Graph connectivity with fixed endpoints in the random-connection model

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Abstract

We consider the count of subgraphs with an arbitrary configuration of endpoints in the random-connection model based on a Poisson point process on $\mathbb{R}^d$. We present combinatorial expressions for the computation of the cumulants and moments of all orders of such subgraph counts, which allow us to estimate the growth of cumulants as the intensity of the underlying Poisson point process goes to infinity. As a consequence, we obtain a central limit theorem with explicit convergence rates under the Kolmogorov distance, and connectivity bounds. Numerical examples are presented using a computer code in SageMath for the closed-form computation of cumulants of any order, for any type of connected subgraph, and for any configuration of endpoints in any dimension $d \geq 1$. In particular, graph connectivity estimates, Gram-Charlier expansions for density estimation, and correlation estimates for joint subgraph counting are obtained.

Keywords: Random-connection model, subgraph count, normal approximation, Kolmogorov distance, cumulant method, Poisson point process, random graphs, connectivity.
Mathematics Subject Classification: 60D05, 05C80, 60G55, 60F05.

1 Introduction

This paper considers the statistics and asymptotic behavior of subgraph counts in a multidimensional random-connection model based on a Poisson point process, which can be used to model physical systems in e.g. statistical mechanics [GGD16], wireless networks [TMA07, MZA10, GBR+15], or cosmology [CZK17, FY20].

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The random-connection model, in which vertices are randomly located and connected with location-dependent probabilities, is a natural generalization of e.g. the Erdős-Rényi random graph or the stochastic block model [SN97]. Namely, given μ a diffuse Radon measure on $\mathbb{R}^d$, the random-connection model $G_H(\eta)$ consists of an underlying Poisson point process η on $\mathbb{R}^d$ with intensity of the form $\lambda\mu(\mathrm{d}x)$, $\lambda > 0$, in which any two vertices x, y in η are connected with the probability $H(x, y)$, where $H : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is a symmetric connection function.

In addition to modeling the random locations of network nodes, many applications of wireless networks require the use of endpoints which are physical devices placed at given fixed locations, such as for example roadside units in vehicular networks such as VANETs, see, e.g., [NZZ+11], [ZCY+12]. The count of subgraphs that connect any single point x in the Poisson process η to m fixed endpoints $y_1, \ldots, y_m \in \mathbb{R}^d$ is known to have a Poisson distribution with mean $\lambda \int_{\mathbb{R}^d} H(x, y_1) \cdots H(x, y_m)\, \mu(\mathrm{d}x)$, see e.g. § 4 in [Pri19]. This Poisson property has been used in [KGKLN23] to derive closed-form estimates of two-hop connectivity in the random-connection model when $m = 2$, see Proposition III.2 therein.

In this paper, we consider the count of general connected subgraphs with a general configuration of fixed endpoints at fixed locations $y_1, \ldots, y_m \in \mathbb{R}^d$ in the random-connection model $G_H(\eta \cup \{y_1, \ldots, y_m\})$ constructed on the union of the Poisson point process η and $\{y_1, \ldots, y_m\}$. In particular, we extend the subgraph count cumulant formulas obtained on $G_H(\eta)$ in [LP24] by taking into account the presence of endpoints in $G_H(\eta \cup \{y_1, \ldots, y_m\})$, and we provide SageMath coding implementations for joint cumulant expressions of any order.

In Proposition 3.4 we derive general expressions for the moments and cumulants of the count $N^G_{y_1,\ldots,y_m}$ of subgraphs with fixed endpoints $y_1, \ldots, y_m$ in $G_H(\eta \cup \{y_1, \ldots, y_m\})$. Such expressions allow us to determine the dominant terms in the growth of cumulants as the intensity λ of the underlying point process tends to infinity, by estimating the counts of vertices and edges in connected partition diagrams as in e.g. [Kho08]. As a consequence, in Theorem 4.1 we obtain growth estimates for the cumulants of the subgraph count $N^G_{y_1,\ldots,y_m}$.

This allows us to show the convergence of renormalized subgraph counts to the normal distribution in Proposition 4.3 as the intensity λ of the underlying Poisson point process on $\mathbb{R}^d$ tends to infinity. Convergence rates under the Kolmogorov distance are then obtained in Proposition 4.4 for the normal approximation of subgraph counts from the combinatorics of cumulants and the Statulevičius condition, see [RSS78, DJS22] and Lemma B.1, extending the results obtained in [LP24] for subgraphs without endpoints. See also [GT18] for other applications of this condition to concentration inequalities, normal approximation and moderate deviations for random polytopes. In Proposition 4.5, connectivity probability estimates and bounds are derived using the second moment method and the factorial moment expansions in Proposition B.2.

In Section 5 we consider several examples of subgraphs with endpoints such as k-hop paths, triangles and trees, for which exact cumulant computations are matched to their Monte Carlo estimates using the Rayleigh connection function $H(x, y) = e^{-\beta\|x-y\|^2}$, $\beta > 0$. In those examples we obtain graph connectivity estimates, and correlation estimates for joint graph counting, which are matched to the outputs of Monte Carlo simulations. In addition, using third order cumulant expressions, we also provide improved fits of probability density functions of renormalized subgraph counts when the Gaussian approximation is not valid, see Figure 9.

Computations are done in closed form using symbolic calculus in the SageMath coding implementations presented in Appendices D-E, and available for download at `https://github.com/nprivaul/random-connection`. We note that although intensive computations may be required, the types of connected subgraphs and associated configurations of endpoints considered is only limited by the available computing power.

This paper is organised as follows. Section 2 introduces some preliminaries on subgraph counting and the computation of moments using summations over partitions in the random-connection model. In Section 3, we use partition diagrams to compute the cumulants of the counts of subgraphs with endpoints in the random-connection model. Subgraph count asymptotics and the associated central limit theorem are given in Section 4, and numerical examples are presented in Section 5. A general derivation of joint cumulant identities is given in Appendix A, extending the construction of [LP24] from the univariate to the multivariate case, for use in Section 5.5. Basic results on Gram-Charlier expansions and probability approximation using cumulant and moment methods are recalled in Appendices B and C. The SageMath codes for the computation of cumulants and joint cumulants are listed in Appendices D and E, and available for download at `https://github.com/nprivaul/random-connection`.

2 Subgraph counts in the random-connection model

In what follows we consider a Radon measure μ on $\mathbb{R}^d$, and we let $\mathbb{P}_\lambda$, $\lambda > 0$, denote the distribution of the Poisson point process η with intensity $\lambda\mu(\mathrm{d}x)$ on the space

$$\mathcal{C} := \{\eta \subset \mathbb{R}^d \ : \ |\eta \cap A| < \infty \text{ for any bounded set } A \subset \mathbb{R}^d\}$$

of locally finite configurations on $\mathbb{R}^d$, whose elements $\eta \in \Omega$ are identified with the Radon point measures $\eta = \sum_{x\in\eta} \epsilon_x$, so that $\eta(B)$ represents the random number of points contained in a Borel set in $\mathbb{R}^d$. In other words,

i) for any relatively compact Borel set $B \subset \mathbb{R}^d$, the distribution of $\eta(B)$ under $\mathbb{P}_\lambda$ is Poisson with parameter $\lambda\mu(B)$;

ii) for any $n \geq 2$ and pairwise disjoint relatively compact Borel sets $B_1, \ldots, B_n \subset \mathbb{R}^d$, the random variables $\eta(B_1), \ldots, \eta(B_n)$ are independent under $\mathbb{P}_\lambda$.

For $n \geq 1$ we let $[n] := \{1, \ldots, n\}$, where n will later on denote the order of the considered moments and cumulants of subgraph counts, and for any set A we denote by $\Pi(A)$ the collection of all set partitions of A. We also let $|A|$ denote the number of elements of any finite set A, and, in particular, $|\sigma|$ represents the number of blocks in a partition $\sigma \in \Pi([n] \times [r])$. Our approach to the computation of moments relies on moment identities on the following form, see Proposition 3.1 in [Pri12], and Proposition A.5 for its multivariate generalization.

Proposition 2.1 *Let $n \geq 1$ and $r \geq 1$, and let $f : (\mathbb{R}^d)^r \to \mathbb{R}$ be a sufficiently integrable measurable function. We have*

$$\mathbb{E}\left[\left(\sum_{(x_1,\ldots,x_r)\in\eta^r} f(x_1, \ldots, x_r)\right)^n\right] = \sum_{\rho\in\Pi([n]\times[r])} \lambda^{|\rho|} \int_{(\mathbb{R}^d)^{|\rho|}} \prod_{k=1}^{n} f\left(x_{\zeta_\rho(k,1)}, \ldots, x_{\zeta_\rho(k,r)}\right) \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_{|\rho|}),$$

where, for $\rho = \{\rho_1, \ldots, \rho_{|\rho|}\}$ a partition of $[n] \times [r]$, we let $\zeta_\rho(k, l)$ denote the index p of the block ρ_p of ρ to which (k, l) belongs.

In particular, Proposition 2.1 will yield cumulant expressions from Möbius inversion and combinatorial arguments based on [MM91], [Kho08] and [LP24], see Propositions 3.4 and A.8.

Definition 2.2 *Given $H : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ a symmetric connection function and $y_1, \ldots, y_m$ fixed points in $\mathbb{R}^d$, the random-connection model $G_H(\eta \cup \{y_1, \ldots, y_m\})$ is the random graph* *built on the union of* $\{y_1,\dots,y_m\}$ *and a Poisson point process sample* η*, in which any two distinct points* $x,y\in\eta\cup\{y_1,\dots,y_m\}$ *are independently connected by an edge with the probability* $H(x,y)$.

In the sequel we will consider a family of connected graphs with endpoints which are described in the following assumption.

Assumption 2.1 *Given* $r\geq 2$ *and* $m\geq 0$*, we consider a connected graph* $G=(V_G,E_G)$ *with edge set* E_G *and vertex set* $V_G=(v_1,\dots,v_r;w_1,\dots,w_m)$*, such that*

i) the subgraph G *induced by* G *on* $\{v_1,\dots,v_r\}$ *is connected, and*

ii) the endpoint vertices $w_1,\dots,w_m$ *are not adjacent to each other in* G.

In case $m=0$*, Condition (ii) is void and* $V_G=(v_1,\dots,v_r)$.

In Figure 1 an example of a graph satisfying Assumption 2.1 is described with $r=4$ and $m=2$.

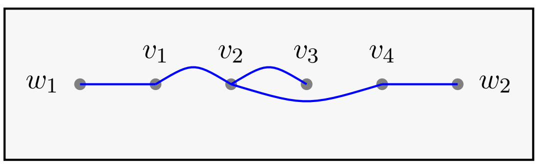


Figure 1: Graph $G=(V_G,E_G)$ with $V_G=(v_1,v_2,v_3,v_4;w_1,w_2)$, $n=3$, $r=4$, $m=2$.

As a convention, in the next definition the sets $\{w_1,\dots,w_m\}$ and $\{y_1,\dots,y_m\}\subset\mathbb{R}^d$ are empty when $m=0$.

Definition 2.3 *Let* G *be a graph satisfying Assumption 2.1. Given* $m\geq 0$ *fixed points* $y_1,\dots,y_m\in\mathbb{R}^d$*, for a.s.* η *we let* $N^G_{y_1,\dots,y_m}$ *denote the count of subgraphs in* $G_H(\eta\cup\{y_1,\dots,y_m\})$ *that are isomorphic to* $G=(V_G,E_G)$ *in the sense that there exists a (random) injection from* V_G *into* $\eta\cup\{y_1,\dots,y_m\}$ *which is one-to-one from* $\{w_1,\dots,w_m\}$ *to* $\{y_1,\dots,y_m\}$*, and preserves the graph structure of* G.

According to Definition 2.3, we express the subgraph count $N^G_{y_1,\dots,y_m}$ as

$$N^G_{y_1,\dots,y_m}=\sum_{(x_1,\dots,x_r)\in\eta^r}f_{y_1,\dots,y_m}(x_1,\dots,x_r),$$

where the random function $f : (\mathbb{R}^d)^r \to \{0,1\}$ defined as

$$f_{y_1,\ldots,y_m}(x_1,\ldots,x_r) := \prod_{\substack{1\le i\le r\\ 1\le j\le m\\ \{v_i,w_j\}\in E_G}} \mathbf{1}_{\{y_j\leftrightarrow x_i\}} \prod_{\substack{1\le k,l\le r\\ \{v_k,v_l\}\in E_G}} \mathbf{1}_{\{x_k\leftrightarrow x_l\}}, \qquad x_1,\ldots,x_r \in \mathbb{R}^d,$$

is independent of the Poisson point process η, and $\mathbf{1}_{\{x\leftrightarrow y\}} = 1$ if and only if $x \neq y$ and $x, y \in \mathbb{R}^d$ are connected in the random-connection model $G_H(\eta \cup \{y_1,\ldots,y_m\})$.

3 Partition diagrams

This section introduces the combinatorial background needed for the derivation of moment and cumulant expressions of subgraph counts. The next definition introduces a notion of connectedness over the rows of partitions of $[n]\times[r]$, and a flatness property which is satisfied when two indices on a same row belong to a given block, see Chapter 4 of [PT11] and Figure 2 below.

Definition 3.1 *Given $n, r \geq 1$, let $\pi := \{\pi_1,\ldots,\pi_n\}$ be the partition in $\Pi([n]\times[r])$ given by*

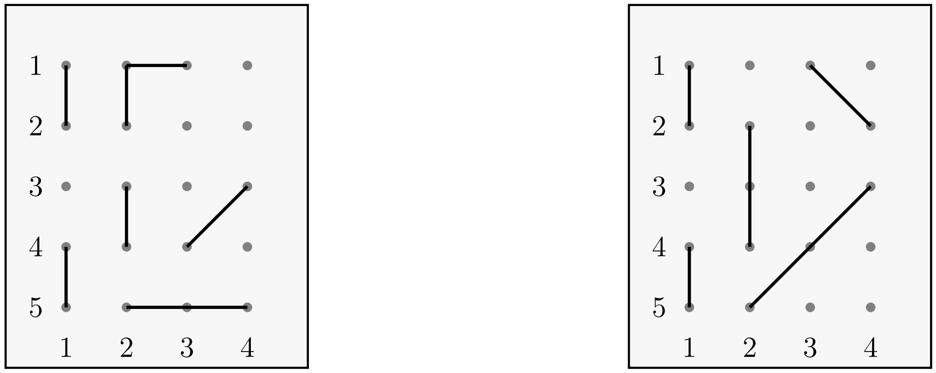
$$\pi_i := \{(i,1),\ldots,(i,r)\}, \quad i = 1,\ldots,n.$$

i) A set partition $\sigma \in \Pi([n]\times[r])$ is connected if $\sigma \vee \pi = \widehat{1}$, where $\sigma\vee\pi$ is the finest set partition which is coarser than both σ and π, and $\widehat{1} = \{[n]\times[r]\}$ is the coarsest partition of $[n]\times[r]$.

ii) A set partition $\sigma \in \Pi([n]\times[r])$ is non-flat if $\sigma\wedge\pi = \widehat{0}$, where $\sigma\wedge\pi$ is the coarsest set partition which is finer than both σ and π, and $\widehat{0}$ is the finest partition of $[n]\times[r]$.

We let $\Pi_{\widehat{1}}([n]\times[r])$ denote the collection of all connected partitions of $[n]\times[r]$.

In the sequel, every partition $\rho \in \Pi(\pi_1\cup\cdots\cup\pi_n)$ will be arranged into a diagram denoted by $\Gamma(\rho,\pi)$, by arranging $\pi_1,\ldots,\pi_n$ into n rows and connecting together the elements of every block of ρ. Figure 2 presents two illustrations of flat non-connected and connected non-flat partition diagrams with $n = 5$ and $r = 4$, in which the partition ρ is represented using line segments.



(a) Flat non-connected diagram $\Gamma(\rho, \pi)$. (b) Connected non-flat diagram $\Gamma(\rho, \pi)$.

Figure 2: Two examples of partition diagrams with $n = 5$ and $r = 4$.

In Definition 3.2, to any graph G and set partition $\rho \in \Pi([n] \times [r])$ we associate a graph ρ_G whose vertices are the blocks of ρ. For this, we use n copies of the graphs induced by the v_i's with addition of the end-points $w_1, \ldots, w_m$, and we merge the nodes obtained in this way on $[n] \times [r]$ according to the partition ρ.

Definition 3.2 *Given ρ a partition of $[n] \times [r]$ and $G = (V_G, E_G)$ a connected graph on $V_G = (v_1, \ldots, v_r; w_1, \ldots, w_m)$, we let ρ_G denote the graph constructed as follows on $[m] \cup [n] \times [r]$:*

i) for all $j_1, j_2 \in [r]$, $j_1 \neq j_2$, and $i \in [n]$, an edge links (i, j_1) to (i, j_2) iff $\{v_{j_1}, v_{j_2}\} \in E_G$;

ii) for all $(j, k) \in [r] \times [m]$ and $i \in [n]$, an edge links (k) to (i, j) iff $\{v_j, w_k\} \in E_G$;

iii) for all $i_1, i_2 \in [n]$ and $j_1, j_2 \in [r]$, merge any two nodes (i_1, j_1) and (i_2, j_2) if they belong to a same block in ρ;

iv) eliminating any redundant edges created by the above construction.

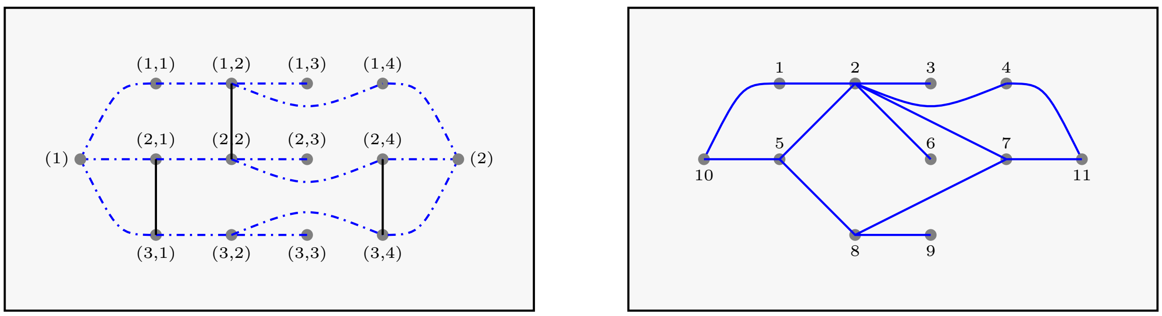
If $\rho \in \Pi([n] \times [r])$ takes the form $\rho = \{b_1, \ldots, b_{|\rho|}\}$, the graph ρ_G forms a connected graph with $|\rho| + m$ vertices, and we reindex the set of vertices V_{ρ_G} of ρ_G as $V_{\rho_G} = [|\rho| + m]$ according to the lexicographic order on $\mathbb{N} \times \mathbb{N}$, followed by the remaining m vertices, indexed as $\{|\rho| + 1, \ldots, |\rho| + m\}$, see Figure 3-$b$) in which we have $|\rho| = 9$, $m = 2$, and $V_{\rho_G} = (1, \ldots, 9; 10, 11)$.

Example. Take $r = 4$, $m = 2$ and $V_G = (v_1, v_2, v_3, v_4; w_1, w_2)$. Figure 3-$b$) shows the graph ρ_G defined from $G = (V_G, E_G)$ of Figure 1 and the 9-block partition $\rho \in \Pi([3] \times [4])$ given by

$$\rho = \big\{\{(1,1)\},$$
$$\{(1,2),(2,2)\},$$

$$
\begin{aligned}
&\{(1,3)\},\\
&\{(1,4)\},\\
&\{(2,1),(3,1)\},\\
&\{(2,3)\},\\
&\{(2,4),(3,4)\},\\
&\{(3,2)\},\\
&\{(3,3)\}\}.
\end{aligned}
$$

In Figure 3-(a) the partition ρ is represented using line segments.



(a) Diagram before merging edges and vertices.

(b) Graph ρ_G after merging edges and vertices.

Figure 3: Example of graph ρ_G with $n = 3$, $r = 4$, and $m = 2$.

Definition 3.3 *For $\rho \in \Pi([n] \times [r])$ of the form $\rho = \{b_1, \dots, b_{|\rho|}\}$ and $j \in [m]$, we let*

$$
\mathcal{A}_j^\rho := \{k \in [|\rho|] \ : \ \exists (s,i) \in b_k \text{ s.t. } (v_i, w_j) \in E_G\}
$$

denote the neighborhood of the vertex $(|\rho| + j)$ in ρ_G, $j = 1, \dots, m$.

For example, in the graph ρ_G of Figure 3 we have $\mathcal{A}_1^\rho = \{1,5\}$ and $\mathcal{A}_2^\rho = \{4,7\}$. The following partition summation formulas extend [LP24, Proposition 5.1] to the counting of subgraphs with endpoints, and they are a special case of Proposition A.8 in appendix, which deals with joint subgraph counting.

Proposition 3.4 *Let $m \geq 0$. The moments and cumulants of $N^G_{y_1,\dots,y_m}$ admit the following expressions:*

$$
\mathbb{E}_\lambda\big[\big(N^G_{y_1,\dots,y_m}\big)^n\big] = \sum_{\substack{\rho \in \Pi([n]\times[r]) \\ \rho\wedge\pi = \widehat{0} \\ (\mathrm{non-flat})}} \lambda^{|\rho|} \int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1\leq j\leq m \\ i \in \mathcal{A}_j^\rho}} H(x_i, y_j) \prod_{\substack{1\leq k,l\leq|\rho| \\ \{k,l\}\in E_{\rho_G}}} H(x_k, x_l)\, \mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{|\rho|}),
$$

and

$$\kappa_n\big(N^G_{y_1,\ldots,y_m}\big) = \sum_{\substack{\rho\in\Pi_{\widehat{1}}([n]\times[r])\\ \rho\wedge\pi=\widehat{0}\\ \text{(non-flat connected)}}} \lambda^{|\rho|}\int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1\le j\le m\\ i\in\mathcal{A}^\rho_j}} H(x_i,y_j) \prod_{\substack{1\le k,l\le|\rho|\\ \{k,l\}\in E_{\rho_G}}} H(x_k,x_l)\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{|\rho|}). \tag{3.1}$$

We note in particular that $N^G_{y_1,\ldots,y_m}$ has positive cumulants, and when $n=1$ the first moment of $N^G_{y_1,\ldots,y_m}$ is given by

$$\mathbb{E}_\lambda\big[N^G_{y_1,\ldots,y_m}\big] = \lambda^r\int_{(\mathbb{R}^d)^r} \prod_{\substack{1\le i\le r\\ 1\le j\le m\\ \{v_i,w_j\}\in E_G}} H(x_i,y_j) \prod_{\substack{1\le k,l\le r\\ \{v_k,v_l\}\in E_G}} H(x_k,x_l)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_r).$$

The cumulant formula of Proposition 3.4 is implemented in the code listed in Appendix D. We also recall the following lemma, see [LP24, Lemma 2.8], in which maximality of connected non-flat partitions refers to maximizing the number of blocks.

Lemma 3.5 *a) The cardinality of the set* $\mathcal{C}(n,r)$ *of connected non-flat partitions of* $[n]\times[r]$ *satisfies*

$$|\mathcal{C}(n,r)| \le n!^r r!^{n-1}, \qquad n,r\ge 1. \tag{3.2}$$

b) The cardinality of the set $\mathcal{M}(n,r)$ *of maximal connected non-flat partition of* $[n]\times[r]$ *satisfies*

$$|\mathcal{M}(n,r)| = r^{n-1}\prod_{i=1}^{n-1}(1+(r-1)i), \qquad n,r\ge 1,$$

with the bounds

$$((r-1)r)^{n-1}(n-1)! \le |\mathcal{M}(n,r)| \le ((r-1)r)^{n-1}n!, \quad n\ge 1,\ r\ge 2. \tag{3.3}$$

4 Subgraph count asymptotics

In this section, we let $m\ge 1$ and investigate the asymptotic behaviour of the cumulants $\kappa_n\big(N^G_{y_1,\ldots,y_m}\big)$ in (3.1) as the intensity λ tends to infinity, which extends the treatment of [LP24] from $m=0$ to $m\ge 1$.

Assumption 4.1 *We assume that*

i) μ *is the Lebesgue measure on* $\mathbb{R}^d$*, and* *ii) the connection function* $H : \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ *is translation invariant, i.e.* $H(x,y) = H(0, y-x)$, $x, y \in \mathbb{R}^d$, *and*

$$\int_{\mathbb{R}^d} H(0,y)\mathrm{d}y < \infty.$$

The following result provides growth estimates for the cumulants of $N^G_{y_1,\ldots,y_m}$.

Theorem 4.1 *Let* $m \geq 1$, $n \geq 1$ *and* $r \geq 2$, *and suppose that Assumption 4.1 is satisfied. We have*

$$0 < \kappa_n\big(N^G_{y_1,\ldots,y_m}\big) \leq n!^r r!^{n-1} (C\lambda)^{1+(r-1)n}, \tag{4.1}$$

and, for $n = 2$,

$$(r-1)rc^{2r}\lambda^{2r-1} \leq \kappa_2\big(N^G_{y_1,\ldots,y_m}\big) \leq r!(C\lambda)^{2r-1}, \tag{4.2}$$

where $c, C > 0$ *are constants independent of* $r \geq 2$ *and* $n \geq 2$.

Proof. According to Proposition 3.4, every non-flat connected partition $\rho \in \Pi([n] \times [r])$ corresponds to a summand of order $O(\lambda^{|\rho|})$. As the cardinality of maximal non-flat connected partitions is $1 + (r-1)n$, the dominating asymptotic order is $O(\lambda^{1+(r-1)n})$. Precisely, by (3.2)-(3.3) and (3.1), letting $j_0 \in \{1, \ldots, m\}$ such that $\mathcal{A}^\rho_{j_0} \neq \emptyset$, for some $i_0 \in \mathcal{A}^\rho_{j_0}$ we have

$$\begin{aligned}
&c^{n|E_G|} C^{1+(r-1)n} ((r-1)r)^{n-1} (n-1)! \lambda^{1+(r-1)n} \\
&\leq \kappa_n\big(N^G_{y_1,\ldots,y_m}\big) \\
&\leq \lambda^{1+(r-1)n} \sum_{\substack{\rho \in \Pi_{\widehat{1}}([n]\times[r]) \\ \rho \wedge \pi = \widehat{0} \\ (\text{non-flat connected})}} \int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1 \leq j \leq m \\ i \in \mathcal{A}^\rho_j}} H(x_i, y_j) \prod_{\substack{1 \leq k,l \leq |\rho| \\ \{k,l\} \in E_{\rho_G}}} H(x_k, x_l) \mathrm{d}x_1 \cdots \mathrm{d}x_{|\rho|} \\
&\leq \lambda^{1+(r-1)n} \sum_{\substack{\rho \in \Pi_{\widehat{1}}([n]\times[r]) \\ \rho \wedge \pi = \widehat{0} \\ (\text{non-flat connected})}} \int_{(\mathbb{R}^d)^{|\rho|}} H(x_{i_0}, y_{j_0}) \prod_{\substack{1 \leq k,l \leq |\rho| \\ \{k,l\} \in E_{\rho_G}}} H(x_k, x_l) \mathrm{d}x_1 \cdots \mathrm{d}x_{|\rho|} \\
&\leq \lambda^{1+(r-1)n} \sum_{\substack{\rho \in \Pi_{\widehat{1}}([n]\times[r]) \\ \rho \wedge \pi = \widehat{0} \\ (\text{non-flat connected})}} \int_{(\mathbb{R}^d)^{|\rho|}} H(x_{i_0}, y_{j_0}) \prod_{\substack{1 \leq k,l \leq |\rho| \\ \{k,l\} \in E_{\rho'_G}}} H(x_k, x_l) \mathrm{d}x_1 \cdots \mathrm{d}x_{|\rho|},
\end{aligned}$$

where for every $\rho \in \Pi_{\widehat{1}}([n] \times [r])$, ρ'_G is a spanning tree contained in ρ_G, with vertices $\{1, \ldots, |\rho|, |\rho| + j\}$ and such that $|\rho| + j_0$ is a leaf. By integrating successively on the variables which correspond to leaves of ρ'_G as in the proofs of e.g. Theorem 7.1 of [LNS21] or Lemma 3.1 of [CT22] and using (3.2), we obtain

$$\kappa_n\big(N^G_{y_1,\ldots,y_m}\big) \leq (C\lambda)^{1+(r-1)n} n!^r r!^{n-1},$$

due to Assumption 4.1-(ii), where $C := \max\big(1, \int_{\mathbb{R}^d} H(0,y)dy\big)$, which yields the right hand side (4.1). In addition, Proposition 3.4 shows that all cumulants are positive, which completes the proof of (4.1). On the other hand, when $r \geq 2$, by (3.3) we have

$$\kappa_2\big(N^G_{y_1,\ldots,y_m}\big) \geq (r-1)rC^{2r}\lambda^{2r-1},$$

where $C > 0$ is a constant independent of $r \geq 2$ and $n \geq 2$, which shows (4.2). □

In what follows, we consider the centered and normalized subgraph count cumulants defined as

$$\widetilde{N}^G_{y_1,\ldots,y_m} := \frac{N^G_{y_1,\ldots,y_m} - \kappa_1\big(N^G_{y_1,\ldots,y_m}\big)}{\sqrt{\kappa_2\big(N^G_{y_1,\ldots,y_m}\big)}}.$$

Corollary 4.2 *Let $m \geq 1$, $n \geq 2$ and $r \geq 2$. We have*

$$\big|\kappa_n\big(\widetilde{N}^G_{y_1,\ldots,y_m}\big)\big| \leq n!^r C_r^{n/2}\lambda^{-(n/2-1)},$$

where $C_r > 0$ is a constant depending only on $r \geq 2$.

As a consequence of Corollary 4.2, the skewness of $\widetilde{N}^G_{y_1,\ldots,y_m}$ satisfies

$$\big|\kappa_3\big(\widetilde{N}^G_{y_1,\ldots,y_m}\big)\big| \leq C_r\lambda^{-1/2}, \tag{4.3}$$

where $C_r > 0$ is a constant depending only on $r \geq 2$.

Proposition 4.3 *Let $m \geq 1$. The renormalized subgraph count $\widetilde{N}^G_{y_1,\ldots,y_m}$ converges in distribution to the standard normal distribution $\mathcal{N}(0,1)$ as λ tends to infinity.*

Proof. From Corollary 4.2 and (4.3) we have $\kappa_1\big(\widetilde{N}^G_{y_1,\ldots,y_m}\big) = 0$, $\kappa_2\big(\widetilde{N}^G_{y_1,\ldots,y_m}\big) = 1$, and

$$\lim_{n\to\infty} \kappa_n\big(\widetilde{N}^G_{y_1,\ldots,y_m}\big) = 0, \qquad n \geq 3,$$

hence the conclusion follows from Theorem 1 in [Jan88]. □

In addition, from Corollary 4.2 and Lemma B.1 the convergence result of Proposition 4.3 can be made more precise via the following convergence bound in the Kolmogorov distance, which extends Corollary 7.1 in [LP24] from $m = 0$ to $m \geq 1$.

Proposition 4.4 *Let $m \geq 1$. We have*

$$\sup_{x\in\mathbb{R}} \big|\mathbb{P}_\lambda\big(\widetilde{N}^G_{y_1,\ldots,y_m} \leq x\big) - \Phi(x)\big| \leq C_r\lambda^{-1/(4r-2)}, \qquad r \geq 2,$$

where $C_r > 0$ is a constant depending only on $r \geq 2$ and Φ is the cumulative distribution function of the standard normal distribution.

By the second moment method, see e.g. (3.4) page 54 of [JŁR00] or Theorem 2.3.2 in [Roc23], we also obtain the following lower bound for endpoint connectivity and subgraph existence.

Proposition 4.5 *Let $m \geq 1$. We have*

$$\mathbb{P}_\lambda\big(N^G_{y_1,\ldots,y_m} > 0\big) \geq \frac{\big(\mathbb{E}_\lambda\big[N^G_{y_1,\ldots,y_m}\big]\big)^2}{\mathbb{E}_\lambda\big[\big(N^G_{y_1,\ldots,y_m}\big)^2\big]}, \qquad \lambda > 0. \tag{4.4}$$

Theorem 4.1 also shows the bounds

$$\frac{C_{r,1}}{\lambda} \leq \frac{\kappa_2\big(N^G_{y_1,\ldots,y_m}\big)}{\big(\mathbb{E}_\lambda\big[N^G_{y_1,\ldots,y_m}\big]\big)^2} \leq \frac{C_{r,2}}{\lambda}, \qquad \lambda > 0, \tag{4.5}$$

for some constants $C_{r,1}, C_{r,2} > 0$ depending only on $r \geq 2$, from which it follows that the lower bound (4.4) converges to 1 as λ tends to infinity.

5 Numerical examples

In this section we assume that H is the Rayleigh connection function

$$H_\beta(x,y) := e^{-\beta\|x-y\|^2}, \qquad x, y \in \mathbb{R}^d,$$

where $\beta > 0$, and μ is the Lebesgue measure on $\mathbb{R}^d$. In this case, Assumption 4.1 is satisfied. In the following examples, the SageMath code listed in Appendix D is run after loading the definitions of Table 1.

`load("cumulants_parallel.sage")`	# Loading the functions definitions
`λ,β = var("λ,β"); assume(β>0)`	# Variable definitions
`def H(x,y,β): return exp(-β*(x-y)**2)`	# Connection function $H(x,y)$
`def mu(x,λ,β): return 1`	# Flat intensity of $\mu(dx)$

Table 1: Functions definitions.

Computations in this and the following examples are run on a standard desktop computer with an 8-core CPU at 4.10GHz. The limitations imposed by this hardware configuration constrain the product $n \times r \times d$ to be below 15 approximately, in order to maintain computation times at a reasonable level. The illustrations of figures 4, 6, 8 and 10 are provided in dimension $d = 2$ for ease of visualization only. Actual computations may be provided in lower dimension d, due to hardware performance constraints when the cumulant order n is beyond 4. Computations in dimension $d = 2$ are presented in Section 5.3 for triangles with endpoints and in Section 5.4 for trees with one endpoint cumulants of orders 2 and 3, while in Section 5.1 for 3-hop paths with two endpoint and in Section 5.2 for four-hop paths with two endpoints we take $d = 1$ in order to reach the cumulant orders $n = 6$ and $n = 4$. In subsequent code inputs, graphs are coded by their edge set E_G, and the set of endpoints is given by the sequence $\mathsf{EP} = [\mathsf{EP}_1, \ldots, \mathsf{EP}_m]$, where EP_i denotes the set of vertices of the subgraph G on $\{v_1, \ldots, v_r\}$ which are attached to the i-th endpoint, $i = 1, \ldots, m$, with $\mathsf{EP} := [\,]$ the empty sequence when G has no endpoint ($m = 0$).

5.1 Three-hop paths with two endpoints

By a k-hop path, we mean a non-self intersecting path having k edges. We take $m = 2$, $r = 2$, and in Table 2 we compute the first three cumulants of $N^G_{y_1,y_2}$ when G is a 3-hop path with two endpoints in dimension $d = 1$, see Figure 4 for an illustration in dimension $d = 2$. Unlike in the two-hop with two endpoints case, this 3-hop count does not have a Poisson distribution. In the following Figures 4, 6, 8 and 10 the endpoints are denoted by red dots, and their edges are denoted by purple dashed lines.

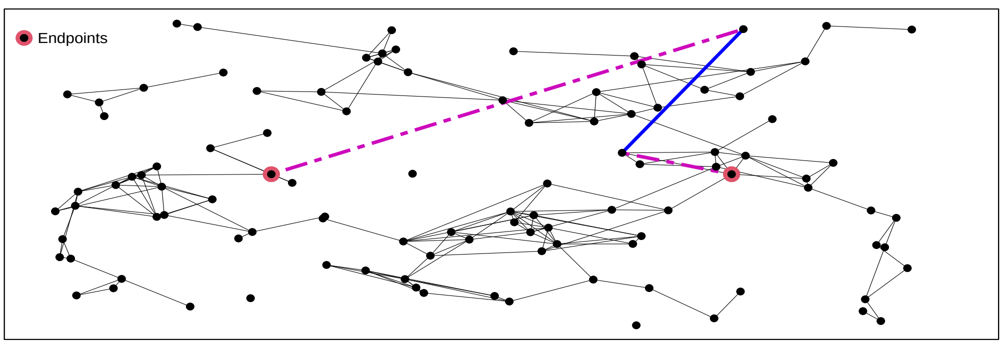


Figure 4: A 3-hop path with two endpoints in dimension $d = 2$.

To make cumulant expressions more compact, the exact formulas in Table 2 are expressed with $y_1 = y_2 = 0$ and $\beta := \pi$, in dimension $d = 1$.

$\mathsf{G} = [[1,2]]$; $\mathsf{EP} = [[1],[2]]$; d=1 # Single edge graph $r = 2$, two endpoint $m = 2$, dimension $d = 1$			
Command	Order	Cumulant output	Connected non-flat partitions
c(1,d,G,EP,mu,H)	1st	$\frac{1}{\sqrt{3}}\lambda^2$	1
c(2,d,G,EP,mu,H)	2nd	$\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\right)\lambda^3 + \left(\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{2}}\right)\lambda^2$	6
c(3,d,G,EP,mu,H)	3rd	$\left(\sqrt{\frac{12}{7}} + \frac{3}{\sqrt{5}} + \frac{3}{\sqrt{7}} + \frac{12}{\sqrt{31}}\right)\lambda^4 + \left(\sqrt{3} + \sqrt{\frac{3}{2}} + \frac{17}{5\sqrt{2}} + \frac{12}{\sqrt{19}}\right)\lambda^3 + \left(\frac{3}{2\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\lambda^2$	68

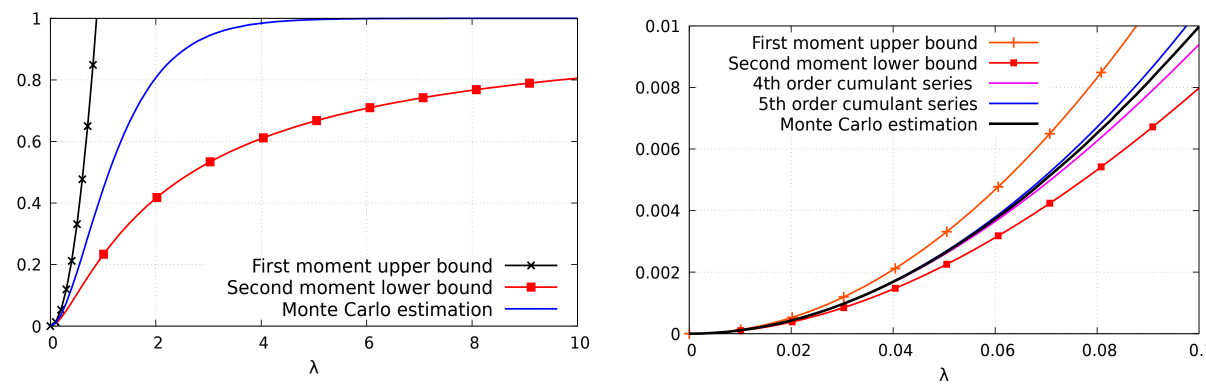
Table 2: Cumulants of the count of 2-hop paths with two endpoints in dimension $d = 1$.

Table 3 lists the counts of connected non-flat partitions and runtimes for the computation of cumulants of orders 1 to 6, and shows that such partitions represent only a fraction (around 25%) of total partition counts.

Order n	2 blocks	3 blocks	4 blocks	5 blocks	6 blocks	7 blocks	Total	$\Pi([n]\times[2])$	Comp. time
1st	1	0	0	0	0	0	1	2	0.5s
2nd	2	4	0	0	0	0	6	15	1s
3rd	4	32	32	0	0	0	68	203	3s
4th	8	208	624	352	0	0	1,192	4,140	1m
5th	16	1,280	8,960	13,904	5,040	0	29,200	115,975	47m
6th	32	7,744	116,160	375,776	351,456	88,544	939,712	4,213,597	29 hours

Table 3: Computation times and counts of connected non-flat *vs.* all partitions in $\Pi([n]\times[2])$.

Figure 5 presents connectivity estimates based on the moment and cumulant formulas of Propositions 4.5 and B.2, in dimension $d = 1$.



(a) First and second moment bounds (4.4). (b) Cumulant approximations (B.1) with $n = 0$.

Figure 5: Connection probabilities.

5.2 Four-hop paths with two endpoints

Here, we take $m = 2$ and $r = 3$, and in Table 4 we compute the first cumulant of $N^G_{y_1,y_2}$ when G is a four-hop path with two endpoints in dimension $d = 1$, see Figure 6 for an illustration in dimension $d = 2$.

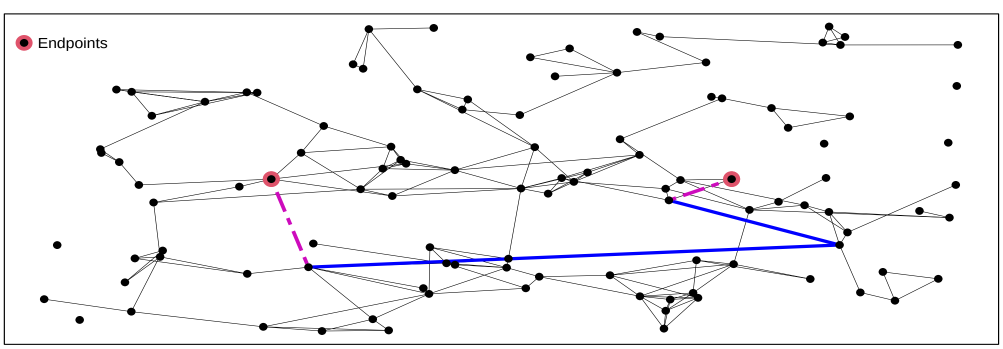


Figure 6: A 4-hop path with two endpoints in dimension $d = 2$.

The closed-form expressions in Table 4 are expressed with $y_1 = y_2 = 0$ and $\beta := \pi$, in dimension $d = 1$.

G = [[1,2],[2,3]]; EP=[[1],[3]]; d=1 # 4-hop path $r = 3$, two endpoints $m = 2$, dimension $d = 1$			
Instruction	Order	Cumulant output	Connected non-flat partitions
c(1,d,G,EP,mu,H)	1st	$\frac{\lambda^3}{2}$	1
c(2,d,G,EP,mu,H)	2nd	$\frac{1}{6}\lambda^3 \left(\left(\sqrt{6} + 4\sqrt{\frac{3}{5}} + \frac{3}{2\sqrt{2}} + \frac{12}{\sqrt{7}} \right) \lambda^2 + \left(3\sqrt{3} + 16\sqrt{\frac{3}{7}} + 8\sqrt{\frac{3}{11}} + \frac{3}{2\sqrt{2}} + \frac{6}{\sqrt{5}} \right) \lambda + \sqrt{3} + \sqrt{6} + 6 \right)$	33

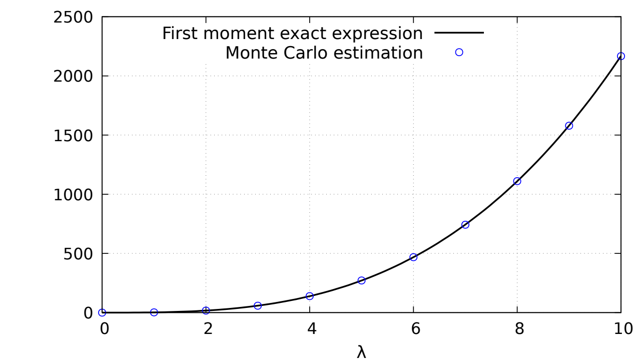
Table 4: First and second cumulants of the count of four-hop paths with two endpoints.

Table 5 presents the counts of connected non-flat partitions at different orders, and shows that such partitions represent only a fraction (around 10%) of total partition counts.

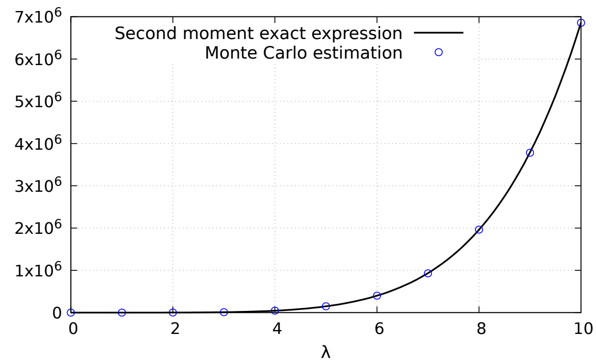
Order n	3 blocks	4 blocks	5 blocks	6 blocks	7 blocks	8 blocks	9 blocks	Total	$\Pi([n] \times [3])$	Comp. time
1st	1	0	0	0	0	0	0	1	5	1s
2nd	6	18	9	0	0	0	0	33	203	2s
3rd	36	540	1242	864	189	0	0	2871	21,147	4m
4th	216	13,608	94,284	186,624	145,908	48,276	5,589	494,500	4,213,597	19 hours

Table 5: Computation times and counts of connected non-flat *vs.* all partitions in $\Pi([n] \times [3])$.

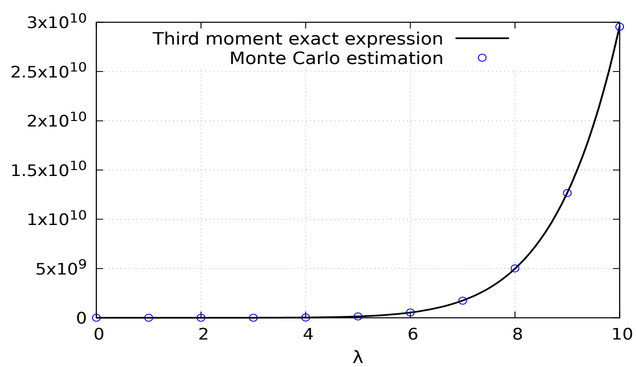
In Figure 7 we plot the corresponding moment expressions *vs.* their Monte Carlo estimates in dimension $d = 1$, with the parameters of Table 4.



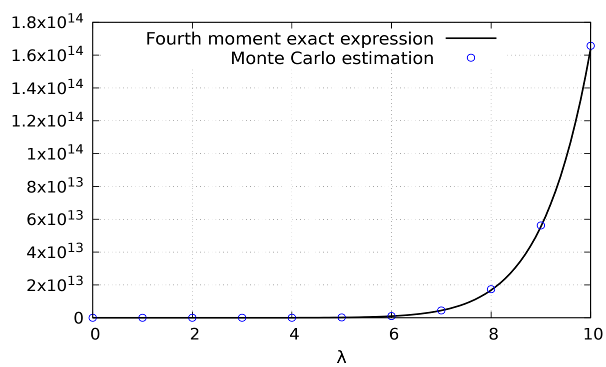
(a) First moment.



(b) Second moment.



(c) Third moment.



(d) Fourth moment.

Figure 7: Moment estimates.

5.3 Triangles with endpoints

Taking $r = 3$ and $m = 3$, in Table 6 we compute the first and second cumulants of $N^G_{y_1,y_1,y_3}$ when G is a triangle with three endpoints in dimension $d = 1$, see Figure 8 for an illustration in dimension $d = 2$.

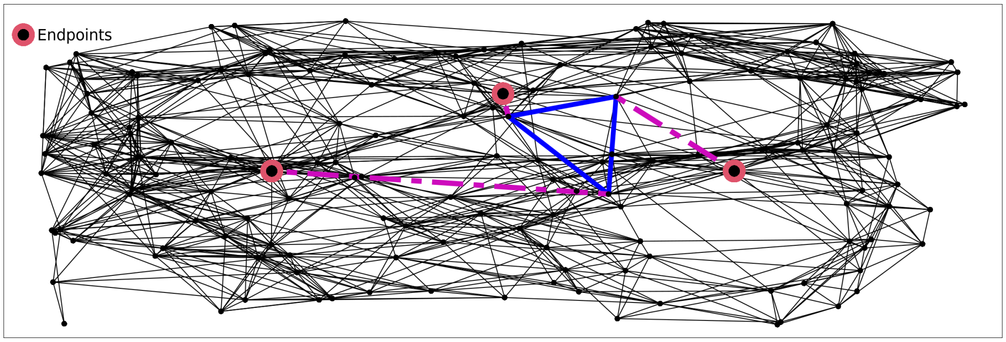


Figure 8: A triangle with three endpoints in dimension $d = 2$.

The closed-form expressions in Table 6 are expressed with $y_1 = y_2 = y_3 = 0$ and $\beta := \pi$, in dimension $d = 1$.

G = [[1,2],[2,3],[3,1]]; EP=[[1],[2],[3]]; d=1; # Triangle graph $r = 3$; three endpoints $m = 3$; dimension $d = 1$			
Instruction	Order	Cumulant output	Connected non-flat partitions
c(1,d,G,EP,mu,H)	1st	$\frac{\lambda^3}{4}$	1
c(2,d,G,EP,mu,H)	2nd	$\left(\frac{\sqrt{3}}{8} + \frac{3}{8}\right)\lambda^5 + \left(\frac{2\sqrt{105}}{35} + \frac{\sqrt{3}}{5} + \frac{3}{4}\right)\lambda^4 + \left(\frac{3\sqrt{35}}{35} + \frac{\sqrt{2}}{5} + \frac{1}{4}\right)\lambda^3$	33

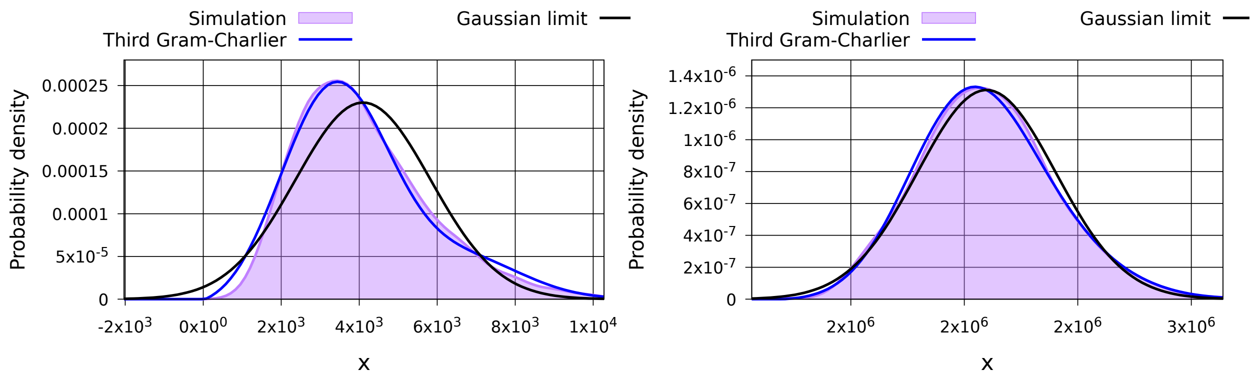
Table 6: First and second cumulants of the count of triangles with three endpoints.

Table 7 presents computation times in dimension $d = 2$.

Order n	3 blocks	4 blocks	5 blocks	6 blocks	7 blocks	Total	$\Pi([n] \times [3])$	Comp. time
1st	1	0	0	0	0	1	5	1s
2nd	6	18	9	0	0	33	203	21s
3rd	36	540	1,242	864	189	2,871	21,147	1 hour

Table 7: Computation times and counts of connected non-flat *vs.* all partitions in $\Pi([n] \times [3])$.

Figure 9 presents second and third order Gram-Charlier expansions (C.1)-(C.2) for the probability density function of the count $N^G_{y_1,y_1,y_3}$ of triangles with three endpoints, based on exact second and third cumulant expressions.



(a) $\lambda = 50$. (b) $\lambda = 400$.

Figure 9: Gram-Charlier density expansions *vs.* Monte Carlo density estimation.

In Figure 9, the purple areas correspond to probability density estimates obtained by Monte Carlo simulations with $\beta = 2$. The second order expansions correspond to the Gaussian diffusion approximation obtained by matching first and second order moments. Figure 9 shows that the actual probability density estimates obtained by simulation can be significantly different from their Gaussian diffusion approximations when skewness takes large absolute values. In addition, in Figure 9 the fourth order Gram-Charlier expansions appear to give the best fit to the actual probability densities, which have positive skewness.

5.4 Trees with one endpoint

Here we take $r = 4$ and $m = 1$, and in Table 8 we compute the first and second cumulants of $N^G_{y_1,y_1,y_3}$ when G is made of a tree and a single endpoint in dimension $d = 2$, see Figure 10 for an illustration.

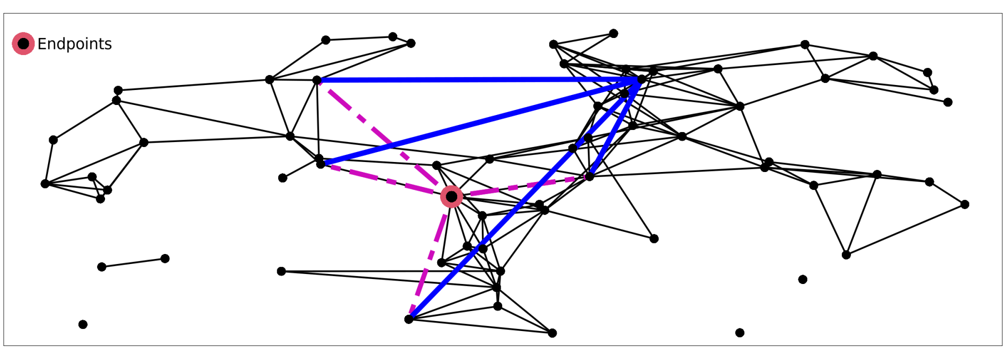


Figure 10: Four trees with a single endpoint in dimension $d = 2$.

In Table 8 the location of the unique endpoint has no impact on cumulant expressions due to space homogeneity of the underlying Poisson point process.

G = [[1,2],[2,3],[2,4]]; EP=[[1,3,4]]; d=2; # Tree $r=4$; single endpoint $m=1$; dimension $d=2$			
Instruction	Order	Cumulant output	Connected non-flat partitions
c(1,d,G,EP,mu,H)	1st	$\frac{\lambda^4}{12}$	1
c(2,d,G,EP,mu,H)	2nd	$\frac{41\lambda^7}{384}+\frac{99039\lambda^6}{165760}+\frac{232885\lambda^5}{175824}+\frac{37\lambda^4}{50}$	208

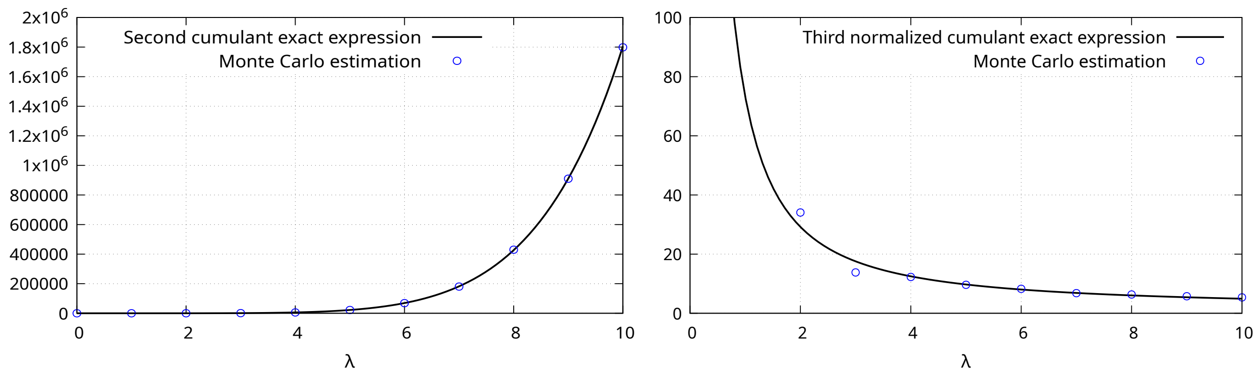
Table 8: First and second cumulants of the count of trees with one endpoint.

The computation times presented in Table 9 are for dimension $d = 2$.

Order n	4 blocks	5 blocks	6 blocks	7 blocks	8 blocks	9 blocks	10 blocks	Total	$\Pi([n]\times[4])$	Comp. time
1st	1	0	0	0	0	0	0	1	15	1s
2nd	24	96	72	15	0	0	0	208	4,140	2m
3rd	576	13,824	50,688	59,904	29,952	6,912	640	162,496	4,213,597	40 hours

Table 9: Computation times and counts of connected non-flat *vs.* all partitions in $\Pi([n]\times[4])$.

In Figure 11 we plot the second cumulant of $N^G_{y_1,\ldots,y_m}$ and the third cumulant of $\widetilde{N}^G_{y_1,\ldots,y_m}$ *vs.* their Monte Carlo estimates in dimension $d = 2$ with $y_1 = y_2 = y_3 = 0$.



(a) Second cumulant. (b) Normalized third cumulant.

Figure 11: Cumulant estimates.

5.5 Correlation of triangles *vs.* four-hop counts

In this example, we run the joint cumulant code provided in Appendix E to compute the correlation of triangle and four-hop counts without endpoints, as a function of the intensity parameter λ. Here, μ is taken to be a finite measure as no endpoints are considered, i.e. we have EP=[ ] and $m = 0$, and the SageMath code listed in Appendix E is run after loading the definitions of Table 10.

`load("cumulants_parallel.sage");load("jointcumulants.sage")`	# Loading the functions definitions
`λ,β = var("λ,β"); assume(β>0)`	# Variable definitions
`def H(x,y,β): return exp(-β*(x-y)**2)`	# Connection function $H(x,y)$
`def mu(x,λ,β): return exp(-β*x**2)`	# Finite intensity measure $\mu(dx)$

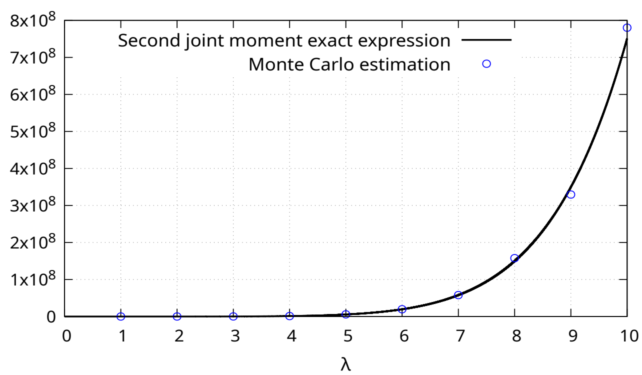
Table 10: Functions definitions.

The closed-form expressions in Table 11 are expressed with $\beta := \pi$, in dimension $d = 2$.

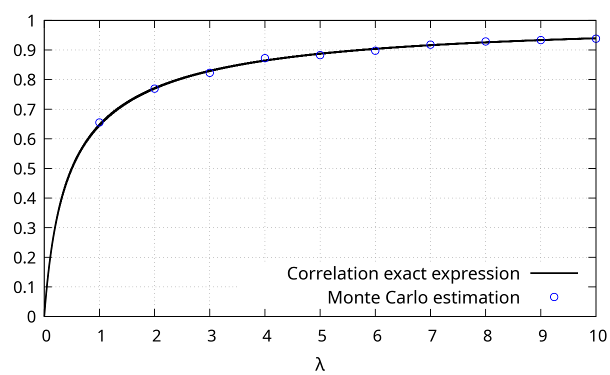
G1 = [[1,2],[2,3],[3,1]]; G2 = [[1,2],[2,3],[3,4],[4,5]]; G2c = [[4,5],[5,6],[6,7],[7,8]]; G = [G1,G2c]; EP=[]; d=2;			
# Triangles G1 and 4-hops G2; $r_1 = 3$, $r_2 = 5$; no endpoints $m = 0$; dimension $d = 2$			
Instruction	Order	Cumulant output	Connected non-flat partitions
c(2,d,G1,EP,mu,H)	2nd	$\frac{3\lambda^5}{64} + \frac{6\lambda^4}{25} + \frac{3\lambda^3}{8}$	33
c(2,d,G2,EP,mu,H)	2nd	$\frac{7344738590701\lambda^9}{687218605505250} + \cdots$	1545
jc(d,G,EP,mu,H)	2nd joint	$\frac{34409\lambda^7}{1537920} + \frac{9101145477\lambda^6}{55004486680} + \frac{10774977\lambda^5}{28148120}$	135

Table 11: Second (joint) moments of triangle counts *vs.* four-hop counts

In Figure 12 we plot the second joint cumulant and correlation of $\left(N^{G_1}, N^{G_2}\right)$ *vs.* their Monte Carlo estimates in dimension $d = 1$.



(a) Second joint cumulant.



(b) Correlation.

Figure 12: Correlation and second joint cumulant estimates.

The limit correlation as λ tends to infinity can be exactly estimated from Table 11 as

$$\frac{34409}{1537920}\sqrt{\frac{64}{3} \times \frac{687218605505250}{7344738590701}} \approx 0.999602.$$

A Multivariate moment and cumulant formulae

In this section, we prove an extension of Proposition 3.4 for the joint moments and cumulants of subgraph counts. The next definition extends Definition 3.1.

Definition A.1 *Given $r_1, \ldots, r_n \geq 1$, we set*

$$\pi_i = \{(i,1), \ldots, (i, r_i)\}, \quad i = 1, \ldots, n,$$

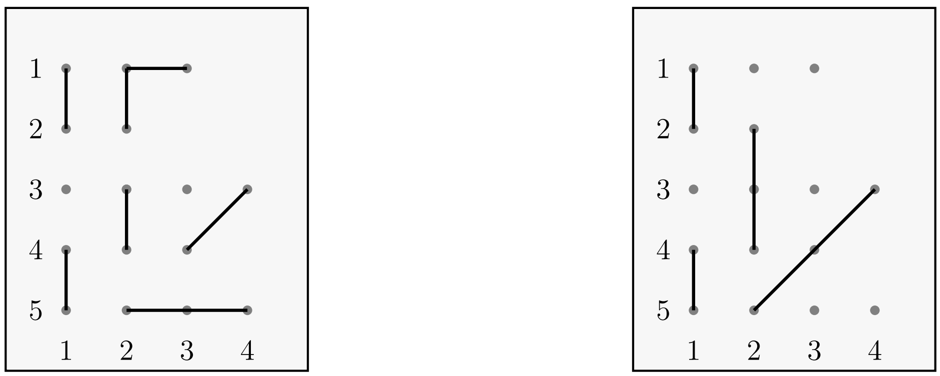
and $\pi := \{\pi_1, \ldots, \pi_n\}$.

i) A set partition $\sigma \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ is connected if $\sigma \vee \pi = \widehat{1}$.

ii) A set partition $\sigma \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ is non-flat if $\sigma \wedge \pi = \widehat{0}$.

We let $\Pi_{\widehat{1}}(\pi_1 \cup \cdots \cup \pi_n)$ denote the collection of all connected partitions of $\pi_1 \cup \cdots \cup \pi_n$.

In what follows, every partition $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ will be arranged into a diagram denoted by $\Gamma(\rho, \pi)$, by arranging $\pi_1, \ldots, \pi_n$ into n rows and connecting together the elements of every block of ρ, see Figure 13 for two illustrations with $n = 5$, $(r_1, r_2, r_3, r_4, r_5) = (3, 2, 4, 3, 4)$.



(a) Non-connected partition diagram $\Gamma(\rho, \pi)$. (b) Connected partition diagram $\Gamma(\rho, \pi)$.

Figure 13: Two examples of partition diagrams.

Definition A.2 extends [LP24, Definition 2.4] to the multivariate setting.

Definition A.2

1) *Given $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$, we let σ_ρ be the partition of $[n]$ defined by the condition*

$$\rho \vee \pi = \left\{ \bigcup_{i \in b} \pi_i : b \in \sigma_\rho \right\}.$$

2) *For any non-empty set $b \subset [n]$, we let*

$$\rho_b := \left\{ c \in \rho : c \subset \bigcup_{i \in b} \pi_i \right\}.$$

As an example, in Figure 14-a), when $b = \{1, 2\}$ we have

$$\rho_{\{1,2\}} = \big\{ \{(1,1), (2,1)\}, \{(1,2), (1,3), (2,2)\} \big\}.$$

(a) Connected subpartition $\rho_{\{1,2\}}$. (b) Splitting ρ into connected subpartitions ρ_{b_1}, ρ_{b_2}.

Figure 14: Diagram $\Gamma(\rho,\pi)$ and splitting of the partition ρ with $\rho\vee\pi = \{\pi_1\cup\pi_2, \pi_3\cup\pi_4\cup\pi_5\}$.

We note that for $b \subset [n]$ we have $\pi_b = \{\pi_i : i \in b\}$, and any partition $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ can be split into subpartitions deduced from the connected components of $\Gamma(\rho,\pi)$, i.e.

$$\rho = \bigcup_{b\in\sigma_\rho} \rho_b,$$

as illustrated in Figure 14-b) with $b_1 = \{1,2\}$, $b_2 = \{3,4,5\}$, and $\sigma_\rho = \{b_1, b_2\}$.

Definition A.3 *For $\sigma \in \Pi([n])$ we let $\Pi_\sigma(\pi_1 \cup \cdots \cup \pi_n)$ denote the collection of partitions $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ such that*

$$\rho \vee \pi = \left\{ \bigcup_{i\in b} \pi_i : b \in \sigma \right\}.$$

In particular, $\Pi_{\widehat{1}}(\pi_1\cup\cdots\cup\pi_n)$ represents the set of connected partitions of $\pi_1\cup\cdots\cup\pi_n$, and $\Pi_{\widehat{0}}(\pi_1\cup\cdots\cup\pi_n)$ represents the partitions of $\pi_1\cup\cdots\cup\pi_n$ that are finer than $\pi := \{\pi_1,\ldots,\pi_n\}$.

Given $F : \Pi'(\pi_1\cup\cdots\cup\pi_n) \to \mathbb{R}$, where $\Pi'(\pi_1\cup\cdots\cup\pi_n)$ is the collection of all subpartitions of $\pi_1 \cup \cdots \cup \pi_n$, we define the mixed moments $\widehat{F} : 2^{[n]} \to \mathbb{R}$ by

$$\widehat{F}(A) = \sum_{\rho\in\Pi(\cup_{i\in A}\pi_i)} F(\rho), \qquad A \subset [n], \tag{A.1}$$

cf. [MM91, p. 33]. The semi-invariants $C_F : 2^{[n]} \to \mathbb{R}$ are defined by the induction formula $C_F(A) = \widehat{F}(A)$ when $|A| = 1$, and

$$C_F(A) = \widehat{F}(A) - \sum_{\substack{\{b_1,\ldots,b_k\}\in\Pi(A)\\ k\geq 2}} \prod_{i=1}^{k} C_F(b_i),$$

for $|A| > 1$, see Relation (16) page 33 of [MM91], where the sum is taken over all partitions $\sigma \in \Pi(A)$ such that $|\sigma| \geq 2$, i.e.

$$C_F(A) = \sum_{\rho \in \Pi(A)} (-1)^{|\rho|}(|\rho|-1)! \prod_{b \in \rho} \widehat{F}(b), \tag{A.2}$$

see Relation (16') in [MM91]. The next proposition generalizes [LP24, Proposition 3.3] to the multivariate case.

Proposition A.4 *Suppose that F satisfies the connectedness factorization property*

$$F(\rho) = \prod_{b \in \sigma_\rho} F(\rho_b), \qquad \rho \in \Pi'(\pi_1 \cup \cdots \cup \pi_n). \tag{A.3}$$

Then, the semi-invariants are given by

$$C_F(A) = \sum_{\rho \in \Pi_{\widehat{1}}(\cup_{i \in A} \pi_i)} F(\rho), \qquad \emptyset \neq A \subset [n]. \tag{A.4}$$

Proof. (i) It is clear that (A.4) holds when $|A| = 1$. When $|A| = 2$, taking $A = \{i, j\} \subset [n]$, $i \neq j$, we have

$$\begin{aligned} C_F(A) &= \widehat{F}(\{i,j\}) - C_F(\{i\})C_F(\{j\}) \\ &= \sum_{\rho \in \Pi(\pi_i \cup \pi_j)} F(\rho) - \widehat{F}(\{i\})\widehat{F}(\{j\}) \\ &= \sum_{\rho \in \Pi_{\widehat{1}}(\pi_i \cup \pi_j)} F(\rho) + \sum_{\rho \in \Pi_{\widehat{0}}(\pi_i \cup \pi_j)} F(\rho) - \left(\sum_{\rho_1 \in \Pi(\pi_i)} F(\rho_1) \right) \left(\sum_{\rho_2 \in \Pi(\pi_j)} F(\rho_2) \right). \end{aligned}$$

By splitting any $\rho \in \Pi_{\widehat{0}}(\pi_i \cup \pi_j)$ into two disjoint subpartitions according to Definition A.2, i.e.

$$\rho = \rho_{\{i\}} \cup \rho_{\{j\}},$$

together with the factorization property (A.3), we find

$$\begin{aligned} \sum_{\rho \in \Pi_{\widehat{0}}(\pi_i \cup \pi_j)} F(\rho) &= \sum_{\substack{\rho \in \Pi_{\widehat{0}}(\pi_i \cup \pi_j) \\ \rho = \rho_{\{i\}} \cup \rho_{\{j\}}}} F(\rho_{\{i\}})F(\rho_{\{j\}}) \\ &= \left(\sum_{\rho_1 \in \Pi(\pi_i)} F(\rho_1) \right) \left(\sum_{\rho_2 \in \Pi(\pi_j)} F(\rho_2) \right), \end{aligned}$$

which shows (A.4).

(ii) Next, suppose that (A.4) holds for any $A \subset [n]$ with $|A| \leq l \leq n-1$. Let $A \subset [n]$ be a subset of $[n]$ with $|A| = l+1$. We have

$$
\begin{aligned}
\widehat{F}(A) &= \sum_{\rho \in \Pi(\cup_{i \in A} \pi_i)} F(\rho) \\
&= \sum_{\substack{\sigma = \{b_1, \ldots, b_k\} \in \Pi(A) \\ k \geq 1}} \sum_{\substack{\rho \in \Pi(\cup_{i \in A} \pi_i) \\ \rho \vee \pi_A = \{\cup_{i \in b_j} \pi_i\}_{j=1}^k}} F(\rho) \\
&= \sum_{\substack{\sigma = \{b_1, \ldots, b_k\} \in \Pi(A) \\ k \geq 1}} \sum_{\substack{\rho \in \Pi(\cup_{i \in A} \pi_i) \\ \rho \vee \pi_A = \{\cup_{i \in b_j} \pi_i\}_{j=1}^k}} \prod_{j=1}^k F(\rho_{b_j}) \\
&= \sum_{\substack{\sigma = \{b_1, \ldots, b_k\} \in \Pi(A) \\ k \geq 1}} \prod_{j=1}^k \sum_{\substack{\rho_j \in \Pi(\cup_{i \in b_j} \pi_i) \\ \rho_j \vee \pi_{b_j} = \widehat{1}}} F(\rho_{b_i}) \\
&= \sum_{\substack{\sigma = \{b_1, \ldots, b_k\} \in \Pi(A) \\ k \geq 1}} \prod_{j=1}^k \sum_{\rho_j \in \Pi_{\widehat{1}}(\cup_{i \in b_j} \pi_i)} F(\rho_{b_i}) \\
&= \sum_{\rho \in \Pi_{\widehat{1}}(\cup_{i \in A} \pi_i)} F(\rho) + \sum_{\substack{\{b_1, \ldots, b_k\} \in \Pi(A) \\ k \geq 2}} \prod_{j=1}^k C_F(b_j),
\end{aligned}
$$

where the last equality follows from the induction hypothesis (A.4) when $|A| \leq l$. The proof is completed by subtracting the last term from both sides. $\square$

Given $n \geq 1$ and $f^{(i)} : (\mathbb{R}^d)^{r_i} \to \mathbb{R}$, $i = 1, \ldots, n$, measurable functions, we let

$$
\left(\bigotimes_{i=1}^n f^{(i)} \right) (x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{n,1}, \ldots, x_{n,r_n}) := \prod_{i=1}^n f^{(i)}(x_{i,1}, \ldots, x_{i,r_i}).
$$

For $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$, we also denote by $\left(\bigotimes_{i=1}^n f^{(i)} \right)_\rho : (\mathbb{R}^d)^{|\rho|} \to \mathbb{R}$ the function obtained by equating any two variables whose indexes belong to a same block of ρ. We refer to [BRSW17, Theorem 3.1] for the next result.

Proposition A.5 *Let $n \geq 1$, $r_1, \ldots, r_n \geq 1$, and let $f^{(i)} : (\mathbb{R}^d)^{r_i} \to \mathbb{R}$ be a sufficiently integrable measurable function for $i = 1, \ldots, n$. We have*

$$
\mathbb{E}\left[\prod_{i=1}^n \sum_{(x_1, \ldots, x_{r_i}) \in \eta^{r_i}} f^{(i)}(x_1, \ldots, x_{r_i}) \right] = \sum_{\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)} \lambda^{|\rho|} \int_{(\mathbb{R}^d)^{|\rho|}} \left(\bigotimes_{i=1}^n f^{(i)} \right)_\rho (\mathbf{x}) \, \mu^{\otimes |\rho|}(\mathrm{d}\mathbf{x}),
$$

Proposition A.5 can be specialized as follows.

Corollary A.6 *Let $r_i \geq 2$, $i = 1, \ldots, n$, and consider $f^{(i)} : (\mathbb{R}^d)^{r_i} \to \mathbb{R}$ measurable functions that vanish on diagonals, i.e. $f^{(i)}(x_1, \ldots, x_{r_i}) = 0$ whenever $x_k = x_l$ for some $1 \leq k \neq l \leq r_i$, $i = 1, \ldots, n$. We have*

$$\mathbb{E}\left[\prod_{i=1}^{n} \sum_{(x_1,\ldots,x_{r_i})\in\eta^{r_i}} f^{(i)}(x_1, \ldots, x_{r_i})\right] = \sum_{\substack{\rho\in\Pi(\pi_1\cup\cdots\cup\pi_n)\\ \rho\wedge\pi=\widehat{0}\\ \text{(non-flat)}}} \lambda^{|\rho|} \int_{(\mathbb{R}^d)^{|\rho|}} \left(\bigotimes_{i=1}^{n} f^{(i)}\right)_{\rho} (\mathbf{x})\, \mu^{\otimes|\rho|}(\mathrm{d}\mathbf{x}). \tag{A.5}$$

For $i = 1, \ldots, n$, let $M_i \subset \{1, \ldots, m\}$, $r_i \geq 2$, and let $G_i = (V_{G_i}, E_{G_i})$ be a connected graph with edge set E_{G_i} and vertex set of the form $V_{G_i} = \big(v_1^{(i)}, \ldots, v_{r_i}^{(i)}; \{w_j^{(i)}\}_{j\in M_i}\big)$, such that

i) the subgraph G_i induced by G_i on $\{v_1^{(i)}, \ldots, v_{r_i}^{(i)}\}$ is connected, and

ii) the endpoint vertices $\{w_j^{(i)}\}_{j\in M_i}$ are not adjacent to each other in G_i,

and let $G := \{G_1, \ldots, G_n\}$. In Definition A.7, for every $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ we build a graph structure induced by $(G_1, \ldots, G_n)$ on the diagram $\Gamma(\rho, \pi)$, analogous to [LP24, Definition 2.2].

Definition A.7 *Given $\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$ a partition of $\pi_1 \cup \cdots \cup \pi_n$, we let $\widetilde{\rho}_G$ denote the multigraph constructed as follows on $[m] \cup \pi_1 \cup \cdots \cup \pi_n$:*

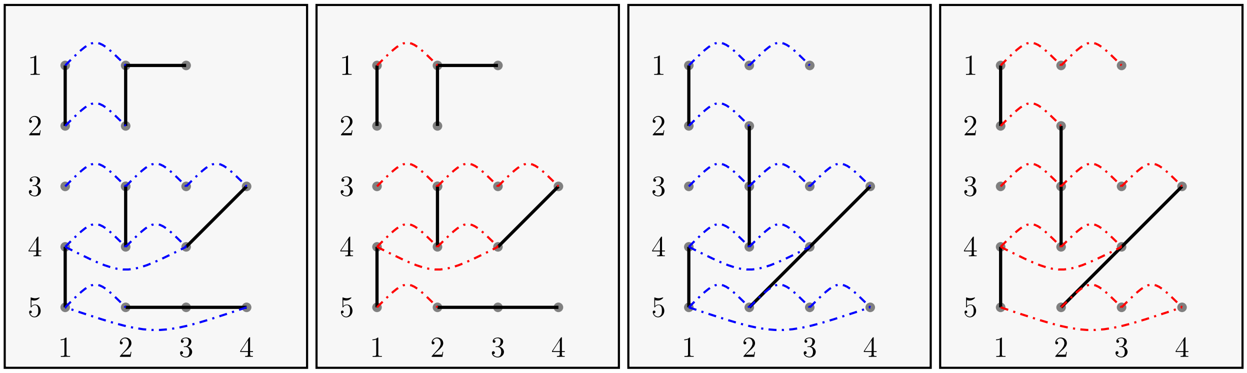
i) for all $j_1, j_2 \in [r_i]$, $j_1 \neq j_2$, and $i \in [n]$, an edge links (i, j_1) to (i, j_2) iff $\{v_{j_1}^{(i)}, v_{j_2}^{(i)}\} \in E_{G_i}$.

ii) for all $(j, k) \in [r_i] \times M_i$ and $i \in [n]$, an edge links (k) to (i, j) iff $\{v_j^{(i)}, w_k^{(i)}\} \in E_{G_i}$;

iii) for all $i_1, i_2 \in [n]$ and $(j_1, j_2) \in [r_{i_1}] \times [r_{i_2}]$, we merge any two nodes (i_1, j_1) and (i_2, j_2) if they belong to a same block in ρ.

In addition, we let ρ_G be the graph constructed from the multigraph $\widetilde{\rho}_G$ by removing any redundant edge in $\widetilde{\rho}_G$.

As in Section 3, the graph ρ_G forms a connected graph with $|\rho| + m$ vertices. Figure 15 presents two examples of multigraphs $\widetilde{\rho}_G$ and graphs ρ_G when G_1, G_2, G_3 are line graphs, G_4 is a triangle, and G_5 is a rectangle on a partition diagram $\Gamma(\rho, \pi)$ with no endpoints, i.e. $M_1 = \cdots = M_n = \emptyset$ here.



(a) Multigraph $\widetilde{\rho}_G$ in blue. (b) Graph ρ_G in red. (c) Multigraph $\widetilde{\rho}_G$ in blue. (d) Graph and ρ_G in red.

Figure 15: Diagram $\Gamma(\rho,\pi)$, multigraph $\widetilde{\rho}_G$, and graph ρ_G.

For each $i = 1, \ldots, n$ denote by $N_{M_i}^{G_i}$ the count of subgraphs in the random-connection model $G_H(\eta \cup \{y_j\}_{j\in M_i})$ with endpoint set $\{y_j\}_{j\in M_i}$, i.e.

$$N_{M_i}^{G_i} = \sum_{(x_1,\ldots,x_{r_i})\in\eta^{r_i}} f_{M_i}^{(i)}(x_1,\ldots,x_{r_i}),$$

where $f_{M_i}^{(i)} : (\mathbb{R}^d)^{r_i} \to \{0,1\}$ is the random function defined as

$$f_{M_i}^{(i)}(x_1,\ldots,x_{r_i}) := \prod_{\substack{1\le l\le r_i,\ j\in M_i\\ \{w_j^{(i)},v_l^{(i)}\}\in E_{G_i}}} \mathbf{1}_{\{y_j\leftrightarrow x_l\}} \prod_{\substack{1\le k,l\le r_i\\ \{v_k^{(i)},v_l^{(i)}\}\in E_{G_i}}} \mathbf{1}_{\{x_k\leftrightarrow x_l\}}, \qquad x_1,\ldots,x_{r_i}\in\mathbb{R}^d.$$

For $\rho = \{b_1,\ldots,b_{|\rho|}\} \in \Pi(\pi_1\cup\cdots\cup\pi_n)$, we also let

$$\mathcal{A}_j^\rho := \{i \in [|\rho|] \ : \ \exists (s,k)\in b_i \text{ s.t. } (v_k^{(s)}, w_j^{(s)}) \in E_{G_s}\}$$

denote the neighborhood of the vertex $(|\rho|+j)$ in ρ_G, $j = 1,\ldots,m$. The next proposition is a consequence of Relation (A.4) and Corollary A.6.

Proposition A.8 *Let $N_{M_i}^{G_i}$ be subgraph counts in the random-connection model $G_H(\eta \cup \{y_1,\ldots,y_m\})$ as defined above, for $i = 1,\ldots,n$. We have*

$$\mathbb{E}\left[\prod_{i=1}^n N_{M_i}^{G_i}\right] = \sum_{\substack{\rho\in\Pi(\pi_1\cup\cdots\cup\pi_n)\\ \rho\wedge\pi=\widehat{0}\\ (\text{non-flat})}} \lambda^{|\rho|}\int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1\le j\le m\\ i\in\mathcal{A}_j^\rho}} H(x_i,y_j) \prod_{(k,l)\in E_{\rho_G}} H(x_k,x_l)\,\mu(\mathrm{d}\mathbf{x}), \tag{A.6}$$

and joint cumulant

$$\kappa(N_{G_1},\ldots,N_{G_n}) = \sum_{\substack{\rho\in\Pi_{\widehat{1}}(\pi_1\cup\cdots\cup\pi_n)\\ \rho\wedge\pi=\widehat{0}\\ (\text{non-flat connected})}} \lambda^{|\rho|}\int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1\le j\le m\\ i\in\mathcal{A}_j^\rho}} H(x_i,y_j) \prod_{(k,l)\in E_{\rho_G}} H(x_k,x_l)\,\mu(\mathrm{d}\mathbf{x}). \tag{A.7}$$

Proof. The moment identity (A.6) is obtained by taking expectation on both sides of (A.5) in Corollary A.6 and using the relation $H(x,y) = \mathbb{E}[\mathbf{1}_{\{x \leftrightarrow y\}}]$, $x, y \in \mathbb{R}^d$. Next, we note that the connectedness factorization property (A.3) is satisfied by

$$F(\rho) := \lambda^{|\rho|} \int_{(\mathbb{R}^d)^{|\rho|}} \prod_{\substack{1 \le j \le m \\ i \in \mathcal{A}_j^\rho}} H(x_i, y_j) \prod_{(k,l) \in E_{\rho_G}} H(x_k, x_l)\, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_{|\rho|}),$$

$\rho \in \Pi(\pi_1 \cup \cdots \cup \pi_n)$, hence (A.7) follows from Relations (A.1), (A.4), (A.6), and the classical cumulant-moment relationship (A.2), see e.g. Relation (3.3) in [Luk55]. □

The cumulant formula of Proposition A.8 is implemented in the code listed in Appendix E.

B Cumulant and factorial moment estimates

The following result can be found in [SS91, Corollary 2.1] or [DJS22, Theorem 2.4].

Lemma B.1 *Let $\{X_\lambda\}$ be a family of random variables with moments of all orders, mean zero and unit variance for all $\lambda > 0$. Suppose that for all $j \ge 3$ and sufficiently large λ, the cumulant of order j of X_λ is bounded by*

$$|\kappa_j(X_\lambda)| \le \frac{(j!)^{1+\gamma}}{(\Delta_\lambda)^{j-2}}$$

where $\gamma \ge 0$ is a constant independent of λ. Then we have the Berry-Esseen bound

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(X_\lambda \le x) - \Phi(x)| \le C_\gamma (\Delta_\lambda)^{-1/(1+2\gamma)},$$

for $C_\gamma > 0$ a constant depending only on γ.

We let $m_n(X) := \mathbb{E}[X(X-1)\cdots(X-n+1)]$ denote the factorial moments of order $n \ge 1$ of a discrete random variable X.

Proposition B.2 (Corollary 1.13 in [Bol01]) *Assume that*

$$\lim_{n \to \infty} m_n(X) \frac{n^k}{n!} = 0, \qquad k \ge 0.$$

Then for any $n \ge 0$, we have

$$\mathbb{P}(X = n) = \frac{1}{n!} \sum_{i \ge 0} \frac{(-1)^i}{i!} m_{n+i}(X). \tag{B.1}$$

C Gram-Charlier expansions

Let $\varphi(x) := e^{-x^2/2}/\sqrt{2\pi}$, $x \in \mathbb{R}$, denote the standard normal probability density function. In addition to the second order expansion Gaussian approximation

$$\phi_X^{(1)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) \tag{C.1}$$

for the probability density $\phi_X(x)$ function of a random variable X, higher order Gram-Charlier expansions of third and fourth order are given by

$$\phi_X^{(3)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\right) \tag{C.2}$$

and

$$\phi_X^{(4)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + c_4 H_4\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + c_6 H_6\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\right).$$

see § 17.6 of [Cra46], where

- $H_0(x) = 1$, $H_1(x) = x$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$ are Hermite polynomials,

- the sequence c_3, c_4, c_5, c_6 is given from the cumulants $(\kappa_n)_{n\geq 1}$ of X as

$$c_3 = \frac{\kappa_3}{3!(\kappa_2)^{3/2}}, \quad c_4 = \frac{\kappa_4}{4!(\kappa_2)^2}, \quad c_5 = \frac{\kappa_5}{5!\kappa_5^{5/2}}, \quad c_6 = \frac{\kappa_6}{6!(\kappa_2)^3} + \frac{(\kappa_3)^2}{2(3!)^2(\kappa_2)^3},$$

  where c_3 and c_4 are expressed from the skewness $\kappa_3/(\kappa_2)^{3/2}$ and the excess kurtosis $\kappa_4/(\kappa_2)^2$.

D Cumulant code

The following code generates closed-form cumulant expressions via symbolic calculations in SageMath for any dimension $d \geq 1$, any connected subgraph G induced by G, and any set of endpoint connections represented as the sequence $\mathsf{EP} = [\mathsf{EP}_1, \ldots, \mathsf{EP}_m]$. When G has no endpoint ($m = 0$) we have $\mathsf{EP} = [\,]$, however, in this case the measure μ should be finite, i.e., the density function `mu(x,λ,β)` should be integrable with respect to the Lebesgue measure on $\mathbb{R}^d$. The choice of SageMath for this implementation is due to its fast handling of symbolic integration via Maxima, which is significantly faster than the Python package

Sympy. This code is also sped up by parallel processing that can distribute the load among different CPU cores. This SageMath code and the next one are available for download at https://github.com/nprivaul/random-connection.

```
from time import time
import datetime
import multiprocessing as mp

global cumulants

def partitions(points):
    if len(points) == 1:
        yield [ points ]
        return
    first = points[0]
    for smaller in partitions(points[1:]):
        for m, subset in enumerate(smaller):
            yield smaller[:m] + [[ first ] + subset]  + smaller[m+1:]
        yield [ [ first ] ] + smaller

def nonflat(partition,r):
    p = []
    for j in partition:
        seq = list(map(lambda x: (x-1)//r,j))
        p.append(len(seq) == len(set(seq)))
    return all(p)

def connected(partition,n,r):
    q = []; c = 0
    if n  == 1: return all([len(j)==1 for j in partition])
    for j in partition:
        jk = list(set(map(lambda x: (x-1)//r,j)))
        if(len(jk)>1):
            if c == 0:
                q = jk; c += 1
            elif(set(q) & set(jk)):
                d=[y for y in (q+jk) if y not in q]
                q = q + d
    return n == len(set(q))

def connectednonflat(n,r):
    points = list(range(1,n*r+1))
    randd = []
    for m, p in enumerate(partitions(points), 1):
        randd.append(sorted(p))
    cnfp = [e for e in randd if (connected(e,n,r) and nonflat(e,r))]
    for rou in range(r,(r-1)*n+2):
        rs = [d for d in cnfp if len(d)==rou]
        print("Connected␣non-flat␣partitions␣with",rou,"blocks:",len(rs))
    print("Connected␣non-flat␣set␣partitions:",len(cnfp))
    return cnfp

def graphs(G,EP,setpartition,n):
    r=len(set(flatten(G)));rhoG = []
    for j in range(n):
        for hop in G: rhoG.append([r*j+hop[0],r*j+hop[1]])
        for l in range(len(EP)):
            F=EP[l]
            for i in F: rhoG.append([j*r+i,n*r+l+1]);
    for i in setpartition:
        if(len(i)>1):
            b = []
            for j in rhoG:
                b.append([i[0] if ele in i else ele for ele in j])
            rhoG = b
    for i in rhoG: i.sort()
    return rhoG

def inner(n,d,G,EP,mu,H,setpartition,z,r):
```

```
    rhoG=graphs(G,EP,setpartition,n)
    for ll in range(len(EP)+1):
        for l in range(1,d+1): z[d*(n*r+ll)+l] = var(str(y)+str(ll)+str('_')+str(l
            ))
    for key in range(1,n*r+1):
        for l in range(1,d+1): z[key*d+l] = var(str(x)+str(key)+str(x)+str(l))
    edgesrhoG = [i for n, i in enumerate(rhoG) if i not in rhoG[:n]]
    vertrhoG = set(flatten(edgesrhoG));
    for ll in range(len(EP)): vertrhoG.remove(n*r+ll+1);
    strr = 'λ*'*len(vertrhoG)
    for i in vertrhoG:
        for l in range(1,d+1): strr = '*mu({},{},{})'.format(z[i*d+l], λ, β) + strr
        for l in range(1,d+1): strr = strr + ').integrate({},-infinity,+infinity)'
            .format(z[i*d+l])
    for i in edgesrhoG:
        for l in range(1,d+1): strr = '*H({},{},{})'.format(z[i[0]*d+l],z[i[1]*d+l
            ], β) + strr
    strr = '('*len(vertrhoG)*d+strr[1:]
    return eval(preparse(strr))

def collect_result(result):
    global cumulants
    global iii
    global tim
    iii=iii+1;
    if (mod(iii,100)==0):
        tim=(time()-t_start2)*(lencnfp-iii)/iii/60
        print('[%d]\r'%(iii),'Est.␣remaining␣time␣(minutes):%d'%(tim),end="")
    cumulants+=result

def c(n,d,G,EP,mu,H):
    global cumulants
    global iii
    global t_start2
    t_start2 = time()
    d_start2 = datetime.datetime.now()
    r=len(set(flatten(G)));
    x,y=var("x,y")
    cumulants = 0; iii = 0
    z = dict(enumerate([str(x)+str(key)+str(x)+str(l) for key in range(0,n*r+1)
       for l in range(1,d+1)], start=1))
    global lencnfp
    cnfp=connectednonflat(n,r)
    lencnfp=len(cnfp)
    pool = mp.Pool(4) # pool = mp.Pool(mp.cpu_count())
    for setpartition in cnfp:
        pool.apply_async(func = inner, args=(n,d,G,EP,mu,H,setpartition,z,r),
            callback=collect_result)
    pool.close()
    pool.join()
    print("\n");
    d_end2 = datetime.datetime.now()
    print("Runtime␣is",(d_end2-d_start2))
    return cumulants._sympy_()
```

E Joint cumulant code

The following code generates closed-form joint cumulant expressions via symbolic calculations in SageMath for any dimension $d \geq 1$, any sequence $(\mathsf{G}_1, \ldots, \mathsf{G}_n)$ of connected subgraphs induced by $(G_1, \ldots, G_n)$. As above, the endpoint connections are represented using represented as the sequence $\mathsf{EP} = [\mathsf{EP}_1, \ldots, \mathsf{EP}_m]$ where EP_i denotes the set of vertices of $\mathsf{G}_1 \cup \cdots \cup \mathsf{G}_n$ which are attached to the i-th endpoint, $i = 1, \ldots, m$.

```
def jpartitions(points):
    if len(points) == 1:
        yield [ points ]
        return
    first = points[0]
    for smaller in jpartitions(points[1:]):
        for m, subset in enumerate(smaller):
            yield smaller[:m] + [[ first ] + subset]  + smaller[m+1:]
        yield [ [ first ] ] + smaller

def jnonflat(partition,rr):
    n=len(rr); p = []
    for j in partition:
        for i in range(n):
            j2 = [l for l in j if l > sum(rr[0:i]) and l<=sum(rr[0:(i+1)])]
            p.append(len(j2) <= 1)
    return all(p)

def jconnected(partition,rr):
    n=len(rr); q = []; c = 0;
    if n  == 1: return True
    for j in partition:
        jk = [i for i in range(n) if len([l for l in j if l > sum(rr[0:i]) and l<=
            sum(rr[0:(i+1)])])>=1]
        if(len(jk)>1):
            if c == 0:
                q = jk; c += 1
            elif(set(q) & set(jk)):
                d=[y for y in (q+jk) if y not in q]
                q = q + d
    return n == len(q)

def jconnectednonflat(rr):
    n=len(rr);
    points = list(range(1,sum(rr)+1))
    randd = []
    for m, p in enumerate(jpartitions(points), 1): randd.append(sorted(p))
    for rou in range(min(rr),sum(rr)-n+2):
        rs = [d for d in randd if (jnonflat(d,rr) and len(d)==rou)]
        rss = [e for e in rs if jconnected(e,rr)]
        print("Connected␣non-flat␣partitions␣with",rou,"blocks:",len(rss))
    cnfp = [e for e in randd if (jconnected(e,rr) and jnonflat(e,rr))]
    print("Connected␣non-flat␣set␣partitions:",len(cnfp))
    return cnfp

def jgraphs(G,EP,setpartition):
    rr=[len(set(flatten(g))) for g in G];
    n=len(G); rhoG = []
    ee=[len(set(flatten(e))) for e in EP];
    for j in range(n):
        for hop in G[j]: rhoG.append([hop[0],hop[1]])
        for l in range(len(EP)):
            F=EP[l]
            for i in F: rhoG.append([i,sum(rr)+l+1]);
    for i in setpartition:
        if(len(i)>1):
            b = []
            for j in rhoG:
                b.append([i[0] if ele in i else ele for ele in j])
            rhoG = b
    for i in rhoG: i.sort()
    return rhoG

def jc(d,G,EP,mu,H):
    rr=[len(set(flatten(g))) for g in G];
    if(sum(rr)!=len(set(flatten(G)))):
        print("Wrong␣G␣format");
        return 0
    n=len(G);
    ee=[len(set(flatten(e))) for e in EP];
    x,y=var("x,y")
```

```
    jcumulants = 0; ii=0
    z = dict(enumerate([str(x)+str(key)+str(x)+str(l) for key in range(1,sum(rr)
        +1) for l in range(1,d+1)], start=1))
    cnfp=jconnectednonflat(rr)
    for setpartition in cnfp:
        ii=ii+1;print('[%d/%d]\r'%(ii,len(cnfp)),end="")
        rhoG=jgraphs(G,EP,setpartition)
        for j in range(n):
            m=len(EP);
            for l in range(m+1):
                for ld in range(1,d+1): z[d*(sum(rr)+l)+ld] = var(str(y)+str(l)+
                    str('_')+str(ld))
        for key in range(1,sum(rr)+1):
            for l in range(1,d+1): z[key*d+l] = var(str(x)+str(key)+str(x)+str(l))
        edgesrhoG = [i for n, i in enumerate(rhoG) if i not in rhoG[:n]]
        vertrhoG = set(flatten(edgesrhoG));
        m=len(EP);
        for l in range(m): vertrhoG.remove(sum(rr)+l+1);
        strr = 'λ*'*len(vertrhoG)
        for i in vertrhoG:
            for l in range(1,d+1): strr = '*mu({},{},{})'.format(z[i*d+l], λ, β) +
                strr
            for l in range(1,d+1): strr = strr + ').integrate({},-infinity,+
                infinity)'.format(z[i*d+l])
        for i in edgesrhoG:
            for l in range(1,d+1): strr = '*H({},{},{})'.format(z[i[0]*d+l],z[i
                [1]*d+l], β) + strr
        strr = '('*len(vertrhoG)*d+strr[1:]
        jcumulants += eval(preparse(strr))
    print("\n");
    jcumulants = simplify(jcumulants).canonicalize_radical().maxima_methods().
        rootscontract().simplify()
    return jcumulants._sympy_()
```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