Semilinear fractional elliptic PDEs with gradient nonlinearities on open balls: existence of solutions and probabilistic representation

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Abstract

We provide sufficient conditions for the existence of classical solutions of fractional semilinear elliptic PDEs of index $\alpha \in (1,2)$ with polynomial gradient nonlinearities on d-dimensional balls, $d \geqslant 2$. Our approach uses a tree-based probabilistic representation of solutions and their partial derivatives using α-stable branching processes, and allows us to take into account gradient nonlinearities not covered by deterministic finite difference methods so far. In comparison with the existing literature on the regularity of solutions, no polynomial order condition is imposed on gradient nonlinearities. Numerical illustrations demonstrate the accuracy of the method in dimension $d = 10$, solving a challenge encountered with the use of deterministic finite difference methods in high-dimensional settings.

Keywords: Elliptic PDEs, semilinear PDEs, fractional Laplacian, gradient nonlinearities, stable processes, branching processes, Monte-Carlo method.

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1 Introduction

The study of solutions of nonlocal and fractional elliptic partial differential equations (PDEs) is an active research topic which has attracted significant attention over the past decades. In

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the case of the classical (local) Laplacian, viscosity solutions of fully nonlinear second-order elliptic PDEs have been constructed in Ishii (1989) by the Perron method.

On the other hand, nonlocal elliptic PDEs can be solved using weak solutions, see Definition 2.1 in Ros-Oton and Serra (2014), or viscosity solutions, see Servadei and Valdinoci (2014) and Remark 2.11 in Ros-Oton and Serra (2014). Weak solutions can be obtained from the Riesz representation or Lax-Milgram theorems as in Felsinger et al. (2015), Ros-Oton (2016). See also Barles et al. (2008) for the use of the Perron method, and Bony et al. (1968) for semi-group methods applied to second-order elliptic integro-differential PDEs.

Given $d \geqslant 1$, let

$$\Delta_\alpha u = -(-\Delta)^{\alpha/2} u = \frac{4^{\alpha/2}\Gamma(d/2+\alpha/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \lim_{r\to 0^+} \int_{\mathbb{R}^d\setminus B(x,r)} \frac{u(\,\cdot\,+z)-u(z)}{|z|^{d+\alpha}} dz,$$

denote the fractional Laplacian on $\mathbb{R}^d$ with parameter $\alpha \in (0,2)$, see, e.g., Kwaśnicki (2017), where $\Gamma(p) := \int_0^\infty e^{-\lambda x}\lambda^{p-1} d\lambda$ is the gamma function and $|z|$ is the Euclidean norm of $z \in \mathbb{R}^d$.

For problems of the form

$$\Delta_\alpha u(x) + f(x) = 0,$$

with $u = \phi$ on $\mathbb{R}^d \setminus D$, where D is an open bounded domain in $\mathbb{R}^d$, the Hölder regularity of viscosity solution has been proved in Kriventsov (2013) when D is a ball and f, ϕ are bounded functions. Existence of viscosity solutions has been derived in Servadei and Valdinoci (2014) under smoothness assumptions on f, ϕ, and the existence of classical Hölder regular solutions has been proved in Serra (2015) when ϕ is bounded continuous and f is Hölder continuous. See also Felsinger et al. (2015), resp. Mou (2017), for the existence of weak solutions, resp. viscosity solutions, with nonlocal operators. Regarding problems of the form

$$\Delta_\alpha u(x) + f(x, u(x)) = 0,$$

existence of non trivial solutions with $u = 0$ outside an open bounded domain D with Lipschitz boundary in $\mathbb{R}^d$ has been considered in Servadei and Valdinoci (2012) using the mountain pass theorem when f is a Carathéodory function on $D \times \mathbb{R}^d$ satisfying a polynomial growth condition of order $m \in (1, (d+\alpha)/(d-\alpha))$.

The regularity of viscosity solutions of semilinear elliptic PDEs of the form

$$\Delta_\alpha u(x) - b(x)\|\nabla u(x)\|_{\mathbb{R}^d}^{\kappa+\tau} - \|\nabla u(x)\|_{\mathbb{R}^d}^{r} = 0, \qquad x \in D, \tag{1.1}$$

where D is an open domain of $\mathbb{R}^d$, b is in the space $C^\tau(\mathbb{R}^d)$ of τ-Hölder continuous functions on $\mathbb{R}^d$ for some $\tau \in (0,1)$, has been considered in § 4.3 in Barles et al. (2011). Namely, from Theorem 3.1 therein, if b is in $C^\tau(\mathbb{R}^d)$ and $\kappa, r \in (0,2)$, then any bounded viscosity solution u of (1.1) is β-Hölder continuous for small enough β, see also § 4.1.2 of Barles et al. (2012) for Lipschitz regularity in the case of mixed local and fractional Laplacians.

More recently, the Lipschitz regularity of viscosity solutions of

$$\Delta_\alpha u(x) + f(x, \nabla u(x)) = 0$$

on $D = B(0,R)$ the open ball of radius $R > 0$ in $\mathbb{R}^d$, has been obtained in Theorem 2.1 of Biswas and Topp (2024), provided that $f \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies a power-type growth condition of order $m \in (0, \alpha+1)$ in $\nabla u(x)$, while this bound can be lifted under an extra coercivity condition on H.

In this paper, we consider the class of semilinear elliptic problems on $B(0,R)$ of the form

$$\begin{cases} \Delta_\alpha u(x) + f(x, u(x), \nabla u(x)) = 0, & x \in B(0,R), \\ u(x) = \phi(x), & x \in \mathbb{R}^d \backslash B(0,R), \end{cases} \tag{1.2}$$

where

- $f(x,y,z)$ is a polynomial nonlinearity on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, of the form

$$f(x,y,z) = \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} c_l(x) y^{l_0} \prod_{i=1}^m (b_i(x)\cdot z)^{l_i}, \tag{1.3}$$

  where $\mathcal{L}_m$ is a finite subset of $\mathbb{N}^{m+1}$ for some $m \geqslant 0$, and $(c_l(x))_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m}$, $(b_i(x))_{i=1,\ldots,m}$ are bounded continuous functions of $x \in \mathbb{R}^d$, with $x \cdot z := x_1 z_1 + \cdots + x_d z_d$,

- $\phi : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function on $\mathbb{R}^d \setminus B(0,R)$.

Using a probabilistic approach, we prove the existence of regular viscosity solutions to (1.2) under the following conditions. We note that, in comparison to the literature quoted above on the regularity of solutions, no coercivity or maximum growth order condition in z is imposed on $f(x,y,z)$.

Assumption (A)

1) The boundary condition ϕ belongs to the fractional Sobolev space

$$H^\alpha(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \ : \ \frac{|u(x)-u(y)|}{|x-y|^{d/2+\alpha/2}} \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \right\}$$

and is bounded on $\mathbb{R}^d \setminus B(0,R)$.

2) *The coefficients* $c_l(x)$, $l \in \mathcal{L}_m$, *are uniformly bounded functions, i.e., we have*

$$\|c_l\|_\infty := \sup_{x\in B(0,R)} |c_l(x)| < \infty, \qquad l = (l_0, \ldots, l_m) \in \mathcal{L}_m.$$

3) *The coefficients* $b_i(x)$, $i = 0, \ldots, m$, *are such that*

$$\sup_{x\in B(0,R)} \frac{|b_i(x)|}{R-|x|} < \infty, \qquad i = 1, \ldots, m.$$

Theorem 1.1 is the main result of this paper. It is implied by Theorem 4.2, in which we prove the existence of a classical solution for fractional elliptic problems of the form (1.2).

Theorem 1.1 *Let* $\alpha \in (1,2)$ *and* $d \geqslant 2$. *Under Assumption (A), the semilinear elliptic PDE (1.2) admits a classical solution in* $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ *for some* $\epsilon > 0$, *provided that* R *and* $\max_{l\in\mathcal{L}_m} \|c_l\|_\infty$ *are sufficiently small.*

Our method of proof relies on the probabilistic representation of PDE solutions using stochastic branching processes, as introduced in Skorokhod (1964) and Ikeda et al. (1968-1969). Probabilistic representations have been applied to the blow-up and existence of solutions for parabolic PDEs in Nagasawa and Sirao (1969), López-Mimbela (1996). They have also been recently extended in Agarwal and Claisse (2020) to treat polynomial nonlinearities in gradient terms in elliptic PDEs with (local) diffusion generators, following the approach of Henry-Labordère et al. (2019) in the parabolic case. In this construction, gradient terms are associated to tree branches to which a Malliavin integration by parts is applied. In Penent and Privault (2022), this approach has been extended to the treatment of nonlocal pseudo-differential operators of the form $-\eta(-\Delta/2)$ using random branching trees constructed from a Lévy subordinator, with application to parabolic PDEs with fractional Laplacians

The existence of viscosity solutions in Theorem 1.1 is obtained through a probabilistic representation of the form

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \qquad x \in B(0,R), \tag{1.4}$$

where $\mathcal{H}_\phi(\mathcal{T}_{x,0})$, see (4.2), is a functional of a random branching tree $\mathcal{T}_{x,0}$ started at $x \in \mathbb{R}^d$, and constructed in Section 3. The proof of Theorem 1.1 also makes use of existence results for nonlinear elliptic PDEs with fractional Laplacians derived in Penent and Privault (2023), see Theorem 1.2 and Proposition 3.5 therein.

To prove Theorem 1.1, in Proposition 4.3 we construct for each $i = 0, \ldots, m$ a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ of a random tree $\mathcal{T}_{x,i}$ such that the probabilistic representation

$$u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\big], \quad x \in \mathbb{R}^d,$$

yields a viscosity solution of (1.2) in $C^1(B(0,R)) \cap C^0(\overline{B}(0,R))$, where the gradients $b_i(x) \cdot \nabla u(x)$, $x \in B(0,R)$, $i = 0, \ldots, m$, can be represented as

$$u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\big], \quad b_i(x) \cdot \nabla u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,i})\big], \quad x \in \mathbb{R}^d,$$

under integrability assumptions on $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$.

Then, in Proposition 4.6 we show that for any $d \geqslant 2$ and $p \geqslant 1$, $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ is bounded in $L^p(\Omega)$ uniformly in $x \in B(0,R)$, and therefore uniformly integrable, $i = 0, \ldots, m$. We conclude the proof of Theorem 1.1 by showing, using results of Kriventsov (2013) and Serra (2015), that the $\mathcal{C}^1$ viscosity solution of (1.2) is in $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ for some $\epsilon > 0$.

For this, we extend arguments of Agarwal and Claisse (2020) from the standard Laplacian Δ and Brownian motion to the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ and its associated stable process. There are, however, significant differences from the Brownian case. In particular, in the stable setting we rely on sharp gradient estimates for fractional Green and Poisson kernel proved in Bogdan et al. (2002), and on integrability results for stable process hitting times, see Bogdan et al. (2015). The behavior of the negative moments of stable processes, see (2.5), requires a more involved treatment of integrability in small time when showing the boundedness of $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ in $L^p(\Omega)$, for $p \geqslant 1$.

In addition, we present a Monte Carlo numerical implementation of the probabilistic representation (1.4) on specific examples. In comparison with deterministic finite difference methods, see e.g. § 6.3 of Huang and Oberman (2014) for the one-dimensional Dirichlet problem, our approach allows us to take into account gradient nonlinearities. We also note that our tree-based Monte Carlo implementation applies to high-dimensional problems, see

Figures 4 and 6 in dimension $d = 10$, whereas the application of deterministic finite difference methods to the fractional Laplacian in higher dimensions is challenging, see e.g. Huang and Oberman (2014), page 3082.

This paper is organized as follows. Section 3 presents the description of the branching mechanism, following the preliminaries on stable processes and kernel introduced in Section 2. In Section 4 we state and prove our main existence result, Theorem 4.2, for the probabilistic representation of the solution of (1.2). Section 5 presents a Monte Carlo numerical implementation of our method on specific examples.

2 Preliminaries and notation

Before proceeding further, we recall some preliminary results on fractional Laplacians on the ball $B(0, R)$ in $\mathbb{R}^d$.

Poisson and Green kernels

Given $(X_t)_{t\geqslant 0}$ an $\mathbb{R}^d$-valued α-stable process, $\alpha \in (0, 2)$, we consider the process

$$X_{t,x} := x + X_t, \qquad t \in \mathbb{R}_+,$$

started at $x \in \mathbb{R}^d$, see e.g. § 1.3.1 in Applebaum (2009), and the first hitting time

$$\tau_R(x) := \inf \big\{ t \geqslant 0,\ X_{t,x} \notin B(0, R) \big\}$$

of $\mathbb{R}^d \setminus B(0, R)$ by $(X_{t,x})_{t\geqslant 0}$. Note that by the bound (1.4) in Bogdan et al. (2015) we have $\mathbb{E}[\tau_R(x)] < \infty$, and therefore $\tau_R(x)$ is almost surely finite for all $x \in B(0, R)$. The Green kernel $G_R(x, y)$ satisfies

$$\mathbb{E}\left[\int_0^{\tau_R(x)} f(X_{t,x}) dt \right] = \int_{B(0,R)} G_R(x, y) f(y) dy, \quad x \in B(0, R), \tag{2.1}$$

for f a nonnegative measurable function on $\mathbb{R}^d$. If $\alpha \in (0, 2) \setminus \{d\}$, we also have

$$G_R(x, y) = \frac{\kappa_\alpha^d}{|x - y|^{d-\alpha}} \int_0^{r_0(x,y)} \frac{t^{\alpha/2-1}}{(1 + t)^{d/2}} dt, \qquad x, y \in B(0, R),$$

see Theorem 3.1 in Bucur (2016), where

$$r_0(x, y) := \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2 |x - y|^2} \quad \text{and} \quad \kappa_\alpha^d := \frac{2^{-\alpha}\Gamma(d/2)}{\pi^{d/2}(\Gamma(\alpha/2))^2}.$$

The Poisson kernel $P_R(x,y)$ of the harmonic measure $\mathbb{P}^x\big(X_{\tau_R(x),x} \in dy\big)$ satisfies

$$\mathbb{E}\big[f\big(X^x_{\tau_R(x)}\big)\big] = \int_{\mathbb{R}^d \setminus B(0,R)} P_R(x,y) f(y) dy, \quad x \in B(0,R), \tag{2.2}$$

for f a nonnegative measurable function on $\mathbb{R}^d$, and is given by

$$P_R(x,y) = \mathcal{A}(d,-\alpha) \int_{B(0,R)} \frac{G_R(x,z)}{|y-z|^{d+\alpha}} dz$$

where

$$\mathcal{A}(d,-\alpha) := \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|}.$$

In particular, when $|x| < R$ and $|y| > R$ we have

$$P_R(x,y) = \frac{\mathcal{C}(\alpha,d)}{|x-y|^d} \left(\frac{R^2 - |x|^2}{|y|^2 - R^2} \right)^{\alpha/2},$$

with $\mathcal{C}(\alpha,d) := \Gamma(d/2) \pi^{-d/2-1} \sin(\pi\alpha/2)$. In addition, we have the bounds

$$|\nabla_x P_R(x,y)| \leqslant (d+\alpha) \frac{P_R(x,y)}{R - |x|}, \quad x \in B(0,R), \ y \in \mathbb{R}^d \setminus \overline{B}(0,R), \tag{2.3}$$

where $\overline{B}(0,R)$ denotes closed ball of radius $R > 0$ in $\mathbb{R}^d$, see Lemma 3.1 in Bogdan et al. (2002), and

$$|\nabla_x G_R(x,y)| \leqslant d \frac{G_R(x,y)}{\min(|x-y|, R-|x|)}, \quad x, y \in B(0,R), \ x \neq y, \tag{2.4}$$

see Corollary 3.3 in Bogdan et al. (2002).

Moments of stable processes

In the sequel we will need to estimate the negative moments $\mathbb{E}[|X_t|^{-p}]$ of an α-stable process $(X_t)_{t \geqslant 0}$ represented as the subordinated Brownian motion $(X_t)_{t\geqslant 0} = (B_{S_t})_{t \geqslant 0}$, where the subordinator $(S_t)_{t\geqslant 0}$ is an $\alpha/2$-stable process with Laplace exponent $\eta(\lambda) = (2\lambda)^{\alpha/2}$, i.e.

$$\mathbb{E}\big[e^{-\lambda S_t}\big] = e^{-t(2\lambda)^{\alpha/2}}, \qquad \lambda, \, t \geqslant 0,$$

see, e.g., Theorem 1.3.23 and pages 55-56 in Applebaum (2009). Using the fact that $B_{S_t}/\sqrt{S_t}$ follows the normal distribution $\mathcal{N}(0,1)$ given S_t, for $d \geqslant 1$ and $p \in (0,d)$ we have

$$\mathbb{E}[|X_t|^{-p}] \;=\; \mathbb{E}[|B_{S_t}|^{-p}]$$

$$
\begin{aligned}
&= \mathbb{E}\left[S_t^{-p/2}\mathbb{E}\left[\frac{S_t^{p/2}}{|B_{S_t}|^p}\,\middle|\, S_t\right]\right] \\
&= \mathbb{E}\left[S_t^{-p/2}\int_{\mathbb{S}^{d-1}}\mu_d(d\sigma)\int_0^\infty r^{d-1-p}\frac{e^{-r^2/2}}{(2\pi)^{d/2}}dr\right] \\
&= 2\frac{2^{(d-p-2)/2}}{2^{d/2}\Gamma(d/2)}\Gamma((d-p)/2)\mathbb{E}\big[S_t^{-p/2}\big] \\
&= \frac{C_{\alpha,d,p}}{t^{p/\alpha}}, \qquad t>0, \quad \alpha\in(1,2),
\end{aligned} \tag{2.5}
$$

where μ_d denotes the surface measure on the d-dimensional sphere $\mathbb{S}^{d-1}$,

$$
C_{\alpha,d,p} := 2^{1-p}\frac{\Gamma(p/\alpha)\Gamma((d-p)/2)}{\alpha\Gamma(p/2)\Gamma(d/2)},
$$

and we used the relation $\mathbb{E}\big[S_t^{-p}\big] = \alpha^{-1}2^{1-p}t^{-2p/\alpha}\Gamma(2p/\alpha)/\Gamma(p)$, $p,t>0$, see, e.g., Relation (1.10) in Penent and Privault (2022).

Integration by parts

The stochastic representation of the gradient $\nabla u(x)$ will rely on an integration by parts argument. For this, we will use the weight functions $\mathcal{W}_{B(0,R)}(x,y)$ and $\mathcal{W}_{\partial B(0,R)}(x,y)$ defined as

$$
\mathcal{W}_{B(0,R)}(x,y) := \frac{\nabla_x G_R(x,y)}{G_R(x,y)} \quad \text{and} \quad \mathcal{W}_{\partial B(0,R)}(x,y) := \frac{\nabla_x P_R(x,y)}{P_R(x,y)}, \qquad x,y\in B(0,R). \tag{2.6}
$$

Lemma 2.1 *Let $\alpha\in(1,2)$ and $d\geqslant 2$.*

a) Given ϕ a bounded measurable function on $\mathbb{R}^d\setminus B(0,R)$, the function

$$
\chi_1^\phi(x) := \mathbb{E}\big[\phi\big(X_{\tau_R(x),x}\big)\big] = \int_{\mathbb{R}^d\setminus B(0,R)} P_R(x,y)\phi(y)dy, \qquad x\in\overline{B}(0,R), \tag{2.7}
$$

belongs to $\mathcal{C}^1(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$, with

$$
\nabla\chi_1^\phi(x) = \mathbb{E}\left[\mathcal{W}_{\partial B(0,R)}\big(x,X_{\tau_R(x),x}\big)\phi\big(X_{\tau_R(x),x}\big)\right], \qquad x\in B(0,R).
$$

b) Given h a bounded continuous function on $\overline{B}(0,R)$, the function

$$
\chi_2^h(x) := \mathbb{E}\left[\int_0^{\tau_R(x)} h(X_{t,x})dt\right] = \int_{B(0,R)} G_R(x,y)h(y)dy, \qquad x\in\overline{B}(0,R),
$$

belongs to $\mathcal{C}^1(B(0,R))\cap\mathcal{C}^0(\overline{B}(0,R))$, with

$$
\nabla\chi_2^h(x) = \mathbb{E}\left[\int_0^{\tau_R(x)} \mathcal{W}_{B(0,R)}(x,X_{t,x})h(X_{t,x})dt\right], \qquad x\in B(0,R).
$$

Proof. (*a*) Using (2.2) and the boundedness of ϕ on $\mathbb{R}^d \setminus B(0,R)$, we differentiate (2.7) under the integral sign, to obtain that χ_1^ϕ is in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$, with

$$\nabla \chi_1^\phi(x) = \int_{\mathbb{R}^d \setminus B(0,R)} \nabla_x P_R(x,y)\phi(y)dy = \mathbb{E}\left[\frac{\nabla_x P_R\big(x, X_{\tau_R(x),x}\big)}{P_R\big(x, X_{\tau_R(x),x}\big)}\phi\big(X_{\tau_R(x),x}\big)\right], \ x \in B(0,R).$$

(*b*) Using (2.1), the condition $d \geqslant 2$ and the relation

$$\begin{aligned}
\chi_2^h(x) &= \int_{B(0,R)} G_R(x,y)h(y)dy \\
&= \int_{B(x,R)} \frac{\kappa_\alpha^d}{|z|^{d-\alpha}} \int_0^{r_0(x,z-x)} \frac{t^{\alpha/2-1}}{(1+t)^{d/2}} dt\, h(z-x)dz, \quad x \in \overline{B}(0,R),
\end{aligned}$$

we differentiate (2.7) under the integral sign and integrate by parts, to obtain

$$\nabla \chi_2^h(x) = \int_{B(0,R)} \nabla_x G_R(x,y)h(y)dy = \mathbb{E}\bigg[\int_0^{\tau_R(x)} \frac{\nabla_x G_R(x, X_{t,x})}{G_R(x, X_{t,x})} h(X_{t,x})dt\bigg],$$

first for h a $\mathcal{C}^1$ function with compact support in $B(0,R)$, then by uniform approximation of h continuous with compact support in $\overline{B}(0,R)$, and finally by pointwise approximation of h bounded continuous on $\overline{B}(0,R)$, using the bound (2.4). □

3 Marked branching process

Let $\rho : \mathbb{R}^+ \to (0,\infty)$ be a probability density function on $\mathbb{R}_+$, and let $(q_{l_0,\dots,l_m})_{(l_0,\dots,l_m)\in\mathcal{L}_m}$ be a strictly positive probability mass function on $\mathcal{L}_m$. We consider

- an i.i.d. family $(\tau^{i,j})_{i,j\geqslant 1}$ of random variables with distribution $\rho(t)dt$ on $\mathbb{R}_+$ and tail distribution function $\overline{F}(t) = \int_t^\infty \rho(ds)ds$, $t \geqslant 0$,

- an i.i.d. family $(I^{i,j})_{i,j\geqslant 1}$ of discrete random variables with distribution

$$\mathbb{P}(I^{i,j} = (l_0,\dots,l_m)) = q_{l_0,\dots,l_m} > 0, \qquad (l_0,\dots,l_m) \in \mathcal{L}_m,$$

- an independent family $(X^{(i,j)})_{i,j\geqslant 1}$ of symmetric α-stable processes.

In addition, the families of random variables $(\tau^{i,j})_{i,j\geqslant 1}$, $(I^{i,j})_{i,j\geqslant 1}$ and $\big(X^{(i,j)}\big)_{i,j\geqslant 1}$ are assumed to be mutually independent.

The probabilistic representation for the solution of (1.2) uses a branching process started from a particle $x \in B(0,R)$ with label $\overline{1} = (1)$ and mark $i \in \{0, \ldots, m\}$, which evolves according to the process $X^{\overline{1}}_{s,x} = x + X^{(1,1)}_s$, $s \in [0, T_{\overline{1}}]$, with $T_{\overline{1}} = \tau^{1,1} \wedge \tau_R(x) = \min\big(\tau^{1,1}, \tau_R(x)\big)$, where in the notation

$$\tau_R(x) := \inf\big\{t \geqslant 0,\ x + X^{(1,1)}_t \notin B(0,R)\big\},$$

we omit the reference to the label $(1,1)$.

If $\tau^{1,1} < \tau_R(x)$, then the process branches at time $\tau^{1,1}$ into new independent copies of $(X_t)_{t\geqslant 0}$, each of them started at $X^{\overline{1}}_{\tau^{1,1},x}$, and determined by a random sample $(l_0, \ldots, l_m) \in \mathcal{L}_m$ of $I^{1,1}$. Namely, $|l| := l_0 + \cdots + l_m$ new branches carrying respectively the marks $i = 0, \ldots, m$ are created with the probability $q_{l_0,\ldots,l_m}$, where:

a) the first l_0 branches carry the mark 0 and are indexed by $(1,1), (1,2), \ldots, (1, l_0)$,

b) for $i = 1, \ldots, m$, the next l_i branches carry the mark i and are indexed by the labels $(1, l_0 + \cdots + l_{i-1} + 1), \ldots, (1, l_0 + \cdots + l_i)$.

Each new particle then follows independently the above mechanism in such a way that particles at generation $n \geqslant 1$ are assigned a label of the form $\overline{k} = (1, k_2, \ldots, k_n) \in \mathbb{N}^n$, and every branch stops when it leaves the domain $B(0,R)$.

Precisely, the particle with label $\overline{k} = (1, k_2, \ldots, k_n) \in \mathbb{N}^n$ is born at the time $T_{\overline{k}-}$, where $\overline{k}- := (1, k_2, \ldots, k_{n-1})$ represents the label of its parent, and its lifetime $\tau^{n,\pi_n(\overline{k})}$ is the element of index $\pi_n(\overline{k})$ in the i.i.d. sequence $(\tau^{n,j})_{j\geqslant 1}$, which defines an injection

$$\pi_n : \mathbb{N}^n \to \mathbb{N}, \qquad n \geqslant 1.$$

The random evolution of the particle of label $\overline{k}$ is given by

$$X^{\overline{k}}_{t,x} := X^{\overline{k}-}_{T_{\overline{k}-},x} + X^{n,\pi_n(\overline{k})}_{t-T_{\overline{k}-}}, \qquad t \in [T_{\overline{k}-}, T_{\overline{k}}],$$

where $T_{\overline{k}} := T_{\overline{k}-} + \tau^{n,\pi_n(\overline{k})} \wedge \tau_R\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$ and

$$\tau_R\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big) := \inf\big\{t \geqslant 0,\ X^{\overline{k}-}_{T_{\overline{k}-},x} + X^{n,\pi_n(\overline{k})}_t \notin B(0,R)\big\}.$$

If $\tau^{n,\pi_n(\overline{k})} < \tau_R\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$, we draw a random sample $(l_0, \ldots, l_m)$ of $I_{\overline{k}} := I^{n,\pi_n(\overline{k})}$ with the probability $q_{l_0,\ldots,l_m}$, and the particle $\overline{k}$ branches into $|I^{n,\pi_n(\overline{k})}| = l_0 + \cdots + l_m$ offsprings, indexed by $(1,\ldots,k_n,j)$, $j=1,\ldots,|I^{n,\pi_n(\overline{k})}|$ and respectively carrying the marks $i=0,\ldots,m$, as in point (b) above. Namely, the particles whose index ends with an integer between 1 and l_0 will carry the mark 0, and those with index ending with an integer between $l_0+\cdots+l_{i-1}+1$ and $l_0+\cdots+l_i$ will carry a mark $i\in\{1,\ldots,m\}$. Finally, the mark of the particle $\overline{k}$ will be denoted by $\theta_{\overline{k}}\in\{0,\ldots,m\}$.

The set of particles dying inside the ball $B(0,R)$ is denoted by $\mathcal{K}^\circ$, whereas those dying outside of $B(0,R)$ form a set denoted by $\mathcal{K}^\partial$. For $n\geqslant 1$, the set *n-th* generation particles that die inside the domain $B(0,R)$ is denoted by $\mathcal{K}_n^\circ$, and the set of n-th generation particles which die outside of $B(0,R)$ is denoted by $\mathcal{K}_n^\partial$, and we let $\mathcal{K}_n=\mathcal{K}_n^\circ\cup\mathcal{K}_n^\partial$.

Definition 3.1 *We denote by $\mathcal{T}_{x,i}$ the marked branching process, or random marked tree constructed above after starting from the position $x\in\mathbb{R}^d$ and mark $i\in\{0,\ldots,m\}$ on its first branch.*

The tree $\mathcal{T}_{x,0}$ will be used for the stochastic representation of the solution $u(x)$ of the PDE (1.2), while the trees $\mathcal{T}_{x,i}$ will be used for the stochastic representation of $b_i(x)\cdot\nabla u(x)$, $i=1,\ldots,m$. Table 1 summarizes the notation introduced so far.

Object	Notation
Initial position	x
Tree rooted at x with initial mark $\theta_{\overline{1}}=i$	$\mathcal{T}_{x,i}$
Particle (or label) of generation $n\geqslant 1$	$\overline{k}=(1,k_2,\ldots,k_n)$
First branching time	$T_{\overline{1}}$
Lifespan of a particle	$T_{\overline{k}}-T_{\overline{k}-}$
Birth time of the particle $\overline{k}$	$T_{\overline{k}-}$
Death time of the particle $\overline{k}\in\mathcal{K}^\circ$	$T_{\overline{k}}=\mathrm{T}_{\overline{k}-}+\tau^{n,\pi_n(\overline{k})}$
Death time of the particle $\overline{k}\in\mathcal{K}^\partial$	$T_{\overline{k}}=T_{\overline{k}-}+\tau_R\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$
Position at birth of the particle $\overline{k}$	$X^{\overline{k}}_{T_{\overline{k}-},x}$
Position at death of the particle $\overline{k}$	$X^{\overline{k}}_{T_{\overline{k}},x}$
Mark of the particle $\overline{k}$	$\theta_{\overline{k}}$
Exit time starting from $x\in B(0,R)$	$\tau_R(x):=\inf\{t\geqslant 0,\ x+X_t\notin B(0,R)\}$

Table 1: Notation.

Figure 1 presents the marking and labeling conventions used for the graphical representation of random marked trees, in which different colors represent different ways of branching.

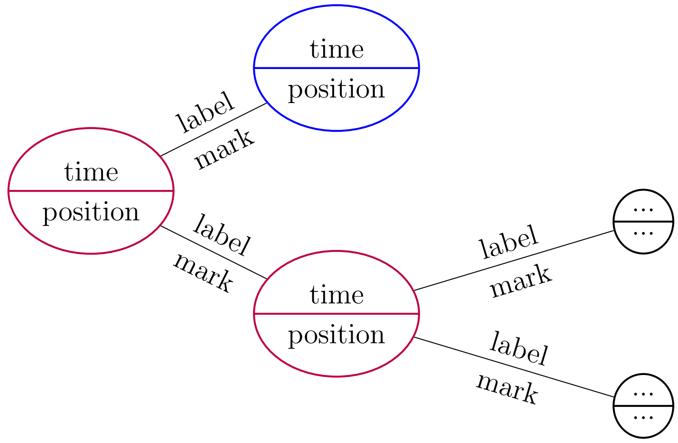


Figure 1: Tree labelling and marking conventions.

A sample tree for the PDE

$$\Delta_\alpha u(t,x) + c_{(0,0)}(x) + c_{(0,1)}(x)u(t,x)\frac{\partial u}{\partial x}(t,x) = 0$$

in dimension $d = 1$ is presented in Figure 2. Absence of branching is represented in blue, branching into two branches, one bearing the mark 0 and the other one bearing the mark 1, is represented in purple, and the black color is used for leaves, i.e. for particles that die outside of the domain $B(0, R)$.

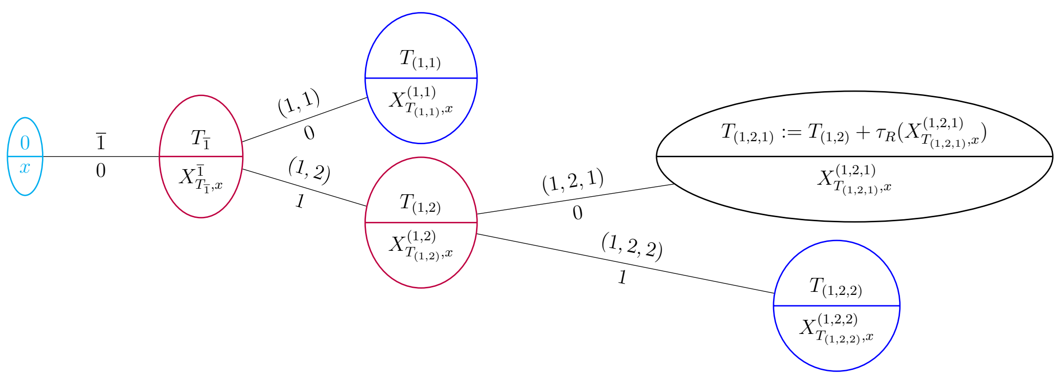


Figure 2: Tree labelling and marking conventions.

In Figure 2 we have $\mathcal{K}^\circ = \{\bar{1}, (1,1), (1,2), (1,2,2)\}$ and $\mathcal{K}^\partial = \{(1,2,1)\}$.

4 Probabilistic representation of PDE solutions

We consider the weight function $\mathcal{W}(t,x,X)$ defined as

$$\mathcal{W}(t,x,X) := \mathcal{W}_{B(0,R)}(x,X_{t,x})\mathbf{1}_{\{X_{t,x}\in B(0,R)\}} + \mathcal{W}_{\partial B(0,R)}(x,X_{\tau_R(x),x})\mathbf{1}_{\{X_{t,x}\notin B(0,R)\}}, \tag{4.1}$$

$x \in B(0,R)$. We note that the products involved in the definition (4.2) of $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ below are almost surely finite since the interbranching times $T_{\overline{k}} - T_{\overline{k}-}$ are identically distributed and the number of offsprings at any branching time is bounded by a constant depending only on the finite set $\mathcal{L}_m$.

Definition 4.1 *We define the functional $\mathcal{H}_\phi$ of the random tree $\mathcal{T}_{x,i}$ with initial mark $\theta_{\overline{1}} = i \in \{0,\ldots,m\}$ as*

$$\mathcal{H}_\phi(\mathcal{T}_{x,i}) := \prod_{\overline{k}\in\mathcal{K}^\circ} \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}}{q_{I_{\overline{k}}}\rho(T_{\overline{k}} - T_{\overline{k}-})} \prod_{\overline{k}\in\mathcal{K}^\partial} \frac{\phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}}{\overline{F}(T_{\overline{k}} - T_{\overline{k}-})}, \quad x \in B(0,R), \tag{4.2}$$

where for $\overline{k} \in \mathcal{K}^\circ \cup \mathcal{K}^\partial$ we let

$$\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x} := \begin{cases} 1 & \text{if } \theta_{\overline{k}} = 0, \\ b_{\theta_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\cdot\mathcal{W}\big(T_{\overline{k}} - T_{\overline{k}-}, X^{\overline{k}}_{T_{\overline{k}-},x}, X^{\overline{k}}\big) & \text{if } \theta_{\overline{k}} = 1,\ldots,m, \end{cases} \tag{4.3}$$

where $\theta_{\overline{k}} \in \{0,\ldots,m\}$ denotes the mark of the particle $\overline{k}$.

Assumption (B) *Let $\alpha \in (1,2)$ and $d \geqslant 2$. We assume that the common probability density function ρ and tail distribution function $\overline{F}$ of the random times $\tau^{i,j}$'s satisfies the conditions*

$$\sup_{t\in(0,1]} \frac{1}{\rho(t)t^{p/\alpha}} < \infty \quad \text{and} \quad \mathbb{E}\Big[\big(\overline{F}(\tau_R(0))\big)^{1-p}\Big] < \infty$$

for some $p \in (1,d)$.

When $\alpha \in (1,2)$ and R is sufficiently small, Assumption (B) is satisfied by any continuous probability density function $\rho(t)$ such that

$$\rho(t) \underset{t\to 0}{\sim} \kappa t^{\delta-1},$$

for some $\delta \in (0, 1-p/\alpha]$ and $\kappa > 0$, and $1/\overline{F}(x) \leqslant e^{\kappa x}$, $x \geqslant 0$, for some $\kappa > 0$, see, e.g., Lemma 6 in Bogdan et al. (2010). This includes for example a gamma distribution with shape parameter $\delta \in (0, 1-p/\alpha]$. The goal of this section is to prove the following result, which implies Theorem 1.1.

Theorem 4.2 *Let $\alpha \in (1,2)$ and $d \geqslant 2$. Under Assumptions (A)-(B), if $R > 0$ and $\max_{l \in \mathcal{L}_m} \|c_l\|_\infty$ are sufficiently small, the semilinear elliptic PDE (1.2) admits a classical solution in $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ for some $\epsilon > 0$, which is the unique viscosity solution of (1.2) and can be represented as*

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \qquad x \in B(0,R). \tag{4.4}$$

Before giving the proof of Theorem 4.2 at the end of this section, we need to state and prove Propositions 4.3 and 4.6 below. First, in Proposition 4.3 we obtain a probabilistic representation for the solutions of semilinear elliptic PDEs of the form (1.2) under uniform integrability conditions on $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x \in B(0,R)}$, $i = 0, \ldots, m$. Then, in Proposition 4.6 we show that such conditions are satisfied under Assumptions (A)-(B).

Proposition 4.3 *Let $\alpha \in (1,2)$ and $d \geqslant 2$, and assume that the family $(\mathcal{H}(\mathcal{T}_{x,i}))_{x \in B(0,R)}$ is uniformly integrable, $i = 0, \ldots, m$. Then, the function $u(x)$ defined as*

$$u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})], \quad x \in \overline{B}(0,R),$$

is a viscosity solution in $C^1(B(0,R)) \cap C^0(\overline{B}(0,R))$ of (1.2). In addition, the gradient $b_i(x) \cdot \nabla u(x)$ can be represented as the expected value

$$b_i(x) \cdot \nabla u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,i})\big], \quad x \in B(0,R), \quad i = 1, \ldots, m.$$

Proof. Let

$$v_i(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})], \quad x \in B(0,R), \quad i = 1, \ldots, m.$$

By considering the first branching at time $T_{\bar{1}}$ and letting $\mathcal{T}^{(j)}_{X^{\bar{1}}_{T_{\bar{1}},x},i}$, $j = 1 + l_0 + \cdots + l_{i-1}, \ldots, l_0 + \cdots + l_i$, denote independent tree copies started at $X^{\bar{1}}_{T_{\bar{1}},x}$ with the mark $i \in \{0, \ldots, m\}$, we have

$$\begin{aligned}
u(x) &= \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})] \\
&= \mathbb{E}\left[\mathbf{1}_{\{T_{\bar{1}} = \tau_R(x)\}} \frac{\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big)}{\overline{F}(T_{\bar{1}})} + \mathbf{1}_{\{T_{\bar{1}} < \tau_R(x)\}} \sum_{l \in \mathcal{L}_m} \mathbf{1}_{\{I_{\bar{1}} = (l_0, \ldots, l_m)\}} \frac{c_{I_{\bar{1}}}\big(X^{\bar{1}}_{T_{\bar{1}},x}\big)}{q_{I_{\bar{1}}} \rho(T_{\bar{1}})} \prod_{i=0}^{m} \prod_{j=1+l_0+\cdots+l_{i-1}}^{l_0+\cdots+l_i} \mathcal{H}_\phi\big(\mathcal{T}^{(j)}_{X^{\bar{1}}_{T_{\bar{1}},x},i}\big)\right] \\
&= \mathbb{E}\left[\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big) + \int_0^{\tau_R(x)} \sum_{l \in \mathcal{L}_m} c_l\big(X^{\bar{1}}_{t,x}\big) u^{l_0}\big(X^{\bar{1}}_{t,x}\big) \prod_{i=1}^{m} v_i^{l_i}\big(X^{\bar{1}}_{t,x}\big) dt\right] \\
&= \mathbb{E}\big[\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big)\big] + \mathbb{E}\left[\int_0^{\tau_R(x)} h\big(X^{\bar{1}}_{t,x}\big) dt\right],
\end{aligned} \tag{4.5}$$

where $u(x)$ and the function

$$h(x) := \sum_{l \in \mathcal{L}_m} c_l(x) u^{l_0}(x) \prod_{i=1}^m v_i^{l_i}(x), \qquad x \in B(0,R),$$

are bounded continuous on $\overline{B}(0,R)$ by Lemma 4.7. Hence by Lemmas 2.1 and 4.7 the function $u(x)$ is differentiable in $x \in B(0,R)$, with

$$\begin{aligned}
\nabla u(x) &= \nabla \mathbb{E}\big[\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big)\big] + \nabla \mathbb{E}\bigg[\int_0^{\tau_R(x)} h\big(X^{\bar{1}}_{t,x}\big)dt\bigg] \\
&= \mathbb{E}\big[\mathcal{W}_{\partial B(0,R)}\big(x, X^{\bar{1}}_{\tau_R(x),x}\big)\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big)\big] + \mathbb{E}\bigg[\int_0^{\tau_R(x)} \mathcal{W}_{B(0,R)}\big(x, X^{\bar{1}}_{t,x}\big)h\big(X^{\bar{1}}_{t,x}\big)dt\bigg] \\
&= \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\mathcal{W}(T_{\bar{1}}, x, X)\big],
\end{aligned}$$

and by (4.3)-(4.2) we have

$$\begin{aligned}
b_i(x) \cdot \nabla u(x) &= \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\ b_i(x) \cdot \mathcal{W}(T_{\bar{1}}, x, X)\big] \\
&= \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})] \\
&= v_i(x), \qquad x \in B(0,R), \quad i = 1, \ldots, m.
\end{aligned}$$

Therefore, using (1.3), Relation (4.5) rewrites as

$$u(x) = \mathbb{E}\bigg[\phi\big(X^{\bar{1}}_{\tau_R(x),x}\big) + \int_0^{\tau_R(x)} f\big(X^{\bar{1}}_{t,x}, u\big(X^{\bar{1}}_{t,x}\big), \nabla u\big(X^{\bar{1}}_{t,x}\big)\big)dt\bigg], \quad x \in B(0,R).$$

It then follows from a classical argument that u is a viscosity solution of (1.2). Indeed, for any $\delta > 0$, by the Markov property we also have

$$u(x) = \mathbb{E}\bigg[u\big(X^{\bar{1}}_{\delta \wedge \tau_R(x),x}\big) + \int_0^{\delta \wedge \tau_R(x)} f\big(X^{\bar{1}}_{t,x}, u\big(X^{\bar{1}}_{t,x}\big), \nabla u\big(X^{\bar{1}}_{t,x}\big)\big)dt\bigg], \quad x \in B(0,R).$$

Next, let $\xi \in \mathcal{C}^2(B(0,R))$ such that x is a maximum point of $u - \xi$ and $u(x) = \xi(x)$. By the Itô-Dynkin formula, we get

$$\mathbb{E}\big[\xi\big(X^{\bar{1}}_{\delta \wedge \tau_R(x),x}\big)\big] = \xi(x) + \mathbb{E}\bigg[\int_0^{\delta \wedge \tau_R(x)} \Delta_\alpha \xi\big(X^{\bar{1}}_{t,x}\big)dt\bigg].$$

Thus, since $u(x) = \xi(x)$ and $u \leqslant \xi$, we find

$$\mathbb{E}\bigg[\int_0^{\delta \wedge \tau_R(x)} \big(\Delta_\alpha \xi\big(X^{\bar{1}}_{t,x}\big) + f\big(X^{\bar{1}}_{t,x}, u\big(X^{\bar{1}}_{t,x}\big), \nabla u\big(X^{\bar{1}}_{t,x}\big)\big)\big)dt\bigg] \geqslant 0.$$

Since $X_{t,x}$ converges in distribution to the constant $x \in \mathbb{R}^d$ as t tends to zero, it admits an almost surely convergent subsequence, hence by continuity and boundedness of $f(\,\cdot\,, u(\,\cdot\,))$ together with the mean-value and dominated convergence theorems, we have

$$\Delta_\alpha \xi(x) + f(x, \xi(x), \nabla \xi(x)) \geqslant 0,$$

hence u is a viscosity subsolution (and similarly a viscosity supersolution) of (1.2). □

The proof of the next lemma uses the filtration $(\mathcal{F}_n)_{n\geqslant 1}$ defined by

$$\mathcal{F}_n := \sigma\left(T_{\bar{k}}, I_{\bar{k}}, X^{\bar{k}}, \bar{k} \in \bigcup_{i=1}^{n} \mathbb{N}^i\right), \qquad n \geqslant 1.$$

Recall that $\mathcal{K}_i^{\circ}$ (resp. $\mathcal{K}_i^{\partial}$), $i = 1, \ldots, n+1$, denotes the set of i-th generation particles which die inside (resp. outside) the domain $B(0,R)$, and $\mathcal{K}_n = \mathcal{K}_n^{\circ} \cup \mathcal{K}_n^{\partial}$.

Lemma 4.4 *Given $p \geqslant 1$, let $v : B(0,R) \to \mathbb{R}_+$ be a bounded measurable function satisfying the inequality*

$$v(x) \geqslant K_1 \mathbb{E}\big[\big(\overline{F}(\tau_R(x))\big)^{1-p}\big] + \mathbb{E}\left[\int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} \frac{v^{|l|}(X_{t,x}^{\bar{1}})}{q_l^{p-1}} dt\right],$$

$x \in B(0,R)$, for some $K_1, K_2, \widetilde{K}_2 > 0$, where $|l| = l_0 + \cdots + l_m$. Then, we have

$$v(x) \geqslant \mathbb{E}\left[\prod_{\bar{k}\in\mathcal{K}^{\partial}} \frac{K_1}{\overline{F}^p(T_{\bar{k}} - T_{\bar{k}-})} \prod_{\substack{\bar{k}\in\mathcal{K}^{\circ} \\ T_{\bar{k}} - T_{\bar{k}-} \leqslant 1}} \frac{K_2^p}{q_{I_{\bar{k}}}^p} \prod_{\substack{\bar{k}\in\mathcal{K}^{\circ} \\ T_{\bar{k}} - T_{\bar{k}-} > 1}} \frac{\widetilde{K}_2^p}{q_{I_{\bar{k}}}^p \rho^p(T_{\bar{k}} - T_{\bar{k}-})}\right], \quad x \in B(0,R). \quad (4.6)$$

Proof. Since $T_{\bar{1}}$ is independent of $\big(X_{s,x}^{\bar{1}}\big)_{s\geqslant 0}$ and has the probability density ρ, letting

$$g(y) := \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} \frac{y^{|l|}}{q_l^{p-1}},$$

we have

$$\begin{aligned}
v(x) &\geqslant \mathbb{E}\left[K_1\big(\overline{F}(\tau_R(x))\big)^{1-p} + \int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) g\big(v\big(X_{t,x}^{\bar{1}}\big)\big) dt\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[K_1\big(\overline{F}(\tau_R(x))\big)^{1-p} + \int_0^{\tau_R(x)} \big(K_2^p \mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p \mathbf{1}_{(1,\infty)}(t)\big) g\big(v\big(X_{t,x}^{\bar{1}}\big)\big) dt \,\middle|\, \big(X_{s,x}^{\bar{1}}\big)_{s\geqslant 0}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{K_1}{\overline{F}^p(\tau_R(x))} \mathbf{1}_{\{T_{\bar{1}} = \tau_R(x)\}} + \int_0^{\tau_R(x)} \left(K_2^p \mathbf{1}_{[0,1]}(t) + \frac{\widetilde{K}_2^p}{\rho(t)} \mathbf{1}_{(1,\infty)}(t)\right) g\big(v\big(X_{t,x}^{\bar{1}}\big)\big)\rho(t) dt \,\middle|\, \big(X_{s,x}^{\bar{1}}\big)_{s\geqslant 0}\right]\right]
\end{aligned}$$

$$= \mathbb{E}\left[\frac{K_1}{\overline{F}^p(\tau_R(x))}\mathbf{1}_{\{T_{\overline{1}}=\tau_R(x)\}} + K_2^p g\big(v\big(X^{\overline{1}}_{T_{\overline{1}},x}\big)\big)\mathbf{1}_{\{T_{\overline{1}}\leqslant \min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho(T_{\overline{1}})} g\big(v\big(X^{\overline{1}}_{T_{\overline{1}},x}\big)\big)\mathbf{1}_{\{1<T_{\overline{1}}<\tau_R(x)\}}\right]$$

$$= \mathbb{E}\left[\frac{K_1}{\overline{F}^p(T_{\overline{1}})}\mathbf{1}_{\{T_{\overline{1}}=\tau_R(x)\}} + \frac{1}{q^p_{I_{\overline{1}}}}\left(K_2^p\mathbf{1}_{\{T_{\overline{1}}\leqslant \min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho(T_{\overline{1}})}\mathbf{1}_{\{1<T_{\overline{1}}<\tau_R(x)\}}\right) v^{|I_{\overline{1}}|}\big(X^{\overline{1}}_{T_{\overline{1}},x}\big)\right],$$

showing that

$$v(x) \geqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}_1^\partial}\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\substack{\overline{k}\in\mathcal{K}_1^\circ\\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}}\frac{K_2^p}{q^p_{I_{\overline{k}}}}\prod_{\substack{\overline{k}\in\mathcal{K}_1^\circ\\ T_{\overline{k}}-T_{\overline{k}-}>1}}\frac{\widetilde{K}_2^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\overline{k}\in\mathcal{K}_2} v\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\right], \quad (4.7)$$

$x \in B(0,R)$. By repeating this argument for the particles in $\overline{k}\in\mathcal{K}_2$, we find

$$v\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big) \geqslant \mathbb{E}\left[\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\mathbf{1}_{\{X^{\overline{k}}_{T_{\overline{k}},x}\notin B(0,R)\}}\right.$$
$$\left. + \frac{1}{q^p_{I_{\overline{k}}}}\left(K_2^p\mathbf{1}_{\{T_{\overline{k}}-T_{\overline{k}-}\leqslant\min(1,\tau_R(x))\}} + \frac{\widetilde{K}_2^p}{\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\mathbf{1}_{\{1<T_{\overline{k}}-T_{\overline{k}-}<\tau_R(x)\}}\right) v^{|I_{\overline{k}}|}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\ \middle|\ \mathcal{F}_1\right].$$

Plugging this expression in (4.7) above and using the tower property of the conditional expectation, we obtain

$$v(x) \geqslant \mathbb{E}\left[\prod_{\overline{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\partial}\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\substack{\overline{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\circ\\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}}\frac{K_2^p}{q^p_{I_{\overline{k}}}}\prod_{\substack{\overline{k}\in\bigcup_{i=1}^2\mathcal{K}_i^\circ\\ T_{\overline{k}}-T_{\overline{k}-}>1}}\frac{\widetilde{K}_2^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\overline{k}\in\mathcal{K}_4} v\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\right],$$

and repeating this process inductively leads to

$$v(x) \geqslant \mathbb{E}\left[\prod_{\overline{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\partial}\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\substack{\overline{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\circ\\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}}\frac{K_2^p}{q^p_{I_{\overline{k}}}}\prod_{\substack{\overline{k}\in\bigcup_{i=1}^n\mathcal{K}_i^\circ\\ T_{\overline{k}}-T_{\overline{k}-}>1}}\frac{\widetilde{K}_2^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\overline{k}\in\mathcal{K}_{n+1}} v\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\right],$$

$n \geqslant 1$. Using Fatou's lemma as n tends to infinity, since all particles become eventually extinct with probability one, we obtain (4.6). □

Lemma 4.5 *Let $\alpha \in (1,2)$, $p \in [1,d)$, $d \geqslant 2$, and*

$$b_{0,\infty} := \max_{1\leqslant i\leqslant m}\sup_{x\in B(0,R)}|b_i(x)|, \quad b_{1,\infty} := \max_{1\leqslant i\leqslant m}\sup_{x\in B(0,R)}\frac{|b_i(x)|}{R-|x|}.$$

Under Assumptions (A)-(B), we have the bound

$$\mathbb{E}\big[\big|\mathcal{H}_\phi(\mathcal{T}_{x,i})\big|^p\big] \leqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\overline{k}} - T_{\overline{k}-})} \prod_{\substack{\overline{k}\in\mathcal{K}^\circ \\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}} \frac{K_4 \max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q^p_{I_{\overline{k}}}} \prod_{\substack{\overline{k}\in\mathcal{K}^\circ \\ T_{\overline{k}}-T_{\overline{k}-}>1}} \frac{K_3 \max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}} - T_{\overline{k}-})}\right], \tag{4.8}$$

$x \in B(0,R)$, $i = 0,\ldots,m$, where

$$K_1 := \|\phi\|_\infty^p(1+(d+\alpha)^p b_{1,\infty}^p), \quad K_3 := 1 + d^p b_{1,\infty}^p + d^p b_{0,\infty}^p C_{\alpha,d,p}, \tag{4.9}$$

and

$$K_4 := \sup_{t\in[0,1]} \frac{1+d^p b_{1,\infty}^p}{\rho^p(t)} + d^p b_{0,\infty}^p \sup_{t\in[0,1]} \frac{C_{\alpha,d,p}}{\rho^p(t)t^{p/\alpha}}. \tag{4.10}$$

Proof. For $x \in B(0,R)$, let

$$w_i(x) := \mathbb{E}\big[\big|\mathcal{H}_\phi(\mathcal{T}_{x,i})\big|^p\big] = \mathbb{E}_i\left[\prod_{\overline{k}\in\mathcal{K}^\circ} \frac{\big|c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\big|^p \big|\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\big|^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}} - T_{\overline{k}-})} \prod_{\overline{k}\in\mathcal{K}^\partial} \frac{\big|\phi\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)\big|^p \big|\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\big|^p}{\overline{F}^p(T_{\overline{k}} - T_{\overline{k}-})}\right], \tag{4.11}$$

where $\mathbb{E}_i$ denotes the conditional expectation given that the tree $\mathcal{T}_{x,i}$ is started with the mark $i \in \{0,\ldots,m\}$. When $\overline{k} \in \mathcal{K}^\circ$ has mark $\theta_{\overline{k}} = 0$ we have $\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x} = 1$, whereas when $\theta_{\overline{k}} \neq 0$, using (2.4), (4.1)-(4.3) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\big|\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\big| &\leqslant \frac{d\big|b_{\theta_k}\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\big|}{\min\big(R - \big|X^{\overline{k}}_{T_{\overline{k}-},x}\big|, \big|X^{\overline{k}}_{T_{\overline{k}},x} - X^{\overline{k}}_{T_{\overline{k}-},x}\big|\big)} \\
&\leqslant d\max\left(\frac{\big|b_{\theta_k}\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\big|}{R - \big|X^{\overline{k}}_{T_{\overline{k}-},x}\big|}, \frac{\big|b_{\theta_k}\big(X^{\overline{k}}_{T_{\overline{k}-},x}\big)\big|}{\big|X^{\overline{k}}_{T_{\overline{k}},x} - X^{\overline{k}}_{T_{\overline{k}-},x}\big|}\right) \\
&\leqslant db_{1,\infty} + \frac{db_{0,\infty}}{\big|X^{\overline{k}}_{T_{\overline{k}},x} - X^{\overline{k}}_{T_{\overline{k}-},x}\big|}.
\end{aligned}$$

Similarly, when $\overline{k} \in \mathcal{K}^\partial$, the definition of $\mathcal{W}_{\partial B(0,R)}(x,y)$ in (2.6), together with the bound (2.3) and the Cauchy-Schwarz inequality, imply

$$\big|\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\big| \leqslant (d+\alpha)b_{1,\infty}. \tag{4.12}$$

Next, by conditional independence given $\mathcal{G} := \sigma\big(\tau^{i,j}, I^{i,j} \ : \ i,j \geqslant 1\big)$ of the terms in the product over $\overline{k} \in \mathcal{K}^\circ$ and $\overline{k} \in \mathcal{K}^\partial$, which involve random terms of the form $X^{\overline{k}}_{T_{\overline{k}},x} - X^{\overline{k}}_{T_{\overline{k}-},x}$ given $T_{\overline{k}} - T_{\overline{k}-}$, by (2.5), and (4.11)-(4.12), we have

$w_i(x)$

$$\leqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\circ} \frac{\|c_{I_{\overline{k}}}\|_\infty^p}{q_{I_{\overline{k}}}^p} \mathbb{E}\left[\frac{2^p}{\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\left(1+d^p b_{1,\infty}^p + \frac{d^p b_{0,\infty}^p}{\left|X_{T_{\overline{k}},x}^{\overline{k}} - X_{T_{\overline{k}-},x}^{\overline{k}}\right|^p}\right) \Bigg| \mathcal{G}\right] \prod_{\overline{k}\in\mathcal{K}^\partial} \mathbb{E}\left[\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})} \Bigg| \mathcal{G}\right]\right]$$

$$= \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\circ}\left(\frac{\|c_{I_{\overline{k}}}\|_\infty^p}{q_{I_{\overline{k}}}^p \rho^p(T_{\overline{k}}-T_{\overline{k}-})}\left(1+d^p b_{1,\infty}^p + \frac{d^p b_{0,\infty}^p C_{\alpha,d,p}}{(T_{\overline{k}}-T_{\overline{k}-})^{p/\alpha}}\right)\right) \prod_{\overline{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\right].$$

Splitting the terms in the product over $\overline{k} \in \mathcal{K}^\circ$ between small and large values of $T_{\overline{k}} - T_{\overline{k}-}$, we get

$$\begin{aligned} &w_i(x) \\ &\leqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\circ} \frac{\|c_{I_{\overline{k}}}\|_\infty^p}{q_{I_{\overline{k}}}^p}\left(\frac{K_3}{\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\mathbf{1}_{\{T_{\overline{k}}-T_{\overline{k}-}>1\}} + K_4 \mathbf{1}_{\{T_{\overline{k}}-T_{\overline{k}-}\leqslant 1\}}\right) \prod_{\overline{k}\in\mathcal{K}^\partial} \frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\right], \end{aligned} \tag{4.13}$$

$x \in B(0,R)$, which yields (4.8), $i = 0, \ldots, m$. □

Proposition 4.6 provides sufficient conditions for the finiteness of the upper bound (4.8), and for $(\mathcal{H}_\phi(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ to be bounded in $L^1(\Omega)$, uniformly in $x \in B(0,R)$, $i = 0, \ldots, m$, as required in Proposition 4.3.

Proposition 4.6 *Let $\alpha \in (1,2)$, $p \in [1,d)$, and $d \geqslant 2$. Under Assumptions (A)-(B), suppose that the boundary condition ϕ is bounded on $\mathbb{R}^d$ and that there exists a bounded strictly positive weak solution $v \in H^{\alpha/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ to the following partial differential inequality:*

$$\begin{cases} \Delta_\alpha v(x) + \widetilde{K}_2 \displaystyle\sum_{l\in\mathcal{L}_m} \frac{v^{|l|}(x)}{q_l^{p-1}} \leqslant 0, \quad x \in B(0,R), \\ v(x) \geqslant \widetilde{K}_1 > 0, \quad x \in \mathbb{R}^d \setminus B(0,R), \end{cases} \tag{4.14}$$

where $\widetilde{K}_1 \geqslant K_1 \mathbb{E}\big[\overline{F}^{1-p}(\tau_R(0))\big]$, $K_1 > 0$ is given by (4.9), and $\widetilde{K}_2 > 0$. Then, for sufficiently small $\max_{l\in\mathcal{L}_m} \|c_l\|_\infty$ we have the bound

$$\mathbb{E}[|\mathcal{H}_\phi(\mathcal{T}_{x,i})|^p] \leqslant v(x) \leqslant \|v\|_\infty < \infty, \qquad x \in B(0,R), \quad i = 0, \ldots, m. \tag{4.15}$$

Proof. We smooth out $v \in H^{\alpha/2}(\mathbb{R}^d)$ as

$$v_\varepsilon(x) := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \psi\left(\frac{x-y}{\varepsilon}\right) v(y)dy, \quad x \in \mathbb{R}, \quad \varepsilon > 0,$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is a mollifier such that $\int_{-\infty}^{\infty} \psi(y)dy = 1$. By (4.14) and Jensen's inequality, we have

$$\Delta_\alpha v_\varepsilon(x) + \widetilde{K}_2 \sum_{l\in\mathcal{L}_m} \frac{v_\varepsilon^{|l|}(x)}{q_l^{p-1}}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Delta_\alpha\psi\left(\frac{x-y}{\varepsilon}\right)v(y)dy + \widetilde{K}_2\sum_{l\in\mathcal{L}_m}\frac{1}{q_l^{p-1}}\left(\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\psi\left(\frac{x-y}{\varepsilon}\right)v(y)dy\right)^{|l|} \\
&\leqslant \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\psi\left(\frac{x-y}{\varepsilon}\right)\Delta_\alpha v(y)dy + \widetilde{K}_2\sum_{l\in\mathcal{L}_m}\frac{1}{\varepsilon q_l^{p-1}}\int_{-\infty}^{\infty}\psi\left(\frac{x-y}{\varepsilon}\right)v^{|l|}(y)dy \\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\psi\left(\frac{x-y}{\varepsilon}\right)\left(\Delta_\alpha v(y) + \widetilde{K}_2\sum_{l\in\mathcal{L}_m}\frac{v^{|l|}(y)}{q_l^{p-1}}\right)dy \\
&\leqslant 0, \quad x\in B(0,R).
\end{aligned}
$$

Applying the Itô-Dynkin formula to $v_\varepsilon(X_{s,x})$ with $v_\varepsilon\in H^\alpha(\mathbb{R}^d)$, by (4.14) we have

$$
\begin{aligned}
v_\varepsilon(x) &= \mathbb{E}\left[v_\varepsilon\big(X^x_{\tau_R(x)}\big) - \int_0^{\tau_R(x)}\Delta_\alpha v_\varepsilon(X_{t,x})dt\right] \\
&\geqslant \mathbb{E}\left[\widetilde{K}_1 + \int_0^{\tau_R(x)}\widetilde{K}_2\sum_{l\in\mathcal{L}_m}\frac{v_\varepsilon^{|l|}(X_{t,x})}{q_l^{p-1}}dt\right], \quad x\in B(0,R).
\end{aligned}
$$

Thus, passing to the limit as ε tends to zero, by dominated convergence and the facts that $\mathbb{E}[\tau_R(x)]<\infty$ and $v(x)$ is upper and lower bounded in $(0,\infty)$, for some sufficiently small $K_2>0$ we have

$$
\begin{aligned}
v(x) &\geqslant \widetilde{K}_1 + \mathbb{E}\bigg[\int_0^{\tau_R(x)}\widetilde{K}_2\sum_{l\in\mathcal{L}_m}\frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}dt\bigg] \\
&\geqslant K_1\mathbb{E}\big[\overline{F}^{1-p}(\tau_R(0))\big] + \mathbb{E}\bigg[\int_0^{\tau_R(x)}\big(K_2^p\mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p\mathbf{1}_{(1,\infty)}(t)\big)\sum_{l\in\mathcal{L}_m}\frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}dt\bigg], \\
&\geqslant K_1\mathbb{E}\big[\overline{F}^{1-p}(\tau_R(x))\big] + \mathbb{E}\bigg[\int_0^{\tau_R(x)}\big(K_2^p\mathbf{1}_{[0,1]}(t)\rho(t) + \widetilde{K}_2^p\mathbf{1}_{(1,\infty)}(t)\big)\sum_{l\in\mathcal{L}_m}\frac{v^{|l|}(X_{t,x})}{q_l^{p-1}}dt\bigg],
\end{aligned}
$$

$x\in B(0,R)$, as the function $\overline{F}^{1-p}$ is non-decreasing. Hence by Lemmas 4.4 and 4.5, for K_3, K_4 given in (4.9)-(4.10) we have, provided that $\max_{l\in\mathcal{L}_m}\|c_l\|_\infty$ is sufficiently small,

$$
\begin{aligned}
v(x) &\geqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\partial}\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\substack{\overline{k}\in\mathcal{K}^\circ\\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}}\frac{K_2^p}{q^p_{I_{\overline{k}}}}\prod_{\substack{\overline{k}\in\mathcal{K}^\circ\\ T_{\overline{k}}-T_{\overline{k}-}>1}}\frac{\widetilde{K}_2^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\right] \\
&\geqslant \mathbb{E}\left[\prod_{\overline{k}\in\mathcal{K}^\partial}\frac{K_1}{\overline{F}^p(T_{\overline{k}}-T_{\overline{k}-})}\prod_{\substack{\overline{k}\in\mathcal{K}^\circ\\ T_{\overline{k}}-T_{\overline{k}-}\leqslant 1}}\frac{K_4\max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q^p_{I_{\overline{k}}}}\prod_{\substack{\overline{k}\in\mathcal{K}^\circ\\ T_{\overline{k}}-T_{\overline{k}-}>1}}\frac{K_3\max_{l\in\mathcal{L}_m}\|c_l\|_\infty^p}{q^p_{I_{\overline{k}}}\rho^p(T_{\overline{k}}-T_{\overline{k}-})}\right] \\
&\geqslant \mathbb{E}\big[\big|\mathcal{H}_\phi(\mathcal{T}_{x,i})\big|^p\big],
\end{aligned}
$$

$x\in B(0,R)$, $i=0,\ldots,m$, which yields (4.15). $\square$

Proof of Theorem 4.2. By Theorem 1.2 in Penent and Privault (2023), the partial differential inequality (4.14) admits a positive (continuous) viscosity solution $v(x)$ on $\mathbb{R}^d$ when $R > 0$ is sufficiently small. In addition, by Proposition 3.5 in Penent and Privault (2023), $v \in H^{\alpha/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and is a weak solution of (4.14). By Propositions 4.3 and 4.6, the PDE (1.2) admits a viscosity solution in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$, which can be represented as (4.4). Hence by Theorem 1.2 in Kriventsov (2013), ∇u and $f(u,\nabla u)$ are in $C^\epsilon(B(0,R))$ for some $\epsilon > 0$, as the kernel of Δ_α satisfies (1.11) therein. Therefore, by Theorem 1.3 in Serra (2015), the (unique) viscosity solution u is in $C^{\alpha+\epsilon}(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$. □

Lemma 4.7 extends Lemma 3.3 in Penent and Privault (2023) from $i = 0$ to $i \in \{1,\ldots,m\}$.

Lemma 4.7 *Let $i \in \{0,\ldots,m\}$, and assume that $(\mathcal{H}(\mathcal{T}_{x,i}))_{x\in B(0,R)}$ is uniformly integrable. Then, the function $v_i(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,i})]$ is continuous in $x \in \overline{B}(0,R)$.*

Proof (given for completeness). Let $x \in \overline{B}(0,R)$. By Lemma 3.2 therein, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $B(0,R)$ converging fast enough to $x \in \overline{B}(0,R)$ we have

$$\mathbb{P}\Big(\lim_{n\to\infty} \tau_R(x_n) = \tau_R(x)\Big) = 1,$$

and letting $\tau_{\overline{k},x} := \tau_R\big(X^{\overline{k}-}_{T_{\overline{k}-},x}\big)$, $\overline{k} \in \mathcal{K}$, the event

$$A_{\overline{k}} := \Big\{\lim_{n\to\infty} \tau_{\overline{k},x_n} = \tau_{\overline{k},x}\Big\} \bigcap \Big\{\lim_{n\to\infty} X^{\overline{k}}_{\cdot,x_n} = X^{\overline{k}}_{\cdot,x}\Big\},$$

has probability one. Again, by Lemma 3.2-*a*) in *ibid*, for some $n_0(\omega)$ large enough we have

$$X^{\overline{k}}_{\tau_{\overline{k},x_n}} = X^{\overline{k}}_{\tau_{\overline{k},x}} + x_n - x,$$

and $\tau_{\overline{k},x_n} = \tau_{\overline{k},x}$, $n \geqslant n_0(\omega)$. Therefore, using the continuity of the functions ϕ and $c_l, l \in \mathcal{L}$, we have

$$\lim_{n\to\infty} \phi\big(X^{\overline{k}}_{\tau_{\overline{k},x_n}}\big)\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x_n}\mathbb{1}_{\{T_{\overline{k}}=\tau_{\overline{k},x_n}\}} = \phi\big(X^{\overline{k}}_{\tau_{\overline{k},x}}\big)\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\mathbb{1}_{\{T_{\overline{k}}=\tau_{\overline{k},x}\}}, \quad \mathbb{P}-\text{a.s.}$$

and

$$\lim_{n\to\infty} c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x_n}\big)\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x_n}\mathbb{1}_{\{T_{\overline{k}}<\tau_{\overline{k},x_n}\}} = \frac{c_{I_{\overline{k}}}\big(X^{\overline{k}}_{T_{\overline{k}},x}\big)}{q_{I_{\overline{k}}}}\mathcal{W}^{\overline{k}}_{T_{\overline{k}-},x}\mathbb{1}_{\{T_{\overline{k}}<\tau_{\overline{k},x}\}}, \quad \mathbb{P}-\text{a.s.}.$$

Hence by (4.2), on the event $A := \bigcap_{\overline{k}\in\mathcal{K}} A_{\overline{k}}$ of probability one, we have

$$\lim_{n\to\infty} \mathcal{H}_\phi(\mathcal{T}_{x_n,i}(\omega)) = \mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega)).$$

Therefore, for any sequence $(x_n)_{n\geqslant 1}$ converging to $x \in \overline{B}(0,R)$ fast enough, we have

$$\mathbb{P}\big(\lim_{n\to\infty} \mathcal{H}_\phi(\mathcal{T}_{x_n,i}) = \mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega))\big) = 1,$$

which yields $\lim_{n\to\infty} v_i(x_n) = v_i(x)$ by uniform integrability of $(\mathcal{H}_\phi(\mathcal{T}_{x,i}(\omega)))_{x\in B(0,R)}$. □

5 Numerical examples

In this section, we consider numerical applications of the probabilistic representation (4.4). The paths of the α-stable process $X_t = B_{S_t}$ are simulated by time discretization, by generating independent random samples of Brownian motion and of the $\alpha/2$-stable process $(S_t)_{t\in\mathbb{R}_+}$ using the identity in distribution

$$S_t \simeq 2t^{2/\alpha}\frac{\sin(\alpha\big(U+\pi/2)/2\big)}{\cos^{2/\alpha}(U)}\left(\frac{\cos(U-\alpha(U+\pi/2)/2)}{E}\right)^{-1+2/\alpha}$$

based on the Chambers-Mallows-Stuck (CMS) method, where U is uniform on $(-\pi/2,\pi/2)$, and E is exponential with unit parameter, see Relation (3.2) in Weron (1996). In order to keep computation times to a reasonable level, the probability density $\rho(t)$ of $\tau^{i,j}$, $i,j\geqslant 1$, is taken to be gamma with shape parameters ranging from 1.5 to 1.7. The C codes used to plot Figures 4 and 6 are available at `https://github.com/nprivaul/fractional_elliptic`.

Given $k\geqslant 0$, we consider the function

$$\Phi_{k,\alpha}(x) := (1-|x|^2)_+^{k+\alpha/2}, \qquad x\in\mathbb{R}^d,$$

which is Lipschitz if $k>1-\alpha/2$, and solves the Poisson problem $\Delta_\alpha\Phi_{k,\alpha} = -\Psi_{k,\alpha}$ on $\mathbb{R}^d$, with

$$\Psi_{k,\alpha}(x) := \begin{cases} \dfrac{\Gamma((d+\alpha)/2)\Gamma(k+1+\alpha/2)}{2^{-\alpha}\Gamma(k+1)\Gamma(d/2)}\,{}_2F_1\left(\dfrac{d+\alpha}{2},-k;\dfrac{d}{2};|x|^2\right), & |x|\leqslant 1 \\ \dfrac{2^{\alpha}\Gamma((d+\alpha)/2)\Gamma(k+1+\alpha/2)}{\Gamma(k+1+(d+\alpha)/2)\Gamma(-\alpha/2)|x|^{d+\alpha}}\,{}_2F_1\left(\dfrac{d+\alpha}{2},\dfrac{2+\alpha}{2};k+1+\dfrac{d+\alpha}{2};\dfrac{1}{|x|^2}\right), & |x|>1 \end{cases}$$

$x\in\mathbb{R}^d$, where ${}_2F_1(a,b;c;y)$ is Gauss's hypergeometric function, see (5.2) in Getoor (1961), Lemma 4.1 in Biler et al. (2015), and Relation (36) in Huang and Oberman (2016).

Linear gradient term

We take $R=1$, $m=1$, $\mathcal{L}_1=\{(0,0),(0,1)\}$, and

$$c_{(0,0)}(x) := \Psi_{k,\alpha}(x) + (2k+\alpha)|x|^2(1-|x|^2)^{k+\alpha/2}, \quad c_{(0,1)}(x) := 1, \quad b_1(x) := (1-|x|^2)x,$$

and consider the PDE

$$\Delta_\alpha u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)|x|^2(1-|x|^2)^{k+\alpha/2} + (1-|x|^2)x\cdot\nabla u(x) = 0, \tag{5.1}$$

$x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,1)$, and explicit solution

$$u(x) = \Phi_{k,\alpha}(x) = (1 - |x|^2)_+^{k+\alpha/2}, \qquad x \in \mathbb{R}^d.$$

The random tree associated to (5.1) starts at the point $x \in B(0,1)$, and branches into 0 branch or 1 branch as in the random tree samples of Figure 3.

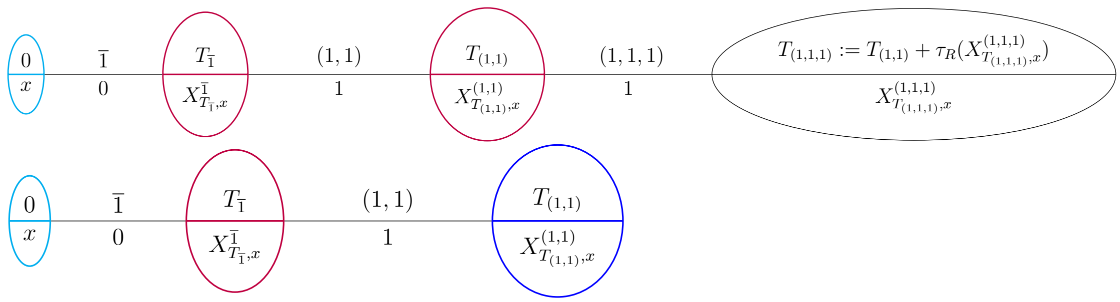
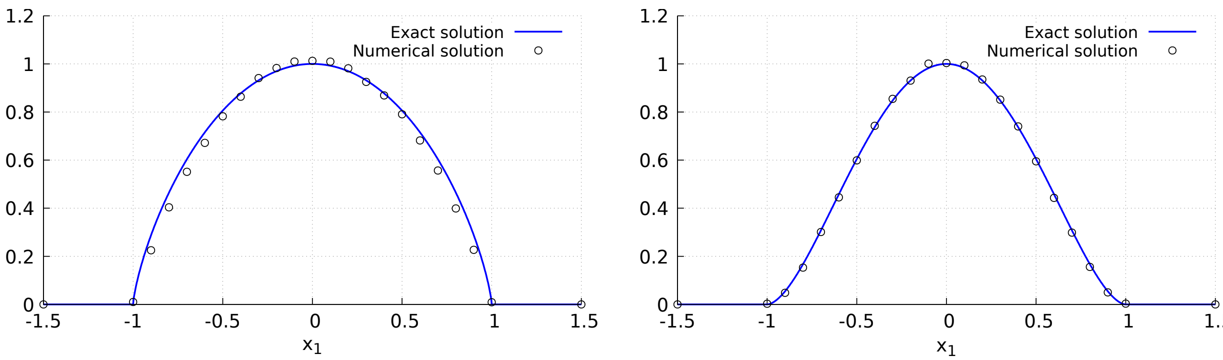


Figure 3: Random tree samples for the PDE (5.1).

The simulations of Figures 4-a) and 4-b) respectively use 10^7 and 2×10^7 Monte Carlo samples.



(a) Numerical solution of (5.1) with $k = 0$.

(b) Numerical solution of (5.1) with $k = 1$.

Figure 4: Numerical solution of (5.1) in dimension $d = 10$ with $\alpha = 1.75$.

Nonlinear gradient term

In this example we take $\mathcal{L}_1 = \{(0,0), (0,2)\}$,

$$c_{(0,0)}(x) := \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha}, \quad c_{(0,2)}(x) := -1, \quad b_1(x) := (1-|x|^2)x,$$

and consider the PDE with nonlinear gradient term

$$\Delta_\alpha u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha} - ((1-|x|^2)x \cdot \nabla u(x))^2 = 0, \qquad (5.2)$$

$x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,R)$, and explicit solution

$$u(x) = \Phi_{k,\alpha}(x) = (1 - |x|^2)_+^{k+\alpha/2}, \qquad x \in \mathbb{R}^d.$$

The random tree associated to (5.2) starts at a point $x \in B(0,1)$ and branches into 0 branch or 2 branches as in the random tree sample of Figure 5.

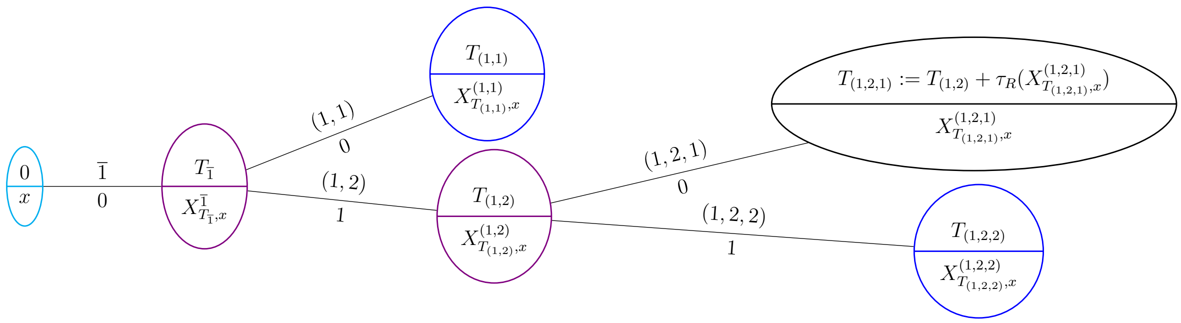
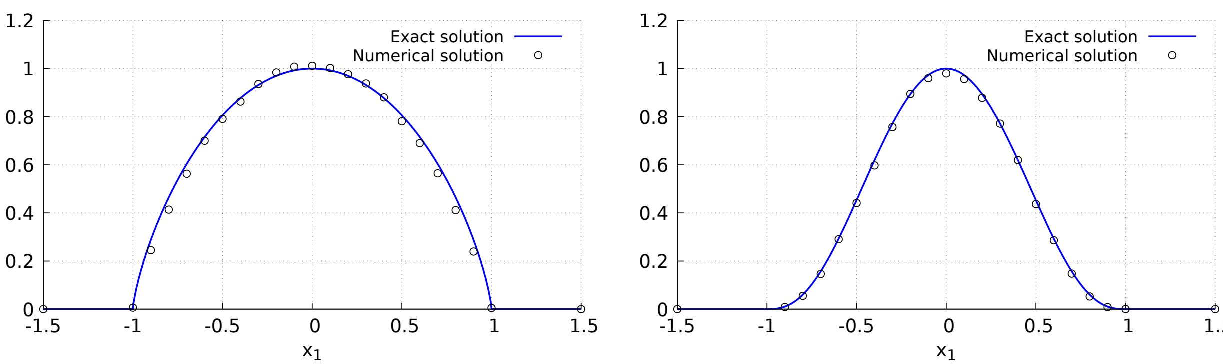


Figure 5: Random tree sample for the PDE (5.2).

The simulations of Figure 6 use five million Monte Carlo samples.



(a) Numerical solution of (5.2) with $k = 0$.

(b) Numerical solution of (5.2) with $k = 2$.

Figure 6: Numerical solution of (5.2) in dimension $d = 10$ with $\alpha = 1.75$.

References

A. Agarwal and J. Claisse. Branching diffusion representation of semi-linear elliptic PDEs and estimation using Monte Carlo method. *Stochastic Processes and their Applications*, 130(8):5006–5036, 2020.

D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

G. Barles, E. Chasseigne, and C. Imbert. On the Dirichlet problem for second-order elliptic integro-differential equations. *Indiana Univ. Math. J.*, 57(1):213–246, 2008.

G. Barles, E. Chasseigne, and C. Imbert. Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)*, 13(1):1–26, 2011.

G. Barles, E. Chasseigne, A. Ciomaga, and C. Imbert. Lipschitz regularity of solutions for mixed integro-differential equations. *J. Differential Equations*, 252(11):6012–6060, 2012.

P. Biler, C. Imbert, and G. Karch. The nonlocal porous medium equation: Barenblatt profiles and other weak solutions. *Archive for Rational Mechanics and Analysis*, 215(2):497–529, 2015.

A. Biswas and E. Topp. Nonlocal ergodic control problem in $\mathbb{R}^d$. *Math. Ann.*, 390(1):45–94, 2024.

K. Bogdan, T. Kulczycki, and A. Nowak. Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes. *Illinois J. Math.*, 46(2):541–556, 2002.

K. Bogdan, T. Grzywny, and M. Ryznar. Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.*, 38(5):1901–1923, 2010.

K. Bogdan, T. Grzywny, and M. Ryznar. Barriers, exit time and survival probability for unimodal Lévy processes. *Probab. Theory Related Fields*, 162(1-2):155–198, 2015.

J.-M. Bony, P. Courrège, and P. Priouret. Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 2):369–521 (1969), 1968.

C. Bucur. Some observations on the Green function for the ball in the fractional Laplace framework. *Commun. Pure Appl. Anal.*, 15(2):657–699, 2016.

M. Felsinger, M. Kassmann, and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4): 779–809, 2015.

R.K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101: 75–90, 1961.

P. Henry-Labordère, N. Oudjane, X. Tan, N. Touzi, and X. Warin. Branching diffusion representation of semilinear PDEs and Monte Carlo approximation. *Ann. Inst. H. Poincaré Probab. Statist.*, 55(1):184–210, 2019.

Y. Huang and A. Oberman. Numerical methods for the fractional Laplacian: a finite difference-quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014.

Y. Huang and A. Oberman. Finite difference methods for fractional Laplacians. Preprint arXiv:1611.00164, 2016.

N. Ikeda, M. Nagasawa, and S. Watanabe. Branching Markov processes I, II, III. *J. Math. Kyoto Univ.*, 8-9: 233–278, 365–410, 95–160, 1968-1969.

H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs. *Comm. Pure Appl. Math.*, 42(1):15–45, 1989.

D. Kriventsov. $C^{1,\alpha}$ interior regularity for nonlinear nonlocal elliptic equations with rough kernels. *Comm. Partial Differential Equations*, 38(12):2081–2106, 2013.

M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7–51, 2017.

J.A. López-Mimbela. A probabilistic approach to existence of global solutions of a system of nonlinear differential equations. In *Fourth Symposium on Probability Theory and Stochastic Processes (Spanish) (Guanajuato, 1996)*, volume 12 of *Aportaciones Mat. Notas Investigación*, pages 147–155. Soc. Mat. Mexicana, México, 1996.

C. Mou. Perron's method for nonlocal fully nonlinear equations. *Analysis and PDE*, 10(5):1227–1254, 2017.

M. Nagasawa and T. Sirao. Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.*, 139:301–310, 1969.

G. Penent and N. Privault. Existence and probabilistic representation of the solutions of semilinear parabolic PDEs with fractional Laplacians. *Stochastics and Partial Differential Equations: Analysis and Computations*, 10:446–474, 2022.

G. Penent and N. Privault. Existence of solutions for nonlinear elliptic PDEs with fractional Laplacians on open balls. *Commun. Pure Appl. Anal.*, 22(8):2646–2660, 2023.

X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016.

X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary. *J. Math. Pures Appl.*, 101(3):275–302, 2014.

J. Serra. $C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels. *Calc. Var. Partial Differential Equations*, 54(4):3571–3601, 2015.

R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

R. Servadei and E. Valdinoci. Weak and viscosity solutions of the fractional Laplace equation. *Publ. Mat.*, 58(1):133–154, 2014.

A.V. Skorokhod. Branching diffusion processes. *Teor. Verojatnost. i. Primenen.*, 9:492–497, 1964.

R. Weron. On the Chambers-Mallows-Stuck method for simulating skewed stable random variables. *Statistics & probability letters*, 28(2):165–171, 1996.