# Moment identities for Skorohod integrals on the Wiener space and applications

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**Abstract**

We prove a moment identity on the Wiener space that extends the Skorohod isometry to arbitrary powers of the Skorohod integral on the Wiener space. As simple consequences of this identity we obtain sufficient conditions for the Gaussianity of the law of the Skorohod integral and a recurrence relation for the moments of second order Wiener integrals. We also recover and extend the sufficient conditions for the invariance of the Wiener measure under random rotations given in [3].

**Key words:** Malliavin calculus, Skorohod integral, Skorohod isometry, Wiener measure, random isometries.
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## 1 Introduction and notation

In [3], sufficient conditions have been found for the Skorohod integral $\delta(Rh)$ to have a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and $R$ is a random isometry of $H$, using an induction argument.

In this paper we state a general identity for the moments of Skorohod integrals, which will allow us in particular to recover the result of [3] by a direct proof and to obtain a recurrence relation for the moments of second order Wiener integrals.

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We refer to [1] and [4] for the notation recalled in this section. Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space $(W,\mu)$ with $W = \mathcal{C}_0(\mathbb{R}_+,\mathbb{R}^d)$. For any separable Hilbert space $X$, consider the Malliavin derivative $D$ with values in $H = L^2(\mathbb{R}_+, X\otimes\mathbb{R}^d)$, defined by

$$D_tF = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\partial_i f(B_{t_1},\ldots,B_{t_n}), \qquad t\in\mathbb{R}_+,,$$

for $F$ of the form

$$F = f(B_{t_1},\ldots,B_{t_n}), \tag{1.1}$$

$f\in\mathcal{C}_b^\infty(\mathbb{R}^n,X)$, $t_1,\ldots,t_n\in\mathbb{R}_+$, $n\geq 1$. Let $I\!D_{p,k}(X)$ denote the completion of the space of smooth $X$-valued random variables under the norm

$$\|u\|_{I\!D_{p,k}(X)} = \sum_{l=0}^k \|D^l u\|_{L^p(W,X\otimes H^{\otimes l})}, \qquad p>1,$$

where $X\otimes H$ denotes the completed symmetric tensor product of $X$ and $H$. For all $p,q>1$ such that $p^{-1}+q^{-1}=1$ and $k\geq 1$, let

$$\delta : I\!D_{p,k}(X\otimes H)\to I\!D_{q,k-1}(X)$$

denote the Skorohod integral operator adjoint of

$$D : I\!D_{p,k}(X)\to I\!D_{q,k-1}(X\otimes H),$$

with

$$E[\langle F,\delta(u)\rangle_X] = E[\langle DF,u\rangle_{X\otimes H}], \qquad F\in I\!D_{p,k}(X), \quad u\in I\!D_{q,k}(X\otimes H).$$

Recall that $\delta(u)$ coincides with the Itô integral of $u\in L^2(W;H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t,$$

when $u$ is square-integrable and adapted with respect to the Brownian filtration.

Each element of $X\otimes H$ is naturally identified to a linear operator from $H$ to $X$ via

$$(a\otimes b)c = a\langle b,c\rangle, \qquad a\otimes b\in X\otimes H, \quad c\in H.$$

For $u\in I\!D_{2,1}(H)$ we identify $Du = (D_tu_s)_{s,t\in\mathbb{R}_+}$ to the random operator $Du : H\to H$ almost surely defined by

$$(Du)v(s) = \int_0^\infty (D_tu_s)v_t dt, \quad s\in\mathbb{R}_+, \quad v\in L^2(W;H),$$

and define its adjoint $D^*u$ on $H \otimes H$ as

$$(D^*u)v(s) = \int_0^\infty (D_s^\dagger u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$.

Recall the Skorohod [2] isometry

$$E[\delta(u)^2] = E[\langle u, u \rangle_H] + E\left[\operatorname{trace}(Du)^2\right], \qquad u \in \mathbb{D}_{2,1}(H), \tag{1.2}$$

with

$$\begin{aligned} \operatorname{trace}(Du)^2 &= \langle Du, D^*u \rangle_{H \otimes H} \\ &= \int_0^\infty \int_0^\infty \langle D_s u_t, D_t^\dagger u_s \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} ds dt, \end{aligned}$$

and the commutation relation

$$D\delta(u) = u + \delta(D^*u), \qquad u \in \mathbb{D}_{2,2}(H). \tag{1.3}$$

# 2 Main results

First we state a moment identity for Skorohod integrals, which will be proved in Section 3.

**Theorem 2.1** *For any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have*

$$E[(\delta(u))^{n+1}] = \sum_{k=1}^n \frac{n!}{(n-k)!} \tag{2.1}$$

$$E\left[ (\delta(u))^{n-k} \left( \langle (Du)^{k-1} u, u \rangle_H + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^k \frac{1}{i} \langle (Du)^{k-i} u, D \operatorname{trace}(Du)^i \rangle_H \right) \right],$$

*where*

$$\operatorname{trace}(Du)^{k+1} = \int_0^\infty \cdots \int_0^\infty \langle D_{t_{k-1}}^\dagger u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_0 \cdots dt_k.$$

For $n = 1$ the above identity coincides with the Skorohod isometry (1.2).

In particular we obtain the following immediate consequence of Theorem 2.1. Recall that $\operatorname{trace}(Du)^k = 0$, $k \geq 1$, when the process $u$ is adapted with respect to the Brownian filtration.

**Corollary 2.2** *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u\rangle_H$ is deterministic and*

$$\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i} u, D \operatorname{trace}(Du)^i \rangle_H = 0, \quad a.s., \qquad 1 \leq k \leq n. \tag{2.2}$$

*Then $\delta(u)$ has the same first $n+1$ moments as the centered Gaussian distribution with variance $\langle u, u\rangle_H$.*

*Proof.* The relation $D\langle u, u\rangle = 2(D^*u)u$ shows that

$$\langle (D^{k-1}u)u, u\rangle = \langle (D^*u)^{k-1}u, u\rangle = \frac{1}{2}\langle u, (D^*)^{k-2} D\langle u, u\rangle\rangle = 0, \qquad k \geq 2, \tag{2.3}$$

when $\langle u, u\rangle$ is deterministic, $u \in \mathbb{D}_{2,1}(H)$. Hence under Condition (2.2), Theorem 2.1 yields

$$E[(\delta(u))^{n+1}] = n\langle u, u\rangle_H E\left[(\delta(u))^{n-1}\right],$$

and by induction

$$E[(\delta(u))^{2m}] = \frac{(2m)!}{2^m m!}\langle u, u\rangle_H^m, \qquad 0 \leq 2m \leq n+1,$$

and $E[(\delta(u))^{2m+1}] = 0$, $0 \leq 2m \leq n$, while $E[\delta(u)] = 0$ for all $u \in \mathbb{D}_{2,1}(H)$. □

We close this section with some applications.

1. Random rotations

   As a consequence of Corollary 2.2 we recover Theorem 2.1-b) of [3], i.e. $\delta(Rh)$ has a centered Gaussian distribution with variance $\langle h, h\rangle_H$ when $u = Rh$, $h \in H$, and $R$ is a random mapping with values in the isometries of $H$, such that $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$ and $\operatorname{trace}(DRh)^{k+1} = 0$, $k \geq 1$. Note that in [3] the condition $Rh \in \cap_{p>1,k\geq 2}\mathbb{D}_{p,k}(H)$ is assumed instead of $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$.

2. Second order Wiener integrals

   Let $d = 1$. The second order Wiener integral $I_2(f_2)$ of a symmetric function $f_2 \in H \otimes H = L^2(\mathbb{R}_+^2)$ can be written as $I_2(f_2) = \delta(u)$ with $u_t = \delta(f_2(\cdot, t))$, $t \in \mathbb{R}_+$. Its law is infinitely divisible with Lévy measure

   $$\nu(dy) = \mathbf{1}_{\{y>0\}} \sum_{k;a_k>0} \frac{1}{2|y|} e^{-y/a_k} dy + \mathbf{1}_{\{y<0\}} \sum_{k;a_k<0} \frac{1}{2|y|} e^{-y/a_k} dy, \tag{2.4}$$

   when $f_2$ is decomposed as

   $$f_2 = \frac{1}{2}\sum_{k=0}^{\infty} a_k h_k \otimes h_k$$

in a complete orthonormal basis $(h_k)_{k\in\mathbb{N}}$ of $H$. Letting

$$g_2^{(k+1)}(s,t) = \int_{\mathbb{R}^k} f_2(s,t_1)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)f_2(t_k,t)dt_1\cdots dt_k,$$

we have $\operatorname{trace}(Du)^{k+1} = \int_{\mathbb{R}^2} g_2^{(k+1)}(s,t)dsdt$, and using the relation

$$\delta(f_1)\delta(g_1) = I_2(f_1\otimes g_1) + \langle f_1, g_1\rangle_H, \qquad f_1, g_1 \in H,$$

we get

$$\begin{aligned}
\langle (Du)^{k-1}u, u\rangle_H &= \int_{\mathbb{R}^{k-1}} \delta(f_2(\cdot,t_1))f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)\delta(f_2(\cdot,t_k))dt_1\cdots dt_k\\
&= \int_{\mathbb{R}^{k-1}} I_2(f_2(\cdot,t_1)\otimes f_2(\cdot,t_k))f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)dt_1\cdots dt_k\\
&\quad + \int_{\mathbb{R}^{k-1}} f_2(t_0,t_1)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)f_2(t_k,t_0)dt_0\cdots dt_k\\
&= I_2(g_2^{(k+1)}) + \operatorname{trace}(Du)^{k+1},
\end{aligned}$$

hence Theorem 2.1 yields the recurrence relation

$$\begin{aligned}
E[(I_2(f_2))^{n+1}] &= \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[(I_2(f_2))^{n-k}(I_2(g_2^{(k+1)}) + 2\operatorname{trace}(Du)^{k+1})\right]\\
&= 2\sum_{k=0}^{n-1}\frac{n!}{k!}\int_{\mathbb{R}^2} g_2^{(n-k+1)}(s,t)dsdtE\left[(I_2(f_2))^k\right]\\
&\quad + \sum_{k=0}^{n-1}\sum_{l=1}^{k}\frac{(-1)^{k+1-l}n!}{(k)!(k+1)!}\binom{k}{l} l^{k+1}E[(I_2(f_2))^{k+1}]\\
&\quad + \sum_{k=0}^{n-1}\sum_{l=1}^{k+1}\frac{(-1)^{k+1-l}n!}{(k)!(k+1)!}\binom{k}{l-1}E\left[I_2\left((l-1)f_2 + g_2^{(n-k+1)}\right)^{k+1}\right],
\end{aligned}$$

for the computation of the moments of second order Wiener integrals, by polarisation of $(I_2(f_2))^{n-k}I_2(g_2^{(n-k+1)})$.

# 3 Proofs

In the sequel, all scalar products will be simply denoted by $\langle\cdot,\cdot\rangle$.

We will need the following lemma.

**Lemma 3.1** *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$. Then for all $1 \leq k \leq n$ we have*

$$E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] - (n-k)E\left[(\delta(u))^{n-k-1}\langle (Du)^k u, D\delta(u)\rangle\right]$$
$$= E\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k}\frac{1}{i}\langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle\right)\right].$$

*Proof.* We have $(Du)^{k-1}u \in \mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}(\mathbb{R})$, and using Relation (1.3) we obtain

$$\begin{aligned}
&E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u+\delta(D^*u)\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, \delta(Du)\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + E\left[\langle D^*u, D((\delta(u))^{n-k}(Du)^{k-1}u)\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + E\left[(\delta(u))^{n-k}\langle D^*u, D((Du)^{k-1}u)\rangle\right]\\
&\quad + E\left[\langle D^*u, ((Du)^{k-1}u)\otimes D(\delta(u))^{n-k}\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right]\\
&\quad + (n-k)E\left[(\delta(u))^{n-k-1}\langle D^*u, ((Du)^{k-1}u)\otimes D\delta(u)\rangle\right]\\
&= E\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right]\\
&\quad + (n-k)E\left[(\delta(u))^{n-k-1}\langle (Du)^k u, D\delta(u)\rangle\right].
\end{aligned}$$

Next,

$$\begin{aligned}
\langle D^*u, D((Du)^{k-1}u)\rangle &= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}}u_{t_k}, D_{t_k}(D_{t_{k-2}}u_{t_{k-1}}\cdots D_{t_0}u_{t_1}u_{t_0})\rangle dt_0\cdots dt_k\\
&= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}}u_{t_k}, D_{t_{k-2}}u_{t_{k-1}}\cdots D_{t_0}u_{t_1}D_{t_k}u_{t_0}\rangle dt_0\cdots dt_k\\
&\quad + \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}}u_{t_k}, D_{t_k}(D_{t_{k-2}}u_{t_{k-1}}\cdots D_{t_0}u_{t_1})u_{t_0}\rangle dt_0\cdots dt_k\\
&= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2}\int_0^\infty \cdots \int_0^\infty\\
&\quad \langle D^\dagger_{t_{k-1}}u_{t_k}, D_{t_k}u_{t_{k+1}}\cdots D_{t_{i+1}}u_{t_{i+2}}(D_{t_i}D_{t_k}u_{t_{i+1}})D_{t_{i-1}}u_{t_i}\cdots D_{t_0}u_{t_1}u_{t_0}\rangle dt_0\cdots dt_k\\
&= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2}\frac{1}{k-i}\int_0^\infty \cdots \int_0^\infty\\
&\quad \langle D_{t_i}\langle D^\dagger_{t_{k-1}}u_{t_k}, D_{t_k}u_{t_{k+1}}\cdots D_{t_{i+1}}u_{t_{i+2}}D_{t_k}u_{t_{i+1}}\rangle, D_{t_{i-1}}u_{t_i}\cdots D_{t_0}u_{t_1}u_{t_0}\rangle dt_0\cdots dt_k\\
&= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2}\frac{1}{k-i}\langle (Du)^i u, D\operatorname{trace}(Du)^{k-i}\rangle.
\end{aligned}$$

□

*Proof of Theorem 2.1.* We decompose

$$E[(\delta(u))^{n+1}] = E[\langle u, D(\delta(u))^n\rangle] = nE[(\delta(u))^{n-1}\langle u, D\delta(u)\rangle]$$
$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left( E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] - (n-k)E\left[(\delta(u))^{n-k-1}\langle (Du)^{k}u, D\delta(u)\rangle\right]\right),$$

as a telescoping sum and then apply Lemma 3.1, which yields (2.1). □

Finally we state some other consequences of Theorem 2.1.

**Corollary 3.2** *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$, and assume that*

$$\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i}\langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle = 0, \qquad 1 \leq k \leq n. \tag{3.1}$$

*Then we have*

$$E[(\delta(u))^{n+1}] = \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right].$$

**Corollary 3.3** *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u\rangle$ is deterministic. We have*

$$E[(\delta(u))^{n+1}] = n\langle u, u\rangle E\left[(\delta(u))^{n-1}\right]$$
$$+ \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[(\delta(u))^{n-k}\left(\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k}\frac{1}{i}\langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle\right)\right].$$

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