Probabilistic representations for the solutions of nonlinear PDEs with fractional Laplacians

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(joint work with Guillaume Penent)

This talk presents tree-based probabilistic algorithms for the existence of solutions of nonlinear fractional PDEs with their numerical implementation.

1. PARABOLIC CASE

Given $\eta : (0, \infty) \to [0, \infty)$ a Bernstein function, consider the (nonlocal) semilinear PDE
\[
\begin{aligned}
&\frac{\partial u}{\partial t}(t, x) - \eta(-\Delta/2)u(t, x) + f(t, x, u(t, x), \frac{\partial u}{\partial x_1}(t, x), \ldots, \frac{\partial u}{\partial x_m}(t, x)) = 0, \\
u(T, x) = \phi(x), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\end{aligned}
\]
where $f(t, x, y, z_1, \ldots, z_m)$ is a polynomial nonlinearity given by
\[
f(t, x, y, z_1, \ldots, z_m) = \sum_{l=(l_0, \ldots, l_m) \in \mathcal{L}_m} c_l(t, x) y^{l_0} z_1^{l_1} \cdots z_m^{l_m},
\]
and $\mathcal{L}_m \subset \mathbb{N}^{m+1}$ is finite. By choosing $\eta(\lambda) := (2\lambda)^{\alpha/2}$, this setting includes the case of the standard fractional Laplacian $\Delta_\alpha$. We assume that the coefficients $c_l(t, x)$ are uniformly bounded and that the terminal condition $\phi$ is Lipschitz and bounded on $\mathbb{R}^d$.

Theorem 1.1. ([8]) Suppose that $\int_{\lambda_0}^{\infty} \frac{1}{\sqrt{\lambda \eta(\lambda)}} d\lambda < \infty$ for some $\lambda_0 > 0$. Then, there exists a small enough $T > 0$ such that the PDE
\[
u(t, x) = \int_{\mathbb{R}^d} \varphi(T - t, y - x) \phi(y) dy
+ \sum_{l=(l_0, \ldots, l_m) \in \mathcal{L}_m} \int_t^T \int_{\mathbb{R}^d} \varphi(s - t, y - x) c_l(s, y) u^{l_0}(s, y) \prod_{j=1}^m \left( \frac{\partial u}{\partial y_j}(s, y) \right)^{l_j} dy ds,
\]
admits an integral solution on $[0, T]$.

To prove the above result, for each $i = 0, 1, \ldots, d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(T_{t,x,i})$ of a random tree $T_{t,x,i}$ driven by a subordinated Lévy process $(Z_t)_{t\in\mathbb{R}_+} := (B_{S_t})_{t\in\mathbb{R}_+}$, where $(B_t)_{t\in\mathbb{R}_+}$ is a multidimensional Brownian motion, such that we have the representations
\[
u(t, x) := \mathbb{E}[\mathcal{H}_\phi(T_{t,x,0})], \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]
and
\[
\frac{\partial u}{\partial x_i}(t, x) := \mathbb{E}[\mathcal{H}_\phi(T_{t,x,i})], \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad i = 1, \ldots, d.
\]
Dealing with gradient terms requires to perform an integration by parts, which is made possible using the Gaussian density of $B_t$ in the subordination $Z_t := B_{S_t}$, as done in [7] in the case of stable processes with $\eta(\lambda) := (2\lambda)^{\alpha/2}$. Related local and
global-in-time existence results have been obtained by deterministic arguments under more technical conditions in e.g. [5], [6].

**Corollary 1.2.** ([8]) Taking $\eta(\lambda) := (2\lambda)^{\alpha/2}$ with $\alpha \in (1, 2)$, under the above assumptions there exists a small enough $T > 0$ such that the PDE

$$\begin{cases}
\frac{\partial u}{\partial t}(t, x) - (-\Delta)^{\alpha/2}u(t, x) + f(t, x, u(t, x), \frac{\partial u}{\partial x_1}(t, x), \ldots, \frac{\partial u}{\partial x_m}(t, x)) = 0, \\
u(T, x) = \phi(x), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\end{cases}$$

with $\alpha$-fractional Laplacian admits an integral solution on $[0, T]$.

2. **Elliptic case**

We consider the class of semilinear elliptic PDEs on the open ball $B(0, R)$ of radius $R > 0$ in $\mathbb{R}^d$, of the form

$$\begin{cases}
\Delta_\alpha u(x) + f(x, u(x), \nabla u(x)) = 0, & x \in B(0, R), \\
u(x) = \phi(x), & x \in \mathbb{R}^d \setminus B(0, R),
\end{cases}$$

(1)

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function on $\mathbb{R}^d \setminus B(0, R)$, $\Delta_\alpha$ denotes the fractional Laplacian with parameter $\alpha \in (0, 2)$, and $f(x, y, z)$ is a polynomial nonlinearity term. Next, we provide existence results and probabilistic representations for the solution of (1) under boundedness and smoothness conditions on polynomial coefficients. Our approach allows us to take into account gradient nonlinearities, which has not been done by deterministic finite difference methods, see e.g. [4].

**Theorem 2.1.** ([9], [10]) Let $d \geq 2$ and $\alpha \in (1, 2)$, assume that the boundary condition $\phi$ belongs to $H^\alpha(\mathbb{R}^d)$ and is bounded on $\mathbb{R}^d \setminus B(0, R)$. Under the above assumptions, the semilinear elliptic PDE

$$\begin{cases}
\Delta_\alpha u(x) + f(x, u(x), \nabla u(x)) = 0, & x \in B(0, R), \\
u(x) = \phi(x), & x \in \mathbb{R}^d \setminus B(0, R),
\end{cases}$$

admits a viscosity solution in $C^1(B(0, R)) \cap C^0(\overline{B}(0, R))$ provided that $R$ and $|c_l|_\infty$, $l \in \mathcal{L}_m$, are sufficiently small.

Existence of solutions are obtained through a probabilistic representation of the form $u(x) := \mathbb{E}[\mathcal{H}_\phi(T_{x,0})]$, $x \in B(0, R)$, where $\mathcal{H}_\phi(T_{x,0})$ is a functional of a random branching tree $T_{x,0}$. For each $i = 0, 1, \ldots, d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(T_{x,i})$ of a random tree $T_{x,i}$ such that we have the representation

$$u(x) = \mathbb{E}[\mathcal{H}_\phi(T_{x,0})], \quad x \in \mathbb{R}^d.$$  (2)

The main difficulty in the proof is to show the uniform integrability required on $\mathcal{H}(T_{x,i})$ for $\mathbb{E}[\mathcal{H}(T_{x,i})]$ to be continuous in $x \in \mathbb{R}^d$ is satisfied for $\alpha \in (1, 2)$, as required in the framework of viscosity solutions. For this, we extend arguments of
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[1] from the standard Laplacian $\Delta$ and Brownian motion to the fractional Laplacian $\Delta^\alpha := (-\Delta)^{\alpha/2}$ and its associated stable process, and use bounds on the fractional Green and Poisson kernel and stable process hitting times from [3], [2].

As an example, consider the elliptic PDE with nonlinear gradient term

$$\Delta^\alpha u(x) + \Psi_{k,\alpha}(x) + (2k + \alpha)^2|x|^4(1 - |x|^2)^{2k+\alpha} + ((1 - |x|^2)x \cdot \nabla u(x))^2 = 0,$$

$x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,R)$, and explicit solution $u(x) = \Phi_{k,\alpha}(x) = (1 - |x|^2)^{k+\alpha/2}$, $x \in \mathbb{R}^d$. Numerical estimates of (2) by the Monte Carlo method are presented in the figure below.

(A) Numerical solution of (3) with $k = 0$.  (B) Numerical solution of (3) with $k = 2$.

**Figure 1.** Numerical solutions with $d = 10$ and $\alpha = 1.75$.

**References**


