Analytic bond pricing for short rate dynamics evolving on matrix Lie groups

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Abstract

We provide closed-form expressions for bond prices in interest rate models based on compact Lie groups. Our approach uses a Doob transform technique and PDE solutions by the Mathieu periodic functions. As a byproduct, we derive formulas for bond option prices as well as new identities for the Laplace transform of periodic functionals of Brownian motion and Brownian diffusion processes.

Keywords: Bond pricing; interest rate modeling; interest rate derivatives; Lie groups; Mathieu equation.
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1 Introduction

Diffusion processes on manifolds and compact Lie groups have attracted significant attention in finance as they exhibit nonlinear and boundedness properties that make them suitable for the modeling of interest rates, cf. [3], [9]. For example, within the framework of the Heath-Jarrow-Morton (HJM) interest rate model, it is shown (cf. [7]) that one can characterize finite-dimensional HJM models that admit arbitrary initial yield curves as invariant manifolds of a separable Hilbert space.

Besides the HJM model, short-rate processes based on Brownian motion on Lie groups have been constructed in [9], [13], [14]. Pricing in these models has so far relied on

Monte Carlo numerical estimates, cf. [13] which deals with several classical matrix Lie groups.

Closed-form computation of bond prices is however generally preferable to Monte Carlo algorithms for calibration and sensitivity analysis purposes, and this is the challenge addressed by this paper. Following [13], we will model the short rate process as

$$r_t = r(g_t), \qquad t \in \mathbb{R}_+,$$

where $(g_t)_{t\in\mathbb{R}_+}$ is Brownian motion on a compact matrix Lie group G, such as $SO(n)$, and $r : G \to \mathbb{R}$ is a smooth function on G. Specifically, we will focus on linear relations of the form

$$r_t = \beta + \gamma \operatorname{tr}(g_t), \quad \gamma, \beta \in \mathbb{R},$$

as in [13]. In this way, the short-rate process is automatically bounded since G is compact, and the parameters can be adjusted to make it positive. This desirable feature is not satisfied by all standard short-term interest rate models available in the literature.

In Propositions 3.1 and 3.2 below, we give closed-form expressions for the zero-coupon bond price

$$F(t, r_t) = P(t,T) = \mathrm{E}\left[e^{-\int_t^T r_s \, \mathrm{d}s} \,\middle|\, \mathcal{F}_t \right], \qquad t \in [0,T], \tag{1.1}$$

on the orthogonal groups $SO(2)$ and $SO(3)$. In section 3.3, we then utilize these expressions to value bond options of the form

$$\mathrm{E}\left[e^{-\int_t^S r_s \, \mathrm{d}s} \rho(P(S,T)) \,\middle|\, \mathcal{F}_t \right],$$

where the option is exercised at time S, with a continuous and piecewise-smooth payoff function ρ on the bond price $P(S,T)$.

In the case of $SO(3)$, we are able to reduce the dimensionality of the problem by showing that with a suitable choice of coordinates, r_t can be rewritten as $r_t = \beta + \alpha\cos(\phi_t)$ where $(\phi_t)_{t\in\mathbb{R}_+}$ is the one-dimensional diffusion process solution of

$$\mathrm{d}\phi_t = \frac{1}{2}\cot\frac{\phi_t}{2}\,\mathrm{d}t + \mathrm{d}W_t, \qquad t \in \mathbb{R}_+, \tag{1.2}$$

and $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, cf. (2.6) below. We then proceed to show that $F(t,y)$ in (1.1) solves the Mathieu PDE

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,y) + \dfrac{h'(y)}{h(y)}\dfrac{\partial F}{\partial y}(t,y) + \omega\dfrac{\partial^2 F}{\partial y^2}(t,y) = (\alpha\cos(2y))\, F(t,y) \\ \\ F(T,y) = 1, \end{cases} \tag{1.3}$$

which originally arises in physics; e.g. for the Schrödinger equation with time dependent periodic potential, cf. [12], [18]. By extending techniques in [2] and using a particular Doob-h transform we simplify the PDE by drift removal in Section 4, and then find an analytical solution to (1.3).

Moreover, in Propositions 3.1 and 3.2 below, we derive closed-form series expressions for the Laplace transforms

$$\mathrm{E}\left[e^{-\int_t^T \cos W_s \, ds} \,\middle|\, \mathcal{F}_t \right] \qquad \text{and} \qquad \mathrm{E}\left[e^{-\int_t^T \cos \phi_s \, ds} \,\middle|\, \mathcal{F}_t \right],$$

using the Mathieu sine and cosine functions $\mathfrak{se}_n$ and $\mathfrak{ce}_n$, where $(\phi_t)_{t\in\mathbb{R}_+}$ is the diffusion process solution of (1.2) above. This complements the existing literature on the Laplace transform of additive functionals of Brownian motion, cf. e.g. [19] and references therein. In Proposition 3.5 we show that the above approximating series converge quadratic exponentially fast.

The remaining of the paper is organized as follows. We begin in Section 2 with the modeling of the short rate process as a function of Brownian motion on $SO(2)$ and on $SO(3)$. In Section 3, we give explicit formulas for zero-coupon bond prices as well as for bond options using the short rate processes constructed in Section 2. Section 3 closes with numerical tests which confirm the efficiency of closed-form solutions when compared with Monte Carlo simulations. In Section 4 we derive the probabilistic representation needed for the solution of Mathieu PDEs. In particular we prove Proposition 4.1 on the solution of the Mathieu PDE, and Proposition 4.3 which is a key step in simplifying bond pricing PDEs by the removal of drift terms via a Doob-h transform argument. Finally, in the Appendix we recall some basic facts on Brownian motion on manifolds which are needed in this paper.

2 Interest rate modeling on $SO(n)$

Following [13] we will model the short rate process as

$$r_t = \beta + \gamma \operatorname{tr}(g_t),$$

where $(g_t)_{t\in\mathbb{R}_+}$ is Brownian motion on a Lie group.

Brownian motion on $SO(2)$

Brownian motion $(g_t)_{t\in\mathbb{R}_+}$ on the (commutative) group $SO(2)$ is given by the matrix stochastic differential equation

$$\mathrm{d}g_t = g_t \circ \mathrm{d}\begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}, \tag{2.1}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion, whose solution is obtained by direct exponentiation

$$g_t = \exp\left(\begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{bmatrix}, \qquad t \in \mathbb{R}_+. \tag{2.2}$$

In this case the interest rate process is simply given by

$$r_t = \beta + \gamma \operatorname{tr}\left(\begin{bmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{bmatrix}\right) = \beta + 2\gamma \cos W_t, \tag{2.3}$$

where the condition $\beta \geq 2|\gamma|$ ensures $r_t \geq 0$, $t \in \mathbb{R}_+$.

Brownian motion on $SO(3)$

The Lie algebra of $SO(3)$ is non-commutative and generated by

$$\xi_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.4}$$

In order to determine Brownian motion $(g_t)_{t\in\mathbb{R}_+}$ on $SO(3)$, we note that by Rodrigues' rotation formula, every $g = e^{x\,\xi_1 + y\,\xi_2 + z\,\xi_3} \in SO(3)$, $x, y, z \in \mathbb{R}$, can be written as

$$g = I_3 + \sin\phi\, A(u_1, u_2, u_3) + (1 - \cos\phi)\, A(u_1, u_2, u_3)^2 \tag{2.5}$$

where

$$A(u_1, u_2, u_3) = \begin{bmatrix} 0 & -u_3 & -u_2 \\ u_3 & 0 & -u_1 \\ u_2 & u_1 & 0 \end{bmatrix},$$

and $\phi = \sqrt{x^2+y^2+z^2}$ is the angle of rotation about the axis

$$(u_1, u_2, u_3) := \frac{1}{\sqrt{x^2+y^2+z^2}}\,(z, -y, x) = (\cos\alpha,\ \sin\alpha\cos\theta,\ \sin\alpha\sin\theta) \in S^2.$$

By (4.17) and Theorem 4.1 in Section 4.2, the Laplacian on $SO(3)$ can be written in spherical coordinates as

$$\begin{aligned}
\Delta_{SO(3)} &= \Delta_{2S^3} \\
&= \frac{\partial^2}{\partial\phi^2} + \cot\frac{\phi}{2}\,\frac{\partial}{\partial\phi} + \frac{1}{4\sin^2\frac{\phi}{2}}\,\Delta_{S^2} \\
&= \frac{\partial^2}{\partial\phi^2} + \cot\frac{\phi}{2}\,\frac{\partial}{\partial\phi} + \frac{1}{4\sin^2\frac{\phi}{2}}\left(\frac{\partial^2}{\partial\alpha^2} + \cot\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}\right),
\end{aligned}$$

which in turn implies that Brownian motion g_t on $SO(3)$ is given in spherical coordinates by

$$g_t = \left(\phi_t,\ \alpha_{\tau(t)},\ \theta_{\tau(t)}\right),$$

where $\tau(t)$ is the random time-change given by

$$\tau(t) = \int_0^t \frac{1}{4\sin^2\frac{\phi_s}{2}}\,\mathrm{d}s,$$

$(\phi_t, \alpha_t, \theta_t)$ is driven by

$$\begin{cases}
\mathrm{d}\phi_t = \dfrac{1}{2}\cot\dfrac{\phi_t}{2}\,\mathrm{d}t + \mathrm{d}B_t^{(1)}, \\[2ex]
\mathrm{d}\alpha_t = \dfrac{1}{2}\cot\alpha_t\,\mathrm{d}t + \mathrm{d}B_t^{(2)}, \\[2ex]
\mathrm{d}\theta_t = \dfrac{1}{\sin\alpha_t}\,\mathrm{d}B_t^{(3)},
\end{cases} \tag{2.6}$$

and (α_t, θ_t) represents standard Brownian motion on S^2 in spherical coordinates. These equations can also be obtained from the fact that S^3 under quaternionic multiplication is isomorphic, as a Lie group, to $SO(3)$, see [17].

In this case the interest rate process is given by

$$\begin{aligned}
r_t &= \beta + \gamma\,\mathrm{tr}(g_t) \qquad (2.7)\\
&= \beta + \gamma\,\mathrm{tr}\left(\exp\left(\begin{bmatrix} 0 & -x_t & y_t \\ x_t & 0 & -z_t \\ -y_t & z_t & 0 \end{bmatrix}\right)\right)
\end{aligned}$$

$$= \quad \beta + \gamma \operatorname{tr}\left(S_t \exp\left(\begin{bmatrix} 0 & -\phi_t & 0 \\ \phi_t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) S_t^{-1} \right)$$

$$= \quad \beta + \gamma \operatorname{tr}\left(S_t \begin{bmatrix} \cos\phi_t & -\sin\phi_t & 0 \\ \sin\phi_t & \cos\phi_t & 0 \\ 0 & 0 & 1 \end{bmatrix} S_t^{-1} \right)$$

$$= \quad (\beta + \gamma) + 2\gamma \cos\phi_t,$$

where S_t is a change of basis matrix and $\beta \geq 2|\gamma| - \gamma$ ensures $r_t \geq 0$, $t \in \mathbb{R}_+$.

3 Bond pricing on $SO(n)$

In the sequel we will use the even (resp. odd) Mathieu cosine (resp. sine) functions

$$\mathfrak{ce}_{n,q}(y) = \mathfrak{ce}_{2m+p,q}(y) = \sum_{r=0}^{\infty} A_{2r+p}^{(2m+p)}(q) \cos((2r+p)\,y), \quad n \geq 0, \quad p = 0, 1,$$

and

$$\mathfrak{se}_{n,q}(y) = \mathfrak{se}_{2m+p,q}(y) = \sum_{r=0}^{\infty} B_{2r+p}^{(2m+p)}(q) \sin((2r+p)\,y), \quad n \geq 1, \quad p = 0, 1,$$

which are the solutions of the eigenvalue problem

$$\mathcal{L}g(y) := g''(y) - (2q\cos(2y))\,g(y) = \lambda g \tag{3.1}$$

on the interval $[0, \pi]$, with boundary conditions $g'(0) = g'(\pi) = 0$, and eigenvalues $\mathfrak{a}_{n,q}$ (resp. $g(0) = g(\pi) = 0$ and $\mathfrak{b}_{n,q}$). See Remark 4.2 in Section 4 for the recursive definitions of the coefficients $A_r^n(q)$ and $B_r^n(q)$ and for further properties of the Mathieu functions and of the coefficients $\mathfrak{a}_{n,q}$, $\mathfrak{b}_{n,q}$.

3.1 Bond pricing on $SO(2)$

In the next Proposition 3.1 we compute the bond price

$$\mathrm{E}\left[e^{-\int_t^T r_s \, \mathrm{d}s} \,\middle|\, \mathcal{F}_t \right] = e^{-\beta(T-t)} \, \mathrm{E}\left[e^{-2\gamma \int_t^T \cos W_s \, \mathrm{d}s} \,\middle|\, \mathcal{F}_t \right]$$

driven by the short rate process $r_t = \beta + 2\gamma \cos W_t$ given in (2.3) with the help of Proposition 4.1 in Section 4 which provides a closed-form expression for the Laplace transform of $\int_t^T \cos W_s \, \mathrm{d}s$, cf. also [14] for a different approach.

Proposition 3.1. *We have*

$$\mathrm{E}\left[e^{-2\gamma\int_t^T \cos W_s\,\mathrm{d}s}\,\Big|\,\mathcal{F}_t\right] = 2\sum_{m=0}^{\infty} A_0^{2m}(8\gamma)\,e^{-\mathfrak{a}_{2m,8\gamma}(T-t)/8}\,\mathfrak{ce}_{2m,8\gamma}\left(\frac{W_t}{2}\right), \qquad t\in[0,T],$$

where $\mathfrak{a}_{2m,q}$ *are the eigenvalues of* (3.1) *under Neumann conditions.*

Proof. We note that by the Feynman-Kac formula, the function

$$u(t,x) := \mathrm{E}\left[e^{-2\gamma\int_t^T \cos W_s\,\mathrm{d}s}\,\Big|\,W_t = x\right]$$

solves the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) = (2\gamma\cos x)\,u(t,x) \\ \\ u(T,x) = 1 \end{cases} \tag{3.2}$$

on $(t,x)\in[0,T]\times\mathbb{R}$. Since $u(T,x)=1$ is an even function we need to solve (3.2) with Neumann boundary conditions on $[0,2\pi]$, and to take the corresponding periodic extension. Thus, Proposition 4.1 in Section 4 shows that

$$\begin{aligned} u(t,x) &= \frac{2}{\pi}\sum_{n=0}^{\infty}\langle 1, \mathfrak{ce}_{n,8\gamma}(\cdot)\rangle_{L^2([0,\pi])}\; e^{-\mathfrak{a}_{n,8\gamma}(T-t)/8}\,\mathfrak{ce}_{n,8\gamma}\left(\frac{x}{2}\right) \\ &= 2\sum_{m=0}^{\infty} A_0^{2m}(8\gamma)\,e^{-\mathfrak{a}_{2m,8\gamma}(T-t)/8}\,\mathfrak{ce}_{2m,8\gamma}\left(\frac{x}{2}\right), \end{aligned}$$

and gives us the representation of $\mathrm{E}\left[e^{-2\gamma\int_t^T \cos W_s\,\mathrm{d}s}\,\Big|\,W_t = x\right]$ for all $x\in\mathbb{R}$. ■

3.2 Bond pricing on $SO(3)$

In Proposition 3.2 below we compute the bond price

$$\mathrm{E}\left[e^{-\int_t^T r_s\,\mathrm{d}s}\,\Big|\,\mathcal{F}_t\right] = e^{-(\beta+\gamma)(T-t)}\mathrm{E}\left[e^{-2\gamma\int_t^T \cos\phi_s\,\mathrm{d}s}\,\Big|\,\mathcal{F}_t\right],$$

for the short rate process on $SO(3)$ given by (2.7), where ϕ_t satisfies

$$\mathrm{d}\phi_t = \frac{1}{2}\cot\frac{\phi_t}{2}\,\mathrm{d}t + \mathrm{d}W_t. \tag{3.3}$$

Proposition 3.2. *Let $\phi_0 \in U = (0, 2\pi)$. We have*

$$\mathrm{E}\left[e^{-2\gamma\int_t^T \cos\phi_s\,\mathrm{d}s}\,\middle|\,\mathcal{F}_t\right] = \mathbb{1}_{[0,\tau_U)}(t)\,\frac{e^{\frac{T-t}{8}}}{\sin\frac{W_t}{2}}\sum_{m=0}^{\infty} B_1^{2m+1}(8\gamma)\,e^{-\mathfrak{b}_{2m+1,8\gamma}(T-t)/8}\,\mathfrak{se}_{2m+1,8\gamma}\left(\frac{W_t}{2}\right),$$

$t \in [0, T]$, where $\tau_U = \inf\{t > 0\ :\ W_t \notin (0, 2\pi)\}$.

Proof. Proposition 4.3 in Section 4 shows that

$$\mathrm{E}\left[e^{-2\gamma\int_t^T \cos\phi_s\,\mathrm{d}s}\,\middle|\,\mathcal{F}_t\right] = \mathrm{E}^{\mathbb{Q}_T}\left[e^{-2\gamma\int_t^T \cos(W_{s\wedge\tau_U})\,\mathrm{d}s}\,\middle|\,\mathcal{F}_t\right] = \mathbb{1}_{[0,\tau_U)}(t)\,v(t, W_t), \qquad (3.4)$$

where the function $v(t, x)$ solves the equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \dfrac{1}{2}\cot\dfrac{x}{2}\dfrac{\partial v}{\partial x}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(t,x) = (2\gamma\cos x)\,v(t,x) \\ \\ v(T,x) = 1, \qquad (t,x) \in [0,T]\times(0,2\pi), \end{cases}$$

and takes the form

$$v(t,x) = \frac{u(t,x)}{h(t,x)} = \frac{e^{-t/8}}{\sin\frac{x}{2}}\,u\,(t,x)\,,$$

where $h(t,x) = e^{t/8}\sin\frac{x}{2}$ is the solution of

$$\begin{cases} \dfrac{\partial h}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 h}{\partial x^2}(t,x) = 0 \\ \\ h(T,x) = e^{T/8}\sin\dfrac{x}{2}, \qquad x \in (0,2\pi), \\ \\ h(t,0) = h(t,2\pi) = 0, \qquad t \in [0,T], \end{cases} \qquad (3.5)$$

and $u(t, x)$ is the solution of the Mathieu PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) = (2\gamma\cos x)\,u(t,x) \\ \\ u(T,x) = e^{T/8}\sin\frac{x}{2}, \qquad x \in (0,2\pi), \\ \\ u(t,0) = u(t,2\pi) = 0, \qquad t \in [0,T]. \end{cases}$$

We conclude again by solving the above Mathieu PDE from Proposition 4.1 as

$$u(t,x) = \frac{2e^{T/8}}{\pi}\sum_{n=1}^{\infty}\langle\sin(\cdot),\ \mathfrak{se}_{n,8\gamma}(\cdot)\rangle_{L^2([0,\pi])}\ e^{-\mathfrak{b}_{n,8\gamma}(T-t)/8}\,\mathfrak{se}_{n,8\gamma}\left(\frac{x}{2}\right)$$

$$= e^{T/8} \sum_{m=0}^{\infty} B_1^{2m+1}(8\gamma)\, e^{-\mathfrak{b}_{2m+1,8\gamma}(T-t)/8}\, \mathfrak{se}_{2m+1,8\gamma}\left(\frac{x}{2}\right).$$

■

Remark: The probability density of $W^x_{t\wedge\tau_{(0,2\pi)}}$ appearing in (3.4) has the form

$$\sum_{n=-\infty}^{\infty} p_t(x, y + 4n\pi)\, \mathrm{d}y,$$

where $p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{(x-y)^2}{2t}} - e^{\frac{(x+y)^2}{2t}}\right)$, cf. [10] and Proposition 4.5 below.

3.3 Valuing bond options

In this section we demonstrate how bond options can be explicitly valued using the formulas in the previous sections. We let S denote the time at which the option is exercised, with payoff of the form $\rho(P(S,T))$ with maturity T and $S < T$, where ρ is a continuous and piecewise-smooth function of the bond price $P(S,T)$. For example we have $\rho(x) = (x-K)^+$ in the case of a European call option with strike price K.

To do so, we will compute the expectation

$$\mathrm{E}\left[e^{-\int_t^S r_s\,\mathrm{d}s}\,\rho(P(S,T))\,\Big|\,\mathcal{F}_t\right] = e^{-\beta(S-t)}\,\mathrm{E}\left[e^{-2\gamma\int_t^S \cos W_s\,\mathrm{d}s}\,\rho(P(S,T))\,\Big|\,\mathcal{F}_t\right]$$

for $SO(2)$, and

$$\mathrm{E}\left[e^{-\int_t^S r_s\,\mathrm{d}s}\,\rho(P(S,T))\,\Big|\,\mathcal{F}_t\right] = e^{-(\beta+\gamma)(S-t)}\mathrm{E}\left[e^{-2\gamma\int_t^S \cos\phi_s\,\mathrm{d}s}\,\rho(P(S,T))\,\Big|\,\mathcal{F}_t\right]$$

in the case of $SO(3)$.

Bond options on $SO(2)$

Proposition 3.3. *We have*

$$\mathrm{E}\left[e^{-2\gamma\int_t^S \cos W_s\,\mathrm{d}s}\,\rho(P(S,T))\Big|\,\mathcal{F}_t\right] = \frac{2}{\pi}\sum_{n=0}^{\infty}\langle\psi(2\cdot), \mathfrak{ce}_{n,8\gamma}(\cdot)\rangle_{L^2([0,\pi])}\, e^{-\mathfrak{a}_{n,8\gamma}(S-t)/8}\,\mathfrak{ce}_{n,8\gamma}\left(\frac{W_t}{2}\right),$$

$t \in [0,S]$, *where*

$$\psi(x) = \rho\left(2e^{-\beta(T-S)}\sum_{m=0}^{\infty} A_0^{2m}(8\gamma)\, e^{-\mathfrak{a}_{2m,8\gamma}(T-S)/8}\,\mathfrak{ce}_{2m,8\gamma}\left(\frac{x}{2}\right)\right). \qquad (3.6)$$

Proof. By Proposition 3.1 we can write $\rho(P(S,T))$ as $\rho(P(S,T)) = \psi(W_S)$ where the function ψ is given by (3.6). By the Feynman-Kac formula,

$$p(t,x) := \mathrm{E}\left[e^{-2\gamma \int_t^S \cos W_s \, \mathrm{d}s} \, \psi(W_S) \,\middle|\, W_t = x \right]$$

solves the PDE

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 p}{\partial x^2}(t,x) = (2\gamma \cos x)\, p(t,x) \\ \\ p(S,x) = \psi(x), \qquad (t,x) \in [0,T] \times \mathbb{R}. \end{cases} \tag{3.7}$$

We note that

$$\psi'(x)|_{x=0,2\pi} = \rho'(u(S,x)) \frac{\partial u}{\partial x}(S,x)_{|x=0,2\pi} = 0, \tag{3.8}$$

where

$$u(S,x) = 2 \sum_{m=0}^{\infty} A_0^{2m}(8\gamma)\, e^{-\mathfrak{a}_{2m,8\gamma}(T-S)/8}\, \mathfrak{ce}_{2m,8\gamma}\left(\frac{x}{2}\right).$$

The above one-sided derivatives are well defined for the continuous and piecewise-smooth payoff function $\rho(x)$. Thus, we can apply Proposition 4.1 (under Neumann conditions) again to obtain the statement of the Proposition. ■

In the case of the European call payoff function $\rho(x) = (x-K)^+$ the derivatives in (3.8) will also be two-sided as long as $u(S,0), u(S,2\pi) \neq K$.

Bond options on $SO(3)$

In the next proposition we deal with the pricing of bond options on $SO(3)$.

Proposition 3.4. *We have*

$$\mathrm{E}\left[e^{-2\gamma \int_t^S \cos \phi_s \, \mathrm{d}s} \, \rho(P(S,T)) \,\middle|\, \mathcal{F}_t \right]$$
$$= \mathbb{1}_{[0,\tau_U)}(t) \, \frac{2e^{\frac{S-t}{8}}}{\pi \sin \frac{W_t}{2}} \sum_{n=1}^{\infty} \langle \sin(\cdot)\psi(2\cdot), \mathfrak{se}_{n,8\gamma}(\cdot) \rangle_{L^2([0,\pi])} \; e^{-\mathfrak{b}_{n,8\gamma}(S-t)/8} \, \mathfrak{se}_{n,8\gamma}\left(\frac{W_t}{2}\right),$$

$t \in [0,S]$, *where*

$$\psi(x) = \rho\left(e^{-(\beta+\gamma)(T-S)} \, \mathbb{1}_{(0,2\pi)}(x) \, \frac{e^{\frac{T-S}{8}}}{\sin \frac{x}{2}} \sum_{m=0}^{\infty} B_1^{2m+1}(8\gamma)\, e^{-\mathfrak{b}_{2m+1,8\gamma}(T-S)/8}\, \mathfrak{se}_{2m+1,8\gamma}\left(\frac{x}{2}\right) \right).$$

Proof. From Proposition 3.2 we have $P(S,T) = e^{-(\beta+\gamma)(T-S)}\, v(S, W_{S\wedge\tau_U})$, with

$$v(S,x) = \mathbb{1}_{(0,2\pi)}(x)\, \frac{e^{\frac{T-S}{8}}}{\sin\frac{x}{2}} \sum_{m=0}^{\infty} B_1^{2m+1}(8\gamma)\, e^{-\mathfrak{b}_{2m+1,8\gamma}(T-S)/8}\, \mathfrak{se}_{2m+1,8\gamma}\left(\frac{x}{2}\right).$$

Hence, by Proposition 4.3,

$$\begin{aligned} \mathrm{E}\left[e^{-2\gamma\int_t^S \cos\phi_s\,\mathrm{d}s}\,\rho(P(S,T))\,\middle|\,\mathcal{F}_t\right] &= \mathrm{E}^{\mathbb{Q}_T}\left[e^{-2\gamma\int_t^S \cos(W_{s\wedge\tau_U})\,\mathrm{d}s}\,\psi(W_{S\wedge\tau_U})\,\middle|\,\mathcal{F}_t\right] \\ &= \mathbb{1}_{[0,\tau_U)}(t)\, p(t, W_t), \end{aligned}$$

where $p(t,x)$ solves the equation

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x) + \dfrac{1}{2}\cot\dfrac{x}{2}\dfrac{\partial p}{\partial x}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 p}{\partial x^2}(t,x) = (2\gamma\cos x)\, p(t,x) \\ \\ p(S,x) = \psi(x), \qquad (t,x)\in[0,T]\times(0,2\pi), \end{cases}$$

and again takes the form

$$p(t,x) = \frac{\tilde{u}(t,x)}{h(t,x)} = \frac{e^{-t/8}}{\sin\frac{x}{2}}\,\tilde{u}\,(t,x)\,.$$

Here, $h(t,x)$ solves (3.5) (with T replaced with S), and $\tilde{u}(t,x)$ is the solution of the Mathieu PDE

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 \tilde{u}}{\partial x^2}(t,x) = (2\gamma\cos x)\,\tilde{u}(t,x) \\ \\ \tilde{u}(S,x) = e^{S/8}\sin\frac{x}{2}\,\psi(x), \qquad x\in(0,2\pi), \\ \\ \tilde{u}(t,0) = \tilde{u}(t,2\pi) = 0, \qquad t\in[0,T]. \end{cases}$$

which can be obtained using Proposition 4.1 as

$$\tilde{u}(t,x) = \frac{2e^{S/8}}{\pi}\sum_{n=1}^{\infty}\langle \sin(\cdot)\psi(2\cdot),\, \mathfrak{se}_{n,8\gamma}(\cdot)\rangle_{L^2([0,\pi])}\; e^{-\mathfrak{b}_{n,8\gamma}(S-t)/8}\,\mathfrak{se}_{n,8\gamma}\left(\frac{x}{2}\right).$$

■

Numerical tests

We close this section with numerical simulations for the $SO(3)$ short-rate model. Figures 1 and 2 below present typical sample paths for the short-rate process

$$r_t = \beta + \gamma\,\mathrm{tr}(g_t) = (\beta+\gamma) + 2\gamma\cos\phi_t.$$

Setting $\beta = \gamma$, we get

$$r_t = 2\beta\left(1 + \cos\phi_t\right),$$

where ϕ_t satisfies

$$\mathrm{d}\phi_t = \frac{1}{2}\cot\frac{\phi_t}{2}\,\mathrm{d}t + \mathrm{d}W_t.$$

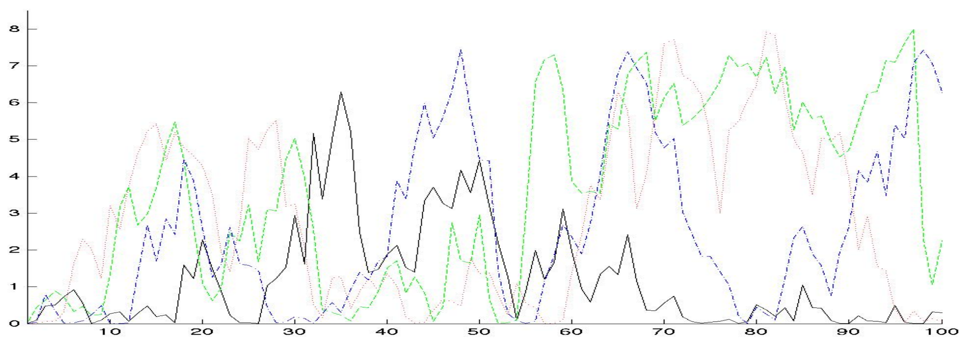

Figure 1: Graph of r_t with $\beta = 2, \phi_0 = \pi$.

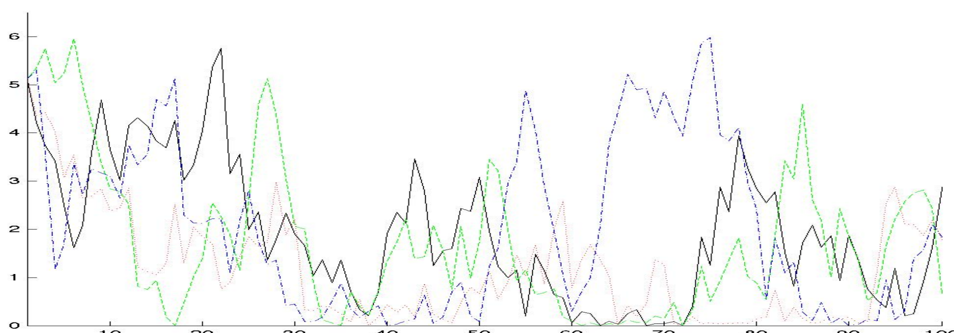

Figure 2: Graph of r_t with $\beta = 1.5, \phi_0 = \pi/4$.

Note that the short-rate process remains bounded in the interval $[0, 4\beta]$. Next, in Figure 3 we compare the evaluation of the bond price given by the explicit formula of Proposition 3.2 with the numerical results obtained by using a Monte-Carlo approximation.

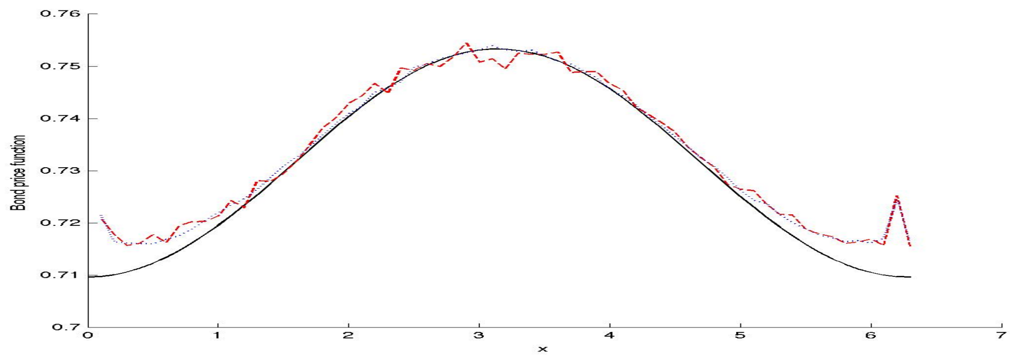

Figure 3: Graph of bond price with $\beta = \gamma = 0.015$, and $T = 10$.

The solid line is given by the explicit formula

$$x \longmapsto \mathrm{E}\left[e^{-\int_0^T r(\phi_s)\,\mathrm{d}s} \middle| W_0 = x \right] = \frac{e^{-(\beta+\gamma)T}}{\sin\frac{x}{2}} \sum_{m=0}^{\infty} B_1^{2m+1}(8\gamma)\, e^{(1-\mathfrak{b}_{2m+1,8\gamma})T/8}\, \mathfrak{se}_{2m+1,8\gamma}\left(\frac{x}{2}\right),$$

while the dotted and dashed lines are the Monte-Carlo estimate of the same quantity with time step $= T/100 = 0.1$ in the discretization of the time integral (dashed: 2000 samples, dotted: 20000 samples).

Our numerical simulations show that typically only five or six terms of the above series have to be computed for good convergence. More precisely, we have the following result:

Proposition 3.5. *For all $q, t > 0$, $x \in \mathbb{R}$ and $\psi \in L^2([0, 2\pi])$, the series*

$$F(t,x) := \sum_{n=0}^{\infty} \langle \psi(2\cdot), \mathfrak{se}_{n+1,q}(\cdot) \rangle_{L^2([0,\pi])}\, e^{-t\mathfrak{b}_{n+1,q}}\, \mathfrak{se}_{n+1,q}\left(\frac{x}{2}\right) \tag{3.9}$$

is absolutely and uniformly convergent, with the bound

$$\sup_{x\in\mathbb{R}} \left| F(t,x) - \sum_{n=0}^{N} \langle \psi(2\cdot), \mathfrak{se}_{n+1,q}(\cdot) \rangle_{L^2([0,\pi])}\, e^{-t\mathfrak{b}_{n+1,q}}\, \mathfrak{se}_{n+1,q}\left(\frac{x}{2}\right) \right| \le C_{q,\psi} \frac{e^{-tN^2}}{2tN}, \tag{3.10}$$

where $C_{q,\psi}$ is some constant depending only on q and the L^2 norm of ψ.

Proof. From the inequalities

$$\left| \langle \psi(2\cdot), \mathfrak{se}_{n+1,q}(\cdot) \rangle_{L^2([0,\pi])} \right| \le \frac{\pi}{2} ||\psi||_{L^2([0,2\pi])},$$

and $\mathfrak{b}_{n+1,q} > \mathfrak{a}_{n,q} \geq \mathfrak{a}_{n,0} = n^2$, cf. [4], and the fact that the Mathieu functions are uniformly bounded (cf. [11] Theorem 2.1), we have

$$\left| \sum_{n=N+1}^{\infty} \langle \psi(2\cdot), \mathfrak{se}_{n+1,q}(\cdot) \rangle_{L^2([0,\pi])} e^{-t\mathfrak{b}_{n+1,q}} \mathfrak{se}_{n+1,q}\left(\frac{x}{2}\right) \right| \leq C_{q,\psi} \sum_{n=N+1}^{\infty} e^{-tn^2}$$
$$\leq C_{q,\psi} \int_N^{\infty} e^{-tx^2} \,\mathrm{d}x \leq C_{q,\psi} \int_N^{\infty} \frac{x}{N} e^{-tx^2} \,\mathrm{d}x = C_{q,\psi} \frac{e^{-tN^2}}{2tN}.$$

■

In addition to the advantage of faster computation, the explicit formulation also gives a more precise evaluation of the bond price when the short rate process starts near 0 and 2π, whereas the Monte-Carlo approach encounters difficulties due to the removable singularities at the boundary points, cf. Figure 3. The calibration of our bond pricing model can be done by minimizing the least squares distance

$$\sum_{i=1}^{n} |P(t_i, T) - M(t_i, T)|^2$$

for a sequence of bond market prices $(M(t_i, T))_{i=1,\ldots,n}$. We refer to [13] for a calibration example of the spot rates on $SO(3)$.

4 Proofs

In this section, we prove the results needed in Section 3.

4.1 Mathieu PDEs

Proposition 4.1. *Given $f \in \mathcal{C}([0,b])$ and piecewise-smooth on $[0,b]$, consider the Mathieu PDE*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \omega \dfrac{\partial^2 u}{\partial x^2}(t,x) = \left(\alpha \cos \dfrac{2\pi x}{b}\right) u(t,x) \\ \\ u(T,x) = f(x). \end{cases} \tag{4.1}$$

on $[0,T] \times [0,b]$, and let $q = \frac{b^2 \alpha}{2\pi^2 \omega}$. Then, *(i) Under Dirichlet boundary conditions $u(t,0) = u(t,b) = 0$, $t \in [0,T]$, the solution to (4.1) is given by*

$$u(t,x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\langle f\left(\frac{b}{\pi}\cdot\right), \mathfrak{se}_{n,q}(\cdot) \right\rangle_{L^2([0,\pi])} \left(e^{-\pi^2 \omega \mathfrak{b}_{n,q}(T-t)/b^2}\right) \mathfrak{se}_{n,q}\left(\frac{\pi x}{b}\right).$$

(ii) Under Neumann boundary conditions $\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,b) = 0$, $t \in [0,T]$, the solution to (4.1) is given by

$$u(t,x) = \frac{2}{\pi} \sum_{n=0}^{\infty} \left\langle f\left(\frac{b}{\pi}\cdot\right), \mathfrak{ce}_{n,q}(\cdot) \right\rangle_{L^2([0,\pi])} \left(e^{-\pi^2 \omega \mathfrak{a}_{n,q}(T-t)/b^2}\right) \mathfrak{ce}_{n,q}\left(\frac{\pi x}{b}\right).$$

In both cases, $u(t,x) \in \mathcal{C}^{1,2}([0,T) \times (0,b)) \cap \mathcal{C}([0,T] \times [0,b])$.

Proof. By separation of variables, we write

$$u(t,x) = \sum_n \frac{\langle f, g_n \rangle_{L^2([0,b])}}{\langle g_n, g_n \rangle_{L^2([0,b])}} e^{-\lambda_n (T-t)} g_n(x), \tag{4.2}$$

where $\{\lambda_n, g_n\}$ satisfy the equation

$$g_n''(x) - \frac{\alpha}{\omega} \cos\left(\frac{2\pi x}{b}\right) g_n(x) = \frac{\lambda_n}{\omega} g_n(x), \quad x \in [0,b]. \tag{4.3}$$

Using a change of variable $y = \pi x/b$, Relation (4.3) can be written in canonical form as

$$g''(y) - (2q\cos(2y))\, g(y) = \lambda\, g(y), \quad y \in [0,\pi]. \tag{4.4}$$

The operator $\mathcal{L}g(y) := g''(y) - (2q\cos(2y))\, g(y)$ is known as the Mathieu Hamiltonian. In addition, when either of the boundary conditions

$$g'(0) = g'(\pi) = 0, \qquad \text{[Neumann]}$$

or

$$g(0) = g(\pi) = 0, \qquad \text{[Dirichlet]}$$

are satisfied, we know from Sturm-Louville theory that $\mathcal{L}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2([0,\pi])}$, which implies the existence of a countably infinite number of eigenvalues (with corresponding eigenfunctions) to (4.4) which accumulate only at infinity, cf. [5]. In the Neumann case, we denote the eigenvalues by $\mathfrak{a}_{n,q}$, $n \geq 0$, and the corresponding eigenfunctions are given by the even Mathieu cosine functions

$$\mathfrak{ce}_{2m+p,q}(y) = \sum_{r=0}^{\infty} A_{2r+p}^{(2m+p)}(q) \cos((2r+p)\, y), \qquad p = 0, 1, \tag{4.5}$$

of period π (when $p = 0$) or 2π (when $p = 1$), cf. [1], [8]. Similarly, in the Dirichlet case, we denote the eigenvalues by $\mathfrak{b}_{n,q}$, $n \geq 1$, and the corresponding eigenfunctions are given by the odd Mathieu sine functions

$$\mathfrak{se}_{2m+p,q}(y) = \sum_{r=0}^{\infty} B_{2r+p}^{(2m+p)}(q) \sin((2r+p)\, y), \qquad p = 0, 1, \tag{4.6}$$

of period π (when $p = 0$) or 2π (when $p = 1$), cf. [1], [8].

These $\mathcal{C}^{\infty}$ eigenfunctions are maximal, uniformly bounded (cf. [11]) and mutually orthogonal with respect to the inner product

$$\langle \mathfrak{ce}_{n,q}, \mathfrak{ce}_{n,q} \rangle_{L^2([0,\pi])} = \langle \mathfrak{se}_{n,q}, \mathfrak{se}_{n,q} \rangle_{L^2([0,\pi])} = \pi/2,$$

cf. [1] and Chapter 6.9 of [8]. Hence, substituting them back into (4.2), we get the statement of the proposition once we verify the regularity of u. For $t < T$, it is simple to check that term-by-term differentiation of (4.2) (with respect to x or t) gives an absolutely convergent series and thus $u(t,x) \in \mathcal{C}^{1,2}([0,T) \times (0,b)) \cap \mathcal{C}([0,T) \times [0,b])$. At $t = T$, we have

$$f(x) = \sum_{n} \frac{\langle f, g_n \rangle_{L^2([0,b])}}{\langle g_n, g_n \rangle_{L^2([0,b])}} \, g_n(x)\,,$$

where the series converges uniformly to f since it is continuous and piecewise-smooth on $[0,b]$. ■

Remark 4.2. *The coefficients in (4.5) and (4.6) depend continuously on q and obey the recursion relations*

$$\mathfrak{a}_{2m} A_0^{(2m)} - q\, A_2^{(2m)} = 0,$$
$$(\mathfrak{a}_{2m+1} - 1 - q)\, A_1^{(2m+1)} - q\, A_3^{(2m+1)} = 0,$$

$$(\mathfrak{a}_{2m}-4)\,A_2^{(2m)}-q\left(2A_0^{(2m)}+A_4^{(2m)}\right)=0,$$
$$\left(\mathfrak{a}_{2m}-4r^2\right)A_{2r}^{(2m)}-q\left(A_{2r-2}^{(2m)}+A_{2r+2}^{(2m)}\right)=0,\quad r\geq 2,$$
$$\left(\mathfrak{a}_{2m+1}-(2m+1)^2\right)A_{2r+1}^{(2m+1)}-q\left(A_{2r-1}^{(2m+1)}+A_{2r+3}^{(2m+1)}\right)=0,\quad r\geq 1,$$

for the even Mathieu cosine functions, and

$$(\mathfrak{b}_{2m+1}-1+q)\,B_1^{(2m+1)}-q\,B_3^{(2m+1)}=0,$$
$$(\mathfrak{b}_{2m+2}-4)\,B_2^{(2m+2)}-q\,B_4^{(2m+2)}=0,$$
$$\left(\mathfrak{b}_{2m+1}-(2m+1)^2\right)B_{2r+1}^{(2m+1)}-q\left(B_{2r-1}^{(2m+1)}+B_{2r+3}^{(2m+1)}\right)=0,\quad r\geq 1,$$
$$\left(\mathfrak{b}_{2m+2}-4m^2\right)B_{2r}^{(2m+2)}-q\left(B_{2r+2}^{(2m+2)}-B_{2r-2}^{(2m+2)}\right)=0,\quad r\geq 2,$$

for the odd Mathieu sine functions, cf. Chapter 20, page 723 of [1].

4.2 Drift removal via the Doob h-transform

In this section we derive a Feynman-Kac formula for a class of PDEs with a particular drift term. Let U be a connected open subset of $\mathbb{R}^n$ with smooth boundary, and let X_t be a diffusion process satisfying

$$\mathrm{d}X_t=b(X_t)\,\mathrm{d}t+\sigma(X_t)\,\mathrm{d}W_t,\quad X_0=x\in U,$$

with transition function given by $p_t(x,\mathrm{d}y)$ and generator

$$\mathcal{L}=\sum_{i=1}^n b_i(x)\,\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j}a_{ij}(x)\,\frac{\partial^2}{\partial x_i\partial x_j},$$

where $a=\sigma\sigma^T$, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with an augmented (therefore right-continuous) Brownian filtration $\mathcal{F}_t\subset\mathcal{F}$, $t\in\mathbb{R}_+$. In the sequel we denote by $h(t,x)$ and $u(t,x)$ the respective solutions in $\mathcal{C}^{1,2}([0,T)\times U)\cap\mathcal{C}([0,T]\times\overline{U})$ to the equations

$$\begin{cases}\dfrac{\partial h}{\partial t}(t,x)+\mathcal{L}h(t,x)=0, & (t,x)\in[0,T]\times U,\\[2ex] h(T,x)\geq 0\ \text{ on }\ U, & \\[2ex] h(t,x)=0\ \text{ on }\ (t,x)\in\partial U\times[0,T], & \end{cases}\tag{4.7}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) = r(x)\,u(t,x), \qquad (t,x) \in [0,T] \times U, \\ \\ u(T,x) = h(T,x)g(x), \\ \\ u(t,x) = 0 \ \text{ on } \ (t,x) \in \partial U \times [0,T]. \end{cases} \tag{4.8}$$

The following result consequence of Proposition 4.4 below will enable us to simplify the computation of the bond price by removing the drift term of these PDEs with domains which have a boundary, while paying close attention to the behaviour of X_t when it reaches a boundary, cf. Propositions 3.2 and 3.4 above.

Proposition 4.3. *Let $D_t := h(t, X_t)/h(0,x)$, $t \in [0,T]$, $x \in U$, and define the probability measure $\mathbb{Q}_T$ by*

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = D_T.$$

(i) The function

$$v(t,x) := \frac{u(t,x)}{h(t,x)}$$

belongs to $\mathcal{C}^{1,2}([0,T) \times U)$ and solves

$$\begin{cases} \dfrac{\partial v}{\partial t} + \left\langle a(x)\,\dfrac{\nabla h(t,x)}{h(t,x)},\, \nabla v \right\rangle + \mathcal{L}v = r(x)\,v, \\ \\ v(T,x) = g(x). \end{cases} \tag{4.9}$$

(ii) If in addition,

$$\sup_{t,\,x \in U} |u(t,x)| < \infty, \tag{4.10}$$

then we have

$$\mathrm{E}^{\mathbb{Q}_T}\left[e^{-\int_t^T r(X_{s\wedge\tau_U})\,\mathrm{d}s} g\left(X_{T\wedge\tau_U}\right) \,\Big|\, \mathcal{F}_t \right] = \frac{u(t,X_t)}{h(t,X_t)}\,\mathbb{1}_{[0,\tau_U)}(t), \qquad \mathbb{Q}_T - a.s. \tag{4.11}$$

Proof. (i) Equation (4.9) is readily verified by differentiating $v(t,x)$.
(ii) Next, we note that by the strong maximum principle, unless $h(t,x)$ is constant, $h(t,x) > 0$ on $U \times [0,T)$. Recall that X_t is the solution to

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t,$$

and let τ_U denote the hitting time

$$\tau_U = \inf\{t > 0 \ : \ X_t \in U^c\}.$$

Thus, $X_{t\wedge\tau_U}$ gives the value of X_t killed at the boundary of U. We note that $h(t, X_{t\wedge\tau_U})$ is a martingale, as given U_n, a sequence of bounded, open subsets of U, such that $\lim_{n\to\infty} U_n = U$, and letting $\tau_{U_n} = \inf\{t > 0 \ : \ X_t \in U_n^c\}$, we have

$$\begin{aligned} h(t\wedge\tau_{U_n}, X_{t\wedge\tau_{U_n}}) &= h(0, X_0) + \int_0^{t\wedge\tau_{U_n}} \langle \nabla h(s, X_s), \, \sigma(X_s)\,\mathrm{d}W_s\rangle \\ &\quad + \int_0^{t\wedge\tau_{U_n}} \frac{\partial h}{\partial s}(s, X_s)\,\mathrm{d}s + \int_0^{t\wedge\tau_{U_n}} \mathcal{L}h(s, X_s)\,\mathrm{d}s \\ &= h(0, x) + \int_0^{t\wedge\tau_{U_n}} \langle \nabla h(s, X_s), \, \sigma(X_s)\,\mathrm{d}W_s\rangle , \end{aligned}$$

and since $\mathrm{E}\left[\mathbb{1}_A\, h\left(t\wedge\tau_{U_n}, X_{t\wedge\tau_{U_n}}\right)\right] \to \mathrm{E}\left[\mathbb{1}_A\, h\left(t\wedge\tau_U, X_{t\wedge\tau_U}\right)\right]$ as n tends to $+\infty$ by the monotone convergence theorem for all $t\in[0,T]$ and $A\in\mathcal{F}_s$, we get

$$\mathrm{E}\left[h(t, X_{t\wedge\tau_U}) \,\middle|\, \mathcal{F}_s\right] = h(s, X_{s\wedge\tau_U}).$$

Applying the Doob transform Proposition 4.4 below, under $\mathbb{Q}_T$, the process X_t killed at the boundary ∂U solves (4.14), i.e.

$$\mathrm{d}X_t = a(X_t)\,\frac{\nabla h(t, X_t)}{h(t, X_t)}\mathrm{d}t + b(X_t)\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x.$$

Now, from Itô's formula we have

$$\begin{aligned} e^{-\int_0^{t\wedge\tau_{U_n}} r(X_{\tilde{s}})\,\mathrm{d}\tilde{s}}\, u(t\wedge\tau_{U_n}, X_{t\wedge\tau_{U_n}}) &= u(0, x) + \int_0^{t\wedge\tau_{U_n}} e^{-\int_t^s r(X_{\tilde{s}})\,\mathrm{d}\tilde{s}} \langle \nabla u(s, X_s), \, \sigma(X_s)\,\mathrm{d}W_s\rangle \\ &\quad + \int_0^{t\wedge\tau_{U_n}} e^{-\int_t^s r(X_{\tilde{s}})\,\mathrm{d}\tilde{s}} \left(\frac{\partial u}{\partial s}(s, X_s) - r\left(X_s\right) + \mathcal{L}u(s, X_s)\right)\mathrm{d}s \\ &= u(0, x) + \int_0^{t\wedge\tau_{U_n}} e^{-\int_t^s r(X_{\tilde{s}})\,\mathrm{d}\tilde{s}} \langle \nabla u(s, X_s), \, \sigma(X_s)\,\mathrm{d}W_s\rangle , \end{aligned}$$

and we get

$$\begin{aligned} \mathrm{E}^{\mathbb{P}}\left[e^{-\int_{t\wedge\tau_U}^{T\wedge\tau_U} r(X_s)\,\mathrm{d}s}\, u\left(T\wedge\tau_U, X_{T\wedge\tau_U}\right) \,\middle|\, \mathcal{F}_{t\wedge\tau_U}\right] &= \mathrm{E}^{\mathbb{P}}\left[\mathbb{1}_{[0,\tau_U)}(T)\, e^{-\int_t^T r(X_s)\,\mathrm{d}s}\, u\left(T, X_T\right) \,\middle|\, \mathcal{F}_{t\wedge\tau_U}\right] \\ &= u\left(t\wedge\tau_U, X_{t\wedge\tau_U}\right) \end{aligned} \tag{4.12}$$

from (4.10) and the bounded convergence theorem. From Lemma 4.6, we have $\mathbb{Q}_T\left(t<\tau_U\right)=1$, and thus,

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}_T}\left[e^{-\int_t^T r(X_{s\wedge\tau_U})\,\mathrm{d}s}\,g(X_{T\wedge\tau_U})\,\Big|\,\mathcal{F}_t\right] &= \mathrm{E}^{\mathbb{Q}_T}\left[\mathbb{1}_{[0,\tau_U)}(t)\,e^{-\int_t^T r(X_{s\wedge\tau_U})\,\mathrm{d}s}\,v(T,X_{T\wedge\tau_U})\,\Big|\,\mathcal{F}_{t\wedge\tau_U}\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[\mathbb{1}_{[0,\tau_U)}(t)\,e^{-\int_t^T r(X_{s\wedge\tau_U})\,\mathrm{d}s}\,\frac{D_T}{D_t}\,v(T,X_{T\wedge\tau_U})\,\Big|\,\mathcal{F}_{t\wedge\tau_U}\right] \\
&= \frac{\mathbb{1}_{[0,\tau_U)}(t)}{h(t,X_{t\wedge\tau_U})}\,\mathrm{E}^{\mathbb{P}}\left[\mathbb{1}_{[0,\tau_U)}(T)\,e^{-\int_t^T r(X_s)\,\mathrm{d}s}\,h(T,X_T)\,v(T,X_T)\,\Big|\,\mathcal{F}_{t\wedge\tau_U}\right],
\end{aligned}$$

where the last line follows from the fact that $h\left(T,X_{\tau_U}\right)=0$. Upon applying (4.12), we get the statement of (4.11). ■

We have the following classical proposition (see § 2.VI.13 of [6], 2.IV-39 of [17], [2], [16], and Theorem 3.2 of [15] in the case of jump processes), whose proof is given here for completeness.

Proposition 4.4. *(Doob transform) The process (X_t) has transition function*

$$\tilde{p}_{s,t}(x,\mathrm{d}y)=\frac{h(t,y)}{h(s,x)}\,p_{t-s}(x,\mathrm{d}y),\qquad s\in[0,t], \tag{4.13}$$

under $\mathbb{Q}_T$, and can be represented as a weak solution of

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+a(X_t)\,\frac{\nabla h(t,X_t)}{h(t,X_t)}\mathrm{d}t+\sigma(X_t)\,\mathrm{d}B_t,\quad X_0=x, \tag{4.14}$$

where (B_t) is a standard Brownian motion under $\mathbb{Q}_T$.

Proof. Recall that $h(t,x)\in\mathcal{C}^{1,2}([0,T)\times U)$ such that $h(t,X_t)$ is a positive martingale. Note that D_t is also a positive martingale with $\mathrm{E}[D_t]=1$, $t>0$, which ensures that the measures defined by $\mathrm{d}\mathbb{Q}_t=D_t\,\mathrm{d}\mathbb{P}$ form a consistent family of probability measures. Relation (4.13) can be proved by noting that for all Borel sets $B\in\mathbb{R}^n$, we have

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}_T}\left[\mathbb{1}_B(X_t)\mid X_s=x\right] &= \mathrm{E}^{\mathbb{P}}\left[\mathbb{1}_B(X_t)\,\frac{D_t}{D_s}\,\Big|\,X_s=x\right] \\
&= \int_B\frac{h(t,y)}{h(s,x)}\,p_{t-s}(x,\mathrm{d}y),
\end{aligned}$$

by Bayes' rule, cf. e.g. § IV.39 of [17]. Now for $\phi\in\mathcal{C}^2(U)$, let M_t^ϕ denote the $\mathbb{P}$-martingale

$$\phi\left(X_t\right)-\phi(x)-\int_0^t\mathcal{L}\phi(X_s)\,\mathrm{d}s=\int_0^t\left\langle\nabla\phi(X_s),\,\sigma(X_s)\,\mathrm{d}W_s\right\rangle.$$

Then by the Girsanov theorem,

$$M_t^\phi - \int_0^t \frac{1}{D_s}\,\mathrm{d}\left\langle M^\phi,\, D\right\rangle_s = M_t^\phi - \int_0^t \left\langle a(\tilde{X}_s)\frac{\nabla h(s,\tilde{X}_s)}{h(s,\tilde{X}_s)},\, \nabla\phi(\tilde{X}_s)\right\rangle \mathrm{d}s$$

$$= \phi(\tilde{X}_t) - \phi(x) - \int_0^t \mathcal{L}\phi(\tilde{X}_s) - \left\langle a(\tilde{X}_s)\, g(s,\tilde{X}_s),\, \nabla\phi(\tilde{X}_s)\right\rangle \mathrm{d}s$$

becomes a local martingale under $\mathbb{Q}_T$, thus giving us a weak solution to (4.14). ■

Example. Proposition 4.4 can be illustrated with the classical example of the representation of 3D Bessel processes by Brownian motion killed at the origin.

Proposition 4.5. *The probability density of BES^3 is given by*

$$\frac{y}{x}\, p_t(x,y), \qquad x,\, y > 0,$$

where

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{(x-y)^2}{2t}} - e^{\frac{(x+y)^2}{2t}}\right). \tag{4.15}$$

Proof. Here we take $U = (0,\infty)$, $\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$, and $h(t,x) = x$, and let $X_t = W^x_{t\wedge\tau_U}$ be Brownian motion killed at 0 with transition sub-probability density (4.15) under $\mathbb{P}$. We can verify that

$$t \longmapsto D_t := \frac{h\,(t, X_t)}{h(t,x)} = \frac{W^x_{t\wedge\tau_U}}{x}$$

is a positive martingale with unit expectation under $\mathbb{P}$. Thus, under the probability

$$\mathrm{d}\mathbb{Q}_T = \frac{W^x_{T\wedge\tau_U}}{x}\,\mathrm{d}\mathbb{P},$$

the process $X_t = W^x_{t\wedge\tau_U}$ is a weak solution to the BES^3 equation

$$\mathrm{d}X_t = \frac{1}{X_t}\,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x.$$

■

Alternatively, one can directly recover (4.15) above by noting that

$$\mathrm{E}^{\mathbb{Q}_T}[f(X_{s+t}) \mid X_s = |a|] = \mathrm{E}^{\mathbb{Q}_T}[f(|B_{s+t}|) \mid B_s = a]$$

$$
\begin{aligned}
&= \frac{1}{(2\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{|a-y|^2}{2t}} f(|y|)\, \mathrm{d}y \\
&= \frac{2\pi}{(2\pi t)^{3/2}} \int_0^\infty f(r) \int_0^\pi r^2\, e^{-\frac{|a|^2+r^2}{2t}+\frac{r|a|}{t}\cos\phi} \sin\phi\, \mathrm{d}\phi\, \mathrm{d}r \\
&= \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{|a|^2+r^2}{2t}} \left(\frac{r}{|a|}\right) \sinh\left(\frac{r|a|}{t}\right) f(r)\, \mathrm{d}r \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^\infty \frac{r}{|a|} \left(e^{\frac{(|a|-r)^2}{2t}} - {}^{\frac{(|a|+r)^2}{2t}}\right) f(r)\, \mathrm{d}r,
\end{aligned}
$$

cf. e.g. [16]. The next lemma has been used in the proof of Proposition 4.3.

Lemma 4.6. *We have* $\mathbb{Q}_T\left(\tau_U \le t\right) = 0$ *for all* $t > 0$.

Proof. We have $\{\tau_U \le t\} \in \mathcal{F}_{t\wedge\tau_U} = \mathcal{F}_t \cap \mathcal{F}_{\tau_U}$, and thus

$$
\begin{aligned}
\mathbb{Q}_T\left(\tau_U \le t\right) &= \mathrm{E}^{\mathbb{P}}\left[D_{t\wedge\tau_U}\, \mathbb{1}_{[\tau_U,\infty)}(t)\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[D_{\tau_U}\, \mathbb{1}_{[\tau_U,\infty)}(t)\right] \\
&= \int_{\{\tau_U \le t\}} \frac{h\left(\tau_U, X_{\tau_U}\right)}{h(0,x)}\, \mathrm{d}\mathbb{P} = 0.
\end{aligned}
$$

■

Appendix - Brownian motion on manifolds and Lie groups

In this appendix, we recall some basic facts on Brownian motion on a Riemannian manifold M of dimension n equipped with its Levi-Civita connection ∇, with application given to $SO(3)$ at the end of the section.

Brownian motion W_t on M is defined to be the Markov process generated by one-half the Laplace-Beltrami operator Δ_M on M, i.e.

$$
M^f(W_t) = f(W_t) - f(W_0) - \int_0^t \frac{1}{2}\Delta_M f(W_s)\, \mathrm{d}s \tag{4.16}
$$

is a local martingale for all $f \in \mathcal{C}^\infty(M)$. Given an orthonormal frame $\{E_i\}_{i=1,\dots,n}$, since

$$
E_i^2 f = \nabla_{E_i} \langle \nabla f, E_i \rangle
$$

$$= \langle \nabla_{E_i} \nabla f, E_i \rangle + \langle \nabla f, \nabla_{E_i} E_i \rangle$$
$$= \nabla^2 f(E_i, E_i) + \nabla_{E_i} E_i \; f,$$

the Laplace-Beltrami operator on M is given by

$$\Delta_M f = \sum_{i=1}^n \nabla^2 f(E_i, E_i)$$
$$= \sum_{i=1}^n E_i^2 f - \nabla_{E_i} E_i f,$$

while in local coordinates we have

$$\Delta_M f = \sum_{i,j} \frac{1}{\sqrt{\det \tilde{G}}} \frac{\partial}{\partial x^i} \left(\sqrt{\det \tilde{G}} \, g^{ij} \frac{\partial f}{\partial x^j} \right), \tag{4.17}$$

where $\{\tilde{G}\}_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$.

In the case of a Lie group G with identity element e, Brownian motion g_t on G is also defined to be the solution to the following equation

$$\mathrm{d}g_t = g_t \circ \mathrm{d}W_t, \tag{4.18}$$

where $g_t \circ \mathrm{d}W_t$ is a shorthand for the Stratonovich differential

$$\sum_{i=1}^n (L_{g_t})_* \, E_i|_e \circ \mathrm{d}W_t^{(i)} = \sum_{i=1}^n E_i|_{g_t} \circ \mathrm{d}W_t^{(i)}.$$

If G is compact, there exists a bi-invariant metric for the elements of its Lie algebra $\mathcal{G}$ and we have $\nabla_X Y = \frac{1}{2}[X, Y]$ whenever X, Y are left-invariant vector fields on G. We denote by $\{E_i\}$ a left-invariant frame on G, orthonormal with respect to this metric. Now let $f : G \to \mathbb{R}$ be a smooth function. We note that

$$f(g_t) = f(g_0) + \sum_{i=1}^n \int_0^t E_i f(g_s) \circ \mathrm{d}W_s^{(i)}$$
$$= f(g_0) + \sum_{i=1}^n \int_0^t E_i f(g_s) \, \mathrm{d}W_s^{(i)} + \frac{1}{2} \sum_{i=1}^n \int_0^t E_i^2 f(g_s) \, \mathrm{d}s$$
$$= f(g_0) + \sum_{i=1}^n \int_0^t E_i f(g_s) \, \mathrm{d}W_s^{(i)} + \frac{1}{2} \int_0^t \Delta_G f(g_s) \, \mathrm{d}s,$$

since

$$\nabla_{E_i} E_i = \frac{1}{2}[E_i, E_i] = 0, \qquad i = 1, \ldots, n.$$

Thus, for a compact Lie group, (4.18) is equivalent (4.16).

$SO(3)$ and $2S^3$

For $A, B \in SO(3)$, let $\langle A, B\rangle$ denote the bi-invariant metric

$$\langle A,\, B\rangle = \frac{1}{2}\,\mathrm{tr}\left(AB^T\right). \tag{4.19}$$

We have the following:

Theorem 4.1. *$SO(3)$ with the metric (4.19) is isometric to $2S^3$.*

Proof. Here, $2S^3$ denotes the sphere in $\mathbb{R}^4$ with radius 2. Using Rodrigues' rotation formula (2.5), one can compute $\frac{\partial}{\partial \nu}\Big|_{I_3} = g^{-1}\left(\frac{\partial g}{\partial \nu}\right)$, for $\nu = \phi, \alpha, \text{or } \theta$, to obtain

$$\begin{aligned}
\frac{\partial}{\partial \phi} &= (\cos\alpha)\ \xi_1 + (\sin\alpha\cos\theta)\ \xi_2 + (\sin\alpha\sin\theta)\ \xi_3,\\
\frac{\partial}{\partial \alpha} &= (\sin\phi\sin\alpha)\ \xi_1 + (\sin\phi\cos\alpha\cos\theta + (\cos\phi - 1)\sin\theta)\,\xi_2\\
&\qquad + ((\cos\phi - 1)\cos\theta) - \sin\phi\cos\alpha\sin\theta)\,\xi_3,\\
\frac{\partial}{\partial \theta} &= \left(2\sin^2\frac{\phi}{2}\sin^2\alpha\right)\ \xi_1 + \sin\alpha((\cos\phi - 1)\cos\alpha\cos\theta - \sin\phi\sin\theta)\,\xi_2\\
&\qquad + \sin\alpha(\cos\theta\sin\phi + (\cos\phi - 1)\cos\alpha\sin\theta)\,\xi_3,
\end{aligned}$$

where $\{\xi_i\}$ are the orthonormal basis elements of the Lie algebra of $SO(3)$ given in (2.4). One can then check that $\frac{\partial}{\partial\phi}$, $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\theta}$ are mutually orthogonal, and that

$$\left\langle\frac{\partial}{\partial\phi},\,\frac{\partial}{\partial\phi}\right\rangle = 1,\quad \left\langle\frac{\partial}{\partial\alpha},\,\frac{\partial}{\partial\alpha}\right\rangle = 4\sin^2\frac{\phi}{2},\quad \left\langle\frac{\partial}{\partial\theta},\,\frac{\partial}{\partial\theta}\right\rangle = 4\sin^2\frac{\phi}{2}\sin^2\alpha.$$

Now we note that in the neighbourhood of every point on $SO(3)$, the map

$$\begin{aligned}
\Phi :\ & SO(3) \longrightarrow 2S^3\\
& (\phi,\alpha,\theta) \longmapsto 2\left(\cos\frac{\phi}{2},\ \sin\frac{\phi}{2}\cos\alpha,\ \sin\frac{\phi}{2}\sin\alpha\cos\theta,\ \sin\frac{\phi}{2}\sin\alpha\sin\theta\right)
\end{aligned}$$

$0 < \phi < 2\pi$, $0 < \alpha < \pi$, $0 < \theta < 2\pi$, gives the required local isometry. ∎

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