An integration by parts formula in a Markovian regime switching model and application to sensitivity analysis

Yue Liu
School of Finance and Economics
Jiangsu University
Zhenjiang 212013
P.R. China

Nicolas Privault
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

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Abstract

We establish an integration by parts formula for the random functionals of a continuous-time Markov chain, based on partial differentiation with respect to jump times. In comparison with existing methods, our approach does not rely on the Girsanov theorem and it imposes less restrictions on the choice of directions of differentiation, while assuming additional continuity conditions on the considered functionals. As an application we compute sensitivities (Greeks) using stochastic weights in an asset price model with Markovian regime switching.

Key words: *Integration by parts, Markov chains, regime switching, sensitivity analysis.*
Mathematics Subject Classification (2010): 60G40; 35R35; 93E20; 60J28; 91G80.

1 Introduction

Integration by parts methods on the Wiener space have been successfully applied to the sensitivity analysis of diffusion models in finance, cf. [10]. This framework has also been implemented for jump processes, using the absolute continuity of jump sizes, cf. [1], or the process jump times as in [13].

More generally, integration by parts formulas for discrete and jump processes can be obtained using multiple stochastic integral expansions and finite difference operators, or the absolute continuity of jump times. In the setting of continuous-time Markov chains, integration by parts formulas have recently been proposed in [19] and [6], based on finite difference and differential operators. The construction of [6] is based on the Girsanov approach as in [3], and it uses time shifts instead of space shifts of the underlying process, while [19] uses the representation of Markov chains as semimartingales, cf. Appendix B of [8]. For discrete-time Markov chains and point processes, multiple stochastic integral expansions for random functionals have been built in e.g. [14], [2], [18], [4], [7].

The result of [19] is stated for functions of the numbers of chain transitions, cf. Theorem 4 therein, using the characterization of pure jump martingales under change of measure. However, this characterization cannot be applied to a shift of a Markov jump process $(\beta_t)_{t\in[0,T]}$ as claimed in Lemma 2 therein, otherwise it would actually entail the absolute continuity of the discrete random variable β_t.

In this paper we derive an integration by parts formula similar to that of [6] for continous-time Markov processes. Our approach uses time changes based on the intensity of the point process as in [15], [16]. In comparison with the construction of [6], we impose less constraints on directions of differentiation as we do not use the Girsanov theorem and assume instead a smoothness condition on random variables.

Namely, we define the gradient DF of a smooth functional F of a continuous-time Markov chain $(\beta_t)_{t\in\mathrm{R}_+}$ with state space $\mathcal{M}$ in such a way that the integration by parts formula

$$\mathbb{E}\left[\int_0^T D_t^\beta F u_t dt\right] = \mathbb{E}\left[F\int_0^T u_t(dN_t^\beta - \alpha_{\beta_t}dt)\right] \tag{1.1}$$

holds for $(u_t)_{t\in[0,T]}$ a square-integrable adapted process, see Proposition 4.2, where

$(N_t^\beta)_{t\in[0,T]}$ is the birth process counting the transitions of $(\beta_t)_{t\in\mathbb{R}_+}$, and $(\alpha_l)_{l\in\mathcal{M}}$ is the set of parameters of the exponentially distributed interjump times of $(\beta_t)_{t\in\mathbb{R}_+}$. In this way, a "partial" Malliavin calculus is established with respect to the absolutely continuous transition times, while the spatial discrete jump component remains unchanged. This approach has some similarities to the partial Malliavin calculus of [5], in which only the Brownian component of a jump-diffusion proces is subjected to a random perturbation.

In comparison with the construction of [6], which is also based on functionals of transition times, we do not constrain $(u_t)_{t\in[0,T]}$ to satisfy the a.s. vanishing condition $\int_0^T u_t dt = 0$. Instead, we assume a continuity condition on the random variables F in (1.1), see the condition (2.3) below, which nevertheless does not prevent us from including β_T in the domain of D, with the relation $D\beta_T = 0$.

As an application, we consider the sensitivity analysis of option prices in a diffusion model with regime-switching. Our method follows the Malliavin calculus approach to the fast computation of Greeks for option hedging [10], and in addition we take both sources of Markovian and Gaussian noises into account. In case the diffusion component vanishes, our result still allows us to estimate sensitivities by using Markovian noise, see Proposition 6.1. In addition, the absence of vanishing requirement on $\int_0^T u_t dt$ makes it easier to satisfy necessary integrability requirements on the associated stochastic weights, cf. Proposition 5.1. Indeed, such conditions are more difficult to satisfy in regime-switching models due to the changing signs of drifts, making it easier for denominators to vanish in the estimation of weights.

In Section 2 we recall some background notation and results on integration by parts and gradient operators defined by infinitesimal perturbation of jump times. Section 3 extends this construction to birth processes, using time changes based on the intensity of the considered birth process. Section 4 treats the general case of finite continuous-time Markov chains by partial differentiation. In Section 5 we apply the construction to sensitivity analysis of option prices in a jump-diffusion model driven by a geometric

Brownian motion with regime switching. Section 6 deals with an extension to non-smooth payoff functions.

2 Integration by parts for the Poisson process

In this section we start by reviewing integration by parts for a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with jump times $(T_n)_{n\geq 1}$, and whose interjump times $(\tau_k)_{k\geq 1}$ are independent exponentially distributed random variables, cf. e.g. § 7.3 of [17].

Definition 2.1 *Let $\mathcal{S}_T$ denote the space of Poisson functionals of the form*

$$F = f_0 \mathbf{1}_{\{N_T=0\}} + \sum_{k=1}^{\infty} \mathbf{1}_{\{N_T=k\}} f_k(T_1,\ldots,T_k), \tag{2.1}$$

where each function f_k is weakly differentiable on the simplex

$$S_T^k := \big\{(t_1,\ldots,t_k) \in [0,T]^k \ : \ 0 \leq t_1 \leq \cdots \leq t_k \leq T\big\},$$

and is such that $\|f_k\|_\infty \leq A^k$ and $\|\partial_l f_k\|_\infty \leq A^k$, $1 \leq l \leq k$, $k \geq 1$, for some constant $A > 0$.

The next statement comes from Definition 7.3.2 in [17]. We let $\partial_l f$ denote the partial derivative of a function f with respect to its l-th variable.

Definition 2.2 *The gradient operator D is defined on $F \in \mathcal{S}_T$ of the form (2.1) by*

$$D_t F := -\sum_{k=1}^{\infty} \mathbf{1}_{\{N_T=k\}} \sum_{l=1}^{k} \mathbf{1}_{[0,T_l]}(t) \partial_l f_k(T_1,\ldots,T_k), \qquad t \in \mathbb{R}_+. \tag{2.2}$$

Let now $\mathcal{S}_T^c$ denote the subspace of $\mathcal{S}_T$ made of Poisson functionals of the form (2.1) that satisfy the continuity condition

$$f_k(t_1,\ldots,t_k) = f_{k+1}(t_1,\ldots,t_k,t_{k+1}), \quad 0 \leq t_1 \leq \cdots \leq t_k \leq T \leq t_{k+1}, \quad k \in \mathbb{N}. \tag{2.3}$$

We note that given $F \in \mathcal{S}_T^c$ of the form (2.1) and such that

$$f_n(t_1,\ldots,t_n) = f_k(t_1,\ldots,t_k), \qquad 0 \leq t_1 \leq \cdots \leq t_k, \quad k \geq n,$$

for some $n \geq 1$, the function g_n defined by

$$g_n(t_1, \ldots, t_n) \tag{2.4}$$
$$:= f_0 \mathbf{1}_{\{T<t_1\}} + \sum_{k=1}^{n-1} \mathbf{1}_{\{0 \leq t_1 \leq \cdots \leq t_k \leq T < t_{k+1}\}} f_k(t_1, \ldots, t_k) + \mathbf{1}_{\{0 \leq t_1 \leq \cdots \leq t_n \leq T\}} f_n(t_1, \ldots, t_n)$$

is weakly differentiable on $\{0 \leq t_1 \leq \cdots \leq t_n\}$. In addition, g_n satisfies the relation

$$F = g_n(T_1, \ldots, T_n), \tag{2.5}$$

and D can be written on $F \in \mathcal{S}_T$ of the form (2.5) as

$$D_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \partial_k g_n(T_1, \ldots, T_n), \qquad t \in \mathbb{R}_+. \tag{2.6}$$

We also define the space of simple processes

$$\mathcal{U}_T^c := \left\{ \sum_{i=1}^{n} G_i h_i \ : \ G_1, \ldots, G_n \in \mathcal{S}_T^c,\ h_1, \ldots, h_n \in \mathcal{C}([0,T]),\ n \geq 1 \right\},$$

where $\mathcal{C}([0,T])$ denotes the space of continuous functions on $[0,T]$, and let the divergence or Skorohod integral operator δ be defined on $\mathcal{U}_T^c$ by

$$\delta(Gh) := G \int_0^T h(t)(dN_t - dt) - \int_0^T h(t) D_t G dt, \tag{2.7}$$

$G \in \mathcal{S}_T$, $h \in \mathcal{C}([0,T])$. The duality relation (2.8) below between D and δ relies on standard integration by parts on $[0,T]^n$ and on the expression

$$\mathbb{E}[F] = \mathrm{e}^{-T} f_0 + \mathrm{e}^{-T} \sum_{k=1}^{\infty} \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} f_k(t_1, \ldots, t_k) dt_1 \cdots dt_k,$$

for $F \in \mathcal{S}_T^c$ of the form (2.1), cf. Proposition 7.3.3 of [17] and the appendix Section 7 for a proof.

Proposition 2.3 *The operators D and δ defined by* (2.2) *and* (2.7) *satisfy the duality relation*

$$\mathbb{E}\left[\int_0^T u_t D_t F dt\right] = \mathbb{E}[F\delta(u)], \quad F \in \mathcal{S}_T^c, \quad u \in \mathcal{U}_T^c. \tag{2.8}$$

As a consequence of the Proposition 2.3, the operators D and δ are closable, and (2.8) extends to their closed domains $\mathrm{Dom}_T^c(D)$ and $\mathrm{Dom}_T^c(\delta)$ defined as the closure of $\mathcal{S}_T^c$ and $\mathcal{U}_T^c$ respectively.

Letting $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ denote the filtration generated by $(N_t)_{t\in\mathbb{R}_+}$, we note that for any $\mathcal{F}_t$-measurable random variable $F \in L^2(\Omega)$ we have

$$D_s F = 0, \qquad s \in [t, \infty),$$

and that δ coincides with the stochastic integral with respect to the compensated Poisson process, i.e.

$$\delta(u) = \int_0^T u_t (dN_t - dt), \tag{2.9}$$

for all $\mathcal{F}_t$-adapted square-integrable process $u \in L^2(\Omega \times [0,T])$, as in e.g. Proposition 7.2.9 of [17].

The condition $F \in \mathcal{S}_T^c$ in the integration by parts (2.8) of Proposition 2.3 can be relaxed to $F \in \mathcal{S}_T$ as in [6] under the additional condition $\int_0^T u_t dt = 0$, a.s., cf. also (7.3.6) in [17], on the space of simple processes

$$\mathcal{U}_T := \left\{ \sum_{i=1}^n G_i h_i \ : \ G_i \in \mathcal{S}_T, \ h_i \in \mathcal{C}([0,T]), \ \int_0^T h_i(t) dt = 0, \ 1 \leq i \leq n \right\}.$$

Proposition 2.4 *The operators D and δ defined by (2.2) and (2.7) satisfy the duality relation*

$$\mathbb{E}\left[\int_0^T u_t D_t F dt \,\Big|\, N_T \geq 1 \right] = \mathbb{E}[F\delta(u) \,|\, N_T \geq 1], \quad F \in \mathcal{S}_T, \quad u \in \mathcal{U}_T.$$

Proof. The same argument as in the proof of Proposition 2.3 in the appendix Section 7 shows that

$$\mathbb{E}\left[\mathbf{1}_{\{N_T \geq 1\}} \int_0^T u_t D_t F dt \right] = \mathbb{E}\left[\int_0^T u_t D_t F dt \right] = \mathbb{E}[F\delta(u)], \quad F \in \mathcal{S}_T, \quad u \in \mathcal{U}_T,$$

and we note that $\delta(u) = 0$ on $\{N_T = 0\}$ due to the condition $\int_0^T u_t dt = 0$. $\square$

From Proposition 2.4 the operators D and δ can be extended to larger domains $\mathrm{Dom}_T(D)$ and $\mathrm{Dom}_T(\delta)$, by completion of the spaces $\mathcal{S}_T$ and $\mathcal{U}_T$. In this case, $\mathrm{Dom}_T(D)$ contains non-smooth functionals such as N_T itself and we retain the equality (2.9) between δ and the (non compensated) Poisson stochastic integral over square-integrable adapted processes, however this requires the additional (a.s.) vanishing condition $\int_0^T u_t dt = 0$.

3 Integration by parts for birth processes

In this section we consider a pure birth process $(N_t^\alpha)_{t\in\mathbb{R}_+}$ whose interjump times $(\tau_k^\alpha)_{k\in\mathbb{N}}$ are independent exponentially distributed random variables with respective parameters $(\alpha_k)_{k\in\mathbb{N}}$ satisfying the bound

$$0 < \alpha_k \le C, \qquad k \in \mathbb{N}, \tag{3.1}$$

for some $C > 0$. In other words, the jump times of $(N_t^\alpha)_{t\in\mathbb{R}_+}$ can be written as

$$T_n^\alpha = \tau_0^\alpha + \cdots + \tau_{n-1}^\alpha = \frac{\tau_0}{\alpha_0} + \cdots + \frac{\tau_{n-1}}{\alpha_{n-1}}, \qquad n \ge 1, \tag{3.2}$$

with the relation

$$N_t^\alpha = \min\left\{n \ge 0 \ : \ t < T_{n+1}^\alpha\right\}, \qquad t \in \mathbb{R}_+.$$

Defining

$$\begin{aligned} \lambda(t) \ &:= \ \int_0^t \alpha_{N_s^\alpha} ds \qquad\qquad (3.3) \\ &= \ \alpha_{N_t^\alpha}(t - T_{N_t^\alpha}^\alpha) + \sum_{k=1}^{N_t^\alpha} \alpha_{k-1}(T_k^\alpha - T_{k-1}^\alpha) \\ &= \ \alpha_{N_t^\alpha}(t - T_{N_t^\alpha}^\alpha) + \sum_{k=1}^{N_t^\alpha} \tau_{k-1} \\ &= \ \alpha_{N_t^\alpha}(t - T_{N_t^\alpha}^\alpha) + T_{N_t^\alpha}, \qquad t \in \mathbb{R}_+, \end{aligned}$$

it follows from (3.2)-(3.3) that $(N_t^\alpha)_{t\in\mathbb{R}_+}$ can be written as the time-changed Poisson process

$$N_t^\alpha = N_{\lambda(t)}, \qquad t \in \mathbb{R}_+,$$

with

$$\lambda(T_n^\alpha) = \sum_{k=1}^n \alpha_{k-1}(T_k^\alpha - T_{k-1}^\alpha) = T_n, \qquad n \geq 1.$$

For all $k \geq 1$ we denote by $\mathcal{C}_b^1(S_T^k)$ the space of functions on $\mathbb{R}^k$ which are continuously differentiable (with bounded derivatives) on the simplex S_T^k.

Definition 3.1 *Let $\mathcal{S}_T^\alpha$ denote the class of functionals of the form*

$$F = f_0^\alpha \mathbf{1}_{\{N_T^\alpha = 0\}} + \sum_{k=1}^\infty \mathbf{1}_{\{N_T^\alpha = k\}} f_k^\alpha(T_1^\alpha, \ldots, T_k^\alpha), \tag{3.4}$$

where $f_0^\alpha \in \mathbb{R}$ and $f_k^\alpha \in \mathcal{C}_b^1(S_T^k)$, are such that $\|f_k^\alpha\|_\infty \leq A^k$ and $\|\partial_l f_k^\alpha\|_\infty \leq A^k$, $1 \leq l \leq k$, $k \geq 1$, for some constant $A > 0$.

We also let $\mathcal{S}_T^{c,\alpha}$ denote the subspace of $\mathcal{S}_T^\alpha$ made of functionals of the form (3.4) that satisfy the continuity condition

$$f_k^\alpha(t_1, \ldots, t_k) = f_{k+1}^\alpha(t_1, \ldots, t_k, t_{k+1}), \quad 0 \leq t_1 \leq \cdots \leq t_k \leq T < t_{k+1}, \quad k \in \mathbb{N}. \tag{3.5}$$

The next lemma will be used for Definition 3.3 and in the proof of Proposition 3.4 below.

Lemma 3.2 *We have the inclusions $\mathcal{S}_T^\alpha \subset \mathcal{S}_{CT}$ and $\mathcal{S}_T^{c,\alpha} \subset \mathcal{S}_{CT}^c$.*

Proof. We proceed in three steps.
(i) Given $F \in \mathcal{S}_T^\alpha$ of the form (3.4), we let

$$t_i^\alpha := \sum_{m=1}^i \frac{1}{\alpha_{m-1}}(t_m - t_{m-1}), \qquad i \geq 1, \tag{3.6}$$

and

$$f_k(t_1, \ldots, t_k) := \sum_{i=0}^k \mathbf{1}_{\{0 \leq t_1^\alpha \leq \cdots \leq t_i^\alpha \leq T < t_{i+1}^\alpha\}} f_i^\alpha(t_1^\alpha, \ldots, t_i^\alpha), \qquad k \geq 1, \tag{3.7}$$

with $f_0 := f_0^\alpha$. Since $N_T^\alpha \leq N_{CT}$, for any $k \in \mathbb{N}$ the condition

$$0 \leq t_1 \leq \cdots \leq t_k \leq CT < t_{k+1}$$

implies $t_{k+1}^\alpha > T$, hence we have

$$f_k(t_1, \ldots, t_k) = \mathbf{1}_{\{0 \leq t_1^\alpha \leq \cdots \leq t_k^\alpha \leq T < t_{k+1}^\alpha\}} f_k^\alpha(t_1^\alpha, \ldots, t_k^\alpha) + \sum_{i=0}^{k-1} \mathbf{1}_{\{0 \leq t_1^\alpha \leq \cdots \leq t_i^\alpha \leq T < t_{i+1}^\alpha\}} f_i^\alpha(t_1^\alpha, \ldots, t_i^\alpha)$$

$$= \mathbf{1}_{\{0\le t_1^\alpha\le\cdots\le t_k^\alpha\le T\}} f_k^\alpha(t_1^\alpha,\dots,t_k^\alpha) + \mathbf{1}_{\{0\le t_1^\alpha\le\cdots\le t_{k-1}^\alpha\le T< t_k^\alpha\}} f_{k-1}^\alpha(t_1^\alpha,\dots,t_{k-1}^\alpha) + \cdots + \mathbf{1}_{\{T<t_1^\alpha\}} f_0^\alpha,$$

under the condition $0 \le t_1 \le \cdots \le t_k \le CT < t_{k+1}$. Hence by the continuity condition (3.5), we find that f_k is weakly differentiable on $\{0 \le t_1 \le \cdots \le t_k \le CT < t_{k+1}\}$ for every $k \ge 1$.

(ii) Next, by (3.4) and the relation $N_T^\alpha \le N_{CT}$ we find

$$\begin{aligned}
& f_0 \mathbf{1}_{\{N_{CT}=0\}} + \sum_{k=1}^{\infty} \mathbf{1}_{\{N_{CT}=k\}} f_k(T_1,\dots,T_k) \\
&= f_0 \mathbf{1}_{\{N_{CT}=0\}} + \sum_{k=1}^{\infty}\sum_{i=0}^{k} \mathbf{1}_{\{N_{CT}=k\}\bigcap\{N_T^\alpha=i\}} f_i^\alpha(T_1^\alpha,\dots,T_i^\alpha) \\
&= f_0 \mathbf{1}_{\{N_{CT}=0\}} + f_0 \mathbf{1}_{\{N_{CT}>0\}\bigcap\{N_T^\alpha=0\}} + \sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathbf{1}_{\{N_{CT}=k\}\bigcap\{N_T^\alpha=i\}} f_i^\alpha(T_1^\alpha,\dots,T_i^\alpha) \\
&= f_0 \mathbf{1}_{\{N_T^\alpha=0\}} + \sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathbf{1}_{\{N_{CT}=k\}\bigcap\{N_T^\alpha=i\}} f_i^\alpha(T_1^\alpha,\dots,T_i^\alpha) \\
&= f_0 \mathbf{1}_{\{N_T^\alpha=0\}} + \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \mathbf{1}_{\{N_{CT}=k\}\bigcap\{N_T^\alpha=i\}} f_i^\alpha(T_1^\alpha,\dots,T_i^\alpha) \\
&= f_0^\alpha \mathbf{1}_{\{N_T^\alpha=0\}} + \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} f_i^\alpha(T_1^\alpha,\dots,T_i^\alpha) \\
&= F,
\end{aligned}$$

hence (2.1) is satisfied and we conclude that $F \in \mathcal{S}_{CT}$.

(iii) Finally, taking $F \in \mathcal{S}_T^{c,\alpha}$, it remains to check that $(f_k)_{k\in\mathbb{N}}$ satisfies the continuity condition (2.3) on the domain $[0, CT]$. For any $k \in \mathbb{N}$ and $0 \le t_1 \le \cdots \le t_k \le CT < t_{k+1}$ we have $t_{k+1}^\alpha > T$, therefore,

$$\begin{aligned}
f_{k+1}(t_1,\dots,t_{k+1}) &= \sum_{i=0}^{k+1} \mathbf{1}_{\{0\le t_1^\alpha\le\cdots\le t_i^\alpha\le T<t_{i+1}^\alpha\}} f_i^\alpha(t_1^\alpha,\dots,t_i^\alpha) \\
&= \sum_{i=0}^{k} \mathbf{1}_{\{0\le t_1^\alpha\le\cdots\le t_i^\alpha\le T<t_{i+1}^\alpha\}} f_i^\alpha(t_1^\alpha,\dots,t_i^\alpha) = f_k(t_1,\dots,t_k).
\end{aligned}$$

□

In the following definition, the gradient D^α of functionals of a point process is defined by a time change based on the intensity $(\lambda(t))_{t\in\mathbb{R}_+}$ of the point process $(N_t^\alpha)_{t\in\mathbb{R}_+}$, cf. Definition 1 of [15] or Definition 2 of [16].

Definition 3.3 *For F in the space $\mathcal{S}_T^\alpha$ of functionals of the form (3.4) we define the time-changed gradient D_t^α as*

$$D_t^\alpha F := \alpha_{N_t^\alpha} D_{\lambda(t)} F, \qquad t \in \mathbb{R}_+, \tag{3.8}$$

where D is applied to $F \in \mathcal{S}_{CT}$ on the time interval $[0, CT]$.

We note that

$$\|D^\alpha F\|^2_{L^2([0,T])} = \int_0^T |D_{\lambda(t)} F|^2 \alpha^2_{N_t^\alpha} dt = \int_0^{\lambda(T)} |D_t F|^2 \alpha_{N_t^\alpha} dt \le C \int_0^{CT} |D_t F|^2 dt, \tag{3.9}$$

$F \in \mathcal{S}_T^\alpha$, hence the definition of D^α extends in particular to all $F \in \mathrm{Dom}^c_{CT}(D)$.

Denoting by $(\mathcal{F}_t^\alpha)_{t\in\mathbb{R}_+}$ the filtration generated by $(N_t^\alpha)_{t\in\mathbb{R}_+}$, the process $(N_t^\alpha - \lambda(t))_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t^\alpha$-martingale, and the stochastic integral

$$\int_0^T u_t (dN_t^\alpha - \alpha_{N_t^\alpha} dt)$$

is defined for square-integrable $(\mathcal{F}_t^\alpha)_{t\in[0,T]}$-adapted processes $(u_t)_{t\in[0,T]}$ via the isometry formula

$$\mathbb{E}\left[\left(\int_0^T u_t (dN_t^\alpha - \alpha_{N_t^\alpha} dt) \right)^2 \right] = \mathbb{E}\left[\int_0^T |u_t|^2 \alpha_{N_t^\alpha} dt \right].$$

Next, we show that under Definition 3.3, D^α is dual to the stochastic integral with respect to $\left(N_t^\alpha - \int_0^t \alpha_{N_s^\alpha} ds \right)_{t\in[0,T]}$, as a consequence of e.g. Proposition 6 of [16] or Proposition 1 of [15].

Proposition 3.4 *Given $F \in \mathrm{Dom}^c_T(D^\alpha)$ and $(u_t)_{t\in[0,T]}$ a square-integrable $(\mathcal{F}_t^\alpha)_{t\in[0,T]}$-adapted process, we have the integration by parts formula*

$$\mathbb{E}\left[\int_0^T D_t^\alpha F u_t dt \right] = \mathbb{E}\left[F \int_0^T u_t (dN_t^\alpha - \alpha_{N_t^\alpha} dt) \right].$$

Proof. Let u be an $(\mathcal{F}_t^\alpha)_{t\in[0,T]}$-adapted simple process in

$$\mathcal{U}_T^{c,\alpha} := \left\{ \sum_{i=1}^n F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}([0,T]), \ F_1, \ldots, F_n \in \mathcal{S}_T^\alpha, \ n \ge 1 \right\}.$$

By Lemma 3.2 we have $u \in \mathcal{U}^c_{CT}$, hence we can apply Proposition 2.3 on $[0, CT]$ to $F \in \mathcal{S}^c_{CT}$ since $\lambda(T) \leq CT$. Hence by (3.3), (3.8), (3.1), we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T D^\alpha_t F u_t dt\right] &= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{[0,T]}(\lambda_t^{-1}) D_t F u_{\lambda_t^{-1}} dt\right] \\
&= \mathbb{E}\left[\int_0^{CT} \mathbf{1}_{[0,\lambda(T)]}(t) D_t F u_{\lambda_t^{-1}} dt\right] \\
&= \mathbb{E}\left[F \int_0^{CT} \mathbf{1}_{[0,\lambda(T)]}(t) u_{\lambda_t^{-1}} (dN_t - dt)\right] \\
&= \mathbb{E}\left[F \int_0^{\infty} \mathbf{1}_{[0,T]}(\lambda_t^{-1}) u_{\lambda_t^{-1}} (dN_t - dt)\right] \\
&= \mathbb{E}\left[F \int_0^{T} u_t (dN_{\lambda(t)} - \alpha_{N^\alpha_t} dt)\right].
\end{aligned}$$

We conclude by the denseness of $\mathcal{U}^{c,\alpha}_T$ in the space of square-integrable $(\mathcal{F}^\alpha_t)_{t\in[0,T]}$-adapted process. □

We denote by $\mathrm{Dom}^c_T(D^\alpha)$ the domain of D^α obtained by completion of $\mathcal{S}^{c,\alpha}_T$ in $\mathrm{Dom}^c_{CT}(D)$. Definition 3.3 can be restated on $\mathcal{S}^\alpha_T$ as in the next proposition whose proof is given in the appendix Section 7.

Proposition 3.5 *For any $F \in \mathcal{S}^\alpha_T$ of the form (3.4) we have*

$$D^\alpha_t F = -\sum_{i=1}^\infty \mathbf{1}_{\{N^\alpha_T = i\}} \sum_{j=1}^i \mathbf{1}_{[0,T^\alpha_j]}(t) \partial_j f^\alpha_i (T^\alpha_1, \ldots, T^\alpha_i), \quad t \in \mathbb{R}_+.$$

Given $n \geq 1$ and $F \in \mathcal{S}^{c,\alpha}_T$ a functional of the form (3.4) such that

$$f^\alpha_n(t_1, \ldots, t_n) = f^\alpha_k(t_1, \ldots, t_k), \qquad 0 \leq t_1 \leq \cdots \leq t_k, \quad k \geq n,$$

we note that similarly to (2.4), the function g^α_n defined by

$$\begin{aligned}
&g^\alpha_n(t_1, \ldots, t_n) \qquad (3.10)\\
&\quad := f^\alpha_0 \mathbf{1}_{\{T < t_1\}} + \sum_{k=1}^{n-1} \mathbf{1}_{\{0 \leq t_1 \leq \cdots \leq t_k \leq T < t_{k+1}\}} f^\alpha_k(t_1, \ldots, t_k) + \mathbf{1}_{\{0 \leq t_1 \leq \cdots \leq t_n \leq T\}} f^\alpha_n(t_1, \ldots, t_n),
\end{aligned}$$

is weakly differentiable on $S^n_T = \{0 \leq t_1 \leq \cdots \leq t_n \leq T\}$, and satisfies

$$F = g^\alpha_n(T^\alpha_1, \ldots, T^\alpha_n). \qquad (3.11)$$

Consequently, similarly to (2.6), Proposition 3.5 admits the following corollary which shows that the gradient operator D^α coincides with that of [6], cf. Proposition 3.1 therein, under certain conditions.

Corollary 3.6 *For $F \in \mathcal{S}_T^{c,\alpha}$ of the form (3.11) we have*

$$D_t^\alpha F = -\sum_{k=1}^n \mathbf{1}_{[0,T_k^\alpha]}(t)\partial_k g_n^\alpha(T_1^\alpha,\ldots,T_n^\alpha), \qquad t \in \mathbb{R}_+. \tag{3.12}$$

4 Integration by parts for Markov chains

Consider a (right-continuous) Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ with state space $\mathcal{M} := \{1,\ldots,m\}$ and transition rate matrix $Q = (q_{i,j})_{1\le i,j\le m}$. Let T_n^β denote the n-th transition time of $(\beta_t)_{t\in\mathbb{R}_+}$ with $T_0^\beta := 0$, and let

$$N_t^\beta := \min\left\{n \in \mathbb{N} \ : \ t < T_{n+1}^\beta\right\}$$

denote the number of transitions of $(\beta_t)_{t\in\mathbb{R}_+}$ up to time $t \in \mathbb{R}_+$. The Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ can be represented as

$$\beta_t = Z_{N_t^\beta}, \qquad t \in \mathbb{R}_+,$$

where $(Z_n)_{n\in\mathbb{N}}$ defined by

$$Z_n := \beta_{T_n^\beta}, \qquad n \in \mathbb{N},$$

denotes the embedded chain of $(\beta_t)_{t\in\mathbb{R}_+}$, with $Z_0 := \beta_0$. It is known in addition that, given $(Z_n)_{n\in\mathbb{N}}$, the interjump times $(T_{n+1}^\beta - T_n^\beta)_{n\in\mathbb{N}}$ form a sequence of independent exponentially distributed random variables with respective parameters $(\alpha_{Z_n})_{n\ge1}$, where

$$\alpha_k := \sum_{\substack{l=1\\ l\neq k}}^r q_{k,l} = -q_{k,k}, \qquad k = 1,\ldots,m, \tag{4.1}$$

cf. e.g. § 4.2 and § 6.4 of [9] for related representations of $(\beta_t)_{t\in\mathbb{R}_+}$. Continuous-time Markov chains can also be represented using stochastic integrals with respect to a Poisson random measure, cf. Chapter II of [20] or Section 3 of [11].

In the sequel we also assume that the sequence $(\alpha_k)_{k\geq 0}$ satisfies the condition (3.1) for some $C>0$. We will also make use of the following Definition 4.1 in order to extend Definition 3.3 to the setting of Markov processes.

Definition 4.1 *Let $\mathcal{S}_T^\beta$ denote the space of functionals of the form*

$$F = f_0^\beta \mathbf{1}_{\{N_T^\beta = 0\}} + \sum_{k=1}^\infty \mathbf{1}_{\{N_T^\beta = k\}} f_k^\beta (T_1^\beta, \ldots, T_k^\beta, Z_0, \ldots, Z_k), \tag{4.2}$$

where $f_0^\beta \in \mathbb{R}$ and $f_k^\beta(\cdot, z_0, z_1, \ldots, z_k) \in \mathcal{C}_b^1(S_T^k)$ are such that $\|f_k^\beta\|_\infty \leq A^k$ and $\|\partial_l f_k^\beta\|_\infty \leq A^k$ for all $z_0, z_1, \ldots, z_k \in \mathcal{M}$, $1 \leq l \leq k \leq n$, for some $A > 0$.

We denote by

$$D_t^\beta F := D_t^\alpha F, \qquad t \in \mathbb{R}_+,$$

the partial gradient D_t^α applied to $F \in \mathrm{Dom}_T(D^\alpha)$. In other words, for $F \in \mathcal{S}_T^\beta$ of the form (4.2), as in Proposition 3.5 we have

$$D_t^\beta F = - \sum_{k=1}^\infty \mathbf{1}_{\{N_T^\beta = k\}} \sum_{l=1}^k \mathbf{1}_{[0,T_l^\beta]}(t) \partial_l f_k^\beta (T_1^\beta, \ldots, T_k^\beta, Z_0, \ldots, Z_k), \quad t \in \mathbb{R}_+. \tag{4.3}$$

In addition, as in (3.9) we have

$$\|D^\beta F\|_{L^2([0,T])}^2 = \int_0^T |D_{\lambda(t)} F|^2 \alpha_{\beta_t}^2 dt = \int_0^{\lambda(T)} |D_t F|^2 \alpha_{\beta_t} dt \leq C \int_0^{CT} |D_t F|^2 dt,$$

hence the definition of D^β extends to all $F \in \mathrm{Dom}_{CT}^c(D)$, with the bound

$$\|D^\beta F\|_{L^2(\Omega \times [0,T])}^2 \leq C\, \mathbb{E} \left[\int_0^{CT} |D_t F|^2 dt \right].$$

Given $n \geq 1$, let $\mathcal{S}_T^{c,\beta}$ denote the subspace of $\mathcal{S}_T^\beta$ made of functionals of the form (4.2) that satisfy the continuity condition

$$f_k^\beta (t_1, \ldots, t_k, z_0, z_1, \ldots, z_k) = f_{k+1}^\beta (t_1, \ldots, t_{k+1}, z_0, z_1, \ldots, z_{k+1}), \tag{4.4}$$

$0 \leq t_1 \leq \cdots \leq t_k \leq T < t_{k+1}$, $z_0, z_1, \ldots, z_{k+1} \in \mathcal{M}$, $k \in \mathbb{N}$.

We will denote by $\mathrm{Dom}_T^c(D^\beta)$ the domain of D^β obtained by completion of $\mathcal{S}_T^{c,\beta}$ in $\mathrm{Dom}_{CT}^c(D)$. From Proposition 3.4 we obtain the integration by parts formula of the next Proposition 4.2, in which $\mathrm{Dom}_T(D^\beta)$ and the filtration $(\mathcal{F}_t^\beta)_{t\in\mathbb{R}_+}$ are defined analogously to $\mathrm{Dom}_T(D^\alpha)$ and $(\mathcal{F}_t^\alpha)_{t\in\mathbb{R}_+}$. As in (2.9) above, the stochastic integral

$$\int_0^T u_t(dN_t^\beta - \alpha_{\beta_t} dt)$$

is defined for square-integrable $(\mathcal{F}_t^\beta)_{t\in\mathbb{R}_+}$-adapted processes $(u_t)_{t\in\mathbb{R}_+}$ via the isometry

$$\mathbb{E}\left[\left(\int_0^T u_t(dN_t^\beta - \alpha_{\beta_t} dt)\right)^2\right] = \mathbb{E}\left[\int_0^T |u_t|^2 \alpha_{\beta_t} dt\right].$$

Proposition 4.2 *Given $F \in \mathrm{Dom}_T^c(D^\beta)$ and $(u_t)_{t\in[0,T]}$ a square-integrable $(\mathcal{F}_t^\beta)_{t\in\mathbb{R}_+}$-adapted process we have the integration by parts formula*

$$\mathbb{E}\left[\int_0^T D_t^\beta F u_t dt\right] = \mathbb{E}\left[F \int_0^T u_t(dN_t^\beta - \alpha_{\beta_t} dt)\right]. \tag{4.5}$$

As in (3.10), the (random) function g_n^β defined by

$$\begin{aligned} & g_n^\beta(t_1, \ldots, t_n, Z_0, \ldots, Z_n) \\ := \quad & f_0^\beta \mathbf{1}_{\{T<t_1\}} + \sum_{k=1}^{n-1} \mathbf{1}_{\{0\le t_1\le\cdots\le t_k\le T<t_{k+1}\}} f_k^\beta(t_1, \ldots, t_k, Z_0, \ldots, Z_k) \\ & + \mathbf{1}_{\{0\le t_1\le\cdots\le t_n\le T\}} f_n^\beta(t_1, \ldots, t_n, Z_0, \ldots, Z_n) \end{aligned}$$

is a.s. weakly differentiable on $S_T^n = \{0 \le t_1 \le \cdots \le t_n \le T\}$ and satisfies

$$F = g_n^\beta(T_1^\beta, \ldots, T_n^\beta, Z_0, \ldots, Z_n). \tag{4.6}$$

Hence Corollary 3.6 yields the following result, see also Proposition 3.1 of [6].

Proposition 4.3 *For $F \in \mathcal{S}_T^{c,\beta}$ of the form (4.6) for which there exists $n \ge 1$ such that*

$$f_n^\beta(t_1, \ldots, t_n, z_0, z_1, \ldots, z_n) = f_k^\beta(t_1, \ldots, t_k, z_0, z_1, \ldots, z_k),$$

$0 \le t_1 \le \cdots \le t_k$, $z_0, z_1, \ldots, z_k \in \mathcal{M}$, $k \ge n$, we have

$$D_t^\beta F = -\sum_{k=1}^n \mathbf{1}_{[0,T_k^\beta]}(t) \partial_k g_n^\beta(T_1^\beta, \ldots, T_n^\beta, Z_0, \ldots, Z_n), \qquad t \in \mathbb{R}_+.$$

5 Sensitivity analysis in a regime switching model

In this section we apply the integration by parts formula (4.5) to the computation of Greeks. Consider a geometric Brownian motion $(Y_t)_{t\in[0,T]}$ with regime switching, given by

$$Y_t = Y_0 \exp\left(\int_0^t \mu_{N_s^\beta}(s,\beta_s)ds + \int_0^t \sigma(\beta_s)dB_s\right), \qquad t \in [0,T], \tag{5.1}$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion independent of the finite-state continuous-time Markov chain $(\beta_t)_{t\in[0,T]}$, and μ, σ are deterministic functions on the state space $\mathcal{M} := \{1,2,\ldots,m\}$ of $(\beta_t)_{t\in[0,T]}$.

Consider the expected value function

$$V(y,z) := \mathbb{E}[\phi(Y_T) \mid Y_0 = y, \beta_0 = z], \qquad y > 0, \quad z \in \mathcal{M},$$

where $r > 0$ denotes the risk-free rate and ϕ is an integrable payoff function. In Proposition 5.1 we compute the sensitivity

$$\Delta(y,z) := \frac{\partial V}{\partial y}(y,z), \qquad y > 0, \quad z \in \mathcal{M},$$

with respect to the initial price y when the payoff function $\phi \in \mathcal{C}_b^1(\mathbb{R})$ has bounded derivative (an extension to non-differentiable payoff functions is given in Section 6). In the sequel we will rewrite (5.1) as $Y_T = Y_0 e^{X_T + W_T}$, where

$$W_T := \int_0^T \sigma(\beta_s)dB_s,$$

and

$$X_T := \int_0^T \mu_{N_s^\beta}(s,\beta_s)ds, \qquad \Sigma_T^2 := \int_0^T \sigma^2(\beta_s)ds \tag{5.2}$$

belong to $\mathcal{S}_T^{c,\beta}$. We also let

$$D_u^\beta F := \int_0^T u_t D_s^\beta F ds, \qquad F \in \mathrm{Dom}_T(D^\beta), \quad u \in L^2(\Omega \times [0,T]).$$

In the next proposition, the absence of hypothesis on the vanishing of $\int_0^T u_t dt$ allows us to keep $D_u X_T$ strictly positive, therefore ensuring integrability of the weight Γ_T together with a better stability of the associated numerical implementation.

Proposition 5.1 *Assume that $\mu_k(\cdot,l) \in \mathcal{C}^1([0,T])$ satisfies*

$$\mu_k(T,l) = 0, \qquad k \geq 0, \quad l = 1, \dots m,$$

and consider $u \in \mathcal{C}([0,T])$ such that $\mathbf{1}_{\{N_T^\beta \geq 1\}}/D_u^\beta X_T$ and $\mathbf{1}_{\{N_T^\beta \geq 1\}} D_u^\beta D_u^\beta X_T / D_u^\beta X_T$ belong to $L^2(\Omega)$. Then we have

$$\Delta(y,z) = \frac{1}{y}\,\mathbb{E}\left[\phi(Y_T)\left(\mathbf{1}_{\{N_T^\beta \geq 1\}}\Gamma_T + \mathbf{1}_{\{N_T^\beta = 0\}}\frac{W_T}{T\sigma^2(z)}\right)\,\Big|\, Y_0 = y, \beta_0 = z\right], \quad y > 0, \tag{5.3}$$

for all $\phi \in \mathcal{C}_b^1(\mathbb{R})$, where the weight Γ_T is given by

$$\Gamma_T := \frac{1}{D_u^\beta X_T}\left(\int_0^T u_t (dN_t^\beta - \alpha_{\beta_t} dt) + \frac{D_u^\beta D_u^\beta X_T}{D_u^\beta X_T} + \frac{\Sigma_T^2 - (W_T)^2}{2\Sigma_T^4} D_u^\beta \Sigma_T^2\right), \tag{5.4}$$

and $(\alpha_l)_{l=1,\dots,m}$ is given in (4.1).

Proof. We write $\Delta(y,z)$ as $\Delta(y,z) = \Delta_1(y,z) + \Delta_2(y,z)$, where

$$\Delta_1(y,z) := \frac{\partial}{\partial y}\,\mathbb{E}\left[\phi(Y_T)\mathbf{1}_{\{N_T^\beta \geq 1\}}\,\Big|\, Y_0 = y, \beta_0 = z\right] \tag{5.5}$$

and

$$\Delta_2(y,z) := \frac{\partial}{\partial y}\,\mathbb{E}\left[\phi(Y_T)\mathbf{1}_{\{N_T^\beta = 0\}}\,\Big|\, Y_0 = y, \beta_0 = z\right]. \tag{5.6}$$

Taking $Y_0 = y$ and $\beta_0 = z$, we have

$$\begin{aligned}
\Delta_1(y,z) &= \frac{\partial}{\partial y}\,\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta \geq 1\}}\phi(Y_T)\,\Big|\, Y_0 = y, \beta_0 = z\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{N_T^\beta \geq 1\}}\frac{\partial}{\partial y}\phi(ye^{X_T + W_T})\right] \qquad (5.7)\\
&= \mathbb{E}\left[\mathbf{1}_{\{N_T^\beta \geq 1\}}\int_{-\infty}^{\infty}\frac{\partial}{\partial y}\phi(ye^{x + X_T})\varphi(x, \Sigma_T)dx\right], \qquad (5.8)
\end{aligned}$$

where X_T is defined in (5.2) and

$$\varphi(x, \Sigma_T) := \frac{1}{\sqrt{2\pi\Sigma_T^2}}\exp\left(-x^2/(2\Sigma_T^2)\right), \quad x \in \mathbb{R},$$

is the probability density function of W_T given Σ_T^2. We note that X_T can be written as

$$X_T = \sum_{i=0}^{N_T^\beta}\int_{T_i^\beta \wedge T}^{T_{i+1}^\beta \wedge T}\mu_i(s, Z_i)ds,$$

where $(Z_n)_{n\in\mathbb{N}}$ is the discrete-time embedded chain of $(\beta_t)_{t\in\mathbb{R}_+}$. Consequently, X_T is an element of $\mathcal{S}_T^{c,\beta}$ which is expressed in the form of (4.2) as

$$X_T = f_0^\beta \mathbf{1}_{\{N_T^\beta=0\}} + \sum_{k=1}^\infty \mathbf{1}_{\{N_T^\beta=k\}} f_k^\beta(T_1^\beta,\ldots,T_k^\beta,Z_0,\ldots,Z_k),$$

where $f_0^\beta = \int_0^T \mu_0(s,Z_0)ds$ and

$$f_k^\beta(t_1,\ldots,t_k,Z_0,\ldots,Z_k) = \sum_{i=0}^k \int_{t_i\wedge T}^{t_{i+1}\wedge T} \mu_i(s,Z_i)ds,$$

$k\geq 1$, satisfy the continuity condition (4.4), and by (4.3) we have

$$D_t^\beta X_T = -\sum_{i=0}^{N_T^\beta-1} \mathbf{1}_{[0,T_{i+1}^\beta]}(t)\mu_i(T_{i+1}^\beta,Z_i) + \sum_{i=1}^{N_T^\beta} \mathbf{1}_{[0,T_i^\beta]}(t)\mu_i(T_i^\beta,Z_i), \quad t\in\mathbb{R}_+,$$

and $D_t^\beta X_T \in \mathcal{S}_T^{c,\beta}$. Similarly, find that the functional e^{X_T} belongs to $\mathcal{S}_T^{c,\beta}$, with

$$D_u^\beta e^{X_T} = e^{X_T} D_u^\beta X_T. \tag{5.9}$$

Hence by (5.9), taking $Y_0=y$ and $\beta_0=z$ we have

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\varphi(x,\Sigma_T)\frac{\partial}{\partial y}\phi(ye^{X_T+x})\right] &= \mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\varphi(x,\Sigma_T)e^{X_T+x}\phi'(ye^{X_T+x})\right]\\ &= \frac{1}{y}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\varphi(x,\Sigma_T)e^{X_T+x}\frac{D_u^\beta\phi(ye^{X_T+x})}{D_u^\beta e^{X_T+x}}\right]\\ &= \frac{1}{y}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\frac{\varphi(x,\Sigma_T)}{D_u^\beta X_T}D_u^\beta\phi(ye^{X_T+x})\right]. \end{aligned}\tag{5.10}$$

Now, for all $\varepsilon>0$ we have

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T)D_u^\beta\phi(ye^{X_T+x})\right]\\ &= \mathbb{E}\left[\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T)D_u^\beta\phi(ye^{X_T+x})\right]\\ &= \mathbb{E}\left[D_u^\beta\left(\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\right)\right] - \mathbb{E}\left[\phi(ye^{X_T+x})D_u^\beta\left(\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T)\right)\right],\end{aligned}$$

where the functional $\dfrac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)$ belongs to $\mathcal{S}_T^{c,\beta}$ for all $x\in\mathbb{R}$. Hence, denoting by $(\mathcal{F}_t^B)_{t\in[0,T]}$ the filtration generated by $(B_t)_{t\in[0,T]}$, by Proposition 4.2

we obtain

$$
\begin{aligned}
&\mathbb{E}\left[D_u^\beta\left(\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\right)\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[D_u^\beta\left(\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\right)\,\middle|\,\mathcal{F}_T^B\right]\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\int_0^T u_t(dN_t^\beta-\alpha_{\beta_t}dt)\,\middle|\,\mathcal{F}_T^B\right]\right]\\
&=\mathbb{E}\left[\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\int_0^T u_t(dN_t^\beta-\alpha_{\beta_t}dt)\right]\\
&=\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\geq 1\}}\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\phi(ye^{X_T+x})\varphi(x,\Sigma_T)\int_0^T u_t(dN_t^\beta-\alpha_{\beta_t}dt)\right],
\end{aligned}
$$

since $D_u^\beta X_T=0$ on $\{N_T^\beta=0\}$. Next, we have

$$
\begin{aligned}
D_u^\beta X_T &= -\sum_{i=0}^{N_T^\beta-1}\int_0^{T_{i+1}^\beta}u_t dt\mu_i(T_{i+1}^\beta,Z_i)+\sum_{i=1}^{N_T^\beta}\int_0^{T_i^\beta}u_t dt\mu_i(T_i^\beta,Z_i)\\
&= \sum_{i=1}^{N_T^\beta}\int_0^{T_i^\beta}u_t dt\left(\mu_i(T_i^\beta,Z_i)-\mu_{i-1}(T_i^\beta,Z_{i-1})\right) \qquad (5.11)\\
&= g_0^\beta\mathbf{1}_{\{N_T^\beta=0\}}+\sum_{k=1}^\infty \mathbf{1}_{\{N_T^\beta=k\}}g_k^\beta(T_1^\beta,\ldots,T_k^\beta,Z_0,\ldots,Z_k),
\end{aligned}
$$

where $g_0^\beta=0$ and

$$
g_k^\beta(t_1,\ldots,t_k,Z_0,\ldots,Z_k)=\sum_{i=1}^k\int_0^{t_i\wedge T}u_t dt\left(\mu_i(t_i,Z_i)-\mu_{i-1}(t_i,Z_{i-1})\right),
$$

$k\geq 1$, satisfy the continuity condition (4.4), hence $D_u^\beta X_T\in\mathcal{S}_T^{c,\beta}$ and by (4.3) we have

$$
\begin{aligned}
D_t^\beta D_u^\beta X_T = &\sum_{i=0}^{N_T^\beta-1}\mathbf{1}_{[0,T_{i+1}^\beta]}(t)u_{T_{i+1}^\beta}\mu_i(T_{i+1}^\beta,Z_i)+\sum_{i=0}^{N_T^\beta-1}\int_0^{T_{i+1}^\beta}u_t dt\mathbf{1}_{[0,T_{i+1}^\beta]}(t)\mu_i'(T_{i+1}^\beta,Z_i)\\
&-\sum_{i=1}^{N_T^\beta}\mathbf{1}_{[0,T_i^\beta]}(t)u_{T_i^\beta}u_{T_i^\beta}\mu_i(T_i^\beta,Z_i)-\sum_{i=1}^{N_T^\beta}\mathbf{1}_{[0,T_i^\beta]}(t)\int_0^{T_i^\beta}u_t dt\mu_i'(T_i^\beta,Z_i),
\end{aligned}
$$

which yields

$$
D_u^\beta D_u^\beta X_T=\sum_{i=0}^{N_T^\beta-1}\int_0^{T_{i+1}^\beta}u_s ds u_{T_{i+1}^\beta}\mu_i(T_{i+1}^\beta,Z_i)+\sum_{i=0}^{N_T^\beta-1}\int_0^{T_{i+1}^\beta}u_t dt\int_0^{T_{i+1}^\beta}u_s ds\mu_i'(T_{i+1}^\beta,Z_i)
$$

$$-\sum_{i=1}^{N_T^\beta} u_{T_i^\beta} \int_0^{T_i^\beta} u_s ds u_{T_i^\beta} \mu_i(T_i^\beta, Z_i) - \sum_{i=1}^{N_T^\beta} \int_0^{T_i^\beta} u_s ds \int_0^{T_i^\beta} u_t dt \mu_i'(T_i^\beta, Z_i).$$

Hence we see that

$$\begin{aligned}
&\mathbb{E}\left[\phi(ye^{X_T+x}) D_u^\beta \left(\frac{D_u^\beta X_T}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T)\right)\right] \\
&= \mathbb{E}\left[\frac{\phi(ye^{X_T+x})}{(D_u^\beta X_T)^2+\varepsilon}\left((D_u^\beta X_T) D_u^\beta \varphi(x,\Sigma_T) - \frac{(D_u^\beta X_T)^2-\varepsilon}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T) D_u^\beta D_u^\beta X_T\right)\right] \\
&= \mathbb{E}\left[\frac{\phi(ye^{X_T+x})}{(D_u^\beta X_T)^2+\varepsilon}\left(\frac{x^2-\Sigma_T^2}{2\Sigma_T^4}\varphi(x,\Sigma_T) D_u^\beta \Sigma_T^2 D_u^\beta X_T - \frac{(D_u^\beta X_T)^2-\varepsilon}{(D_u^\beta X_T)^2+\varepsilon}\varphi(x,\Sigma_T) D_u^\beta D_u^\beta X_T\right)\right].
\end{aligned} \tag{5.12}$$

Combining (5.5), (5.8) and (5.10)-(5.12) and letting ε tend to zero we find, by dominated convergence,

$$\begin{aligned}
&\Delta_1(y,z) \\
&= \int_{-\infty}^{\infty} \mathbb{E}\Bigg[\mathbf{1}_{\{N_T^\beta \geq 1\}} \\
&\qquad \times \frac{\phi(ye^{X_T+x})}{D_u^\beta X_T}\varphi(x,\Sigma_T)\left(\int_0^T u_t(dN_t^\beta - \alpha_{\beta_t} dt) + \frac{D_u^\beta D_u^\beta X_T}{D_u^\beta X_T} + \frac{\Sigma_T^2 - x^2}{2\Sigma_T^4} D_u^\beta \Sigma_T^2\right)\Bigg]\frac{dx}{y} \\
&= \frac{1}{y}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta \geq 1\}} \frac{\phi(Y_T)}{D_u^\beta X_T}\left(\int_0^T u_t(dN_t^\beta - \alpha_{\beta_t} dt) + \frac{D_u^\beta D_u^\beta X_T}{D_u^\beta X_T} + \frac{\Sigma_T^2 - (W_T)^2}{2\Sigma_T^4} D_u^\beta \Sigma_T^2\right)\right].
\end{aligned} \tag{5.13}$$

Regarding the computation of $\Delta_2(y,z)$, we note that by the classical Malliavin calculus for Brownian motion, or by a standard integration by parts with respect to the Gaussian density, we have

$$\Delta_2(y,z) = \frac{\partial}{\partial y}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta = 0\}}\phi(Y_T)\right] = \frac{1}{y}\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta = 0\}}\phi(Y_T)\frac{W_T}{T\sigma^2(z)}\right], \tag{5.14}$$

as in e.g. [10]. The proof is completed by (5.13)-(5.14). □

The integrability assumptions in Proposition 5.1 can be satisfied by choosing $\mu_k(t,l)$ so that $D_u^\beta X_T$ in (5.11) remains strictly positive. However, positivity of $D_u^\beta X_T$ is not necessary, as shown in the following numerical illustration in which we consider a two-state Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ with values in $\{1,2\}$, with

$$\mu_k(t,l)L = (k+1)(T-t)^\gamma \eta_l, \qquad k \geq 0, \quad l = 1,2,$$

and the parameters

$$\gamma = 1,\ T = 1,\ Y_0 = 1,\ \alpha_1 = 40,\ \alpha_2 = 20,\ \eta_1 = 1.5,\ \eta_2 = 1,\ \sigma_1 = 0.2,\ \sigma_2 = 0.5.$$

Figure 1 shows the faster convergence of (5.3) for a digital option with payoff $\phi(x) = \mathbf{1}_{[K,\infty)}(x)$, strike $K = 150$, and $u_t := 1$, $t \in [0,T]$, compared to a standard finite difference scheme and to an application of the partial Wiener-Malliavin calculus as
$$\Delta(y,z) = \frac{\partial}{\partial y}\mathbb{E}\left[\phi(Y_T)\,\middle|\,N_T^\beta \geq 1, Y_0 = y, \beta_0 = z\right] = \frac{1}{y}\mathbb{E}\left[\phi(Y_T)\frac{W_T}{\Sigma_T^2}\,\middle|\,Y_0 = y, \beta_0 = z\right],$$
cf. e.g. [5].

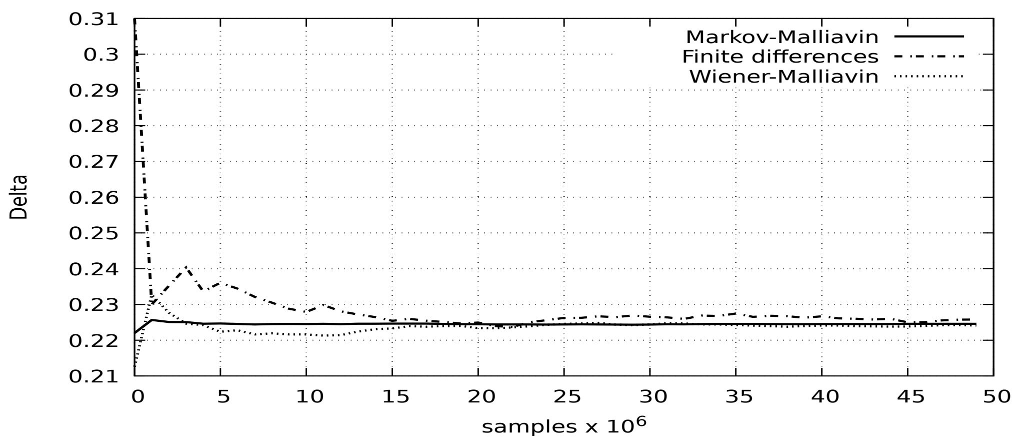

Figure 1: Monte Carlo convergence graph.

In case the diffusion term σ vanishes, Proposition 6.1 still allows us to estimate the conditional sensitivity given that $N_T^\beta \geq 1$, as
$$\frac{\partial}{\partial y}\mathbb{E}\left[\phi(Y_T)\,\middle|\,N_T^\beta \geq 1, Y_0 = y, \beta_0 = z\right]$$
$$= \frac{1}{y}\mathbb{E}\left[\phi(Y_T)\frac{\mathbf{1}_{\{N_T^\beta \geq 1\}}}{D_u^\beta X_T}\left(\int_0^T u_t dN_t^\beta - \int_0^T u_t \alpha_{\beta_t} dt + \frac{D_u^\beta D_u^\beta X_T}{D_u^\beta X_T}\right)\,\middle|\,N_T^\beta \geq 1, Y_0 = y, \beta_0 = z\right].$$

6 Extension to non-differentiable payoff functions

In this section, following the approach of in [12] we show that Proposition 5.1 can be extended to non-differentiable payoff functions in the class
$$\Lambda(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \ :\ f = \sum_{i=1}^n f_i \mathbf{1}_{A_i},\ f_i \in C_L(\mathbb{R}),\ A_i \text{ intervals of } \mathbb{R},\ n \geq 1 \right\}, \tag{6.1}$$

where

$$C_L(\mathbb{R}) := \big\{ f \in C(\mathbb{R}) \ : \ |f(x) - f(y)| \leq k|x-y| \ \text{ for some } k \geq 0 \big\}. \tag{6.2}$$

Proposition 6.1 *Under the hypotheses of Proposition 5.1 we have*

$$\Delta(y,z) = \frac{1}{y}\,\mathbb{E}\left[\phi(Y_T) \left(\mathbf{1}_{\{N_T^\beta \geq 1\}} \Gamma_T + \mathbf{1}_{\{N_T^\beta = 0\}} \frac{W_T}{\Sigma_T^2} \right) \Big| Y_0 = y, \beta_0 = z \right], \tag{6.3}$$

for all $\phi \in \Lambda(\mathbb{R})$, *where* Γ_T *is given in* (5.4).

Proof. Since $\phi \in \Lambda(\mathbb{R})$, there exists $N \geq 1$ and a sequence $(k_1, \ldots, k_N) \subset \mathbb{R}_+^N$ and a family $(A_1, \ldots, A_N)$ of disjoint intervals such that

$$\phi(x) = \sum_{i=1}^N f_i(x) \mathbf{1}_{A_i}(x), \qquad x \in \mathbb{R},$$

where $f_i(x) \in C_L(\mathbb{R})$ with

$$|f(x) - f(y)| \leq k_i |x - y|, \qquad x, y \in A_i, \quad i = 1, \ldots, N.$$

We denote $A_i = (a_{i-1}, a_i]$, $i = 1, \ldots, N$ with $a_0 = -\infty$ and $a_N = \infty$. Let

$$Y_t^\varepsilon := \frac{y + \varepsilon}{y} Y_t \qquad t \in [0,T], \quad \varepsilon \in \mathbb{R},$$

where $(Y_t)_{t\in[0,T]}$ is defined in (5.1) with $Y_0 = y > 0$.
(i) Assuming that $\phi \in \Lambda(\mathbb{R}) \cap C^1(\mathbb{R})$ we show that $\phi(Y_T)$ is integrable, with

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\phi(Y_T^\varepsilon) - \phi(Y_T)}{\varepsilon} \Big| Y_0 = y, \beta_0 = z \right] = \mathbb{E}\left[\lim_{\varepsilon \to 0} \frac{\phi(Y_T^\varepsilon) - \phi(Y_T)}{\varepsilon} \Big| Y_0 = y, \beta_0 = z \right]. \tag{6.4}$$

Since ϕ is continuous, we see that

$$\begin{aligned}
&|\,\phi(Y_T) - \phi(Y_0)\,| \\
&\leq \max_{i=1,\ldots,N} k_i \max_{i=1,\ldots,N-1} |Y_T - a_i| + \max_{i=1,\ldots,N} k_i \max_{i=1,\ldots,N-1} |y - a_i| + \sum_{i,j=1,\ldots,N-1} |\phi(a_i) - \phi(a_j)| \\
&\leq \max_{i=1,\ldots,N} k_i (Y_T + \max_{i=1,\ldots,N-1} |a_i|) + \max_{i=1,\ldots,N} k_i \max_{i=1,\ldots,N-1} |y - a_i| + \sum_{i,j=1,\ldots,N-1} |\phi(a_i) - \phi(a_j)|,
\end{aligned}$$

which is integrable, hence the integrability of $\phi(Y_T)$ is proved. On the other hand,

$$\lim_{\varepsilon\to 0}\left|\frac{\phi(Y_T^\varepsilon)-\phi(Y_T)}{\varepsilon}\right| \le \left(\max_{1\le i\le N} k_i\right)\lim_{\varepsilon\to 0}\left|\frac{Y_T^\varepsilon - Y_T}{\varepsilon}\right| = \frac{1}{y}\max_{1\le i\le N} k_i,$$

which is uniformly bounded. Therefore, (6.4) follows by dominated convergence.

(ii) Next, we note that (6.3) holds for $\phi\in\Lambda(\mathbb{R})\cap C^1(\mathbb{R})$, as by (6.4) we have

$$\begin{aligned}
&\frac{\partial V}{\partial y}(y,z) = \lim_{\varepsilon\to 0}\mathbb{E}\left[\frac{\phi(Y_T^\varepsilon)-\phi(Y_T)}{\varepsilon}\,\Big|\, Y_0=y,\beta_0=z\right]\\
&= \mathbb{E}\left[\lim_{\varepsilon\to 0}\frac{\phi(Y_T^\varepsilon)-\phi(Y_T)}{\varepsilon}\,\Big|\, Y_0=y,\beta_0=z\right]\\
&= \mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\ge 1\}}\frac{\partial}{\partial y}\phi(Y_T)\,\Big|\, Y_0=y,\beta_0=z\right] + \mathbb{E}\left[\mathbf{1}_{\{N_T^\beta=0\}}\frac{\partial}{\partial y}\phi(Y_T)\,\Big|\, Y_0=y,\beta_0=z\right],
\end{aligned} \tag{6.5}$$

which shows that (6.3) holds for $\phi\in\Lambda(\mathbb{R})\cap C^1(\mathbb{R})$ by repeating the arguments from (5.7) to the end of proof of Proposition 5.1.

(iii) Finally, we extend the result from an increasing sequence $(\phi_n)_{n\in\mathbb{N}}\subset\Lambda(\mathbb{R})\cap C^1(\mathbb{R})$ to its pointwise limit $\phi\in\Lambda(\mathbb{R})$. By e.g. (3.6)-(3.7) in [12], it suffices to show that for all compact subsets $K\subset(0,\infty)$ we have

$$\lim_{n\to\infty}\sup_{y\in K}\Big|\mathbb{E}[\phi_n(Y_T)\mid Y_0=y,\beta_0=z]-\mathbb{E}[\phi(Y_T)\mid Y_0=y,\beta_0=z]\Big| = 0, \tag{6.6}$$

and

$$\lim_{n\to\infty}\sup_{y\in K}\left|\frac{\partial}{\partial y}\mathbb{E}[\phi_n(Y_T)\mid Y_0=y,\beta_0=z]-\mathbb{E}[\phi(Y_T)J\mid Y_0=y,\beta_0=z]\right| = 0. \tag{6.7}$$

where the weight J given by

$$J := \mathbf{1}_{\{N_T^\beta\ge 1\}}\frac{1}{D_u^\beta X_T}\left(\int_0^T u_t dN_t^\beta - \int_0^T v_t dt\right) + \mathbf{1}_{\{N_T^\beta=0\}}\frac{B_T}{\sigma(z)}$$

is square-integrable under the condition $\mathbb{E}\left[\mathbf{1}_{\{N_T^\beta\ge 1\}}|D_u^\beta X_T|^{-2}\right]<\infty$. Since $\phi\in\Lambda(\mathbb{R})$ is continuous on every interval (a_{i-1},a_i), $i=1,\ldots,N$, there exists a pointwise increasing sequence $(\phi_n)_{n\in\mathbb{N}}\in\Lambda(\mathbb{R})\cap\ C^1(\mathbb{R})$ such that

$$\lim_{n\to\infty}\phi_n(x)=\phi(x),\qquad x\in\mathbb{R}\setminus\{a_1,\ldots,a_{N-1}\}.$$

The increasing sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions defined by

$$f_n(y) := \mathbb{E}\left[\phi_n(Y_T) \,\middle|\, Y_0 = y, \beta_0 = z\right], \quad n \in \mathbb{N}, \quad y \in K,$$

satisfies

$$\lim_{n\to\infty} f_n(y) = \mathbb{E}\left[\phi(Y_T) \,\middle|\, Y_0 = y, \beta_0 = z\right]$$

uniformly in $y \in K$ by dominated convergence and Dini's theorem, which proves (6.6). Regarding (6.7), since $(\phi_n)_{n\in\mathbb{N}} \in \Lambda(\mathbb{R}) \cap \ C^1(\mathbb{R})$, by point (ii) above we have

$$\begin{aligned}
&\Big| \frac{\partial}{\partial y} \mathbb{E}[\phi_n(Y_T) \mid Y_0 = y, \beta_0 = z] - \mathbb{E}[\phi(Y_T)J \mid Y_0 = y, \beta_0 = z] \Big| \\
&= \ | \ \mathbb{E}[\phi_n(Y_T)J \mid Y_0 = y, \beta_0 = z] - \mathbb{E}[\phi(Y_T)J \mid Y_0 = y, \beta_0 = z] \ | \\
&\le \ \mathbb{E}[(\phi(Y_T) - \phi_n(Y_T))|J| \mid Y_0 = y, \beta_0 = z] \\
&\le \ \sqrt{\mathbb{E}[(\phi(Y_T) - \phi_n(Y_T))^2 \mid Y_0 = y, \beta_0 = z]}\sqrt{\mathbb{E}[|J|^2 \mid Y_0 = y, \beta_0 = z]}.
\end{aligned}$$

Similarly to the above we conclude by noting that the sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions defined by

$$g_n(y) := \mathbb{E}\left[(\phi(Y_T) - \phi_n(Y_T))^2 \,\middle|\, Y_0 = y, \beta_0 = z\right], \quad n \in \mathbb{N}, \quad y \in K,$$

is decreasing to 0 uniformly on K. □

7 Appendix

In this appendix we provide a proof of Proposition 2.3 that does not rely on the symmetry condition on the functions $f_k(t_1, \ldots, t_k)$ assumed in Proposition 7.3.3 of [17].

Proof of Proposition 2.3. First, taking $h \in \mathcal{C}([0,T])$ we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T h(t) D_t^\beta F dt\right] &= \mathbb{E}\left[\mathbf{1}_{\{N_T \ge 1\}} \int_0^T h(t) D_t^\beta F dt\right] \qquad (7.1) \\
&= - \ \mathbb{E}\left[\sum_{k=1}^\infty \mathbf{1}_{\{N_T = k\}} \sum_{l=1}^k \partial_l f_k(T_1, \ldots, T_k) \int_0^{T_l} h(t) dt\right] \\
&= -\mathrm{e}^{-T} \sum_{k=1}^\infty \frac{T^k}{k!} \ \mathbb{E}\left[\sum_{l=1}^k \partial_l f_k(T_1, \ldots, T_k) \int_0^{T_l} h(t) dt \,\middle|\, N_T = k\right]
\end{aligned}$$

$$= -\mathrm{e}^{-T} \sum_{k=1}^{\infty} \sum_{l=1}^{k} \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \partial_l f_k(t_1, \ldots, t_k) \int_0^{t_l} h(t) dt dt_1 \cdots dt_k. \tag{7.2}$$

We start with the first term when $l = 1$ and apply the chain rule of derivation in the following integration by parts on $[0, t_2]$:

$$\begin{aligned}
&\int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \partial_1 f_k(t_1, \ldots, t_k) \int_0^{t_1} h(t) dt dt_1 \cdots dt_k \\
&= \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \frac{\partial}{\partial t_1} \left(\int_0^{t_1} h(t) dt f_k(t_1, \ldots, t_k) \right) dt_1 \cdots dt_k \\
&\quad - \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} h(t_1) f_k(t_1, \ldots, t_k) dt_1 \cdots dt_k \\
&= \int_0^T \int_0^{t_k} \cdots \int_0^{t_3} \int_0^{t_2} h(t) dt f_k(t_2, t_2, t_3, \ldots, t_k) dt_2 \cdots dt_k \\
&\quad - \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} h(t_1) f_k(t_1, \ldots, t_k) dt_1 \cdots dt_k.
\end{aligned} \tag{7.3}$$

Similarly we have, by integration by parts on $[t_{l-1}, t_{2+1}]$, $l \in \{2, \ldots, k-1\}$,

$$\begin{aligned}
&\int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \partial_l f_k(t_1, \ldots, t_k) \int_0^{t_l} h(t) dt dt_1 \cdots dt_k \\
&= \int_0^T \int_0^{t_k} \cdots \int_0^{\hat{t}_l} \cdots \int_0^{t_2} \int_0^{t_{l+1}} h(t) dt f_k(t_1, \ldots, t_{l-1}, t_{l+1}, t_{l+1}, \ldots, t_k) dt_1 \cdots d\hat{t}_l \cdots dt_k \\
&\quad - \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} h(t_l) f_k(t_1, \ldots, t_k) dt_1 \cdots dt_k \\
&\quad - \int_0^T \int_0^{t_k} \cdots \int_0^{\hat{t}_l} \cdots \int_0^{t_2} \int_0^{t_l} h(t) dt f_k(t_1, \ldots, t_l, t_l, t_{l+1}, \ldots, t_k) dt_1 \cdots d\hat{t}_{l-1} \cdots dt_k,
\end{aligned}$$

where $\int_0^{\hat{t}_l}$ denotes the absence of $\int_0^{t_l}$, $d\hat{t}_l$ denotes the absence of dt_l. Finally, by integration by parts on $[t_k, T]$ we find

$$\begin{aligned}
&\int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \partial_k f_k(t_1, \ldots, t_k) \int_0^{t_k} h(t) dt dt_1 \cdots dt_k \qquad (7.4) \\
&= \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_2} f_k(t_1, \ldots, t_{k-1}, T) dt_1 \cdots dt_{k-1} \int_0^T h(t) dt \\
&\quad - \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} h(t_k) f_k(t_1, \ldots, t_k) dt_1 \cdots dt_k \\
&\quad - \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_2} f_k(t_1, \ldots, t_{k-2}, t_k, t_k) dt_1 \cdots dt_{k-2} dt_k \int_0^{t_k} h(t) dt.
\end{aligned}$$

Plugging (7.3)-(7.4) into (7.1) yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T h(t) D_t^\beta F dt\right] \\
&= \mathrm{e}^{-T} \sum_{k=1}^\infty \left(\int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \sum_{l=1}^k h(t_l) f_l(t_1, \ldots, t_k) dt_1 \cdots dt_k \right. \\
&\qquad \left. - \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_2} f_k(t_1, \ldots, t_{k-1}, T) dt_1 \cdots dt_{k-1} \int_0^T h(t) dt \right) \\
&= -\sum_{k=1}^\infty \mathbb{E}\left[\mathbf{1}_{\{N_T = k-1\}} f_{k-1}(T_1, \ldots, T_{k-1})\right] \int_0^T h(t) dt + \mathbb{E}\left[F \int_0^T h(t) dN_t\right] \\
&= \mathbb{E}\left[F \int_0^T h(t)(dN_t - dt)\right],
\end{aligned}$$

where we applied the continuity condition (2.3). Next, if $u = Gh \in \mathcal{U}_T^c$, by (2.7) we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{N_T \geq 1\}} \int_0^T u_t D_t^\beta F dt\right] &= \mathbb{E}\left[\int_0^T u_t D_t^\beta F dt\right] \\
&= \mathbb{E}\left[G \int_0^T h(t) D_t^\beta F dt\right] \\
&= \mathbb{E}\left[\int_0^T h(t) D_t^\beta (FG) dt - F \int_0^T h(t) D_t^\beta G dt\right] \\
&= \mathbb{E}\left[FG \int_0^T h(t)(dN_t - dt) - F \int_0^T h(t) D_t^\beta G dt\right] \\
&= \mathbb{E}[F \delta(u)].
\end{aligned}$$

□

Proof of Proposition 3.5. By Definition 2.2, Lemma 3.2 and the relations $N_T^\alpha \leq N_{CT}$ and

$$f_k(t_1, \ldots, t_k) = \sum_{i=0}^k \mathbf{1}_{\{0 \leq t_1^\alpha \leq \cdots \leq t_i^\alpha \leq T < t_{i+1}^\alpha\}} f_i^\alpha(t_1^\alpha, \ldots, t_i^\alpha), \quad k \in \mathbb{N},$$

cf. (3.7), where t_i^α is defined by (3.6), we have

$$\begin{aligned}
D_t^\alpha F &= \alpha_{N_t^\alpha} D_{\lambda(t)}^\beta F \\
&= -\alpha_{N_t^\alpha} \sum_{k=1}^\infty \mathbf{1}_{\{N_{CT} = k\}} \sum_{l=1}^k \mathbf{1}_{[0, T_l]}(\lambda(t)) \partial_l f_k(T_1, \ldots, T_k)
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha_{N_t^\alpha} \sum_{k=1}^{\infty} \mathbf{1}_{\{N_{CT}=k\}} \sum_{l=1}^{k} \mathbf{1}_{[0,T_l^\alpha]}(t) \sum_{i=0}^{k} \mathbf{1}_{\{N_T^\alpha=i\}} \frac{\partial}{\partial t_l} f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \mathbf{1}_{\{N_{CT}=k\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \mathbf{1}_{\{N_T^\alpha=i\}} \frac{\partial}{\partial t_l} f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_{CT}\geq i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \mathbf{1}_{\{N_T^\alpha=i\}} \frac{\partial}{\partial t_l} f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \frac{\partial}{\partial t_l} f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \\
&\quad \times \left(\frac{1}{\alpha_{l-1}} \partial_l f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) + \left(\frac{1}{\alpha_{l-1}} - \frac{1}{\alpha_l} \right) \sum_{j=l+1}^{i} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \right) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \frac{1}{\alpha_{l-1}} \partial_l f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&\quad -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \left(\frac{1}{\alpha_{l-1}} - \frac{1}{\alpha_l} \right) \sum_{j=l+1}^{i} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{l=1}^{i} \mathbf{1}_{[0,T_l^\alpha]}(t) \frac{1}{\alpha_{l-1}} \partial_l f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&\quad -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \sum_{l=1}^{j-1} \mathbf{1}_{[0,T_l^\alpha]}(t) \left(\frac{1}{\alpha_{l-1}} - \frac{1}{\alpha_l} \right) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{j=1}^{i} \mathbf{1}_{[0,T_j^\alpha]}(t) \frac{1}{\alpha_{j-1}} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \\
&\quad -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \sum_{l=1}^{j-1} \frac{1}{\alpha_{l-1}} \mathbf{1}_{[0,T_l^\alpha]}(t) \\
&\quad +\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha) \sum_{l=2}^{j} \frac{1}{\alpha_{l-1}} \mathbf{1}_{[0,T_{l-1}^\alpha]}(t) \\
&= -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha=i\}} \mathbf{1}_{[0,T_1^\alpha]}(t) \frac{1}{\alpha_0} \partial_1 f_i^\alpha(T_1^\alpha,\ldots,T_i^\alpha)
\end{aligned}
$$

$$
\begin{aligned}
& -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \sum_{l=1}^{j} \frac{1}{\alpha_{l-1}} \mathbf{1}_{[0,T_l^\alpha]}(t) \\
& +\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \sum_{l=1}^{j} \frac{1}{\alpha_{l-1}} \mathbf{1}_{[0,T_{l-1}^\alpha]}(t) \\
= \; & -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \mathbf{1}_{[0,T_1^\alpha]}(t) \frac{1}{\alpha_0} \partial_1 f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \\
& -\alpha_{N_t^\alpha} \sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \sum_{l=1}^{j} \frac{1}{\alpha_{l-1}} \mathbf{1}_{\{N_t^\alpha = l-1\}} \\
= \; & -\sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \mathbf{1}_{[0,T_1^\alpha]}(t) \partial_1 f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \\
& -\sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \sum_{l=1}^{j} \mathbf{1}_{\{N_t^\alpha = l-1\}} \\
= \; & -\sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \mathbf{1}_{[0,T_1^\alpha]}(t) \partial_1 f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \\
& -\sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=2}^{i} \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha) \mathbf{1}_{[0,T_j^\alpha]}(t) \\
= \; & -\sum_{i=1}^{\infty} \mathbf{1}_{\{N_T^\alpha = i\}} \sum_{j=1}^{i} \mathbf{1}_{[0,T_j^\alpha]}(t) \partial_j f_i^\alpha (T_1^\alpha, \ldots, T_i^\alpha), \quad t \in \mathbb{R}_+.
\end{aligned}
$$

□

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