla u(e, s), dB_s)_{\mathcal{G}}(\nabla v(e, t), dB_t)_{\mathcal{G}} + \int_0^\infty (\nabla u(e, t), \nabla v(e, t))_{\mathcal{G}} dt \\
& = \int_{\mathbf{G} \times \mathbf{R}_+} v(\gamma, t) \int_{\mathbf{G}} \int_0^t u(\sigma, s)d\tilde{M}_{\sigma, s}d\tilde{M}_{\gamma, t} + \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t) \int_{\mathbf{G}} \int_0^t v(\sigma, s)d\tilde{M}_{\sigma, s}d\tilde{M}_{\gamma, t} \\
& + \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, t)v(\gamma, t)d\tilde{M}_{\gamma, t} + (u, v)_{L^2(\mathbf{R}_+, \mathcal{H})},
\end{aligned}$$

since

$$\int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, s)v(\gamma, s)d\tilde{M}_{\gamma, s} = \int_{\mathbf{G} \times \mathbf{R}_+} u(\gamma, s)v(\gamma, s)d\tilde{N}_{\gamma, s},$$

because  $\nabla(uv)(e, t) = u(e, t)\nabla v(e, t) + v(e, t)\nabla u(e, t) = 0$ ,  $t \in \mathbf{R}_+$ , given that  $u, v \in L^2(\mathbf{R}_+, \mathcal{H})$ , a.s. □

Using the Itô formula, the multiplication formula for multiple stochastic integrals is formally the same with respect to  $d\tilde{M}$  and  $d\tilde{N}$ .

**Proposition 9** *Let  $f, g \in L^2(\mathbf{R}_+, \mathcal{H})$ . We have*

$$I_n(f^{\circ n})I_1(g) = I_{n+1}(g \circ f^{\circ n}) + n(f, g)_{L^2(\mathbf{R}_+, \mathcal{H})}I_{n-1}(f^{\circ(n-1)}) + nI_n((fg) \circ f^{\circ(n-1)}). \quad (20)$$

*Proof.* From Prop. 8, the formula holds for  $n = 1$ . Let  $n \geq 2$ . By induction, assuming that (20) holds for  $n - 1$ , we have

$$\begin{aligned}
& I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) \int_{\mathbf{G}} \int_0^t g(\sigma, s)d\tilde{M}_{\sigma, s} = I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)})I_1(\pi_t g) \\
& = I_n(\pi_t^{\otimes n} f^{\circ(n-1)} \circ g) + (n-1)I_{n-1}(\pi_t^{\otimes(n-1)}(fg) \circ f^{\circ(n-2)}) \\
& \quad + (n-1)(\pi_t f, \pi_t g)_{L^2(\mathbf{R}_+, \mathcal{H})}I_{n-2}(\pi_t^{\otimes(n-2)} f^{\circ(n-2)}),
\end{aligned}$$

hence

$$\begin{aligned}
I_n(f^{\circ n})I_1(g) & = n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t)d\tilde{M}_{\gamma, t} \int_{\mathbf{G} \times \mathbf{R}_+} f(\gamma, t)I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)})d\tilde{M}_{\gamma, t} \\
& = n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t) \int_{\mathbf{G}} \int_0^t f(\sigma, s)I_{n-1}(\pi_s^{\otimes(n-1)} f^{\circ(n-1)})d\tilde{M}_{\sigma, s}d\tilde{M}_{\gamma, t} \\
& \quad + n \int_{\mathbf{G} \times \mathbf{R}_+} f(\gamma, t)I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) \int_{\mathbf{G}} \int_0^t g(\sigma, s)d\tilde{M}_{\sigma, s}d\tilde{M}_{\gamma, t} \\
& \quad + n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t)f(\gamma, t)I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)})d\tilde{M}_{\sigma, t}
\end{aligned}$$

$$\begin{aligned}
& +n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t) f(\gamma, t) I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) d\mu_{\sigma, t} \\
& +n \int_{\mathbf{G} \times \mathbf{R}_+} (\nabla g(e, t), \nabla f(e, t))_{\mathcal{G}} I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) dt \\
= & n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t) \int_{\mathbf{G}} \int_0^t I_{n-1}(\pi_s^{\otimes(n-1)} f^{\circ n}) d\tilde{M}_{\sigma, s} d\tilde{M}_{\gamma, t} \\
& +n \int_{\mathbf{G} \times \mathbf{R}_+} f(\gamma, t) \left( I_n(\pi_t^{\otimes n} f^{\circ(n-1)} \circ g) + (n-1) I_{n-1}(\pi_t^{\otimes(n-1)} (fg) \circ f^{\circ(n-2)}) \right. \\
& \left. + (n-1) (f, g)_{L^2(\mathbf{R}_+, \mathcal{H})} I_{n-2}(\pi_t^{\otimes(n-2)} f^{\circ(n-2)}) \right) d\tilde{M}_{\gamma, t} \\
& +n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t) f(\gamma, t) I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) d\tilde{M}_{\gamma, t} \\
& +n \int_{\mathbf{G} \times \mathbf{R}_+} g(\gamma, t) f(\gamma, t) I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) d\mu(\gamma, t) \\
& +n \int_{\mathbf{G} \times \mathbf{R}_+} (\nabla g(e, t), \nabla f(e, t))_{\mathcal{G}} I_{n-1}(\pi_t^{\otimes(n-1)} f^{\circ(n-1)}) dt \\
= & I_{n+1}(g \circ f^{\circ n}) + n(f, g)_{L^2(\mathbf{R}_+, \mathcal{H})} I_{n-1}(f^{\circ(n-1)}) + nI_n((fg) \circ f^{\circ(n-1)}). \quad \square
\end{aligned}$$

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