Measure invariance on the Lie-Wiener path space

Nicolas Privault

Abstract In this paper we extend some recent results on moment identities, Hermite polynomials, and measure invariance properties on the Wiener space, to the setting of path spaces over Lie groups. In particular we prove the measure invariance of transformations having a quasi-nilpotent covariant derivative via a Girsanov identity and an explicit formula for the expectation of Hermite polynomials in the Skorohod integral on path space.

Key words: Malliavin calculus, Skorohod integral, measure invariance, covariant derivatives, quasi-nilpotence, path space, Lie groups.

Mathematics Subject Classification: 60H07, 58G32.

1 Introduction

The Wiener measure is known to be invariant under random isometries whose Malliavin gradient satisfies a quasi-nilpotence condition, cf. [12]. In particular, the Skorohod integral $\delta(Rh)$ is known to have a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H such that DRh is a.s. a quasi-nilpotent operator. Such results can be proved using the Skorohod integral operator δ and its adjoint the Malliavin derivative D on the Wiener space, and have been recently recovered under simple conditions and with short proofs in [5] using moment identities and in [6] via an exact formula for the expectation of random Hermite polynomials. Indeed it is well known that the Hermite polynomial defined by its generating function

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$$e^{xt-t^2\mu^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,\mu), \qquad x,t \in \mathbb{R},$$

satisfies the identity

$$E[H_n(X,\sigma^2)] = 0, \tag{1}$$

when $X \simeq \mathcal{N}(0,\sigma^2)$ is a centered Gaussian random variable with variance $\sigma^2 \geq 0$, and that the generating function can be used to characterize the gaussianity of X. In [6], conditions on the process $(u_t)_{t\in\mathbb{R}_+}$ have been deduced for the expectation $E[H_n(\delta(u),\|u\|^2)]$, $n \geq 1$, to vanish. Such conditions cover the quasi-nilpotence condition of [12] and include the adaptedness of $(u_t)_{t\in\mathbb{R}_+}$, which recovers the above invariance result using the characteristic function of $\delta(u)$.

On the other hand, the Skorohod integral and Malliavin gradient can also be defined on the path space over a Lie group, cf. [1], [3], [10]. In this paper we prove an extension of (1) to the path space case, by computing in Theorem 1 the expectation

$$E[H_n(\delta(u),\|u\|^2)], \qquad n \geq 1,$$

of the random Hermite polynomial $H_n(\delta(u),\|u\|^2)$, where $\delta(u)$ is the Skorohod integral of a possibly anticipating process $(u_t)_{t\in\mathbb{R}_+}$. This result also recovers the above conditions for the invariance of the path space measure, and extends the results of [6] and [5] to path spaces over Lie group.

In Corollaries 4 and 5 below we summarize our results in the derivation formula

$$\begin{aligned}\frac{\partial}{\partial\lambda} E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] &= \lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\,\mathrm{trace}\,(\nabla u)(I-\lambda\nabla u)^{-1}(Du)\right] \\ &\quad -\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle (I-\lambda\nabla u)^{-1}u, D\log\det\nolimits_2(I-\lambda\nabla u)\rangle\right],\end{aligned} \tag{2}$$

for λ in a neighborhood of 0, in which D, ∇ respectively denote the Malliavin gradient and covariant derivative on path space, and $\det_2(I-\lambda\nabla u)$ denotes the Carleman-Fredholm determinant of $I-\lambda\nabla u$. When ∇u is quasi-nilpotent in the sense of (16) below we have $\det_2(I-\lambda\nabla u) = 1$, cf. Theorem 3.6.1 of [13], or [14], and the derivative (2) vanishes, which yields

$$E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right] = 1,$$

for λ in a neighborhood of 0, cf. Corollary 3. If in addition $\langle u,u\rangle$ is a.s. constant, this implies

$$E\left[e^{\lambda\delta(u)}\right] = e^{-\frac{\lambda^2}{2}\|u\|^2}, \qquad \lambda \in \mathbb{R},$$

showing that $\delta(u)$ is centered Gaussian with variance $\|u\|^2$.

This paper is organized as follows. In Section 2 we review some notation on closable gradient and divergence operators, and associated commutation relations. In Section 3 we derive moment identities for the Skorohod integral on path spaces. In Section 4 we consider the expectation of Hermite polynomials, and in Section 5 we derive Girsanov identities on path space.

2 The Lie-Wiener path space

In this section we recall some notation on the Lie-Wiener path space, cf. [1], [3], [10], [11], and we prove some auxiliary results. Let G denote either $\mathbb{R}^d$ or a compact connected d-dimensional Lie group with associated Lie algebra $\mathscr{G}$ identified to $\mathbb{R}^d$ and equipped with an Ad-invariant scalar product on $\mathbb{R}^d \simeq \mathscr{G}$, also denoted by $(\cdot,\cdot)$. The commutator in $\mathscr{G}$ is denoted by $[\cdot,\cdot]$. Let $\mathrm{ad}\,(u)v = [u,v]$, $u,v \in \mathscr{G}$, with $\mathrm{Ad}\,e^u = e^{\mathrm{ad}\,u}$, $u \in \mathscr{G}$.

The Brownian motion $(\gamma(t))_{t\in\mathbb{R}_+}$ on G with paths in $\mathscr{C}_0(\mathbb{R}_+,\mathscr{G})$ is constructed from $(B_t)_{t\in\mathbb{R}_+}$ via the Stratonovich differential equation

$$\begin{cases} d\gamma(t) = \gamma(t) \odot dB_t \\ \\ \gamma(0) = \mathrm{e}, \end{cases}$$

where e is the identity element in G. Let $\mathbf{P}(\mathsf{G}) = \mathscr{C}_0(\mathbb{R}_+,\mathscr{G})$ denote the space of continuous G-valued paths starting at e, with the image measure of the Wiener measure by $I : (B_t)_{t\in\mathbb{R}_+} \mapsto (\gamma(t))_{t\in\mathbb{R}_+}$. Here we take

$$\mathscr{S} = \{F = f(\gamma(t_1),\dots,\gamma(t_n)) \quad : \quad f \in \mathscr{C}_b^\infty(\mathsf{G}^n)\},$$

and

$$\mathscr{U} = \left\{ \sum_{i=1}^n u_i F_i \quad : \quad F_i \in \mathscr{S},\ u_i \in L^2(\mathbb{R}_+;\mathscr{G}),\ i = 1,\dots,n,\ n \geq 1 \right\}.$$

Next is the definition of the right derivative operator D.

Definition 1. For $F = f(\gamma(t_1),\dots,\gamma(t_n)) \in \mathscr{S}$, $f \in \mathscr{C}_b^\infty(\mathsf{G}^n)$, we let $DF \in L^2(\Omega \times \mathbb{R}_+;\mathscr{G})$ be defined as

$$\langle DF, v\rangle = \frac{d}{d\varepsilon} f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} v_s ds},\dots,\gamma(t_n)e^{\varepsilon\int_0^{t_n} v_s ds}\right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+,\mathscr{G}).$$

For $F \in \mathscr{S}$ of the form $F = f(\gamma(t_1),\dots,\gamma(t_n))$ we also have

$$D_t F = \sum_{i=1}^n \partial_i f(\gamma(t_1),\dots,\gamma(t_n)) \mathbf{1}_{[0,t_i]}(t), \qquad t \geq 0.$$

The operator D is known to be closable and to admit an adjoint δ that satisfies

$$E[F\delta(v)] = E[\langle DF, v\rangle], \quad F \in \mathscr{S},\ v \in \mathscr{U}, \tag{3}$$

cf. e.g. [1]. Let $I\!D_{p,k}(X)$, $k \geq 1$, denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{I\!D_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(W, X\otimes H^{\otimes l})}, \qquad p \in [1,\infty],$$

where $H = L^2(\mathbb{R}_+, \mathscr{G})$, and $X \otimes H$ denotes the completed symmetric tensor product of X and H. We also let $I\!D_{p,k} = I\!D_{p,k}(\mathbb{R})$, $p \in [1,\infty]$, $k \geq 1$.

Next we turn to the definition of the covariant derivative on the path space $\mathbf{P}(\mathsf{G})$, cf. [1].

Definition 2. Let the operator ∇ be defined on $u \in I\!D_{2,1}(H)$ as

$$\nabla_s u_t = D_s u_t + \mathbf{1}_{[0,t]}(s) \mathrm{ad}\, u_t \in \mathscr{G} \otimes \mathscr{G}, \qquad s,t \in \mathbb{R}_+. \tag{4}$$

When $h \in H$ we have

$$\nabla_s h_t = \mathbf{1}_{[0,t]}(s) \mathrm{ad}\, h_t, \qquad s,t \in \mathbb{R}_+,$$

and $\mathrm{ad}\, v \in \mathscr{G} \otimes \mathscr{G}$, $v \in \mathscr{G}$, is the matrix

$$(\langle e_j, \mathrm{ad}\,(e_i) v\rangle)_{1\leq i,j\leq d} = (\langle e_j, [e_i, v]\rangle)_{1\leq i,j\leq d}.$$

The operator $\mathrm{ad}\,(v)$ is antisymmetric on $\mathscr{G}$ because $(\cdot,\cdot)$ is Ad-invariant. In addition if $u = hF$, $h \in H$, $F \in I\!D_{2,1}$, we have

$$D_s u_t = D_s F \otimes h(t), \qquad \mathrm{ad}\, u_t = F \,\mathrm{ad}\, h(t), \qquad s,t \in \mathbb{R}_+,$$

and

$$\begin{aligned}
\langle e_i \otimes e_j, \nabla_s u_t\rangle &= \langle e_i \otimes e_j, \nabla_s (hF)(t)\rangle \\
&= \langle e_i \otimes e_j, D_s F \otimes h(t)\rangle + \mathbf{1}_{[0,t]}(s) F \langle e_i \otimes e_j, \mathrm{ad}\, h(t)\rangle \\
&= \langle h(t), e_j\rangle \langle e_i, D_s F\rangle + \mathbf{1}_{[0,t]}(s) F \langle e_j, \mathrm{ad}\,(e_i) h(t)\rangle \\
&= \langle h(t), e_j\rangle \langle e_i, D_s F\rangle + \mathbf{1}_{[0,t]}(s) F \langle e_j, [e_i, h(t)]\rangle,
\end{aligned}$$

$i,j = 1,\ldots,d$. In the commutative case we have $\mathrm{ad}\,(v) = 0$, $v \in \mathscr{G}$, hence $\nabla = D$.

By (4) we have

$$(\nabla_v u)(t) := (\nabla u) v_t = \int_0^t (\nabla_s u_t) v_s ds, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathscr{U}$ in the direction $v \in L^2(\mathbb{R}_+;\mathscr{G})$, with $\nabla_v u \in L^2(\mathbb{R}_+;\mathscr{G})$, cf. [1] and Lemma 3.4 in [4].

It is known that D and ∇ satisfy the commutation relation

$$D\delta(u) = u + \delta(\nabla^* u), \tag{5}$$

for $u \in \mathbb{D}_{2,1}(H)$ such that $\nabla^* u \in \mathbb{D}_{2,1}(H \otimes H)$, cf. e.g. [1]. On the other hand, the commutation relation (5) shows that the Skorohod isometry [9]

$$E[\delta(u)\delta(v)] = E[\langle u, v\rangle] + E\left[\operatorname{trace}(\nabla u)(\nabla v)\right], \qquad u, v \in \mathbb{D}_{2,1}(H), \tag{6}$$

holds as a consequence of (5), cf. [1] and Theorem 3.3 in [4], where

$$\operatorname{trace}(\nabla u)(\nabla v) = \langle \nabla u, \nabla^* v\rangle_{H\otimes H} = \int_0^\infty \int_0^\infty \langle \nabla_s u_t, \nabla_t^\dagger v_s\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} ds dt,$$

and $\nabla_t^\dagger v_s$ denotes the transpose of the matrix $\nabla_t v_s$, $s,t \in \mathbb{R}_+$. Note also that we have

$$\nabla_s u_t = D_s u_t, \qquad s > t, \tag{7}$$

Note that for $u \in \mathbb{D}_{2,1}(H)$ and $v \in H$ we have

$$(\nabla u)^k v(t) = \int_0^\infty \cdots \int_0^\infty (\nabla_{t_k} u_t \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}) v_{t_1} dt_1 \cdots dt_k, \qquad t \in \mathbb{R}_+,$$

and

$$\begin{aligned}\operatorname{trace}(\nabla u)^k &= \langle \nabla^\dagger u, (\nabla u)^{k-1}\rangle \\ &= \int_0^\infty \cdots \int_0^\infty \langle \nabla_{t_k}^\dagger u_{t_1}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}\rangle dt_1 \cdots dt_k,\end{aligned}$$

$k \geq 2$.

In addition we have the following lemma, which will be used to apply our invariance results to adapted processes.

Lemma 1. *Assume that the process $u \in \mathbb{D}_{2,1}(H)$ is adapted with respect to the Brownian filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Then we have*

$$\operatorname{trace}(\nabla u)^k = \operatorname{trace}(\nabla u)^{k-1}(Du) = 0, \qquad k \geq 2.$$

Proof. For almost all $t_1, \ldots, t_{k+1} \in \mathbb{R}_+$ there exists $i \in \{1, \ldots, k+1\}$ such that $t_i > t_{i+1 \bmod k+1}$, and (7) yields

$$\begin{aligned}\nabla_{t_i} u_{t_{i+1 \bmod k+1}} &= D_{t_i} u_{t_{i+1 \bmod k+1}} + \mathbf{1}_{[0,t_{i+1 \bmod k+1}]}(t_i) \\ &= D_{t_i} u_{t_{i+1 \bmod k+1}} \\ &= 0,\end{aligned}$$

since $(u_t)_{t\in\mathbb{R}_+}$ is adapted.

We close this section with three lemmas that will be used in the sequel.

Lemma 2. *For any $u \in \mathbb{D}_{2,1}(H)$ we have*

$$\langle (\nabla u)v, u\rangle = \frac{1}{2}\langle v, D\langle u,u\rangle\rangle, \qquad v \in H.$$

Proof. We have

$$\begin{aligned}
(\nabla^* u)u_t &= \int_0^\infty (\nabla_t u_s)^\dagger u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds + \int_0^\infty \mathbf{1}_{[0,s]}(t)(\mathrm{ad}\, u_s)^\dagger u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds - \int_0^\infty \mathbf{1}_{[0,s]}(t)\mathrm{ad}\,(u_s) u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds \\
&= (D^* u)u_t.
\end{aligned}$$

Next, the relation $D\langle u,u\rangle = 2(D^*u)u$ shows that

$$\begin{aligned}
\langle (\nabla u)v, u\rangle &= \langle (\nabla^* u)u, v\rangle \\
&= \langle (D^* u)u, v\rangle \\
&= \frac{1}{2}\langle v, D\langle u,u\rangle\rangle.
\end{aligned}$$

Lemma 3. *For all $u \in \mathbb{D}_{2,2}(H)$ and $v \in \mathbb{D}_{2,1}(H)$ we have*

$$\langle \nabla^* u, D((\nabla u)^k v)\rangle = \mathrm{trace}\,((\nabla u)^{k+1} D v) + \sum_{i=2}^{k+1} \frac{1}{i}\langle (\nabla u)^{k+1-i} v, D\,\mathrm{trace}\,(\nabla u)^i\rangle, \qquad k \in \mathbf{N}.$$

Proof. Note that we have the commutation relation $\nabla D = D\nabla$, and as a consequence for all $1 \le k \le n$ we have

$$\begin{aligned}
&\langle \nabla^* u, D((\nabla u)^k v)\rangle = \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, D_{t_{k+1}}(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} v_{t_0})\rangle dt_0 \cdots dt_{k+1} \\
&= \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} D_{t_{k+1}} v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&\quad + \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, D_{t_{k+1}}(\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1}) v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&= \mathrm{trace}\,((\nabla u)^{k+1} D v) + \sum_{i=0}^{k-1} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} (\nabla_{t_i} D_{t_{k+1}} u_{t_{i+1}}) \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1}
\end{aligned}$$

$$= \operatorname{trace}((\nabla u)^{k+1} D v) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \int_0^\infty \cdots \int_0^\infty$$
$$\langle \nabla_{t_i} \langle \nabla_{t_k}^\dagger u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} \nabla_{t_{k+1}} u_{t_{i+1}} \rangle, \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0} \rangle dt_0 \cdots dt_{k+1}$$
$$= \operatorname{trace}((\nabla u)^{k+1} D v) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (\nabla u)^i v, D \operatorname{trace}(\nabla u)^{k+1-i} \rangle.$$

Lemma 4. *For all $u \in I\!D_{2,2}(H)$ and $v \in I\!D_{2,1}(H)$ such that $\|\nabla u\|_{L^\infty(\Omega; H \otimes H)} < 1$ we have*

$$\langle \nabla^* u, D((I - \nabla u)^{-1} v) \rangle = \operatorname{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) - \langle (I - \nabla u)^{-1} v, D \log \det{}_2(I - \nabla u) \rangle.$$

Proof. By Lemma 3 we have

$$\langle \nabla^* u, D((I - \nabla u)^{-1} v) \rangle = \sum_{n=0}^{\infty} \langle \nabla^* u, D((\nabla u)^n v)) \rangle$$
$$= \sum_{n=0}^{\infty} \operatorname{trace}((\nabla u)^{n+1} D v) + \sum_{n=0}^{\infty} \sum_{i=2}^{n+1} \frac{1}{i} \langle (\nabla u)^{n+1-i} v, D \operatorname{trace}(\nabla u)^i \rangle$$
$$= \operatorname{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) + \sum_{i=2}^{\infty} \frac{1}{i} \sum_{n=0}^{\infty} \langle (\nabla u)^n v, D \operatorname{trace}(\nabla u)^i \rangle$$
$$= \operatorname{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) + \sum_{i=2}^{\infty} \frac{1}{i} \langle (I - \nabla u)^{-1} v, D \operatorname{trace}(\nabla u)^i \rangle$$
$$= \operatorname{trace}(\nabla u)(I - \nabla u)^{-1}(Dv) - \langle (I - \nabla u)^{-1} u, D \log \det{}_2(I - \nabla u) \rangle,$$

since $\det_2(I - \lambda \nabla u)$ satisfies

$$\det{}_2(I - \lambda \nabla u) = \exp\left(-\sum_{i=2}^{\infty} \frac{\lambda^i}{i} \operatorname{trace}(\nabla u)^i \right), \tag{8}$$

cf. [8] page 108, which shows that

$$D \log \det{}_2(I - \lambda \nabla u) = -\sum_{i=2}^{\infty} \frac{\lambda^i}{i} D \operatorname{trace}(\nabla u)^i.$$

3 Moment identities on path space

The following moment identity extends Theorem 2.1 of [5] to the path space setting. The Wiener case is obtained by taking $\nabla = D$.

Proposition 1. *For any $n \geq 1$ and $u \in I\!D_{n+1,2}(H)$, $v \in I\!D_{n+1,1}(H)$ we have*

$$E[\delta(u)^n \delta(v)] = nE\left[\delta(u)^{n-1} \langle u, v \rangle\right] \tag{9}$$

$$+\frac{1}{2}\sum_{k=2}^{n}\frac{n!}{(n-k)!}E\left[\delta(u)^{n-k}\langle(\nabla u)^{k-2}v,D\langle u,u\rangle\rangle\right]$$
$$+\sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[\delta(u)^{n-k}\left(\mathrm{trace}\,((\nabla u)^{k+1}Dv)+\sum_{i=2}^{k}\frac{1}{i}\langle(\nabla u)^{k-i}v,D\,\mathrm{trace}\,(\nabla u)^{i}\rangle\right)\right].$$

For $n=1$ the above identity (9) coincides with the Skorohod isometry (6).

When $\langle u,u\rangle$ is deterministic, $u\in\mathbb{D}_{2,1}(H)$, and $\mathrm{trace}\,(\nabla u)^k=0$ a.s., $k\geq 2$, Proposition 1 yields

$$E[\delta(u)^{n+1}]=n\langle u,u\rangle E\left[\delta(u)^{n-1}\right],\qquad n\geq 1,$$

and by induction we have

$$E[\delta(u)^{2m}]=\frac{(2m)!}{2^m m!}\langle u,u\rangle^m,\qquad 0\leq 2m\leq n+1,$$

and $E[\delta(u)^{2m+1}]=0$, $0\leq 2m\leq n$, while $E[\delta(u)]=0$ for all $u\in\mathbb{D}_{2,1}(H)$, hence the following corollary of Proposition 1.

Corollary 1. *Let $u\in\mathbb{D}_{\infty,2}(H)$ such that $\langle u,u\rangle$ is deterministic and*

$$\mathrm{trace}\,(\nabla u)^k=\ \mathrm{trace}\,(\nabla u)^{k-1}(Du)=0,\quad a.s.,\quad k\geq 2. \tag{10}$$

Then $\delta(u)$ has a centered Gaussian distribution with variance $\langle u,u\rangle$.

In particular, under the conditions of Corollary 1, $\delta(Rh)$ has a centered Gaussian distribution with variance $\langle h,h\rangle$ when $u=Rh$, $h\in H$, and R is a random mapping with values in the isometries of H, such that $Rh\in\cap_{p>1}\mathbb{D}_{p,2}(H)$ and $\mathrm{trace}\,(DRh)^k=0$, $k\geq 2$. In the Wiener case this recovers Theorem 2.1-b) of [12], cf. also Corollary 2.2 of [5].

In addition, Lemma 1 shows that Condition (10) holds when the process u is adapted with respect to the Brownian filtration.

Next we prove Proposition 1 based on Lemmas 2, 3 and Lemma 5 below.

Proof of Proposition 1. Let $n\geq 1$ and $u\in\mathbb{D}_{n+1,2}(H)$. We show that for any $n\geq 1$ and $u\in\mathbb{D}_{n+1,2}(H)$, $v\in\mathbb{D}_{n+1,1}(H)$, we have

$$E[\delta(u)^n\delta(v)]=\sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[\delta(u)^{n-k}\left(\langle(\nabla u)^{k-1}v,u\rangle+\langle\nabla^* u,D((\nabla u)^{k-1}v)\rangle\right)\right]. \tag{11}$$

We have $(\nabla u)^{k-1}v\in\mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u)\in\mathbb{D}_{(n+1)/(n-k+1),1}$, and by Lemma 5 below applied to $F=1$ we get

$$E\left[\delta(u)^l\langle(\nabla u)^i v,D\delta(u)\rangle\right]-lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1}v,D\delta(u)\rangle\right]$$

$$\begin{aligned}
&= E\left[\delta(u)^l\langle(\nabla u)^i v,u\rangle\right] + E\left[\delta(u)^l\langle(\nabla u)^i v,\delta(\nabla^* u)\rangle\right]\\
&\quad -lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1} v,u\rangle\right] - lE\left[\delta(u)^{l-1}\langle(\nabla u)^{i+1} v,\delta(\nabla^* u)\rangle\right]\\
&= E\left[\delta(u)^l\langle(\nabla u)^i v,u\rangle\right] + E[\delta(u)^l\langle\nabla^* u, D((\nabla u)^i v)\rangle],
\end{aligned}$$

and applying this formula to $l = n-k$ and $i = k-1$ yields

$$\begin{aligned}
&E[\delta(u)^n\delta(v)] = E[\langle v, D\delta(u)^n\rangle] = nE[\delta(u)^{n-1}\langle v, D\delta(u)\rangle]\\
&= \sum_{k=1}^n \frac{n!}{(n-k)!}\left(E\left[\delta(u)^{n-k}\langle(\nabla u)^{k-1} v, D\delta(u)\rangle\right] - (n-k)E\left[\delta(u)^{n-k-1}\langle(\nabla u)^k v, D\delta(u)\rangle\right]\right)\\
&= \sum_{k=1}^n \frac{n!}{(n-k)!}\left(E\left[\delta(u)^{n-k}\langle(\nabla u)^{k-1} v, u\rangle\right] + E\left[\delta(u)^{n-k}\langle\nabla^* u, D((\nabla u)^{k-1} v)\rangle\right]\right).
\end{aligned}$$

We conclude by applying Lemmas 2 and 3. The next lemma extends the argument of Lemma 3.1 in [5] pages 120-121 to the path space case, including an additional random variable $F \in \mathbb{D}_{2,1}$.

Lemma 5. *Let $F \in \mathbb{D}_{2,1}$, $u \in \mathbb{D}_{n+1,2}(H)$, and $v \in \mathbb{D}_{n+1,1}(H)$. For all $k, i \geq 0$ we have*

$$\begin{aligned}
&E[F\delta(u)^k\langle(\nabla u)^i v, \delta(\nabla^* u)\rangle] - kE[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle]\\
&= kE[F\delta(u)^{k-1}\langle(\nabla u)^{i+1} v, u\rangle] + E[\delta(u)^k\langle(\nabla u)^{i+1} v, DF\rangle] + E[F\delta(u)^k\langle\nabla^* u, D((\nabla u)^i v)\rangle].
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&E[F\delta(u)^k\langle(\nabla u)^i v, \delta(\nabla^* u)\rangle] - iE[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle]\\
&= E[\langle\nabla^* u, D(F\delta(u)^k(\nabla u)^i v)\rangle] - kE[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle]\\
&= kE[F\delta(u)^{k-1}\langle\nabla^* u, (\nabla u)^i v \otimes D\delta(u)\rangle] - kE[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle]\\
&\quad + E[\delta(u)^k\langle\nabla^* u, D(F(\nabla u)^i v)\rangle]\\
&= kE[F\delta(u)^{k-1}\langle\nabla^* u, (\nabla u)^i v \otimes u\rangle] + kE[F\delta(u)^{k-1}\langle\nabla^* u, (\nabla u)^i v \otimes \delta(\nabla^* u)\rangle]\\
&\quad - kE[F\delta(u)^{k-1}\langle(\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle] + E[\delta(u)^k\langle\nabla^* u, D(F(\nabla u)^i v)\rangle]\\
&= kE[F\delta(u)^{k-1}\langle(\nabla u)^{i+1} v, u\rangle] + E[\delta(u)^k\langle(\nabla u)^{i+1} v, DF\rangle]\\
&\quad + E[F\delta(u)^k\langle\nabla^* u, D((\nabla u)^i v)\rangle],
\end{aligned}$$

where we used the commutation relation (5).

The case of the left derivative D^L defined as

$$\langle D^L F, v\rangle = \frac{d}{d\varepsilon} f\left(e^{\varepsilon\int_0^{t_1} v_s ds}\gamma(t_1), \ldots, e^{\varepsilon\int_0^{t_n} v_s ds}\gamma(t_n)\right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+, \mathscr{G}),$$

for $F = f(\gamma(t_1), \ldots, \gamma(t_n)) \in \mathscr{S}$, $f \in \mathscr{C}_b^\infty(\mathsf{G}^n)$, can be dealt with by application of the existing results on the flat Wiener space, using the expression of its adjoint the left divergence δ^L which can be written as

$$\delta^L(u) = \hat{\delta}(\mathrm{Ad}\,\gamma.u.)$$

using the Skorohod integral operator $\hat{\delta}$ on the flat space $\mathbb{R}^d$, cf. [3], [10], and § 13.1 of [11].

4 Random Hermite polynomials on path space

In this section we extend the results of [6] on the expectation of Hermite polynomials to the path space framework. This also allows us to recover the invariance results of Section 3 in Corollary 2 and to derive a Girsanov identity in Corollary 3 as a consequence of the derivation formula stated in Proposition 2.

It is well known that the Gaussianity of X is not required for $E[H_n(X,\sigma^2)]$ to vanish when σ^2 is allowed to be random. Indeed, such an identity also holds in the random adapted case under the form

$$E\left[H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)\right] = 0, \tag{12}$$

where $(u_t)_{t\in\mathbb{R}_+}$ is a square-integrable process adapted to the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$, since $H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)$ is the n-th order iterated multiple stochastic integral of $u_{t_1}\cdots u_{t_n}$ with respect to $(B_t)_{t\in\mathbb{R}_+}$, cf. [7] and page 319 of [2].

In Theorem 1 below we extend Relations (1) and (12) by computing the expectation of the random Hermite polynomial $H_n(\delta(u),\|u\|^2)$ in the Skorohod integral $\delta(u)$, $n\geq 1$. This also extends Theorem 3.1 of [6] to the setting of path spaces over Lie groups.

Theorem 1. *For any $n\geq 0$ and $u\in \mathbb{D}_{n+1,2}(H)$ we have*

$$E[H_{n+1}(\delta(u),\|u\|^2)] = \sum_{l=0}^{n-1}\frac{n!}{l!}E\left[\delta(u)^l \sum_{0\leq 2k\leq n-1-l}\frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle\nabla^* u, D((\nabla u)^{n-2k-l-1}u)\rangle\right].$$

Clearly, it follows from Theorem 1 that if $u\in\mathbb{D}_{n,2}(H)$ and

$$\langle\nabla^* u, D((\nabla u)^k u)\rangle = 0, \qquad 0\leq k\leq n-2, \tag{13}$$

then we have

$$E[H_n(\delta(u),\|u\|^2)] = 0, \qquad n\geq 1, \tag{14}$$

which extends Relation (12) to the anticipating case. In addition, from Theorem 1 and Lemma 2 we have

$E[H_{n+1}(\delta(u),\|u\|^2)]$

$$= \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \mathrm{trace}\left((\nabla u)^{n-2k-l}(Du)\right)\right] \tag{15}$$
$$+ \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \sum_{i=2}^{n-2k-l} \frac{1}{i} \langle (\nabla u)^{n-2k-l-i} u, D\,\mathrm{trace}\,(\nabla u)^i\rangle\right].$$

As a consequence, Lemma 1 leads to the following corollary of Theorem 1, which extends Corollary 3.3 of [6] to the path space setting.

Corollary 2. *Let $u \in \mathbb{D}_{n,2}(H)$ such that $\nabla u : H \to H$ is a.s. quasi-nilpotent in the sense that*

$$\mathrm{trace}\,(\nabla u)^k = \mathrm{trace}\,(\nabla u)^{k-1}(Du) = 0, \qquad k \ge 2, \tag{16}$$

or more generally that (13) *holds. Then for any $n \ge 1$ we have*

$$E[H_n(\delta(u), \|u\|^2)] = 0.$$

As above, Lemma 1 shows that Corollary 2 holds when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted with respect to the Brownian filtration, and this shows that (12) holds for the stochastic integral $\delta(u)$ on path space when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted.

We now turn to the proof of Theorem 1, which follows the same steps as the proof of Theorem 3.1 in [6], the main change being in the different roles played here by ∇ and D.

Proof of Theorem 1. Step 1. We show that for any $n \ge 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$E[H_{n+1}(\delta(u), \|u\|^2)] = \sum_{0\le 2k\le n-1} \frac{(-1)^k n!}{k! 2^k (n-2k-1)!} E[\delta(u)^{n-2k-1} \langle u,u\rangle^k \langle u, \delta(\nabla^* u)\rangle]$$
$$+ \sum_{1\le 2k\le n} \frac{(-1)^k n!}{k! 2^k (n-2k)!} E[\delta(u)^{n-2k} \langle u, D\langle u,u\rangle^k\rangle]. \tag{17}$$

For $F \in \mathbb{D}_{2,1}$ and $k,l \ge 1$ we have

$$E[F\delta(u)^{l+1}] = \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[F\delta(u)^{l+1}]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[\langle u, D(\delta(u)^l F)\rangle]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1} \langle u, D\delta(u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l \langle u, DF\rangle]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l(l+1)}{2k} E[F\delta(u)^{l-1} \langle u,u\rangle]$$
$$- \frac{l(l+1)}{2k} E[F\delta(u)^{l-1} \langle u, \delta(\nabla^* u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l \langle u, DF\rangle],$$

i.e.

$$
\begin{aligned}
&E[F\delta(u)^{n-2k+1}]+\frac{(n-2k)(n-2k+1)}{2k}E[F\delta(u)^{n-2k-1}\langle u,u\rangle]\\
&=\frac{n+1}{2k}E[F\delta(u)^{n-2k+1}]-\frac{(n-2k)(n-2k+1)}{2k}E[F\delta(u)^{n-2k-1}\langle u,\delta(\nabla^*u)\rangle]\\
&\quad-\frac{n-2k+1}{2k}E[\delta(u)^{n-2k}\langle u,DF\rangle].
\end{aligned}
$$

Hence, taking $F=\langle u,u\rangle^k$, we get

$$
\begin{aligned}
E[\delta(u)^{n+1}] &= E[\langle u,D\delta(u)^n\rangle]\\
&= nE[\delta(u)^{n-1}\langle u,D\delta(u)\rangle]\\
&= nE[\delta(u)^{n-1}\langle u,u\rangle]+nE[\delta(u)^{n-1}\langle u,\delta(\nabla^*u)\rangle]\\
&= nE[\delta(u)^{n-1}\langle u,\delta(\nabla^*u)\rangle]\\
&\quad-\sum_{1\le 2k\le n+1}\frac{(-1)^k n!}{(k-1)!2^{k-1}(n+1-2k)!}\Big(E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]\\
&\qquad\qquad+\frac{(n-2k+1)(n-2k)}{2k}E[\delta(u)^{n-2k-1}\langle u,u\rangle^{k+1}]\Big)\\
&= nE[\delta(u)^{n-1}\langle u,\delta(\nabla^*u)\rangle]\\
&\quad-\sum_{1\le 2k\le n+1}\frac{(-1)^k n!}{(k-1)!2^{k-1}(n+1-2k)!}\Big(\frac{n+1}{2k}E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]\\
&\qquad\qquad-\frac{(n-2k)(n-2k+1)}{2k}E[\delta(u)^{n-2k-1}\langle u,u\rangle^k\langle u,\delta(\nabla^*u)\rangle]\\
&\qquad\qquad-\frac{n-2k+1}{2k}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle]\Big)\\
&=-\sum_{1\le 2k\le n+1}\frac{(-1)^k (n+1)!}{k!2^k(n+1-2k)!}E[\delta(u)^{n-2k+1}\langle u,u\rangle^k]\\
&\quad+\sum_{0\le 2k\le n-1}\frac{(-1)^k n!}{k!2^k(n-2k-1)!}E[\delta(u)^{n-2k-1}\langle u,u\rangle^k\langle u,\delta(\nabla^*u)\rangle]\\
&\quad+\sum_{1\le 2k\le n}\frac{(-1)^k n!}{k!2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle],
\end{aligned}
$$

which yields (17) after using the identity (19).

Step 2. For $F\in\mathbb{D}_{2,1}$ and $k,i\ge 0$, by Lemma 5 we have

$$
\begin{aligned}
&E[F\delta(u)^k\langle(\nabla u)^i u,\delta(\nabla^*u)\rangle]-kE[F\delta(u)^{k-1}\langle(\nabla^*u)^{i+1}u,\delta(\nabla^*u)\rangle]\\
&=kE[F\delta(u)^{k-1}\langle(\nabla u)^{i+1}u,u\rangle]+E[\delta(u)^k\langle(\nabla u)^{i+1}u,DF\rangle]+E[F\delta(u)^k\langle\nabla^*u,D((\nabla u)^i u)\rangle].
\end{aligned}
$$

Hence, replacing k above with $l-i$, we get

$$
E[F\delta(u)^l\langle u,\delta(\nabla^*u)\rangle]=l!E[F\langle(\nabla u)^l u,\delta(\nabla^*u)\rangle]
$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}\Big(E[F\delta(u)^{l-i}\langle(\nabla u)^i u,\delta(\nabla^* u)\rangle]-(l-i)E[F\delta(u)^{l-i-1}\langle(\nabla^* u)^{i+1}u,\delta(\nabla^* u)\rangle]\Big)$$

$$= l!E[F\langle(\nabla u)^l u,\delta(\nabla^* u)\rangle]+\sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle(\nabla u)^{i+1}u,u\rangle]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[\delta(u)^{l-i}\langle(\nabla u)^{i+1}u,DF\rangle]+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle\nabla^* u,D((\nabla u)^i u)\rangle]$$

$$= l!E[\langle(\nabla u)^{l+1}u,DF\rangle]+\sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle(\nabla u)^{i+1}u,u\rangle]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E[\delta(u)^{l-i}\langle(\nabla u)^{i+1}u,DF\rangle]+\sum_{i=0}^{l}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle\nabla^* u,D((\nabla u)^i u)\rangle]$$

$$= l!E[\langle(\nabla u)^{l+1}u,DF\rangle]+\sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E[F\delta(u)^{l-i-1}\langle(\nabla u)^{i+1}u,u\rangle]$$

$$+\sum_{i=1}^{l}\frac{l!}{(l-i+1)!}E[\delta(u)^{l-i+1}\langle(\nabla u)^i u,DF\rangle]+\sum_{i=0}^{l}\frac{l!}{(l-i)!}E[F\delta(u)^{l-i}\langle\nabla^* u,D((\nabla u)^i u)\rangle],$$

thus, letting $F=\langle u,u\rangle^k$ and $l=n-2k-1$ above, and using (17) in Step 1, we get

$$E[H_{n+1}(\delta(u),\|u\|^2)]=\sum_{0\le 2k\le n-1}\frac{(-1)^k n!}{k!2^k(n-2k-1)!}E[\delta(u)^{n-2k-1}\langle u,u\rangle^k\langle u,\delta(\nabla^* u)\rangle]$$

$$+\sum_{1\le 2k\le n}\frac{(-1)^k n!}{k!2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle]$$

$$=\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!2^k}E[\langle(\nabla u)^{n-2k}u,D\langle u,u\rangle^k\rangle]$$

$$+\sum_{0\le 2k\le n-2}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2(k+1)-i}\langle(\nabla u)^{i+1}u,u\rangle]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-i)!}E[\delta(u)^{n-2k-i}\langle(\nabla u)^i u,D\langle u,u\rangle^k\rangle]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^* u,D((\nabla u)^i u)\rangle]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k n!}{k!2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle]$$

$$=\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!2^k}E[\langle(\nabla u)^{n-2k}u,D\langle u,u\rangle^k\rangle]$$

$$-\sum_{0\le 2k\le n-2}\frac{(-1)^{k+1}}{(k+1)!2^{k+1}}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E[\delta(u)^{n-2(k+1)-i}\langle(\nabla u)^i u,D\langle u,u\rangle^{k+1}\rangle]$$

$$+\sum_{1\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-i)!}E[\delta(u)^{n-2k-i}\langle(\nabla u)^i u, D\langle u,u\rangle^k\rangle]$$
$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^* u, D((\nabla u)^i u)\rangle]$$
$$=-\sum_{0\le 2k\le n-2}\frac{(-1)^{k+1}}{(k+1)!2^{k+1}}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E[\delta(u)^{n-2(k+1)-i}\langle(\nabla u)^i u, D\langle u,u\rangle^{k+1}\rangle]$$
$$+\sum_{1\le 2k\le n}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k}\frac{n!}{(n-2k-i)!}E[\delta(u)^{n-2k-i}\langle(\nabla u)^i u, D\langle u,u\rangle^k\rangle]$$
$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^* u, D((\nabla u)^i u)\rangle]$$
$$=\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!2^k}\sum_{i=0}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle\nabla^* u, D((\nabla u)^i u)\rangle],$$

where we applied Lemma 2 with $v=(\nabla u)^i u$, which shows that

$$\langle u,u\rangle^k\langle(\nabla u)^{i+1}u,u\rangle=\frac{1}{2}\langle u,u\rangle^k\langle(\nabla u)^i u, D\langle u,u\rangle\rangle=\frac{1}{2(k+1)}\langle(\nabla u)^i u, D\langle u,u\rangle^{k+1}\rangle.$$

5 Girsanov identities on path space

In the sequel we let $\mathbb{D}_{\infty,2}(H)=\bigcap_{n\ge 1}\mathbb{D}_{n,2}(H)$. The next result follows from Theorem 1 and extends Corollary 4.1 of [6] with the same proof, which is omitted here.

Corollary 3. *Let $u\in\mathbb{D}_{\infty,2}(H)$ with $E[e^{|\delta(u)|+\|u\|^2/2}]<\infty$, and such that $\nabla u: H\to H$ is a.s. quasi-nilpotent in the sense of* (16) *or more generally that* (13) *holds. Then we have*

$$E\left[\exp\left(\delta(u)-\frac{1}{2}\|u\|^2\right)\right]=1. \tag{18}$$

Again, Relation (18) shows in particular that if $u\in\mathbb{D}_{\infty,2}(H)$ is such that $\|u\|$ is deterministic and (16) or more generally (13) holds, then we have

$$E\left[e^{\delta(u)}\right]=e^{-\frac{1}{2}\|u\|^2},$$

i.e. $\delta(u)$ has a centered Gaussian distribution with variance $\|u\|^2$.

As a consequence of Theorem 1 we also have the following derivation formula.

Proposition 2. *Let $u\in\mathbb{D}_{\infty,2}(H)$ such that $E[e^{a|\delta(u)|+a^2\|u\|^2}]<\infty$ for some $a>0$. Then we have*

$$\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]=\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle\nabla^*u,D((I-\lambda\nabla u)^{-1}u)\rangle\right],$$

for all $\lambda\in(-a/2,a/2)$ *such that* $|\lambda|<\|\nabla u\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. From the identity

$$H_n(x,\mu)=\sum_{0\le 2k\le n}\frac{n!(-\mu/2)^k}{k!(n-2k)!}x^{n-2k},\qquad x,\mu\in\mathbb{R}, \tag{19}$$

we get the bound

$$|H_n(x,\sigma^2)|\le\sum_{0\le 2k\le n}\frac{(-1)^k}{k!2^k}\frac{n!}{(n-2k)!}|x|^{n-2k}(-\sigma^2)^k=H_n(|x|,-\sigma^2),$$

hence

$$\begin{aligned}E\left[\sum_{n=0}^\infty\frac{|\lambda|^n}{n!}|H_{n+1}(\delta(u),\|u\|^2)|\right]&\le E\left[\sum_{n=0}^\infty\frac{|\lambda|^n}{n!}H_{n+1}(|\delta(u)|,-\|u\|^2)\right]\\&=E\left[(|\delta(u)|+\lambda\|u\|^2)e^{|\lambda\delta(u)|+\lambda^2\|u\|^2/2}\right]\\&=E\left[e^{2|\lambda\delta(u)|+4\lambda^2\|u\|^2}\right]\\&<\infty,\end{aligned}$$

hence by the Fubini theorem we can exchange the infinite sum and the expectation to obtain

$$\begin{aligned}&\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]=\sum_{n=0}^\infty\frac{\lambda^n}{n!}E[H_{n+1}(\delta(u),\|u\|^2)]\\&=\sum_{n=0}^\infty\frac{\lambda^n}{n!}\sum_{l=0}^{n-1}\frac{n!}{l!}E\left[\delta(u)^l\sum_{0\le 2k\le n-1-l}\frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle\nabla^*u,D((\nabla u)^{n-2k-l-1}u)\rangle\right]\\&=\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle\nabla^*u,D((I-\lambda\nabla u)^{-1}u)\rangle\right].\end{aligned}$$

In addition, Relation (15) yields the following result, in which $\det_2(I-\lambda\nabla u)$ denotes the Carleman-Fredholm determinant of $I-\lambda\nabla u$.

Corollary 4. *Let* $u\in\mathbb{D}_{\infty,2}(H)$ *such that* $E[e^{a|\delta(u)|+a^2\|u\|^2}]<\infty$ *for some* $a>0$. *Then we have*

$$\begin{aligned}\frac{\partial}{\partial\lambda}E\left[e^{\lambda\delta(u)-\frac{\lambda^2}{2}\|u\|^2}\right]&=\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\operatorname{trace}(\nabla u)(I-\lambda\nabla u)^{-1}(Du)\right]\\&\quad-\lambda E\left[e^{\lambda\delta(u)-\lambda^2\langle u,u\rangle/2}\langle(I-\lambda\nabla u)^{-1}u,D\log\det{}_2(I-\lambda\nabla u)\rangle\right],\end{aligned}$$

for all $\lambda\in(-a/2,a/2)$ *such that* $|\lambda|<\|\nabla u\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. From Lemma 4 we have

$$\lambda \langle \nabla^* u, D((I-\lambda \nabla u)^{-1} v) \rangle$$
$$= \lambda \operatorname{trace}(\nabla u)(I-\lambda \nabla u)^{-1}(Du) - \lambda \langle (I-\lambda \nabla u)^{-1} u, D \log \det_2 (I-\lambda \nabla u) \rangle.$$

When (16) or more generally (13) holds, Proposition 2 and Corollary 4 show that

$$\frac{\partial}{\partial \lambda} E\left[e^{\lambda \delta(u) - \frac{\lambda^2}{2}\|u\|^2} \right] = 0,$$

for λ in a neighborhood of 0, which recovers the result of Corollary 3.

On the Wiener space we have $\nabla = D$ and we obtain the following corollary.

Corollary 5. *Let $u \in \mathbb{D}_{\infty,2}(H)$ such that $E[e^{a|\delta(u)|+a^2\|u\|^2}] < \infty$ for some $a > 0$. Then we have*

$$\frac{\partial}{\partial \lambda} E\left[e^{\lambda \delta(u) - \frac{\lambda^2}{2}\|u\|^2} \right] = -E\left[e^{\lambda \delta(u) - \lambda^2 \langle u,u \rangle /2} \frac{\partial}{\partial \lambda} \log \det_2 (I - \lambda Du) \right]$$
$$-\lambda E\left[e^{\lambda \delta(u) - \lambda^2 \langle u,u \rangle /2} \langle (I-\lambda Du)^{-1} u, D \log \det_2 (I - \lambda Du) \rangle \right],$$

for all $\lambda \in (-a/2, a/2)$ such that $|\lambda| < \|Du\|^{-1}_{L^\infty(\Omega;H\otimes H)}$.

Proof. We note that (8) shows that

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \log \det_2 (I - \lambda Du) &= -\sum_{n=2}^{\infty} \lambda^{n-1} \operatorname{trace}(Du)^n \\
&= -\lambda \sum_{n=0}^{\infty} \lambda^n \langle D^* u, (Du)^{n+1} \rangle \\
&= -\lambda \langle D^* u, (I - \lambda Du)^{-1} Du \rangle \\
&= -\lambda \operatorname{trace}(Du)(I-\lambda Du)^{-1}(Du),
\end{aligned}$$

and apply Corollary 4.

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