Linear Skorohod stochastic differential equations on Poisson space

Nicolas Privault

Equipe d'Analyse et Probabilités, Université d'Evry-Val d'Essonne
Boulevard des Coquibus, 91025 Evry Cedex, France

Abstract

We study the absolute continuity of transformations defined by anticipative flows on Poisson space, and show that the process of densities associated to those transformations allows to solve anticipative linear stochastic differential equations on the Poisson space.

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1 Introduction

Linear Skorohod stochastic differential equations have been studied on the Wiener space, cf. [3], using the anticipative stochastic calculus developed in [6]. It has been shown in particular that the solutions of such equations are associated to the density induced by absolutely continuous transformations defined by flows on the Wiener space. Such absolute continuity results have been extended in [10]. Our goal here is to investigate the Poisson space case. The anticipative stochastic calculus on Poisson space, cf. [5], [8], permits to introduce anticipative stochastic differential equations by means of an extension of the compensated Poisson stochastic integral, also called the Skorohod integral. We study the absolute continuity of some anticipative flows on Poisson space and show that their associated densities allow to solve Skorohod stochastic differential equations. Let us describe the Poisson space interpretation that we are working with, cf. [8]. Let B be a space of sequences with a probability measure P such that the coordinate functionals

$$\tau_k : B \longrightarrow \mathbb{R} \quad k \in \mathbb{N},$$

are independent identically distributed exponential random variables. The space B is endowed with the norm $\| x \|_B = \sup_{n\in\mathbb{N}} | x_n | /(n+1)$ such that P is defined on the Borel σ-algebra of B. Let $T_k = \sum_{i=0}^{i=k-1} \tau_i$, $k \geq 0$, denote the k-th jump time of the Poisson process (N_t) defined as $N_t = \sum_{k\geq 0} 1_{[T_k,\infty[}(t)$, $t \in \mathbb{R}_+$. Denote by $(e_k)_{k\in\mathbb{N}}$ the canonical basis of the space of square-summable sequences $H = l^2(\mathbb{N})$. We define an operator i that turns any discrete time stochastic process $u = (u_k)_{k\in\mathbb{N}}$ into a continuous time process $i(u)$ by $i_t(u) = u_{N_{t^-}}$, or

$$i_t(u) = \sum_{k\geq 0} u_k 1_{]T_k,T_{k+1}]}(t), \quad t \in \mathbb{R}_+. \tag{1}$$

The flow that we will consider is the family $(\mathcal{T}_t)_{t\in[0,1]}$ of transformations $\mathcal{T}_t : B \to B$, defined by

$$\mathcal{T}_t(\omega) = \omega + \left(\int_0^t i_s(e_k)(\mathcal{T}_s(\omega))\sigma_s(\mathcal{T}_s(\omega))ds \right)_{k\geq 0},$$

where σ is a process satisfying some boundedness conditions. If the transformation $\mathcal{T}_t$, $t \in [0,1]$, is absolutely continuous, then the process of densities $\left(\frac{d(\mathcal{T}_t^{-1})_* P}{dP} \right)_{t\in[0,1]}$ solves the anticipative stochastic differential equation

$$X_t = 1 + \int_0^t \sigma_s(\omega) X_s \delta\tilde{N}_s,$$

where $\int_0^t u_s \delta\tilde{N}_s = \tilde{\delta}\left(u 1_{[0,t]}\right)$ is the Skorohod integral of $u1_{[0,t]}$ on the Poisson space, as defined in [1], [5], [8]. This integral is an extension to anticipative integrands of the stochastic integral with respect to the compensated Poisson process. It is the adjoint of a derivation operator defined by shifting the Poisson process jump times, and has in particular the property of being an integral with zero expectation. As a consequence, we will be able to solve the anticipative stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma_s X_s \delta\tilde{N}_s + \int_0^t b_s X_s ds \quad t \in [0,1], \tag{2}$$

where X_0 and b are bounded random variables. In case the processes b and σ are predictable, the equation defining the inverse $(\mathcal{A}_t)_{t\in[0,1]}$ of $(\mathcal{T}_t)_{t\in[0,1]}$ becomes

$$\mathcal{A}_t(\omega) = \omega + \left(\int_0^t i_s(e_k)\sigma_s(\omega)ds \right)_{k\geq 0},$$

and we retrieve a classical result, cf. for instance [2].

We proceed as follows. In Sect. 2 the definitions and main results of the anticipative stochastic calculus on the Poisson space as introduced in [5], [8] are recalled. Sect. 3 is devoted to the definition of the flow $(\mathcal{T}_t)_{t\in[0,1]}$ of anticipative transformations of the Poisson process trajectories and to the study of its absolute continuity. Those results are applied in Sect. 4, where the solution of the linear Skorohod stochastic differential equation (2) is given.

2 Anticipative stochastic calculus on the Poisson space

Let $\mathcal{S}$ denote the set of functionals of the form

$$F = f(\tau_0, \ldots, \tau_n),$$

with $n \in \mathbb{N}$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}_+^{n+1})$. We define a gradient operator $D : L^2(B) \to L^2(B) \otimes H$ by

$$DF = (\partial_k f(\tau_0, \ldots, \tau_n))_{k\in\mathbb{N}}, \quad F \in \mathcal{S}. \tag{3}$$

We also define $\tilde{D} : L^2(B) \to L^2(B) \otimes L^2(\mathbb{R}_+)$ as

$$\tilde{D}F = -i \circ DF, \quad F \in \mathcal{S}.$$

The operators $\tilde{D}$ and D are closable, cf. [8]. Denote by $\mathbf{D}_{1,2}$ the domain of the closed extension of D. Let $\tilde{\delta}$ be the adjoint of $\tilde{D}$, which is also closable and can be extended to a closed operator

$$\tilde{\delta} : L^1(B \times [0,1]) \longrightarrow L^1(B),$$

of domain $Dom(\tilde{\delta})$. Let $\mathcal{V}$ denote the class of processes of the form

$$v = \sum_{i=1}^{i=n} 1_{\Delta_i} f_i(\tau_0, \ldots, \tau_n),$$

where $f_i \in \mathcal{C}_c^\infty(\mathbb{R}_+^{n+1})$, $1 \leq i \leq n$, and $\Delta_1, \ldots, \Delta_n \subset [0,1]$. We have the following formula, cf. [5], [8]:

$$\tilde{\delta}(v) = \int_0^\infty v(s) d(N_s - s) - \int_0^\infty \tilde{D}_s v(s) ds, \quad v \in \mathcal{V}. \tag{4}$$

The interpretation of $\tilde{\delta}$ as an extension of the stochastic integral with respect to the compensated Poisson process comes from the following proposition, cf. [5], [8].

Proposition 1 *Let $v \in L^2(B) \otimes L^2(\mathbb{R}_+)$ be predictable with respect to the filtration generated by the Poisson process (N_t). We have*

$$\tilde{\delta}(v) = \int_0^\infty v(s) d(N_s - s).$$

Denote by $\mathbf{D}_{1,\infty}$ the subset of $D_{1,2}$ made of the random variables F for which

$$\| F \|_{L^\infty(B)} + \| | DF |_H \|_{L^\infty(B)}$$

is bounded and let $L_{1,\infty} = L^2([0,1], \mathbf{D}_{1,\infty})$, $L_{1,2} = L^2([0,1], D_{1,2})$. If $\mathcal{T} : B \longrightarrow B$ is a measurable mapping, we denote by $\mathcal{T}_* P$ the image measure of P by $\mathcal{T}$, and say that $\mathcal{T}$ is absolutely continuous if $\mathcal{T}_* P$ is absolutely continuous with respect to P. A flow $(\phi_{s,t})_{0 \le s < t \le 1}$ of transformations of B is said to be absolutely continuous if $\phi_{s,t}$ is absolutely continuous, $0 \le s < t \le 1$. We end this section with four propositions which will be useful in the sequel. Their statements and proofs are adapted from [4]. Proofs are given in the appendix.

Proposition 2 *Let $F \in \mathbf{D}_{1,2}$. For any $\varepsilon > 0$, there is a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ that converges to F in $\mathbf{D}_{1,2}$ and such that*

1. *ess inf $F < F_n <$ ess sup F, $n \in \mathbb{N}$.*
2. *$\| | DF_n |_H \|_\infty \le \| | DF |_H \|_\infty + \varepsilon$, $n \in \mathbb{N}$.*

We obtain in the same way the following result.

Proposition 3 *Let $\sigma \in L_{1,\infty}$ with $\sigma > -1$ a.s. and $\int_0^1 \| \frac{1}{1+\sigma_r} \|_\infty^2 dr < \infty$. For any $\varepsilon > 0$, there is a sequence $(\sigma^n)_{n \in \mathbb{N}} \subset \mathcal{V}$ that converges to σ in $L_{1,2}$ and such that for $n \in \mathbb{N}$,*

1. *$\sigma^n > -1$.*
2. *$\int_0^1 | \sigma_s^n |_\infty^2 ds \le \int_0^1 | \sigma_s |_\infty^2 ds$.*
3. *$\left(\int_0^1 \| | D\sigma_s^n |_H \|_\infty^2 ds \right)^{1/2} \le \varepsilon + \left(\int_0^1 \| | D\sigma_s |_H \|_\infty^2 ds \right)^{1/2}$.*
4. *$\int_0^1 \| \frac{1}{1+\sigma_r^n} \|_\infty^2 \le \int_0^1 \| \frac{1}{1+\sigma_r} \|_\infty^2 dr$.*
5. *$\| \sigma^n \|_{L^\infty(B \times [0,1])} \le \| \sigma \|_{L^\infty(B \times [0,1])}$.*
6. *$\| D\sigma^n \|_{L^\infty(B \times [0,1] \times \mathbb{N})} \le \varepsilon + \| D\sigma \|_{L^\infty(B \times [0,1] \times \mathbb{N})}$.*

If σ has continuous trajectories a.s., then $(\sigma_{T_k}^n)_{n \in \mathbb{N}}$ converges in $L^2(B)$ to σ_{T_k}, $k \ge 1$.

Proposition 4 *Let $\mathcal{T}^1, \mathcal{T}^2$ be two absolutely continuous transformations, respectively defined by*

$$\mathcal{T}^1(\omega) = \omega + \left(\int_0^1 i_s(e_k) \sigma_s^1(\omega) ds \right)_{k \in \mathbb{N}}$$

and

$$\mathcal{T}^2(\omega) = \omega + \left(\int_0^1 i_s(e_k) \sigma_s^2(\omega) ds \right)_{k \in \mathbb{N}},$$

$\omega \in B$, with $\sigma^1, \sigma^2 \in L^2(B \times [0,1])$. Let $F \in D_{1,\infty}$. We have

$$| F \circ \mathcal{T}^1(\omega) - F \circ \mathcal{T}^2(\omega) | \leq \| | DF |_H \|_\infty | \sigma^1(\omega) - \sigma^2(\omega) |_{L^2([0,1])} .$$

If $F \in \mathcal{S}$, then

$$| F(\omega) - F(\omega + h) | \leq \| | DF |_H \|_\infty \| h \|_H \qquad h \in H, \ \omega \in B.$$

Proposition 5 *Let $(\mathcal{T}^n)_{n \in \mathbb{N}}$ be a sequence of absolutely continuous transformations with*

$$\mathcal{T}^n \omega = \omega + \left(\int_0^\infty i_s(e_k) \sigma_s^n(\omega) ds \right)_{k \in \mathbb{N}},$$

defined by a sequence $(\sigma^n)_{n \in \mathbb{N}}$ of processes that converges in $L^2(B) \otimes L^2([0,1])$ to a process σ, such that the sequence of densities $(L^n)_{n \in \mathbb{N}} = \left(\frac{d\mathcal{T}_*^n P}{dP} \right)_{n \in \mathbb{N}}$ is uniformly integrable. If $(F_n)_{n \in \mathbb{N}}$ converges to F in $L^2(B)$, then $(F_n \circ \mathcal{T}^n)_{n \in \mathbb{N}}$ converges to $F \circ \mathcal{T}$ in probability, where $\mathcal{T}$ is defined by

$$\mathcal{T} \omega = \omega + \left(\int_0^\infty i_s(e_k) \sigma_s(\omega) ds \right)_{k \in \mathbb{N}}.$$

Moreover, $\mathcal{T}$ is absolutely continuous.

3 Absolute continuity of anticipative flows

Proposition 6 *Let $\sigma \in \mathcal{V}$. The equation*

$$\mathcal{T}_t \omega = \omega - \left(\int_0^t i_s(e_k)(\mathcal{T}_s \omega) \sigma_s(\mathcal{T}_s \omega) ds \right)_{k \geq 0}, \quad t \in [0,1], \tag{5}$$

has a unique solution which is invertible. For $s, t \in [0,1]$, $s < t$, let $\mathcal{A}_t = \mathcal{T}_t^{-1}$ and $\phi_{s,t} = \mathcal{T}_s \circ \mathcal{A}_t$, $s \leq t$. Then $\phi_{s,t}$ satisfies to

$$\phi_{s,t} \omega = \omega + \left(\int_s^t i_r(e_k)(\omega) \sigma_r(\phi_{r,t} \omega) dr \right)_{k \in \mathbb{N}} \quad \omega \in B, \quad 0 \leq s < t \leq 1. \tag{6}$$

Let $\psi_{s,t} = \mathcal{T}_t \circ \mathcal{A}_s$*,* $s \leq t$*. We have*

$$\psi_{s,t}\omega = \omega - \left(\int_s^t i_r(e_k)(\psi_{s,r}\omega)\sigma_r(\psi_{s,r}\omega)dr \right)_{k\in\mathbb{N}} \quad \omega \in B, \quad 0 \leq s < t \leq 1.$$

Proof. The equations (5) and (6) can be solved as differential equations in finite dimension since σ_r is Lipschitz, $r \in [0,1]$, cf. Prop. 4. Denote by $\mathcal{A}_t$ and $\mathcal{T}_t$ the solutions of (5) and (6) for $s = 0$. It remains to show that $\phi_{s,t} \circ \mathcal{T}_t = \mathcal{T}_s$, $s \leq t$. Let us show that for $r \leq t$, $i_r(e_k)(\mathcal{T}_t\omega) = i_r(e_k)(\mathcal{T}_r\omega)$, $k \in \mathbb{N}$. For $s \leq t$, we notice from (1) that $T_k(\mathcal{T}_s\omega) \leq s \Rightarrow T_k(\mathcal{T}_s\omega) = T_k(\mathcal{T}_t\omega)$, and $T_k(\mathcal{T}_s) \geq s \Rightarrow T_k(\mathcal{T}_t) \geq s$. Hence

$$\begin{aligned} T_k(\mathcal{T}_s\omega) \leq s \leq T_{k+1}(\mathcal{T}_s\omega) &\iff T_k(\mathcal{T}_t\omega) \leq s \leq T_{k+1}(\mathcal{T}_s\omega) \\ &\iff T_k(\mathcal{T}_t\omega) \leq s \leq T_{k+1}(\mathcal{T}_t\omega). \end{aligned}$$

This gives

$$\phi_{s,t}(\mathcal{T}_t\omega) = \mathcal{T}_t\omega + \left(\int_s^t i_r(e_k)(\mathcal{T}_t\omega)\sigma(r, \mathcal{T}_r\omega)dr \right)_{k\in\mathbb{N}} = \mathcal{T}_s\omega,$$

which implies (5). Finally, $i_r(e_k)(\psi_{s,t}) = i_r(e_k)(\psi_{s,r})$, $0 \leq s < r < t \leq 1$, and

$$\begin{aligned} \omega &= \phi_{s,t} \circ \psi_{s,t}\omega = \psi_{s,t}\omega + \left(\int_s^t i_r(e_k)(\psi_{s,t}\omega)\sigma_r(\phi_{r,t} \circ \psi_{s,t}\omega)dr \right)_{k\in\mathbb{N}} \\ &= \psi_{s,t} + \left(\int_s^t i_r(e_k)(\psi_{s,r}\omega)\sigma_r(\psi_{s,r}\omega)dr \right)_{k\in\mathbb{N}}. \end{aligned}$$

□

Theorem 1 *For* $\sigma \in L_{1,\infty}$ *with* $\sigma > -1$ *a.s., and* $\int_0^1 \| \frac{1}{1+\sigma_r} \|_\infty dr < \infty$*, Eq. (5) has a unique absolutely continuous solution which is invertible and whose inverse flow* $\{\psi_{s,t} : 0 \leq s \leq t \leq 1\}$ *satisfies to (6). Assume that* σ *has continuous trajectories, a.s. Then*

$$\begin{aligned} L_{s,t} &= \frac{d(\phi_{s,t})_* P}{dP} \qquad (7) \\ &= \exp\left(-\int_s^t \tilde{D}_r\sigma_r(\phi_{r,t})dr - \int_s^t \sigma_r(\phi_{r,t})dr \right) \prod_{s\leq T_k\leq t} (1 + \sigma_{T_k}(\phi_{T_k,t})), \end{aligned}$$

$0 \leq s \leq t \leq 1$.

Remark. $\tilde{D}_r\sigma_r(\phi_{r,t})$ is here interpreted as $i_r(D\sigma_r(\phi_{r,t}))$.
Proof. We start by assuming that $\sigma \in \mathcal{V}$ and depends only on $\tau_0, \ldots, \tau_n$ for some $n \in \mathbb{N}$.

Lemma 1 *If $\sigma \in \mathcal{V}$, we have with ϕ given by Prop. 6:*

$$\det(D\phi_{s,t}) = \exp\left(-\int_s^t \tilde{D}_r\sigma_r(\phi_{r,t})dr\right) \prod_{s\le T_k\le t}(1+\sigma_{T_k}(\phi_{T_k,t})) \quad 0 \le s \le t \le 1.$$

Proof. We have that $\phi_{s,t}(l)$ is differentiable since it is expressed with the solution of a differential equation with $\mathcal{C}^\infty$ coefficients, and

$$\begin{aligned} &D_k\phi_{s,t}(l) \\ &= 1_{\{k=l\}} + \sigma_{T_{l+1}}(\phi_{T_{l+1},t})1_{\{k\le l\}}1_{\{s<T_{l+1}<t\}} - \sigma_{T_l}(\phi_{T_l,t})1_{\{k<l\}}1_{\{s<T_l<t\}} \\ &\quad + \int_s^t \sum_{i=0}^{i=n} i_r(e_l)D_k\phi_{r,t}(i)D_i\sigma_r(\phi_{r,t})dr \quad k,l\in\mathbb{N}. \end{aligned}$$

Letting $U_{s,t} = (D_k\phi_{s,t}(l))_{0\le k,l\le n}$, this gives the following differential equation in the space of $(n+1)\times(n+1)$ matrices:

$$U_{s,t} = A_{s,t} + \int_s^t U_{r,t}B_{r,t}dr, \tag{8}$$

where

$$A_{s,t}(k,l) = 1_{\{k=l\}} + \sigma_{T_{l+1}}(\phi_{T_{l+1},t})1_{\{k\le l\}}1_{\{s<T_{l+1}<t\}} - \sigma_{T_l}(\phi_{T_l,t})1_{\{k<l\}}1_{\{s<T_l<t\}}$$

$0 \le k,l \le n$, and $B_{s,t} = (i_s(e_l)D_k\sigma_s(\phi_{s,t}))_{0\le k,l\le n}$. Solving this differential equation in $s \in [0,t]$ for fixed ω on the intervals $]T_l, T_{l+1}[\bigcap[0,t]$, $k \in \mathbb{N}$, we get

$$\begin{aligned} &(D_k\phi_{s,t}(l))_{0\le k,l\le n} \\ &= \exp\left(\int_s^{t\wedge T_{N_s+1}} B_{r,t}dr\right) \prod_{s<T_l<t}\left(\exp\left(\int_{s\wedge T_l}^{t\wedge T_{l+1}} B_{r,t}dr\right) + C_l\right), \end{aligned} \tag{9}$$

$0 \le s \le t \le 1$, where C_l, $l \ge 1$, is a matrix such that $C_l(l,l) = \sigma_{T_l}(\phi_{T_l,t})$ and $C_l(i,j) = 0$ if $i \ne l$ or $j > i$. Since $B_{r,t}(i,j) = 0$ if $r < T_i$, $j = 0,\ldots,n$, we have

$$\begin{aligned} &\det\left(\exp\left(\int_{s\wedge T_l}^{t\wedge T_{l+1}} B_{r,t}dr\right) + C_l\right) \\ &= (1+\sigma_{T_l}(\phi_{T_l,t}))\det\left(\exp\left(\int_{s\wedge T_l}^{t\wedge T_{l+1}} B_{r,t}dr\right)\right). \end{aligned} \tag{10}$$

Hence

$$\det(U_{s,t}) = \exp\left(\int_s^t trace(B_{r,t})dr\right) \prod_{s<T_k<t,\ k\le n}(1+\sigma_{T_k}(\phi_{T_k,t})).$$

Noticing that for $k > n$,

$$
\begin{aligned}
& D_k \phi_{s,t}(l) \\
& \quad = 1_{\{k=l\}} + \sigma_{T_{l+1}}(\phi_{T_{l+1},t}) 1_{\{k \leq l\}} 1_{\{s < T_{l+1} < t\}} - \sigma_{T_l}(\phi_{T_l,t}) 1_{\{k < l\}} 1_{\{s < T_l < t\}}
\end{aligned}
$$

and $trace(B_{r,t}) = \sum_{k=0}^{k=n} D_k \sigma_r(\phi_{r,t}) i_r(e_k) = \tilde{D}_r \sigma_r(\phi_{r,t})$, we obtain

$$
\det (D\phi_{s,t}) = \exp \left(- \int_s^t \tilde{D}_r \sigma_r(\phi_{r,t}) dr \right) \prod_{s \leq T_k \leq t} (1 + \sigma_{T_k}(\phi_{T_k,t})) .
$$

□

Define for $k \in \mathbb{N}$ $\pi_k : B \longrightarrow H$ by $\pi_k(w) = (1_{\{k \leq n\}} \tau_k)_{k \in \mathbb{N}}$. Let $\Phi_{s,t} = \phi_{s,t} - I_B$, $0 \leq s \leq t \leq 1$, and $F_k = \pi_k \Phi_{s,t}$ for $k \geq n$. The mapping $I_B - F_k$ is a diffeomorphism of $B^+ = \{\omega \in B \ : \ \omega_k \geq 0, \ k \in \mathbb{N}\}$, and we have for $f \in \mathcal{C}_b^+(B)$, from the finite dimensional Jacobi theorem:

$$
\begin{aligned}
E[f] &= E \left[f(I_B + F_k) \mid \det(I_H + DF_k) \mid \exp \left(- \sum_{i=0}^{i=k} F_k(i) \right) \right] \\
&= E[f(I_B + F_k) \mid \Lambda_k \mid], \quad k \geq n,
\end{aligned}
$$

with from (9):

$$
\Lambda_k = \exp \left(- \int_s^t \tilde{D}_r \sigma_r(\phi_{r,t}) dr - \int_s^t \sigma_r(\phi_{r,t}) dr \right) \prod_{s \leq T_i \leq t, \ i \leq k} (1 + \sigma_{T_i}(\phi_{T_i,t})).
$$

Now,

$$
\begin{aligned}
& E \left[\Lambda_k \mid \log \Lambda_k \mid \right] \\
& \quad = \ E \left[\mid \log \Lambda_k \circ (I_B + F_k)^{-1} \mid \right] \\
& \quad \leq \ E \left[\sum_{i=1}^{i=k} \parallel \sigma_v \parallel_\infty |_{v=T_i} + \parallel \frac{1}{1+\sigma_r} \parallel_\infty |_{r=T_i} \right] + \int_0^1 \parallel\mid D\sigma_r \mid_H \parallel_\infty dr \\
& \quad \leq \ \int_0^1 \parallel \sigma_r \parallel_\infty^2 dr + \int_0^1 \parallel \frac{1}{1+\sigma_r} \parallel_\infty^2 dr + \int_0^1 \parallel\mid D\sigma_r \mid_H \parallel_\infty^2 dr, \quad k \geq n.
\end{aligned}
$$

Hence by uniform integrability of $(\Lambda_k)_{k \in \mathbb{N}}$, we obtain

$$
E[f] = E[f \circ \phi_{s,t} L_{s,t}]
$$

for $f \in \mathcal{C}_b^+(B)$. We now return to the case of a general σ. From Prop. 3, we can choose a sequence $(\sigma^n)_{n \in \mathbb{N}} \subset \mathcal{V}$ that converges to σ in $L_{1,2}$, with

$\sigma_n > -1$, $n \in \mathbb{N}$. The sequence $(\sigma^n)_{n\in\mathbb{N}}$ defines a sequence of transformations $\left(\phi^n_{s,t}\right)_{n\in\mathbb{N}}$, $\left(\psi^n_{s,t}\right)_{n\in\mathbb{N}}$ and density functions $(L^n_{s,t})_{n\in\mathbb{N}}$. The uniform integrability of the sequence $(L^n_{s,t})_{n\in\mathbb{N}}$ is shown as above:

$$
\begin{aligned}
& E\left[L^n_{s,t} \mid \log L^n_{s,t} \mid\right] = E\left[\mid \log L^n_{s,t}(\psi^n_{s,t}) \mid\right] \\
\leq\ & E\left[\sum_{k\geq 1} \mid \sigma^n_{T_k}(\psi^n_{T_k,t}) \mid + \mid \frac{1}{1+\sigma^n_{T_k}(\psi^n_{T_k,t})} \mid \right. \\
& \left. + \int_s^t \mid \tilde{D}_r \sigma^n_r(\psi^n_{s,r}) \mid dr + \int_s^t \mid \sigma^n_r(\psi^n_{s,r}) \mid dr\right] \\
\leq\ & E\left[\sum_{k\geq 1} \| \sigma^n_v \|_\infty|_{v=T_k} + \| \frac{1}{1+\sigma^n_v} \|^2_\infty|_{v=T_k} \right. \\
& \left. + \int_0^t \mid \tilde{D}_r \sigma^n_r(\psi^n_{s,r}) \mid dr\right] + \int_0^t \| \sigma^n_r \|^2_\infty \, dr \\
\leq\ & 2\int_0^1 \| \sigma_r \|^2_\infty \, dr + \int_0^1 \| \frac{1}{1+\sigma_r} \|^2_\infty \, dr + \int_0^1 \||\, D\sigma_r \,|_H\|^2_\infty \, dr + \varepsilon.
\end{aligned}
$$

where ε does not depend on n. Let $\Phi^n_{s,t} = \phi^n_{s,t} - I_B$, and let us show that $(\Phi^n_{r,t})_{n\in\mathbb{N}}$ converges in $L^2(B)\otimes H$. We have

$$
\begin{aligned}
& E\left[\mid \phi^n_{s,t} - \phi^m_{s,t} \mid^2_H\right] \\
\leq\ & E\left[\int_s^t \mid \sigma^n_r(r, \phi^n_{r,t}) - \sigma^m_r(r, \phi^m_{r,t}) \mid^2 dr\right] \\
\leq\ & 2E\left[\int_s^t \mid \sigma^n_r - \sigma^m_r \mid^2 L^n_{r,t} dr + \int_s^t \mid \sigma^m_r(\phi^n_{r,t}) - \sigma^m_r(\phi^m_{r,t}) \mid^2 dr\right] \\
\leq\ & 2E\left[\int_s^t \mid \sigma^n_r - \sigma^m_r \mid^2 L^n_{r,t} dr \right. \\
& \left. + \int_s^t (\||\, D\sigma_r \,|_H\|_\infty + 1)^2 \int_s^r \mid \sigma^n_u(\phi^n_{u,r}) - \sigma^m_u(\phi^m_{u,r}) \mid^2 du dr\right] \\
\leq\ & 2E\left[\int_s^t \mid \sigma^n_r - \sigma^m_r \mid^2 L^n_{r,t} dr\right] \exp\left(\int_s^t (\||\, D\sigma_r \,|_H\|_\infty + 1)^2 \, dr\right)
\end{aligned}
$$

$n, m \in \mathbb{N}$, $0 \leq s \leq t \leq 1$, by the Gronwall lemma and Prop. 4. This converges to 0 by uniform integrability. Denote by $\phi_{s,t}$ the limit of $(\phi^n_{s,t})_{n\in\mathbb{N}}$. From Prop. 5, the sequence $(\sigma^n_r(\phi^n_{r,t}))_{n\in\mathbb{N}}$ converges to $\sigma_r(\phi_{r,t})$ in $L^2(B)$, for $r \in [0,1]$, hence by boundedness of σ the limit $\phi_{s,t}$ solves Eq. (5). Moreover, $\phi_{s,t}$ is absolutely continuous from Prop. 5 and is the only absolutely continuous solution from Prop. 4. We can now show that

$\left(\tilde{D}_{\cdot}\sigma^n_{\cdot}(\phi^n_{\cdot,t})\right)_{n\in\mathbb{N}}$ converges to $\tilde{D}_{\cdot}\sigma_{\cdot}(\phi_{\cdot,t})$ in $L^2(B)\otimes L^2([0,t])$:

$$\begin{aligned}
&E\left[\int_0^t \mid \tilde{D}_r\sigma^n_r(\phi^n_{r,t}) - \tilde{D}_r\sigma(\phi_{r,t}) \mid^2 dr\right] \\
&\quad\leq 2E\left[\int_0^t \left(\mid D_r\sigma^n_r(\phi^n_{r,t}) - D_r\sigma_r(\phi^n_{r,t}) \mid^2_H + \mid D_r\sigma_r(\phi^n_{r,t}) - D_r\sigma(\phi_{r,t}) \mid^2_H\right) dr\right] \\
&\quad\leq 2E\left[\int_0^t \mid D(\sigma^n_r - \sigma_r) \mid^2_H L^n_{r,t} dr + \int_0^t \mid D\sigma_r(\phi^n_{r,t}) - D\sigma_r(\phi_{r,t}) \mid^2_H dr\right],
\end{aligned}$$

which converges to 0 as n goes to infinity since $\mid D\sigma_r(\phi^n_{r,t}) \mid_H \leq \| \mid D\sigma_r \mid_H \|_\infty$, $r\in[0,1]$. We also have that $(\sigma^n_{T_k}(\phi^n_{T_k,t}))_{n\in\mathbb{N}}$ converges to $\sigma_{T_k}(\phi_{T_k,t})$ in $L^2(B)$, $k\in\mathbb{N}$, from Prop. 3. Hence by unifom integrability and convergence in probability of $\left\{L^n_{s,t} \ : \ n\in\mathbb{N}\right\}$ to $L_{s,t}$, we have for $f\in\mathcal{C}^+_b(B)$:

$$E[f] \ = \ \lim_{n\to\infty} E\left[f(\phi^n_{s,t})L^n_{s,t}\right] = E\left[f(\phi_{s,t})L_{s,t}\right].$$

Since $1+\sigma>0$ a.s., it is not difficult to see that $\phi_{s,t}$ is bijective and that its inverse $\psi_{s,t}$ satisfies (6). □

Remark. The expression of the density can be written in a form which is closer to its expression on Wiener space, cf. [4], [10], i.e.

$$\begin{aligned}
L_{s,t} &= \frac{d(\phi_{s,t})_* P}{dP} \\
&= \exp\left(\int_s^t \tilde{D}_r(\sigma_r(\phi_{r,t}\omega))dr - \int_s^t \tilde{D}_r\sigma_r(\phi_{r,t}\omega)dr + \tilde{\delta}(1_{[0,t]}\sigma_{\cdot}(\phi_{\cdot,t}))\right) \\
&\quad \times \prod_{s\leq T_k\leq t}(1+\sigma_{T_k}(\phi_{T_k,t}))\exp(-\sigma_{T_k}(\phi_{T_k,t})),
\end{aligned}$$

using (4). From [9] and (4), we obtain the following formal expression for the Carleman-Fredholm determinant of $D\phi_{s,t}$:

$$\begin{aligned}
\det{}_2(D\phi_{s,t}) &= \exp\left(\int_s^t \tilde{D}_r(\sigma_r(\phi_{r,t}\omega))dr - \int_s^t \tilde{D}_r\sigma_r(\phi_{r,t}\omega)dr\right) \\
&\quad \times \prod_{s\leq T_k\leq t}(1+\sigma_{T_k}(\phi_{T_k,t}))\exp(-\sigma_{T_k}(\phi_{T_k,t}))
\end{aligned}$$

Lemma 2 *If $F\in\mathcal{S}$ depends only on $\tau_0,\ldots,\tau_m$ and $\sigma\in\mathcal{V}$, then*

$$\begin{aligned}
&\mid D(F(\mathcal{A}_t)) \mid_H \\
&\quad\leq \ 2(m+1) \parallel \sigma \parallel_{L^\infty(B\times[0,1])} \\
&\qquad \times\left(1+\int_0^1 \|\mid D\sigma_r \mid_H\|_\infty \, dr \exp\left(\int_0^1 \|\mid D\sigma_r \mid_H\|_\infty \, dr\right)\right) \mid DF \mid_H,
\end{aligned}$$

$t\in[0,1]$.

Proof. We have from (8) and the Gronwall lemma, since $A_{0,t}(k,l)=0$ if $k>l$:

$$\begin{aligned}
&\mid D(F(\mathcal{A}_t)) \mid_H \\
&\leq \left(\int_0^t \| \mid D\sigma_r \mid_H \|_\infty \mid DA_{0,r} \mid_{\mathrm{R}^{m+1}\otimes\mathrm{R}^{m+1}} dr \right.\\
&\quad \left. \times \exp\left(\int_0^t \| \mid D\sigma_r \mid_H \|_\infty \, dr\right) + \mid DA_{0,t} \mid_{\mathrm{R}^{m+1}\otimes\mathrm{R}^{m+1}} \right) \mid DF \mid_H \\
&\leq 2(m+1) \parallel \sigma \parallel_{L^\infty(B\times[0,1])} \\
&\quad \times \left(1+\int_0^1 \| \mid D\sigma_r \mid_H \|_\infty \, dr \exp\left(\int_0^1 \| \mid D\sigma_r \mid_H \|_\infty \, dr\right)\right) \mid DF \mid_H,
\end{aligned}$$

□

4 Solution of a linear Skorohod equation

We need the following lemma.

Lemma 3 *Let $F\in\mathcal{S}$ and let $(\mathcal{T}_t)_{t\in[0,1]}$ be the flow defined by $\sigma\in L_{1,2}$, $\sigma>-1$ a.s. We have*

$$\frac{d}{dt} F\circ\mathcal{T}_t = \sigma_t(\mathcal{T}_t)\left(\tilde{D}_t F\right)\circ\mathcal{T}_t. \tag{11}$$

If moreover $\sigma\in\mathcal{V}$, then

$$\frac{d}{dt} F\circ\mathcal{A}_t = -\sigma_t \tilde{D}_t (F\circ\mathcal{A}_t).$$

Proof. Eq. (11) comes from (3) and (5). We also have if $\sigma\in\mathcal{V}$

$$\begin{aligned}
0 &= \frac{d}{dt} F\circ\mathcal{A}_t\circ\mathcal{T}_t = \frac{d}{dt}(F\circ\mathcal{A}_t)\circ\mathcal{T}_t + \frac{d}{ds} F\circ\mathcal{A}_t\circ\mathcal{T}_s \mid_{s=t} \\
&= \frac{d}{dt}(F\circ\mathcal{A}_t)\circ\mathcal{T}_t + \sigma_t(\mathcal{T}_t)\tilde{D}_t(F\circ\mathcal{A}_t)\circ\mathcal{T}_t.
\end{aligned}$$

□

Theorem 2 *Let $\sigma\in L_{1,\infty}$ with continuous trajectories a.s., such that $\sigma>-1$ and $\int_0^1 \parallel \frac{1}{1+\sigma_r} \parallel_\infty < \infty$, $b\in L^2([0,1],L^\infty(B))$ and $\eta\in L^\infty(B)$. The anticipative stochastic differential equation*

$$X_t = \eta + \int_0^t \sigma_r X_r \delta\tilde{N}_r + \int_0^t b_s X_s ds \quad t\in[0,1] \tag{12}$$

has for solution

$$X_t = \eta(\mathcal{T}_t^{-1}) \exp\left(-\int_0^t \tilde{D}_s \sigma_s(\phi_{s,t}\omega) ds - \int_0^t \sigma_s(\phi_{s,t}) ds + \int_0^t b_s(\phi_{s,t}) ds \right)$$
$$\prod_{0 \le T_k \le t} (1 + \sigma_{T_k}(\phi_{T_k,t})), \quad t \in [0,1].$$

If moreover $\| b \|_{L^\infty(B\times[0,1])}$, $\| \sigma \|_{L^\infty(B\times[0,1])}$, $\| D\sigma \|_{L^\infty(B\times[0,1]\times\mathbb{N})}$ *are finite, then* X *is the unique solution of (12) in* $L^1(B \times [0,1])$.

Proof. The proof is close to [3], [7]. We have $X \in L^1(B \times [0,1])$ by integrability of the density $L_{0,t}$. Let $G \in \mathcal{S}$.

$$\begin{aligned}
& E\left[\int_0^t \sigma_s X_s \tilde{D}_s G ds \right] \\
&= E\left[\int_0^t \sigma_s(\mathcal{T}_s) \eta \exp\left(\int_0^s b_r(\mathcal{T}_r) dr \right) \tilde{D}_s G(\mathcal{T}_s) ds \right] \\
&= E\left[\int_0^t \eta \exp\left(\int_0^s b_r(\mathcal{T}_r) dr \right) \frac{d}{ds} G(\mathcal{T}_s) ds \right] \\
&= E\left[\exp\left(\int_0^t b_s(\mathcal{T}_s) ds \right) G(\mathcal{T}_t)\eta - \eta G \right. \\
&\quad \left. - \int_0^t \eta b_s(\mathcal{T}_s) \exp\left(\int_0^s b_r(\mathcal{T}_r) dr \right) G(\mathcal{T}_s) ds \right] \\
&= E\left[\eta(A_t) \exp\left(\int_0^t b_s(\phi_{s,t}) ds \right) L_{0,t} G - \eta G \right. \\
&\quad \left. - \int_0^t \eta(\mathcal{A}_s) b_s \exp\left(\int_0^s b_r(\phi_{r,s}) ds \right) L_{0,s} G ds \right] \\
&= E\left[\left(X_t - \eta - \int_0^t b_s X_s ds \right) G \right],
\end{aligned}$$

and $X_t - \eta - \int_0^1 b_s X_s ds \in L^1(B)$. Hence $\sigma X 1_{[0,t]} \in Dom(\tilde{\delta})$, $t \in [0,1]$, and $(X_t)_{t\in[0,1]}$ is solution to Eq. (12). We now show the uniqueness of the solution in $L^1(B \times [0,1])$. Let $(\sigma^n)_{n\in\mathbb{N}}$ be a sequence given by Prop. 3, and let $(Y_t)_{t\in[0,1]}$ be the difference of two solutions, which satisfies

$$Y_t = \int_0^t b_s Y_s ds + \int_0^t \sigma_s Y_s \delta \tilde{N}_s.$$

Let $F \in \mathcal{S}$.

$$E\left[Y_t F(\mathcal{A}_t^n)\right] = E\left[\int_0^t \sigma_s Y_s \tilde{D}_s \left(F(\mathcal{A}_t^n) \right) ds + \int_0^t b_s Y_s F(\mathcal{A}_t^n) ds \right]$$

$$
\begin{aligned}
&= E\left[\int_0^t \sigma_s Y_s \tilde{D}_s \left(F(\mathcal{A}_s^n) - \int_s^t \sigma_r^n \tilde{D}_r \left(F(\mathcal{A}_r^n)\right) dr\right) ds \right.\\
&\quad \left. + \int_0^t b_s Y_s \left(F(\mathcal{A}_s^n) - \int_s^t \sigma_r^n \tilde{D}_r \left(F(\mathcal{A}_r^n)\right) dr\right) ds\right].
\end{aligned}
$$

We have for $u \in \mathcal{V}$

$$
\begin{aligned}
E\left[\int_0^t \sigma_s Y_s \tilde{D}_s \int_s^t u_r dr ds\right] &= E\left[\int_0^t \int_0^r \sigma_s Y_s \tilde{D}_s u_r ds dr\right] \\
&= E\left[\int_0^t \int_0^r \sigma_s Y_s \delta \tilde{N}_s u_r dr\right].
\end{aligned}
$$

This relation can be extended by density to the process $u = \sigma^n \tilde{D}(F(\mathcal{A}^n))$ since $\int_s^t \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n))dr = F(\mathcal{A}_s^n) - F(\mathcal{A}_t^n) \in D_{2,1}$ and gives

$$
\begin{aligned}
&E\left[\int_0^t \sigma_s Y_s \tilde{D}_s \int_s^t \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n)) dr ds\right] \\
&= E\left[\int_0^t \int_0^r \sigma_s Y_s \delta \tilde{N}_s \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n)) dr\right] \\
&= E\left[\int_0^t \left(Y_r - \int_0^r b_s Y_s ds\right) \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n)) dr\right] \\
&= E\left[\int_0^t Y_r \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n)) dr - \int_0^t b_s Y_s \int_s^t \sigma_r^n \tilde{D}_r(F(\mathcal{A}_r^n)) dr ds\right].
\end{aligned}
$$

Hence

$$
E\left[Y_t F(\mathcal{A}_t^n)\right] = E\left[\int_0^t (\sigma_s - \sigma_s^n) Y_s \tilde{D}_s(F(\mathcal{A}_s^n)) ds + \int_0^t b_s Y_s F(\mathcal{A}_s^n) ds\right].
$$

From Lemma 2, $\mid D(F(\mathcal{A}_s^n)) \mid_H$ is uniformy bounded in n and ω, hence letting n go to infinity we get

$$
E\left[Y_t F(\mathcal{A}_t)\right] = E\left[\int_0^t b_s Y_s F(\mathcal{A}_s) ds\right].
$$

Then

$$
E\left[Y_t(\mathcal{T}_t) F \mathcal{L}_t\right] = E\left[\int_0^t \mathcal{L}_s b_s(\mathcal{T}_s) Y_s(\mathcal{T}_s) F ds\right],
$$

with $\mathcal{L}_s = (L_{0,s}(\mathcal{T}_t))^{-1}$, which is satisfied by density for

$$
F = sign(\mathcal{L}_t Y_t(\mathcal{T}_t)).
$$

This gives

$$
E\left[\mid Y_t \mid\right] \leq \int_0^t E\left[\mid Y_s \mid\right] ds
$$

and $Y = 0$ by the Gronwall lemma. Consequently the solution is unique.

□

Remark. If moreover the processes σ and b are $(\mathcal{F}_t)$-adapted and $\eta = 1$, then the solution coincides with the usual result, i.e.

$$X_t = \exp\left(\int_0^t b_s ds - \int_0^t \sigma_s ds\right) \prod_{0\leq T_k\leq t} (1+\sigma_{T_k}) \quad 0\leq t\leq 1,$$

since $\phi_{s,t}(k) = \tau_k$ if $T_{k+1} < s$ and σ_s, b_s depend on τ_k only if $T_{k+1} < s$, $s\in[0,1]$.

Appendix.

Let $\mathcal{F}_n$ denote the σ-algebra generated by $\tau_0,\ldots,\tau_n$, $n\in\mathbb{N}$.

Proof of Prop. 2. Let $F_n = (1-\frac{1}{n})E[F \mid \mathcal{F}_n]$, $n\in\mathbb{N}$. We have $ess\ inf\ F < F_n < ess\ sup\ F$, $n\in\mathbb{N}$. If $(G_k)_{k\in\mathbb{N}}\subset\mathcal{S}$ converges to F in $D_{2,1}$, then

$$\begin{aligned}
\||\ E[DG_k \mid \mathcal{F}_n]\ |_H\|_\infty &\leq \|\left(\sum_{i=0}^{i=n}(D_iE[G_k\mid\mathcal{F}_n])^2\right)^{1/2}\|_\infty \\
&\leq \|\left(\sum_{i=0}^{\infty}(E[D_iG_k\mid\mathcal{F}_n])^2\right)^{1/2}\|_\infty \\
&\leq \|\left(\sum_{i=0}^{\infty}E[(D_iG_k)^2\mid\mathcal{F}_n]\right)^{1/2}\|_\infty \leq \||\ DG_k\ |_H\|_\infty .
\end{aligned}$$

This gives $\||\ DF_n\ |_H\|_\infty \leq \||\ DF\ |_H\|_\infty$. We also have the convergence of $(F_k)_{k\in\mathbb{N}}$ to F in $\mathbf{D}_{1,2}$. Hence it suffices to prove the result for $F\in\mathbf{D}_{1,2}$ of the form $F = f(\tau_0,\ldots,\tau_n)$. Assume first that f has a compact support in $\mathbb{R}_+^{n+1}$. Let $\Psi\in\mathcal{C}_c^\infty(\mathbb{R}_+^{n+1})$ with $\int_{\mathbb{R}_+^{n+1}}\Psi(x)dx = 1$, $\Psi\geq 0$, and let $f_k(y) = \frac{1}{k^{n+1}}\int_{\mathbb{R}_+^{n+1}}\Psi(kx)f(y+x)dx$, $k>0$, $y\in\mathbb{R}_+^{n+1}$. With $F_k = f_k(\tau_0,\ldots,\tau_n)$, we still have $ess\ inf\ F\leq F_k\leq ess\ sup\ F$, $k\in\mathbb{N}$, and

$$\||\ DF_k\ |_H\|_\infty \leq \||\ DF\ |_H\|_\infty .$$

If f does not have a compact support, let $\Phi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\Phi(x) = 1$ for $|\ x\ |< 1$ and $0\leq\Phi\leq 1$ on $\mathbb{R}^n$. Let $F_k = E[F\mid\mathcal{F}_n]\Phi(\tau_0/k,\ldots,\tau_m/k)$. Then $(F_k)_{k\in\mathbb{N}}$ converges to F in $\mathbf{D}_{1,2}$ and

$$\begin{aligned}
\||\ DF_k\ |_H\|_\infty &= \||\ \frac{1}{k}E[F\mid\mathcal{F}_n]D\Phi + \phi E[DF\mid\mathcal{F}_n]\ |_H\|_\infty \\
&\leq \|\ \Phi\mid DF\ |_H\|_\infty + \frac{1}{k}\|\ F_k\ \|_\infty \||\ D\Phi\ |_H\|_\infty \\
&\leq \||\ DF\ |_H\|_\infty + \frac{1}{k}\|\ F\ \|_\infty \sup\sum_{i=0}^{i=n}(\partial_i\Phi)^2 \\
&\leq \||\ DF\ |_H\|_\infty + \varepsilon
\end{aligned}$$

for k great enough. ☐

Proof of Prop. 3. For $\pi = \{\Delta_1, \dots, \Delta_n\}$ a partition of $[0,1]$, let

$$\sigma^\pi = \sum_{i=1}^{i=n} 1_{\Delta_i} \int_{\Delta_i} \sigma_r dr / \mid \Delta_i \mid .$$

Let $(\pi_n)_{n\in\mathbf{N}}$ be a sequence of partitions of $[0,1]$, mutually increasing with $\max_{1\le i\le n} \mid \Delta_i^n \mid$ converging to 0 as n goes to infinity. We have that $(\sigma^{\pi_n})_{n\in\mathbf{N}}$ converges to σ in $L_{1,2}$ with

$$\int_0^1 \parallel \sigma_s^{\pi_n} \parallel_\infty^2 ds \le \int_0^1 \parallel \sigma_s \parallel_\infty^2 ds,$$

$$\int_0^1 \parallel \frac{1}{1+\sigma_s^{\pi_n}} \parallel_\infty^2 ds \le \int_0^1 \parallel \frac{1}{1+\sigma_s} \parallel_\infty^2 ds,$$

and

$$\int_0^1 \||\, D\sigma_s^{\pi_n} \,|_H\|_\infty^2 ds \le \int_0^1 \||\, D\sigma_s \,|_H\|_\infty^2 ds.$$

We can apply Prop. 2 to $\frac{1}{|\Delta_i|}\int_{\Delta_i} \sigma_s ds$, $1 \le i \le n$. ☐

Proof of Prop. 4. Assume $F = f(\tau_0, \dots, \tau_n)$.

$$\begin{aligned}
&\mid F \circ \mathcal{T}^1(\omega) - F \circ \mathcal{T}^2(\omega) \mid \\
&= \ \mid f\left(\int_0^{T_1} \sigma_s^1(\omega)ds, \dots, \int_{T_n}^{T_{n+1}} \sigma_s^1(\omega)ds\right) \\
&\quad -f\left(\int_0^{T_1} \sigma_s^2(\omega)ds, \dots, \int_{T_n}^{T_{n+1}} \sigma_s^2(\omega)ds\right) \mid \\
&\le \ \||\, DF \,|_H\|_\infty \left(\sum_{i=0}^{i=n}\left(\int_{T_i}^{T_{i+1}} \sigma_s^1(\omega)ds - \int_{T_i}^{T_{i+1}} \sigma_s^2(\omega)ds\right)^2\right)^{1/2} \\
&\le \ \||\, DF \,|_H\|_\infty \mid \sigma^1(\omega) - \sigma^2(\omega) \mid_{L^2([0,1])}, \quad \omega \in B.
\end{aligned}$$

The same argument holds for the second part. If $F \in D_{1,\infty}$, then there is a sequence $(F_n)_{n\in\mathbf{N}} \subset \mathcal{S}$ that converges to F in $D_{1,2}$ and

$$\||\, DF \,|_H\|_\infty \le \||\, DF \,|_H\|_\infty + \varepsilon.$$

Since $\mathcal{T}^1$ and $\mathcal{T}^2$ are absolutely continuous, $P(\mid F_n \circ \mathcal{T}^1 - F \circ \mathcal{T}^1 \mid \ge \delta)$ goes to 0 as n goes to ∞, for any $\delta > 0$. The same is true for $\mathcal{T}^2$. This gives

$$\mid F \circ \mathcal{T}^2 - F \circ \mathcal{T}^2 \mid \le (\||\, DF \,|_H\|_\infty + \varepsilon) \mid \sigma^1 - \sigma^2 \mid_{L^2([0,1])},$$

where ε is arbitrary.

□

Proof of Prop. 5. For any $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that

$$\sup_{n\in\mathbb{N}} E\left[L^n 1_{\{L^n > M_\varepsilon\}}\right] \leq \varepsilon/2.$$

For any $\delta > 0$, there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\begin{aligned} P(\mid F(\mathcal{T}^n) - F_n(\mathcal{T}^n) \mid\geq \delta) &= E\left[1_{\{|F-F_n|\geq\delta\}} L^n\right] \\ &\leq E\left[1_{\{L^2 > M_\varepsilon\}} L^n\right] + M_\varepsilon P(\mid F - F_n \mid\geq \delta) \\ &\leq \varepsilon/2 + M_\varepsilon P(\mid F - F_n \mid\geq \delta) \leq \varepsilon. \end{aligned}$$

Let $(G^n)_{n\in\mathbb{N}} \subset \mathcal{S}$ be a sequence that converges to F in $L^2(B)$. The density L of $\mathcal{T}$ is the weak limit of $(L^n)_{n\in\mathbb{N}}$. For any $\varepsilon, \delta > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} P(\mid F \circ \mathcal{T}^n - G^{k_0} \circ \mathcal{T}^n \mid\geq \delta) &+ P(\mid F \circ \mathcal{T} - G^{k_0} \circ \mathcal{T} \mid\geq \delta) \\ &\leq E\left[1_{\left\{|F-G^{k_0}|\geq\delta\right\}}(L^n + L)\right] \\ &\leq \varepsilon + 2M_\varepsilon \leq 2\varepsilon \end{aligned}$$

for any $n \in \mathbb{N}$. We also have

$$\begin{aligned} P(\mid G^{k_0} \circ \mathcal{T} - G^{k_0} \circ \mathcal{T}^n \mid\geq \delta)& \\ &\leq \frac{1}{\delta} \||| DG^{k_0} |_H\|_\infty| \sigma - \sigma^n |_{L^2([0,1])} \leq \varepsilon \end{aligned}$$

for n great enough, from Prop. 4. Hence there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$P(\mid F \circ \mathcal{T} - F \circ \mathcal{T}^n \mid 3 \geq \delta) \leq 3\varepsilon.$$

□

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