Asymptotic analysis of k-hop connectivity in the 1D unit disk random graph model

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Abstract

We propose an algorithm for the closed-form recursive computation of joint moments and cumulants of all orders of k-hop counts in the 1D unit disk random graph model with Poisson distributed vertices. Our approach uses decompositions of k-hop counts into multiple Poisson stochastic integrals. As a consequence, using the Stein and cumulant methods we derive Berry-Esseen bounds for the asymptotic convergence of renormalized k-hop path counts to the normal distribution as the density of Poisson vertices tends to infinity. Computer codes for the recursive symbolic computation of moments and cumulants of any orders are provided as an online resource.

Key words: Random graph, 1D unit disk model, k-hop counts, Poisson process, multiple stochastic integrals, moments, cumulants.

Mathematics Subject Classification (2020): 05C80, 60G55, 60F05, 60B10.

1 Introduction

The Poisson random-connection model (RCM) $G_H(\eta)$ is a random graph whose vertex set is given by a Poisson point process η with intensity Λ on $\mathbb{R}^d$, $d \geqslant 1$, and in which every pair of vertices is randomly connected with a location-dependent probability given by a

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connection function $H : \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$. This setting has the ability to model physical systems in e.g. wireless networks, complex networks, and statistical mechanics.

When H is a function of the distance between pairs of points of η, i.e. $H(x,y) := \phi(\|x-y\|)$ for some measurable function $\phi : \mathbb{R}_+ \to [0,1]$, the resulting graph is also known as a soft random geometric graph, see [Pen91, Pen16, LNS21]. When ϕ takes the form $\phi(u) = \mathbf{1}_{\{u \leqslant r_0\}}$, for some $r_0 > 0$, the random-connection model becomes a random geometric graph, c.f. the monograph [Pen03], in which a pair of vertices is connected by an edge if and only if the distance between them is less than the fixed threshold r_0, see also [WDG20] for the soft connection model.

In this paper, we focus on the one-dimensional unit disk random connection model with $d = 1$ and connection radius $r > 0$ on a finite interval, see [Dro97], as illustrated Figure 1.

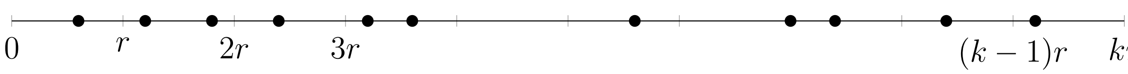


Figure 1: Unit disk random connection model.

Here, the nodes are distributed on $[0, kr]$, $k \geqslant 1$, according to a Poisson point process $(N_t)_{t\in[0,kr]}$ with intensity $\lambda(ds)$ of the form

$$\lambda(ds) = \sum_{l=1}^{k} \mathbf{1}_{((l-1)r,lr]}(s)\lambda_l(lr - ds), \tag{1.1}$$

where $\lambda_1(ds) = \lambda_1(s)ds, \ldots, \lambda_k(ds) = \lambda_k(s)ds$ are absolutely continuous intensity measures on $[0, r]$, $l = 1, \ldots, k$. In addition, two nodes located $s, t \in [0, kr]$ are said to be connected if and only if $|t - s| \leqslant r$, and for $k \geqslant 2$, a k-hop is a path connecting two fixed points $x, y \in [0, kr]$ via $k - 1$ nodes, see Figure 2 for an example with $k = 5$, $x = 0$, $y = 4.5$ and $r = 1$.

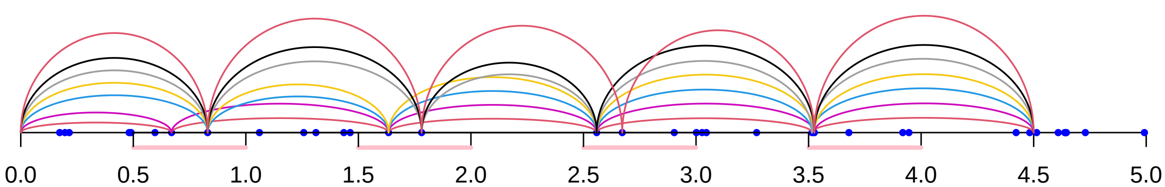


Figure 2: Graph of seven 5-hop paths linking $x = 0$ to $y = 4.5$ with $r = 1$.

We are interested in the count $\sigma_k(t)$ of k-hops connecting the additional nodes located respectively at 0 and t for some $t \in [0, kr]$. In this context, the distribution of k-hop counts has been expressed by a combinatorial approach in [KGKP21]. We will consider the statistics and asymptotic behavior of k-hop connectivity as the intensity of the underlying Poisson process tends to infinity. For this, we will use the Stein method, see [RS13], [ET14], [LRR16], [DP17], [PS22], and the cumulant method, see [RSS78], [SS91], [Kho08], [GT18], [DE13], [Jan19], [DJS22], which has been recently applied to moderate deviation in random geometric graphs and weighted random-connection models in [ST23], [HHO23], and to normal approximation for subgraph counts in the multidimensional random-connection model in [LP24].

The moments of k-hop counts in the random-connection model have been expressed in [Pri19] as summations over non-flat partition diagrams, however, those expressions are difficult to apply to the derivation of explicit bounds. In this paper we use a different approach based on the representation of k-hop counts in terms of multiple Poisson stochastic integrals, which allows us to derive explicit expressions for moments and cumulants of all orders by recursive formulas.

In Proposition 4.1 we provide a combinatorial expression for the computation of the joint moments of k-hop counts at different endpoint locations within $[(k-1)r, kr]$. This expression is then specialized to the computation of variance in Proposition 4.2 and Corollary 4.3.

In Proposition 5.1, a recursive algorithm on k-hop orders is derived for the closed-form computation of joint moments, using the representation of k-hop counts as multiple Poisson stochastic integrals. A related recursion formula is derived in Proposition 6.1 for the computation of joint cumulants, and yields a cumulant bound in Proposition 7.2.

As a consequence, denoting by $c_{k,n}^{(\lambda)}(\underbrace{kr - t; \ldots; kr - t}_{n \text{ times}})$ the cumulant of order $n \geqslant 2$ of $\sigma_k(t)$, for $t \in [(k-1)r, kr)$, we obtain the bound

$$\frac{c_{k,n}^{(\lambda)}(kr - t; \ldots; kr - t)}{\left(c_{k,2}^{(\lambda)}(kr - t; kr - t)\right)^{n/2}} \leqslant (n!)^{k-2} O(\lambda^{1-n/2}), \qquad n \geqslant 2.$$

Next, denoting by $\mathbb{P}_\lambda$ the distribution of the 1D unit disk model with constant Poisson intensity $\lambda > 0$, by the cumulant method this implies the Komogorov distance bound (8.3) for the convergence of the renormalized k-hop count

$$\widetilde{\sigma}_k(t) := \frac{\sigma_k(t) - \mathbb{E}_\lambda[\sigma_k(t)]}{\sqrt{\mathrm{Var}_\lambda[\sigma_k(t)]}}$$

to the normal distribution $\mathcal{N}$ as λ tends to infinity. In Proposition 8.1 we also obtain the Berry-Esseen bound

$$\sup_{x \in \mathbb{R}} |\mathbb{P}_\lambda(\widetilde{\sigma}_k(t) \leqslant x) - \mathbb{P}(\mathcal{N} \leqslant x)| \leqslant \frac{C(k,r)}{\sqrt{\lambda}}$$

using the Stein method, together with a bound of same order for the Wasserstein distance.

The content of this paper can be summarized as follows. In Section 2 we show that k-hop counts can be represented in terms of multiple Poisson stochastic integrals. In Section 3 we specialize those expressions when the k-hops are made of a single node per cell. Section 4 presents moment expressions in terms of sums over non-flat partitions, and Sections 5-6 develop recursive expressions for the explicit calculation of joint moments and cumulants of any order. In Sections 7 and 8 we derive moment and cumulant bounds with application to Berry-Esseen rates for the convergence of normalized k-hop counts to the normal distribution using the Stein and cumulant methods. The online resources contain specific moment and cumulant computations, background results on moment computations for Poisson point processes based on [Pri12, Pri16], and an implementation of moment and cumulant recursions in Mathematica.

Set partitions, moments, cumulants, and Möbius inversion

This section gathers some preliminary facts on the relationships between joint moments, cumulants, and sums over partitions that will be useful in the sequel. We let $\Pi[n]$ denote the set of partitions of $\{1, \ldots, n\}$, and given a symmetric function $f(\pi) = f(\pi_1, \ldots, \pi_l)$ where $\pi = \{\pi_1, \ldots, \pi_l\} \in \Pi[n]$ is a partition of $\{1, \ldots, n\}$ of size $l \geqslant 1$ we will use the notation

$$\sum_{\pi \in \Pi[n]} f(\pi) = \sum_{l=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} f(\{\pi_1, \ldots, \pi_l\}).$$

We will also use the Möbius transform $\widehat{G}$ of a function G on partitions π of $\{1, \ldots, n\}$, defined as

$$\widehat{G}(\sigma) := \sum_{\pi \preceq \sigma} G(\pi), \qquad \sigma \in \Pi[n], \tag{1.2}$$

where the sum (1.2) runs over all partitions π of $\{1, \ldots, n\}$ that are finer than σ, i.e. $\pi \preceq \sigma$. The Möbius inversion formula, see e.g. [Rot75] or § 2.5 of [PT11], states that the function G in (1.2) can be recovered from its Möbius transform $\widehat{G}$ as

$$G(\pi) = \sum_{\sigma \preceq \pi} \mu(\sigma, \pi) \widehat{G}(\sigma), \tag{1.3}$$

where $\mu(\sigma, \pi)$ is the Möbius function, with $\mu(\sigma, \widehat{\mathbf{1}}) = (|\sigma| - 1)!(-1)^{|\sigma|}$, where $|\sigma|$ denotes the cardinality of the block $\sigma \in \Pi[n]$ and $\widehat{\mathbf{1}} := \{\{1, \ldots, n\}\}$ is the one-block partition of $\{1, \ldots, n\}$. By (1.2) and (1.3) we also have the relation

$$G(\pi) = \sum_{\sigma \preceq \pi} \mu(\sigma, \pi) \sum_{\eta \preceq \sigma} G(\eta) = \sum_{\eta \preceq \sigma \preceq \pi} \mu(\sigma, \pi) G(\eta), \qquad \pi \in \Pi[n]. \tag{1.4}$$

Given X a random variable, its cumulants of order $l_1 \geqslant 1$ are the coefficients $\kappa_l(X)$ appearing in the log-moment generating (MGF) expansion

$$\log \mathbb{E}\left[\mathrm{e}^{tX}\right] = \sum_{l \geqslant 1} \frac{t^l}{l!} \kappa_l(X),$$

for t in a neighborhood of zero. The moments of X are given from its cumulants by the joint moment-cumulant relation

$$\mathbb{E}[X^n] = \sum_{\pi \in \Pi[n]} \prod_{A \in \pi} \kappa_{|A|}(X) = \sum_{l=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \prod_{j=1}^{l} \kappa_{|\pi_j|}(X), \tag{1.5}$$

see Theorem 1 in [Luk55] or Relation (2.9) in [McC87]. The Möbius inversion relation (1.3) allows us to recover cumulants of X from its moments as

$$\begin{aligned} \kappa_n(X) &= \sum_{\sigma \in \Pi[n]} \mu(\sigma, \widehat{\mathbf{1}}) \prod_{A \in \sigma} \mathbb{E}[X^{|A|}] \\ &= \sum_{l=1}^{n} (l-1)!(-1)^{l-1} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \prod_{j=1}^{l} \mathbb{E}[X_i^{|\pi_j|}], \end{aligned} \tag{1.6}$$

see Theorem 1 of [Luk55] or Corollary 5.1.6 in [Sta99].

The above relations can be extended to random vectors $X = (X_1, \ldots, X_n)$ whose joint cumulants of order $(l_1, \ldots, l_n)$ are the coefficients $\kappa\big(X_1^{l_1}; \ldots; X_n^{l_n}\big)$ appearing in the joint log-MGF expansion

$$\log \mathbb{E}\left[\mathrm{e}^{t_1 X_1 + \cdots + t_n X_n}\right] = \sum_{l_1, \ldots, l_n \geqslant 1} \frac{t_1^{l_1} \cdots t_n^{l_n}}{l_1! \cdots l_n!} \kappa\big(X_1^{l_1}; \ldots; X_n^{l_n}\big),$$

for $(t_1, \ldots, t_n)$ in a neighborhood of zero in $\mathbb{R}^n$. By polarization of (1.5)-(1.6), the joint moments of $(X_1, \ldots, X_n)$ are given from its cumulants by the joint moment-cumulant relation

$$\mathbb{E}[X_1 \cdots X_n] = \sum_{\pi \in \Pi[n]} \prod_{A \in \pi} \kappa\big((X_i)_{i \in A}\big) = \sum_{l=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \prod_{j=1}^{l} \kappa\big((X_i)_{i \in \pi_j}\big),$$

and

$$\begin{aligned} \kappa(X_1; \ldots; X_n) &= \sum_{\sigma \in \Pi[n]} \mu(\sigma, \widehat{\mathbf{1}}) \prod_{A \in \sigma} \mathbb{E}\left[\prod_{i \in A} X_i\right] \\ &= \sum_{l=1}^{n} (l-1)!(-1)^{l-1} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \prod_{j=1}^{l} \mathbb{E}\left[\prod_{i \in \pi_j} X_i\right], \end{aligned} \tag{1.7}$$

see [LS59], [Mal80], and Proposition 3.2.1 in [PT11]. In particular, the cumulant of order $n \geqslant 1$ of a Poisson distributed random variable X coincides with its intensity parameter $\lambda > 0$ for all $n \geqslant 1$, with

$$\mathbb{E}[X^n] = \sum_{l=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \lambda^l = \sum_{l=1}^{n} S(n, l) \lambda^l$$

where $S(n, l)$ the Stirling number of the second kind, i.e. the number of ways to partition a set of n objects into l non-empty subsets, $1 \leqslant l \leqslant n$. For $\lambda = 1$ this yields the Bell number

$$B_n = \sum_{l=1}^{n} S(n, l) \tag{1.8}$$

which is the number of partitions of $\{1, \ldots, n\}$, i.e. the cardinality of $\Pi[n]$.

2 Multiple stochastic integral representation of k-hop counts

In what follows, we use the notations $u \wedge v := \min(u, v)$ and $u \vee v := \max(u, v)$, $u, v \geqslant 0$. Our approach to the recursive computation of moments and cumulants relies on the following stochastic integral representation of k-hop counts with respect to the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity (1.1).

Proposition 2.1 *Let $k \geqslant 2$. The number of k-hops joining 0 to $t \in [0, kr]$ can be written as the (non-compensated) multiple Poisson stochastic integral*

$$\sigma_k(t) = \int_0^t \cdots \int_0^t f_k(s_1, \ldots, s_{k-1}) dN_{s_1} \cdots dN_{s_{k-1}}, \quad t \in [0, kr], \tag{2.1}$$

where f_k is the function of $k-1$ variables defined as

$$f_k(s_1, \ldots, s_{k-1}) := \prod_{l=0}^{k-1} \mathbf{1}_{\{s_{l+1} < s_l + r\}}, \tag{2.2}$$

$s_1, \ldots, s_{k-1} \in [0, t]$, with $s_0 := 0$ and $s_k := t$.

Proof. When $k = 2$ we note that if a node is present at $s \in [0, r]$ it connects to every node inside $[0, s)$, and therefore it generates $\sigma_1(s^-)$ new nodes, where $\sigma_1(s^-)$ denotes the almost sure left limit $\sigma_1(s^-) := \lim_{u \nearrow s} \sigma_1(u)$. When $s \in (r, 2r]$, if a node is present at $s - r \in [0, r]$ then the count of 1-hops linking 0 to $s - r$ has to be deducted, which yields the evolution

$$d\sigma_2(s) = \begin{cases} \mathbf{1}_{[0,r]}(s)\sigma_1(s^-)dN_s, & s \in [0, r], \\ \\ -\mathbf{1}_{[r,2r]}(s)\sigma_1(s^- - r)dN_{s-r}, & s \in (r, 2r]. \end{cases}$$

hence

$$\begin{aligned} \sigma_2(t) &= \int_0^{r\wedge t} \sigma_1(s^-)dN_s - \int_r^{r\vee t} \sigma_1(s^- - r)dN_{s-r} \\ &= \int_0^{r\wedge t} \sigma_1(s^-)dN_s - \int_0^{0\vee(t-r)} \sigma_1(s^-)dN_s \\ &= \int_{0\vee(t-r)}^{r\wedge t} \sigma_1(s^-)dN_s \\ &= \begin{cases} \displaystyle\int_0^t \sigma_1(s^-)dN_s, & t \in [0, r], \\ \\ \displaystyle\int_{t-r}^r \sigma_1(s^-)dN_s, & t \in (r, 2r]. \end{cases} \end{aligned}$$

More generally, applying this argument by iterations to any $k \geqslant 3$ leads to the system of jump stochastic differential equations

$$d\sigma_k(s) = \begin{cases} \mathbf{1}_{[0,(k-1)r]}(s)\sigma_{k-1}(s^-)dN_s, & s \in [0, (k-1)r], \\ \\ -\mathbf{1}_{[r,kr]}(s)\sigma_{k-1}(s^- - r)dN_{s-r}, & s \in ((k-1)r, kr], \end{cases}$$

or

$$d\sigma_k(s) = \mathbf{1}_{[0,(k-1)r]}(s)\sigma_{k-1}(s^-)dN_s - \mathbf{1}_{[r,kr]}(s)\sigma_{k-1}(s^- - r)dN_{s-r}, \quad s \in [0, kr],$$

hence the recurrence relation

$$\begin{aligned}
\sigma_k(t) &= \int_0^{((k-1)r)\wedge t} \sigma_{k-1}(s^-)dN_s - \int_r^{r\vee t} \sigma_{k-1}(s^- - r)dN_{s-r} \\
&= \int_0^{((k-1)r)\wedge t} \sigma_{k-1}(s^-)dN_s - \int_0^{0\vee(t-r)} \sigma_{k-1}(s^-)dN_s \\
&= \int_{0\vee(t-r)}^{((k-1)r)\wedge t} \sigma_{k-1}(s^-)dN_s \\
&= \begin{cases} \displaystyle\int_{t-r}^{(k-1)r} \sigma_{k-1}(s^-)dN_s, & t \in [(k-1)r, kr], \\ \\ \displaystyle\int_{t-r}^{t} \sigma_{k-1}(s^-)dN_s, & t \in [r, (k-1)r], \\ \\ \displaystyle\int_0^t \sigma_{k-1}(s^-)dN_s, & t \in [0, r], \end{cases}
\end{aligned}$$

for $k \geqslant 3$. Finally, by induction we obtain

$$\begin{aligned}
\sigma_k(t) &= \int_{0\vee(t-r)}^{((k-1)r)\wedge t} \int_{0\vee(s_{k-1}^- - r)}^{((k-2)r)\wedge s_{k-1}^-} \cdots \int_{0\vee(s_2^- - r)}^{r\wedge s_2^-} dN_{s_1} \cdots dN_{s_{k-1}} \\
&= \int_{0\vee(t-r)}^{((k-1)r)\wedge t} \int_{0\vee(s_{k-1}^- - r)}^{((k-2)r)\wedge t} \cdots \int_{0\vee(s_2^- - r)}^{r\wedge t} dN_{s_1} \cdots dN_{s_{k-1}}, \qquad (2.3)
\end{aligned}$$

and we conclude by letting

$$f_k(s_1, \ldots, s_{k-1}) := \mathbf{1}_{\{t-r<s_{k-1}<(k-1)r\}} \mathbf{1}_{\{s_{k-1}-r<s_{k-2}<(k-2)r\}} \cdots \mathbf{1}_{\{s_2-r<s_1<r\}},$$

$s_1, \ldots, s_{k-1} \in [0, t]$. □

In particular, the 2-hop count is given by

$$\sigma_2(t) = \int_{0\vee(t-r)}^{r\wedge t} dN_s = N_t \mathbf{1}_{[0,r]}(t) + (N_r - N_{t-r})\mathbf{1}_{[r,2r]}(t). \qquad (2.4)$$

In the case of 3-hops, we have

$$\sigma_3(t) = \int_{0\vee(t-r)}^{(2r)\wedge t} \sigma_2(s^-)dN_s$$

$$
= \begin{cases} \displaystyle\int_{t-r}^{2r} \sigma_2(s^-) dN_s = \int_{t-r}^{2r} \int_{s^- - r}^{s^-} dN_u dN_s = \int_{t-r}^{2r} (N_{s^-} - N_{s^- - r}) dN_s, \quad t \in [2r, 3r], \\[2ex] \displaystyle\int_{t-r}^{t} \sigma_2(s^-) dN_s = \int_{t-r}^{t} N_{s^-} dN_s - \int_{r}^{t} N_{s^- - r} dN_s \\ \displaystyle\qquad = \frac{1}{2}(N_t - 1) N_t - \frac{1}{2}(N_{t-r} - 1) N_{t-r} - \sum_{l=1+N_r}^{N_t} N_{T_l^- - r}, \quad t \in [r, 2r], \\[2ex] \displaystyle\int_0^t N_{s^-} dN_s = \frac{1}{2}(N_t - 1) N_t, \qquad t \in [0, r], \end{cases}
$$

where $(T_n)_{n \geqslant 1}$ denotes the jump times sequence of the Poisson process $(N_t)_{t \in [0, kr]}$.

More generally, when $t \in [(l-1)r, lr]$ for some $l \in \{1, \ldots, k-1\}$, by (2.3) we have

$$
\sigma_k(t) = \int_{0 \vee (t-r)}^{t} \int_{0 \vee (s_{k-1}^- - r)}^{t} \cdots \int_{0 \vee (s_{l+1}^- - r)}^{t} \int_{s_l^- - r}^{(l-1)r} \cdots \int_{s_2^- - r}^{r} dN_{s_1} \cdots dN_{s_{k-1}},
$$

hence the identity in distribution

$$
\sigma_k(t) \overset{d}{\simeq} \int_{r \vee t}^{t+r} \int_{0 \vee (s_{k-1}^- - r)}^{t+r} \cdots \int_{0 \vee (s_{l+1}^- - r)}^{t+r} \int_{s_l^- - r}^{lr} \int_{s_{l-1}^- - r}^{(l-1)r} \cdots \int_{s_2^- - r}^{2r} dN_{s_1} \cdots dN_{s_{k-1}}.
$$

The multiple compensated Poisson stochastic integral of order $n \geqslant 1$ of a deterministic symmetric function $f_n \in L^2(\mathbb{R}_+^n, \lambda^{\otimes n})$ is defined by

$$
I_n(f_n) := n! \int_0^\infty \int_0^{t_{n-1}} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) d(N_{t_1} - \lambda(dt_1)) \cdots d(N_{t_n} - \lambda(dt_n)),
$$

with the isometry property

$$
\mathbb{E}[(I_n(f_n))^2] = (n!)^2 \int_0^\infty \int_0^{t_{n-1}} \cdots \int_0^{t_2} f_n^2(t_1, \ldots, t_n) \lambda(dt_1) \cdots \lambda(dt_n), \quad n \geqslant 1, \tag{2.5}
$$

see e.g. Propositions 2.7.1 and 6.2.4 in [Pri09], and references therein. The next corollary of Proposition 2.1 gives the chaos decomposition of k-hop counts in terms of multiple Poisson stochastic integrals.

Corollary 2.2 *For any $t \in [0, kr]$, the number $\sigma_k(t)$ of k-hops linking $x = 0$ to $y = t$ can be represented as the sum of multiple compensated Poisson stochastic integrals*

$$
\sigma_k(t) = \frac{1}{(k-1)!} \sum_{l=0}^{k-1} \binom{k-1}{l} I_l \left(\mathbf{1}_{\{* \in [0,t]^l\}} \int_0^t \cdots \int_0^t \widetilde{f}_k(*, s_{l+1}, \ldots, s_{k-1}) ds_{l+1} \cdots ds_{k-1} \right), \tag{2.6}
$$

where $\widetilde{f}_k$ is the symmetrization in $k-1$ variables of the function f_k defined in (2.2).

Proof. This is a direct consequence of Proposition 2.1 and the binomial theorem applied to $(dN_{t_1} - \lambda(dt_1)) \cdots (dN_{t_{k-1}} - \lambda(dt_{k-1}))$. □

3 Single node per cell

In the case where k-hop paths are constrained to have a single node per cell $[(l-1)r, lr]$, $l = 1, \ldots, k$, we must have $t \in [(k-1)r, kr]$, and $f_k(s_1, \ldots, s_{k-1}) = 0$ unless

$$(s_1, \ldots, s_{k-1}) \in [0, r] \times [r, 2r] \times \cdots \times [(k-2)r, (k-1)r].$$

Multiple Poisson stochastic integral expression

In particular, when $t \in [(k-1)r, kr]$ with $k \geqslant 2$, Relation (2.3) yields the following identity in distribution.

Proposition 3.1 *Let $k \geqslant 2$. For $\tau \in [0, r]$ we have the identity in distribution*

$$\sigma_k(kr - \tau) \overset{d}{\simeq} \int_0^\tau \int_0^{s_{k-1}^-} \cdots \int_0^{s_2^-} dN_{s_1}^{(1)} \cdots dN_{s_{k-1}}^{(k-1)}, \tag{3.1}$$

where $\big(N_s^{(l)}\big)_{s \in [0,r]}$ is a family of independent Poisson processes with respective intensities $\lambda_l(ds) = \lambda_l(s)ds$, $l = 1, \ldots, k-1$.

Proof. When $t \in [(k-1)r, kr]$, by the change of variables $u_l := lr - s_l$, $l = 1, \ldots, k-1$, we have

$$\begin{aligned} \sigma_k(t) &= \int_{t-r}^{(k-1)r} \int_{s_{k-1}^- - r}^{(k-2)r} \cdots \int_{s_2^- - r}^{r} dN_{s_1} \cdots dN_{s_{k-1}} \\ &= \int_0^{kr-t} \int_0^{u_{k-1}^-} \cdots \int_0^{u_2^-} dN_{r-u_1} \cdots dN_{(k-1)r - u_{k-1}}. \end{aligned}$$

We note that in the above integral, for $l = 1, \ldots, k-1$ we have $0 \leqslant u_l \leqslant kr - t \leqslant r$ and $(l-1)r \leqslant lr - u_l \leqslant lr$, hence the integration intervals are disjoint. Therefore, by (1.1), letting $\big(N_{u_l}^{(l)}\big)_{u_l \in [0,r]} := (N_{lr} - N_{lr-u_l})_{u_l \in [0,r]}$ define $k-1$ independent Poisson processes with respective intensities $\lambda_l(ds) = \lambda_l(s)ds$, $l = 1, \ldots, k-1$, and

$$\sigma_k(t) = \int_0^{kr-t} \int_0^{u_{k-1}^-} \cdots \int_0^{u_2^-} dN_{u_1}^{(1)} \cdots dN_{u_{k-1}}^{(k-1)}, \qquad (k-1)r \leqslant t \leqslant kr.$$

□

U-Statistics formulation

As noted in e.g. [KGKP21], when $\tau \in [0, r]$, any node contributing to a k-hop path linking $x_0 := 0$ to $x_{k+1} := kr - \tau$ must belong to one of the lenses pictured in pink in Figure 2, and defined as the intervals

$$L_j := [jr - \tau, jr] = [0, \tau] + jr - \tau, \qquad j = 1, \ldots, k-1,$$

of identical length τ. In particular, any k-hop path linking $x_0 := 0$ to $x_{k+1} := kr - \tau$ should have a single node per cell $[(j-1)r, jr]$, $j = 1, \ldots, k$, hence it must be realized using a sequence $(x_1, \ldots, x_{k-1})$ of nodes such that

$$x_{i+1} < x_i + r, \qquad i = 0, 1, \ldots, k,$$

with $x_0 := 0$ and $x_k := kr - \tau$. Therefore, any k-hop path $(x_1, \ldots, x_{k-1})$ can be mapped to a sequence $(y_1, \ldots, y_{k-1}) \in [0, \tau]^{k-1}$ by the relation

$$y_j := x_j - (jr - \tau), \qquad j = 1, \ldots, k-1,$$

with $y_1 > \cdots > y_{k-1}$. Based on the above description, we can model the random graph using a Poisson point process ω on $X := [0, r] \times \{1, \ldots, k-1\}$, with intensity μ of the form

$$\mu(ds, \{i\}) := \lambda_l(ds) = \lambda_l(s)ds, \qquad l = 1, \ldots, k-1,$$

and (3.1) can be rewritten as in the next proposition.

Proposition 3.2 *When $\tau \in [0, r]$, the count $\sigma_k(kr - \tau)$ of k-hop paths can be represented as the U-statistics*

$$\sigma_k(kr - \tau) = \sum_{\substack{((x_1,l_1),\ldots,((x_{k-1},l_{k-1}))\in\omega^{k-1} \\ (x_i,l_i)\neq(x_j,l_j),1\leqslant i\neq j\leqslant d}} f_\tau(x_1, l_1; \ldots; x_{k-1}, l_{k-1}) \tag{3.2}$$

of order $k-1$, where $f_\tau : ([0, r] \times \{1, \ldots, k-1\})^{k-1} \to \{0, 1\}$ is the function of $k-1$ variables in $[0, r] \times \{1, \ldots, k-1\}$ given by

$$f_\tau(x_1, l_1; \ldots; x_{k-1}, l_{k-1}) = \prod_{i=0}^{k-1} \mathbf{1}_{\{x_i < x_{i+1},\ l_i < l_{i+1}\}} = \mathbf{1}_{\{l_1=1,\ldots,l_{k-1}=k-1\}} \mathbf{1}_{\{0 < x_1 < \cdots < x_{k-1} < \tau\}}$$

with $(x_0, l_0) := (0, 0)$ and $(x_k, l_k) := (t, k)$.

4 Joint moments of k-hop counts

Proposition 4.1 provides a combinatorial expression for the joint moments of $(\sigma_k(kr - \tau_1), \ldots, \sigma_k(kr - \tau_n))$ for any $\tau_1, \ldots, \tau_n \in [0, r]$, using sums over partitions of $\{1, \ldots, n\}$.

Proposition 4.1 *Let $n \geqslant 1$. For any $\tau_1, \ldots, \tau_n \in [0, r]$, letting $\widehat{\tau}_\pi := \min_{i \in \pi} \tau_i$ for $\pi \subset \{1, \ldots, n\}$, we have*

$$\mathbb{E}[\sigma_k(kr - \tau_1) \cdots \sigma_k(kr - \tau_n)] \\ = \sum_{\pi^1, \ldots, \pi^{k-1} \in \Pi[n]} \int_{\prod_{l=1}^{k-1} \prod_{j=1}^{|\pi^l|} [0, \widehat{\tau}_{\pi^l_j}]} \prod_{\substack{1 \leqslant l < k \\ 1 \leqslant j \leqslant |\pi^l| \\ i \in \pi^l_j}} \mathbf{1}_{\{z^1_{\zeta^1_i} < \cdots < z^{k-1}_{\zeta^{k-1}_i}\}} \lambda_1(dz^1_{\pi^1}) \cdots \lambda_{k-1}(dz^{k-1}_{\pi^{k-1}}), \tag{4.1}$$

where ζ^j_i denotes the block of π^j that contains the index $i \in \{1, \ldots, n\}$, and $dz^j_{\pi^j} := (dz^j_i)_{i \in \pi^j}$, $j = 1, \ldots, k-1$.

Proof. For any $\tau_1, \ldots, \tau_n \in [0, r]$, by (3.2) and Corollary B.4 in Online Resource B, we have

$$\mathbb{E}[\sigma_k(kr - \tau_1) \cdots \sigma_k(kr - \tau_n)] \\ = \sum_{\substack{\pi \in \Pi[n \times (k-1)] \\ \pi \wedge \rho = \hat{0}}} \int_{[0,r]^{|\pi|}} \sum_{\substack{1 \leqslant l_q \leqslant k-1 \\ 1 \leqslant q \leqslant |\pi|}} \prod_{\substack{1 \leqslant j \leqslant |\pi| \\ i \in \pi_j}} f_{\tau_i}\big(z_{\zeta^\pi_{i,1}}, l_{\zeta^\pi_{i,1}}; \ldots; z_{\zeta^\pi_{i,k-1}}, l_{\zeta^\pi_{i,k-1}}\big) \lambda_{\bar{\pi}_1}(dz_1) \cdots \lambda_{\bar{\pi}_{|\pi|}}(dz_{|\pi|}),$$

where $\widehat{\mathbf{0}} := \{\{1\}, \ldots, \{n\}\}$ is the n-block partition of $\{1, \ldots, n\}$, $\bar{\pi}_i$ denotes the index $j \in \{1, \ldots, k-1\}$ of the unique block $\eta_j = ((i,j))_{i=1,\ldots,n}$ containing π_i, $i = 1, \ldots, |\pi|$, and the sum is taken over the set $\mathrm{NC}[n \times (k-1)]$ of partitions π in $\Pi[n \times (k-1)]$ that are non-flat, and non-crossing in the sense that if (k, l) and (k', l') belong to a same block of π then we should have $l = l'$. This yields

$$\mathbb{E}[\sigma_k(kr - \tau_1) \cdots \sigma_k(kr - \tau_n)] \\ = \sum_{\substack{\pi \in \Pi[n \times (k-1)] \\ \pi \wedge \rho = \hat{0}}} \int_0^{\widehat{\tau}_{\pi_1}} \cdots \int_0^{\widehat{\tau}_{\pi_{|\pi|}}} \sum_{\substack{1 \leqslant l_q \leqslant k-1 \\ 1 \leqslant q \leqslant |\pi|}} \prod_{\substack{1 \leqslant j \leqslant |\pi| \\ i \in \pi_j}} f_{\tau_i}\big(z_{\zeta^\pi_{i,1}}, l_{\zeta^\pi_{i,1}}; \ldots; z_{\zeta^\pi_{i,k-1}}, l_{\zeta^\pi_{i,k-1}}\big) \lambda_{\bar{\pi}_1}(dz_1) \cdots \lambda_{\bar{\pi}_{|\pi|}}(dz_{|\pi|}) \\ = \sum_{\substack{\pi \in \Pi[n \times (k-1)] \\ \pi \wedge \rho = \hat{0}}} \mathbf{1}_{\{\bar{\pi}_1 \leqslant \cdots \leqslant \bar{\pi}_{|\pi|}\}} \sum_{l_1 \leqslant \cdots \leqslant l_{|\pi|}}$$

$$\int_0^{\widehat{\tau}_{\pi_1}} \cdots \int_0^{\widehat{\tau}_{\pi_{|\pi|}}} \prod_{\substack{1 \leqslant j \leqslant |\pi| \\ i \in \pi_j}} f_{\tau_i}\big(z_{\zeta^\pi_{i,1}}, l_{\zeta^\pi_{i,1}}; \ldots ; z_{\zeta^\pi_{i,k-1}}, l_{\zeta^\pi_{i,k-1}}\big) \lambda_{\bar{\pi}_1}(dz_1) \cdots \lambda_{\bar{\pi}_{|\pi|}}(dz_{|\pi|})$$

$$= \sum_{\substack{\pi \in \mathrm{NC}[n \times (k-1)] \\ \pi \wedge \rho = \hat{0}}} \mathbf{1}_{\{\bar{\pi}_1 \leqslant \cdots \leqslant \bar{\pi}_{|\pi|}\}} \int_0^{\widehat{\tau}_{\pi_1}} \cdots \int_0^{\widehat{\tau}_{\pi_{|\pi|}}} \prod_{\substack{1 \leqslant j \leqslant |\pi| \\ i \in \pi_j}} \mathbf{1}_{\{z_{\zeta^\pi_{i,1}} < \cdots < z_{\zeta^\pi_{i,k-1}}\}} \lambda_{\bar{\pi}_1}(dz_1) \cdots \lambda_{\bar{\pi}_{|\pi|}}(dz_{|\pi|}).$$

We conclude to (4.1) by noting that any non-flat and non-crossing partition π in $\mathrm{NC}[n \times (k-1)]$ can be written as

$$\pi = \{\pi_1, \ldots, \pi_{|\pi|}\} = \bigcup_{l=1}^{k-1} \pi^l,$$

where $\pi^l \in \Pi[n]$ is a partition of $\{1, \ldots, n\}$ for every $l = 1, \ldots, k-1$. □

Next, we present the application of Proposition 4.1 in the particular cases of first and second moments.

First moment

When $n = 1$ there is only one non-flat and non-crossing partition of $1 \times (k-1)$, which is given as $\rho = \{\{(1,1)\}, \ldots, \{(1, k-1)\}\}$ and can be represented as in Figure 3 for $k = 9$.

Figure 3: Single non-flat and non-crossing partition of $1 \times (k-1)$ with $k = 9$.

This yields

$$\begin{aligned} \mathbb{E}[\sigma_k(kr - \tau)] &= \int_0^t \cdots \int_0^t \mathbf{1}_{\{z_1 < \cdots < z_{k-1}\}} \lambda_1(dz_1) \cdots \lambda_{k-1}(dz_{k-1}) \\ &= \int_0^t \int_0^{z_{k-1}} \cdots \int_0^{z_2} \lambda_1(dz_1) \cdots \lambda_{k-1}(dz_{k-1}), \end{aligned} \tag{4.2}$$

and when $\lambda_i(s)$ is the constant density $\lambda_i > 0$ on cell i, $i = 1, \ldots, k-1$, we find

$$\mathbb{E}[\sigma_k(kr - \tau)] = \lambda_1 \cdots \lambda_{k-1} \frac{\tau^{k-1}}{(k-1)!}.$$

Second moment

When $n = 2$ the count of blocks of non-flat and non-crossing partitions of $[2 \times (k-1)]$ ranges from $k-1$ to $2k-2$, each block has size either one or two, as in Figure 4 for $k = 9$.

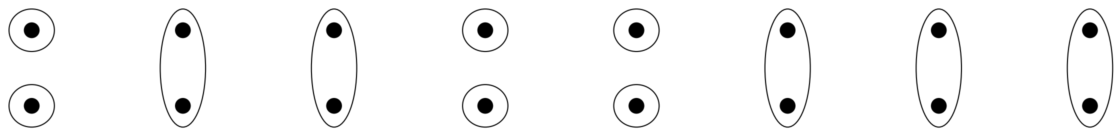


Figure 4: Example of non-flat and non-crossing partition of $2 \times (k-1)$ with $k=9$.

As a consequence, in Proposition 4.2 we obtain the second moment of the count $\sigma_k(t)$ of k-hop paths. Higher cumulants and moments of $\sigma_k(t)$ may also be computed by this method using Corollary 2.2 above and Corollary 7.4.1 of [PT11].

Proposition 4.2 *The variance of the k-hop path count $\sigma_k(kr-\tau)$, $\tau \in [0,r]$, is given by*

$$\mathrm{Var}_\lambda[\sigma_k(kr-\tau)] = \sum_{l=1}^{k-1} \tau^{2k-2-l} \frac{\lambda_1^2 \cdots \lambda_{k-1}^2}{(2k-2-l)!} \sum_{\substack{j_0+\cdots+j_l=k-1-l \\ j_0,\ldots,j_l \geqslant 0}} \prod_{q=1}^{l} \frac{1}{\lambda_{j_0+\cdots+j_{q-1}+q}} \prod_{p=0}^{l} \binom{2j_p}{j_p}.$$

Proof. We apply Proposition 4.1 by noting that the blocks of size one are in even number, and denoting by $i_1,\ldots,i_l$ their locations with $i_1=2, i_2=3, i_3=6, i_4=7, i_5=8$ in the above example, when $t=t_1=t_2$, and letting $z_0:=0$ and $z_k:=\tau$, we obtain

$$\mathbb{E}[\sigma_k^2(kr-\tau)] = \lambda_1 \cdots \lambda_{k-1} \sum_{l=0}^{k-1} \sum_{0=i_0<i_1<\cdots<i_l<i_{l+1}=k} \left(\prod_{\substack{1\leqslant q<k \\ q\notin\{i_1,\ldots,i_l\}}} \lambda_q \right)$$
$$\times \int_0^\tau \int_0^{z_{i_l}} \cdots \int_0^{z_{i_2}} \prod_{p=0}^{l} \left(\frac{(z_{i_{p+1}}-z_{i_p})^{(i_{p+1}-i_p-1)}}{(i_{p+1}-i_p-1)!} \right)^2 dz_{i_1}\cdots dz_{i_l},$$

where we let $z_0:=0$ and $z_k:=\tau$, To conclude, we check that

$$\begin{aligned} & \int_0^{a_k} \int_0^{z_{i_l}} \cdots \int_0^{z_{i_2}} \prod_{p=0}^{l} \frac{(z_{i_{p+1}}-z_{i_p})^{2(i_{p+1}-i_p-1)}}{((i_{p+1}-i_p-1)!)^2} dz_{i_1}\cdots dz_{i_l} \\ &= \frac{1}{((i_1-1)!)^2} \prod_{p=1}^{l} \int_0^1 \frac{(1-y)^{2(i_{p+1}-i_p-1)} y^{2i_p-p-1}}{((i_{p+1}-i_p-1)!)^2} dy \\ &= \frac{1}{((i_1-1)!)^2} \prod_{p=1}^{l} \frac{B(2i_p-p, 2(i_{p+1}-i_p)-1)}{((i_{p+1}-i_p-1)!)^2} \\ &= \frac{1}{(2k-2-l)!} \prod_{p=0}^{l} \frac{(2(i_{p+1}-i_p-1))!}{((i_{p+1}-i_p-1)!)^2}, \end{aligned}$$

where

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{(x-1)!(y-1)!}{(x+y-1)!}, \qquad x,y>0,$$

is the beta function, and

$$\frac{2^{-(i_{l+1}-i_l-1)}(2(i_{l+1}-i_l-1))!}{(i_{l+1}-i_l-1)!}$$

is the number of pair-partitions of $2(i_{l+1}-i_l-1)$. □

Alternatively, Proposition 4.2 can be proved as a consequence of the isometry formula for multiple Poisson stochastic integrals. For this, we can use the expression

$$\sigma_k(kr-\tau) = \int_{(k-1)r-\tau}^{(k-1)r} \int_{s_{k-1}^- - r}^{(k-2)r} \cdots \int_{s_2^- - r}^{r} dN_{s_1} \cdots dN_{s_{k-1}}$$
$$= \sum_{l=0}^{k-1} \sum_{0=i_0<i_1<\cdots<i_l<i_{l+1}=k} \int_0^{(k-1)r} \cdots \int_0^{(k-1)r} f_k(s_1,\ldots,s_{k-1}) \prod_{\substack{1\leqslant q\leqslant d\\ q\notin\{i_1,\ldots,i_l\}}} (\lambda_q ds_q) \prod_{p=1}^{l} (dN_{s_{i_p}} - \lambda_{i_p} ds_{i_p})$$

that follows from Corollary 2.2 and the isometry and orthogonality property (2.5) of multiple Poisson stochastic integrals, to show that

$$\mathbb{E}[\sigma_k^2(kr-\tau)]$$
$$= \sum_{l=0}^{k-1} \sum_{0=i_0<i_1<\cdots<i_l<i_{l+1}=k} \int_{[0,(k-1)r]^l} \left(\int_{[0,(k-1)r]^{d-l}} f_k(s_1,\ldots,s_{k-1}) \prod_{\substack{1\leqslant q<k\\ q\notin\{i_1,\ldots,i_l\}}} (\lambda_q ds_q) \right)^2 \prod_{p=1}^{l} (\lambda_{i_p} ds_{i_p}).$$

The following table provides variance formulas of k-hop counts computed from Proposition 4.2 by taking $\tau = 1$ for simplicity.

	Variance
2-hops	λ_1
3-hops	$\frac{\lambda_1\lambda_2}{2} + 2\frac{\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2}{3!}$
4-hops	$\frac{\lambda_1\lambda_2\lambda_3}{3!} + 2\frac{\lambda_1^2\lambda_2\lambda_3 + \lambda_1\lambda_2^2\lambda_3 + \lambda_1\lambda_2\lambda_3^2}{4!} + \frac{4\lambda_1^2\lambda_2\lambda_3^2 + 6\lambda_1^2\lambda_2^2\lambda_3 + 4\lambda_1\lambda_2^2\lambda_3^2}{5!}$

Table 1: Variances of k-hop counts.

In case the Poisson intensities are identical on all cells, we obtain the following result.

Corollary 4.3 *Assume that $\lambda = \lambda_1 = \cdots = \lambda_{k-1}$ and let $\tau \in [0,r]$. Then, the variance of the k-hop path count $\sigma_k(kr-\tau)$ is given by*

$$\mathrm{Var}_\lambda[\sigma_k(kr-\tau)] = \frac{1}{(k-1)!} \sum_{l=0}^{k-2} \binom{k-1}{l} (\lambda\tau)^{k-1+l} \frac{\Gamma((k-1-l)/2+1)}{\Gamma((k-1+l)/2+1)}, \quad (4.3)$$

where Γ *denotes the gamma function defined as*

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx, \qquad z > 0.$$

Proof. Let $X_0, \ldots, X_l$ be independent standard normal random variables. From the moment relation $\mathbb{E}\left[X_p^{2j_p}\right] = 2^{-j_p}(2j_p)!/j_p!$ and the fact that the sum $X_0^2 + \cdots + X_l^2$ has a Chi square distribution, we have

$$\begin{aligned}
\sum_{\substack{j_0+\cdots+j_l=k-1-l \\ j_0,\ldots,j_l \geqslant 0}} \prod_{p=0}^{l} \binom{2j_p}{j_p} &= \frac{2^{k-1-l}}{(k-1-l)!} \mathbb{E}\left[\sum_{\substack{j_0+\cdots+j_l=k-1-l \\ j_0,\ldots,j_l \geqslant 0}} \frac{(k-1-l)!}{j_0! \cdots j_l!} \prod_{p=0}^{l} X_p^{2j_p} \right] \\
&= \frac{2^{2(k-1-l)}}{(k-1-l)!} \mathbb{E}\left[\left(\frac{X_0^2 + \cdots + X_l^2}{2} \right)^{k-1-l} \right] \\
&= 2^{2(k-1-l)} \frac{\Gamma(k-1-l+(l+1)/2)}{(k-1-l)!\Gamma((l+1)/2)}.
\end{aligned}$$

Hence, from Proposition 4.2 we find

$$\begin{aligned}
\mathbb{E}[\sigma_k^2(kr-\tau)] &= \sum_{l=0}^{k-1} \frac{(\lambda\tau)^{2k-2-l}}{(2k-2-l)!} \frac{2^{2(k-1-l)}}{(k-1-l)!} \frac{\Gamma(k-1+(1-l)/2)}{\Gamma((l+1)/2)} \\
&= \sum_{l=0}^{k-1} \frac{(\lambda\tau)^{k-1+l}}{(k-1+l)!} \frac{2^{2l}}{l!} \frac{\Gamma((k-1+l+1)/2)}{\Gamma((k-1-l+1)/2)} \\
&= \frac{1}{(k-1)!} \sum_{l=0}^{k-1} \binom{k-1}{l} (\lambda\tau)^{k-1+l} \frac{((k-1-l)/2)!}{((k-1+l)/2)!}.
\end{aligned}$$

□

The following table provides variance formulas of k-hop counts obtained from Corollary 4.3, by taking $\lambda = 1$ for simplicity.

	Variance
2-hops	τ
3-hops	$\frac{\tau^2}{2} + 2\frac{\tau^3}{3}$
4-hops	$\frac{\tau^3}{3!} + \frac{\tau^4}{4} + 2\frac{\tau^5}{15}$
5-hops	$\frac{\tau^4}{4!} + \frac{\tau^5}{15} + \frac{\tau^6}{24} + \frac{4\tau^7}{315}$
6-hops	$\frac{\tau^5}{5!} + \frac{\tau^6}{72} + \frac{\tau^7}{105} + \frac{\tau^8}{288} + \frac{2\tau^9}{2835}$

Table 2: Variances of k-hop counts.

Using the Legendre duplication formula $(2k-3)!\Gamma(3/2) = 2^{2k-4}(k-2)!\Gamma(k-1/2)$, Corollary 4.3 also yields the following asymptotic variance. For f and g two nonvanishing functions on $\mathbb{R}_+$ we write $f(\lambda) \approx g(\lambda)$ if $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = 1$.

Proposition 4.4 *We have the equivalence*

$$\mathrm{Var}_\lambda[\sigma_k(kr-\tau)] \approx \frac{(2\lambda\tau)^{2k-3}}{2(2k-3)!} \tag{4.4}$$

as λ tends to infinity, for $k \geqslant 2$ and $\tau \in [0,r]$.

5 Joint moments recursion

In the one-hop case we simply have $\sigma_1(t) = 1$ and $m_{1,n}^{(\lambda)} = 1$, $n \geqslant 0$. As for the two-hop count, (2.4) yields the joint Poisson moments formula

$$m_{2,n}^{(\lambda)}(\tau_1,\ldots,\tau_n) = \mathbb{E}[\sigma_2(2r-\tau_1)\cdots\sigma_2(2r-\tau_n)] = \sum_{l=1}^{n}\lambda_1^l \sum_{\eta_1\cup\cdots\cup\eta_l=\{1,\ldots,n\}} \prod_{j=1}^{l} \widehat{\tau}_{\eta_j} \tag{5.1}$$

that follows from the expression $\lambda_1 \min(\tau_1,\ldots,\tau_n)$ of the joint Poisson cumulants of $(\sigma_2(2r-\tau_1),\ldots,\sigma_2(2r-\tau_n))$. The direct application of Proposition 4.1 to the evaluation of higher order joint moments of k-hop counts is not an easy task due to the complexity of the summations over partitions involved in (4.1). In Proposition 5.1 we propose to compute joint moments by a recursion argument using the multiple Poisson stochastic integral representation (3.1) instead of the U-statistics expression (3.2). Particular cases are considered with explicit computations for $n = 1,2,3$ in Online Resource C.

Proposition 5.1 *For $k \geqslant 1$, the joint moments*

$$m_{k,n}^{(\lambda)}(\tau_1, \ldots, \tau_n) := \mathbb{E}_\lambda[\sigma_k(kr-\tau_1)\cdots\sigma_k(kr-\tau_n)], \qquad 0 \leqslant \tau_1, \ldots, \tau_n \leqslant r,$$

satisfy the recursion

$$m_{k+1,n}^{(\lambda)}(\tau_1, \ldots, \tau_n) = \sum_{l=1}^{n} \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_0^{\widehat{\tau}_{\pi_l}} \cdots \int_0^{\widehat{\tau}_{\pi_1}} m_{k,n}^{(\lambda)}(\bar{u}_{\pi_1}, \ldots, \bar{u}_{\pi_l}) \lambda_k(du_1)\cdots\lambda_k(du_l), \tag{5.2}$$

$0 \leqslant \tau_1, \ldots, \tau_n \leqslant r$, *where* $\bar{u}_{\pi_i} := (\underbrace{u_i, \ldots, u_i}_{|\pi_i| \text{ times}})$ *and* $\widehat{\tau}_\pi := \min_{i\in\pi} \tau_i$ *for* $\pi \subset \{1, \ldots, n\}$.

Proof. By Proposition 3.1, when $\tau \in [0, r]$ we have $\sigma^{(k)}(kr-\tau) \overset{d}{\simeq} Z_\tau^{(k)}$ in distribution, where $Z_\tau^{(k)}$ satisfies $Z_\tau^{(0)} = 1$ and the recursion

$$Z_\tau^{(k+1)} = \int_0^\tau Z_u^{(k)} dN_u^{(k)}, \qquad \tau \in [0, r], \quad k \geqslant 1, \tag{5.3}$$

and $(N_u^{(l)})_{u\in[0,r]}$ is a family of independent Poisson processes with respective intensities $\lambda_l(ds) := \lambda_l(s)ds$, $l = 1, \ldots, k$. Hence, by Proposition B.1 in Online Resource B, for $\tau_1, \ldots, \tau_n \in [0, r]$ we have

$$\begin{aligned}
&m_{k+1,n}^{(\lambda)}(\tau_1, \ldots, \tau_n) = \mathbb{E}\left[Z_{\tau_1}^{(k+1)} \cdots Z_{\tau_n}^{(k+1)}\right] \\
&= \mathbb{E}\left[\int_0^{\tau_1} Z_u^{(k)} dN_u^{(k)} \cdots \int_0^{\tau_n} Z_u^{(k)} dN_u^{(k)}\right] \\
&= \sum_{l=1}^{n} \lambda_k^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_{\mathbb{R}_+^l} \mathbb{E}\left[\prod_{j=1}^{l} \prod_{i\in\pi_j} \left(Z_{u_j}^{(k)} \mathbf{1}_{[0,\tau_i]}(u_j)\right)\right] \lambda_k(du_1)\cdots\lambda_k(du_l) \\
&= \sum_{l=1}^{n} \lambda_k^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_{\mathbb{R}_+^l} \mathbb{E}\left[\prod_{j=1}^{l} \left(\left(Z_{u_j}^{(k)}\right)^{|\pi_j|} \mathbf{1}_{[0,\widehat{\tau}_{\pi_j}]}(u_j)\right)\right] \lambda_k(du_1)\cdots\lambda_k(du_l) \\
&= \sum_{l=1}^{n} \lambda_k^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_0^{\widehat{\tau}_{\pi_l}} \cdots \int_0^{\widehat{\tau}_{\pi_1}} \mathbb{E}\left[\prod_{j=1}^{l} \left(Z_{u_j}^{(k)}\right)^{|\pi_j|}\right] \lambda_k(du_1)\cdots\lambda_k(du_l) \\
&= \sum_{l=1}^{n} \lambda_k^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_0^{\widehat{\tau}_{\pi_l}} \cdots \int_0^{\widehat{\tau}_{\pi_1}} m_{k,n}^{(\lambda)}(\bar{u}_{\pi_1}, \ldots, \bar{u}_{\pi_l}) \lambda_k(du_1)\cdots\lambda_k(du_l).
\end{aligned} \tag{5.4}$$

□

Table 3 lists the first four joint moments $m_{2,n}^{(\lambda)}(\tau_1,\ldots,\tau_n)$ of the two-hop counts, computed as an application of Proposition 5.1 from the command mk[$\{\tau_1,\ldots,\tau_n\},\{\lambda_1\}$] in the Mathematica code 1 in Online Resource A for $n=1,2,3,4$, when $\lambda_1(ds)=\lambda_1 ds=ds$. The case $\lambda_1(ds)=\lambda_1 ds$, $\lambda_1>0$, is obtained by replacing τ_i with $\lambda_1\tau_i$, $i=1,2,3$.

	Joint moments of 2-hop counts
First	τ_1
Second	$\tau_1+\tau_1\tau_2$
Third	$\tau_1+\tau_1\tau_3+2\tau_1\tau_2+\tau_1\tau_2\tau_3$
Fourth	$\tau_1+\tau_1\tau_4+2\tau_1\tau_3+4\tau_1\tau_2+\tau_1\tau_3\tau_4+2\tau_1\tau_2\tau_4+3\tau_1\tau_2\tau_3+\tau_1\tau_2\tau_3\tau_4$

Table 3: Joint moments $m_{2,n}^{(\lambda)}(\tau_1,\ldots,\tau_n)$ of 2-hop counts of orders $n=1,2,3,4$.

Tables 4 and 5 list the first four moments of the three-hop and four-hop counts computed from the commands mk[$\{\tau_1,\ldots,\tau_n\},\{\lambda_1,\lambda_2\}$] and mk[$\{\tau_1,\ldots,\tau_n\},\{\lambda_1,\lambda_2,\lambda_3\}$] in Mathematica for $n=1,2,3,4$, where for simplicity we take $\lambda_1=\lambda_2=\lambda_3=1$ and $\tau_1=\tau_2=\tau_3=\tau$.

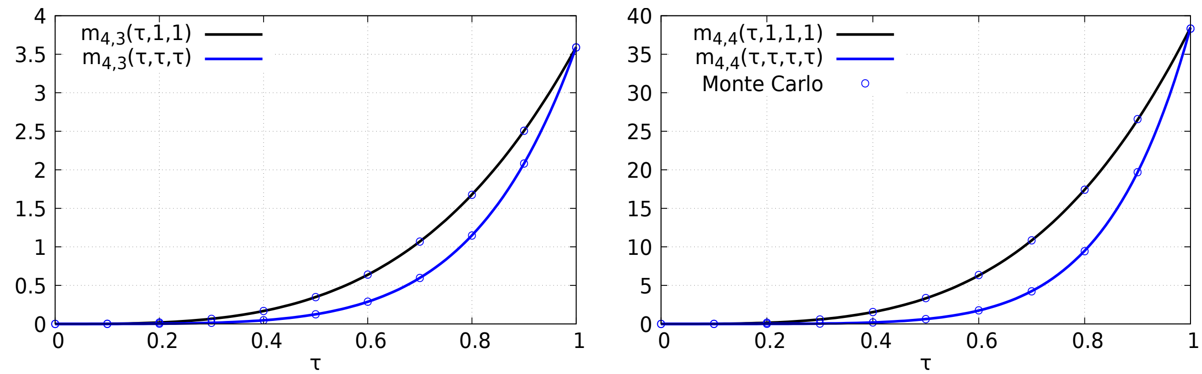
	Moments of 3-hop counts
First	$\frac{\tau^2}{2}$
Second	$\frac{\tau^2}{2}+2\frac{\tau^3}{3}+\frac{\tau^4}{4}$
Third	$\frac{\tau^2}{2}+2\tau^3+5\frac{\tau^4}{2}+\tau^5+\frac{\tau^6}{8}$
Fourth	$\frac{\tau^2}{2}+14\frac{\tau^3}{3}+53\frac{\tau^4}{4}+66\frac{\tau^5}{5}+67\frac{\tau^6}{12}+\tau^7+\frac{\tau^8}{16}$

Table 4: Moments $m_{3,n}^{(\lambda)}(\tau,\ldots,\tau)$ of 3-hop counts of orders $n=1,2,3,4$.

	Moments of 4-hop counts
First	$\frac{\tau^3}{3!}$
Second	$\frac{\tau^3}{3!}+\frac{\tau^4}{4}+2\frac{\tau^5}{15}+\frac{\tau^6}{36}$
Third	$\frac{\tau^3}{3!}+3\frac{\tau^4}{4}+5\frac{\tau^5}{4}+\frac{59\tau^6}{60}+\frac{13\tau^7}{35}+\frac{\tau^8}{15}+\frac{\tau^9}{216}$

Table 5: Moments $m_{4,n}^{(\lambda)}(\tau,\ldots,\tau)$ of 4-hop counts of orders $n=1,2,3$.

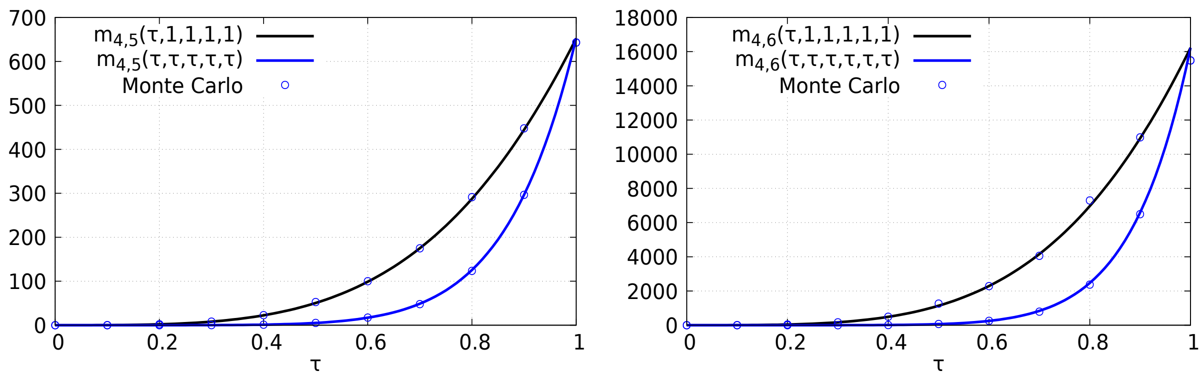
Higher joint moment formulas up to the order six are plotted in Figures 5-6, together with their confirmations by Monte Carlo simulations.



(a) Third moments.

(b) Fourth moments.

Figure 5: Third and fourth joint moments of 4-hop counts.



(a) Fifth moments.

(b) Sixth moments.

Figure 6: Fifth and sixth joint moments of 4-hop counts.

6 Joint cumulants recursion

In the sequel, given $\{\pi_1, \ldots, \pi_l\}$ a partition of $\{1, \ldots, n\}$ and $\tau_1, \ldots, \tau_l \in [0, r]$ we denote by

$$c_{k,n}^{(\lambda)}(\bar{\tau}_{\pi_1}; \ldots; \bar{\tau}_{\pi_l}) := \kappa_\lambda\big((\sigma_k(kr - \tau_1))^{|\pi_1|}, \ldots, (\sigma_k(kr - \tau_l))^{|\pi_l|}\big)$$

the joint cumulant of $\big((\sigma_k(kr - \tau_1))^{|\pi_1|}, \ldots, (\sigma_k(kr - \tau_l))^{|\pi_l|}\big)$, and we let $c_{k,n}^{(\lambda)}(\tau_1; \ldots; \tau_n)$ denote the joint cumulant $\kappa_\lambda(\sigma_k(kr - \tau_1), \ldots, \sigma_k(kr - \tau_n))$. The following proposition is the counterpart of the moment recursion of Proposition 5.1 obtained by the Möbius inversion relation (1.3). Particular cases are considered with explicit computations for $n = 2, 3, 4$ in Online Resource D.

Proposition 6.1 *Let $\tau \in [0,r]$ and $k,n \geqslant 1$. The cumulant of order n of $\sigma_{k+1}(kr-\tau)$ satisfies the recursion*

$$c^{(\lambda)}_{k+1,n}(\tau_1;\ldots;\tau_n) = \sum_{l=1}^n \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \int_0^{\widehat{\tau}_{\pi_1}} \cdots \int_0^{\widehat{\tau}_{\pi_l}} c^{(\lambda)}_{k,l}(\bar{s}_{\pi_1};\ldots;\bar{s}_{\pi_l})\lambda_k(ds_1)\cdots\lambda_k(ds_l), \tag{6.1}$$

where we let $\widehat{\tau}_\pi := \min_{i\in\pi}\tau_i$ for $\pi \subset \{1,\ldots,n\}$.

Proof. When $k=1$ we have $\sigma_1(\tau)=1$, $\tau\in[0,r]$, hence $c^{(\lambda)}_{1,n}(\tau_1;\ldots;\tau_n) = \mathbf{1}_{\{n=1\}}$, and by (2.4) the cumulants of the two-hop count $\sigma_2(2r-\tau) = N_r - N_{r-\tau} \stackrel{d}{\simeq} N^{(1)}_\tau$, $\tau\in[0,r]$, are the joint Poisson cumulants

$$c^{(\lambda)}_{2,n}(\tau_1;\ldots;\tau_n) = \min(\tau_1,\ldots,\tau_n), \qquad n\geqslant 1, \tag{6.2}$$

which is consistent with the joint Poisson moments formula (5.1), and shows that (6.1) holds at the rank $k=1$. Next, assuming that (6.1) holds at the rank $k\geqslant 1$, by the joint cumulant-moment inversion relation (1.7) we have

$$\begin{aligned}
c^{(\lambda)}_{k+1,n}(\tau_1;\ldots;\tau_n) &= \sum_{l=1}^n (l-1)!(-1)^{l-1} \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{q=1}^l m^{(\lambda)}_{k+1,|\pi_q|}(\tau_{\pi_q}) \\
&= \sum_{l=1}^n (l-1)!(-1)^{l-1} \\
&\quad \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{q=1}^l \sum_{\eta^q\preceq\pi_q} \int_0^{\widehat{\tau}_{\eta^q_1}} \cdots \int_0^{\widehat{\tau}_{\eta^q_{|\eta^q|}}} m^{(\lambda)}_{k,|\pi_q|}\big(\bar{u}_{\eta^q_1};\ldots;\bar{u}_{\eta^q_{|\eta^q|}}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\eta^q|}), \\
&= \sum_{l=1}^n (l-1)!(-1)^{l-1} \\
&\quad \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \sum_{\substack{\eta^i\preceq\pi_i \\ 1\leqslant i\leqslant l}} \prod_{q=1}^l \int_0^{\widehat{\tau}_{\eta^q_1}} \cdots \int_0^{\widehat{\tau}_{\eta^q_{|\eta^q|}}} m^{(\lambda)}_{k,|\eta^q|}\big(\bar{u}_{\eta^q_1};\ldots;\bar{u}_{\eta^q_{|\eta^q|}}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\eta^q|}), \\
&= \sum_{l=1}^n (l-1)!(-1)^{l-1} \\
&\quad \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \sum_{\substack{\eta^i\preceq\pi_i \\ 1\leqslant i\leqslant l}} \prod_{q=1}^l \sum_{\psi^q\preceq\eta^q} \int_0^{\widehat{\tau}_{\eta^q_1}} \cdots \int_0^{\widehat{\tau}_{\eta^q_{|\eta^q|}}} \prod_{m=1}^{|\psi^q|} c^{(\lambda)}_{k,n}\big(\big(\bar{u}_{\eta^q_i}\big)_{i\in\psi^q_m}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\eta^q|}) \\
&= \sum_{l=1}^n (l-1)!(-1)^{l-1}
\end{aligned}$$

$$\sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \sum_{\substack{\psi^i\preceq\eta^i\preceq\pi_i\\ 1\leqslant i\leqslant l}} \int_0^{\widehat{\tau}_{\eta_1^q}} \cdots \int_0^{\widehat{\tau}_{\eta^q_{|\eta^q|}}} \prod_{m=1}^{|\psi^q|} \prod_{q=1}^{l} c_{k,n}^{(\lambda)}\big(\big(\bar{u}_{\eta_i^q}\big)_{i\in\psi_m^q}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\psi^q|})$$

$$= \sum_{l=1}^{n} \mu(\sigma,\widehat{\mathbf{1}}) \sum_{\eta\preceq\sigma\preceq\widehat{\mathbf{1}}} \sum_{\psi\preceq\eta} \int_0^{\widehat{\tau}_{\psi_1}} \cdots \int_0^{\widehat{\tau}_{\psi_{|\psi|}}} \prod_{A\in\eta} c_{k,n}^{(\lambda)}\big(\big(\bar{u}_{A\cap B}\big)_{B\in\psi}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\psi|})$$

$$= \sum_{l=1}^{n} \mu(\sigma,\widehat{\mathbf{1}}) \sum_{\eta\preceq\sigma\preceq\widehat{\mathbf{1}}} G(\eta)$$

$$= G(\widehat{\mathbf{1}})$$

$$= \sum_{\eta\preceq\widehat{\mathbf{1}}} \int_0^{\widehat{\tau}_{\eta_1}} \cdots \int_0^{\widehat{\tau}_{\eta_{|\eta|}}} c_{k,n}^{(\lambda)}(\bar{u}_{\eta_1};\ldots;\bar{u}_{\eta_{|\eta|}})\lambda_k(du_1)\cdots\lambda_k(du_{|\eta|}),$$

where we applied Relation (1.4) with $\pi := \widehat{\mathbf{1}}$ and

$$G(\eta) := \sum_{\psi\preceq\eta} \int_0^{\widehat{\tau}_{\psi_1}} \cdots \int_0^{\widehat{\tau}_{\psi_{|\psi|}}} \prod_{A\in\eta} c_{k,n}^{(\lambda)}\big(\big(\bar{u}_{A\cap B}\big)_{B\in\psi}\big)\lambda_k(du_1)\cdots\lambda_k(du_{|\psi|})$$

which shows (6.1) by induction on $k \geqslant 1$. □

In order to use Proposition 6.1 as an induction relation, the higher order cumulants $c_{k,n}^{(\lambda)}(\widehat{s}_{\pi_1};\ldots;\widehat{s}_{\pi_l})$ appearing in (6.1) can be computed by recurrence using the following proposition.

Proposition 6.2 *For any sequence $(X_1,\ldots,X_{n+1})$ of random variables we have the cumulant relation*

$$\kappa(X_1,\ldots,X_nX_{n+1}) = \kappa(X_1,\ldots,X_n,X_{n+1}) + \sum_{\substack{\eta_1\cup\eta_2=\{1,\ldots,n+1\}\\ \eta_1\ni n,\ \eta_2\ni n+1,\ \eta_1\cap\eta_2=\emptyset}} \kappa(X_{\eta_1},X_n)\kappa(X_{\eta_2},X_{n+1}).$$

Proof. This relation is a particular case of the cumulant-moment relationship

$$\kappa\big(Z_1,\ldots,Z_n\big) = \sum_{l=1}^{n}(l-1)!(-1)^{l-1} \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{j=1}^{l} \mathbb{E}\left[\prod_{i\in\pi_j} Z_i\right]$$

which yields, taking $Z_1 := X_1, Z_2 := X_2, \ldots, Z_n := X_nX_{n+1}$,

$$\kappa(X_1,\ldots,X_nX_{n+1}) = \sum_{l=1}^{n}(l-1)!(-1)^{l-1} \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{j=1}^{l} \mathbb{E}\left[\prod_{i\in\pi_j} X_i \prod_{\pi_j\ni n}(X_nX_{n+1})\right]$$

$$= \sum_{l=1}^{n}(l-1)!(-1)^{l-1} \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{j=1}^{l} \sum_{p=1}^{|\pi_j|+1} \sum_{\eta_1\cup\cdots\cup\eta_p=\pi_j\cup\{n,n+1\}} \prod_{v=1}^{p} \kappa\big((X_u)_{u\in\eta_p}\big)$$

$$= \kappa(X_1,\ldots,X_n,X_{n+1}) + \sum_{\substack{\eta_1\cup\eta_2=\{1,\ldots,n+1\}\\ \eta_1\ni n,\ \eta_2\ni n+1,\ \eta_1\cap\eta_2=\emptyset}} \kappa(X_{\eta_1},X_n)\kappa(X_{\eta_2},X_{n+1}),$$

□

As an application of Proposition 6.2, Tables 6 and 7 present the first five and first three cumulants $c^{(\lambda)}_{3,n}(\tau;\ldots;\tau)$ of the three-hop count and $c^{(\lambda)}_{4,n}(\tau;\ldots;\tau)$ of the four-hop count computed by the commands ck[$\{\tau_1,\ldots,\tau_n\}$, $\{1,\ldots,1\}$, $\{\lambda_1,\lambda_2,\lambda_2\}$] and ck[$\{\tau_1,\ldots,\tau_n\}$, $\{1,\ldots,1\}$, $\{\lambda_1,\lambda_2,\lambda_3\}$] in the Mathematica codes 2-3 in Online Resource A for $n=1,2,3,4,5$, where for simplicity we take $\lambda_i=1$, $i=1,2,3$ and $\tau_j=\tau$, $j=1,\ldots,5$.

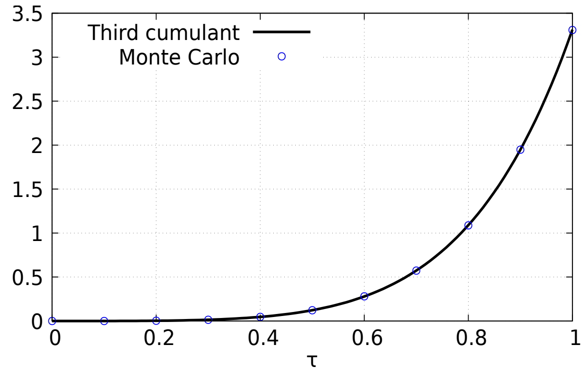
	Cumulants of 3-hop counts
First	$\frac{\tau^2}{2}$
Second	$\frac{\tau^2}{2}+\frac{2\tau^3}{3}$
Third	$\frac{\tau^2}{2}+2\tau^3+\frac{7\tau^4}{4}$
Fourth	$\frac{\tau^2}{2}+\frac{14\tau^3}{3}+\frac{23\tau^4}{2}+\frac{36\tau^5}{5}$
Fifth	$\frac{\tau^2}{2}+10\tau^3+\frac{215}{4}\tau^4+86\tau^5+41\tau^6$

Table 6: Cumulants $c^{(\lambda)}_{3,n}(\tau;\ldots;\tau)$ of 3-hop counts of orders $n=1,2,3$.

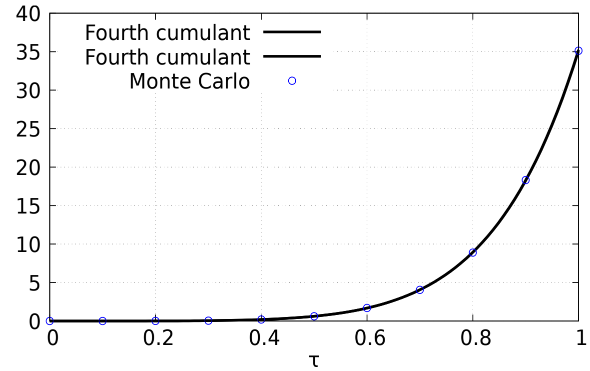
	Cumulants of 4-hop counts
First	$\frac{\tau^3}{3!}$
Second	$\frac{\tau^3}{3!}+\frac{\tau^4}{4}+\frac{2\tau^5}{15}$
Third	$\frac{\tau^3}{3!}+\frac{3\tau^4}{4}+\frac{5\tau^5}{4}+\frac{9\tau^6}{10}+\frac{69\tau^7}{280}$

Table 7: Cumulants $c^{(\lambda)}_{4,n}(\tau;\ldots;\tau)$ of 4-hop counts of orders $n=1,2,3$.

Figures 7-8 present third and fourth order cumulant plots for 4-hop counts, together with their confirmations by Monte Carlo simulations.

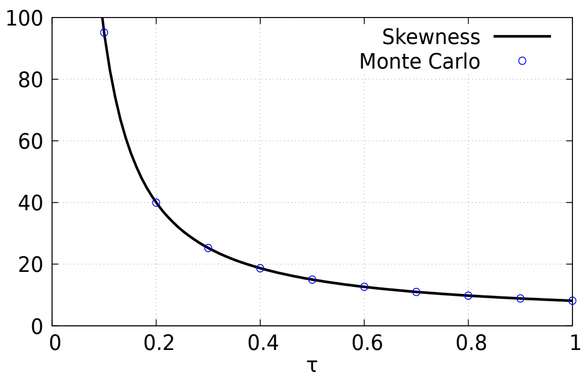


(a) Third cumulant $c_{4,3}^{(\lambda)}(\tau;\tau;\tau)$.

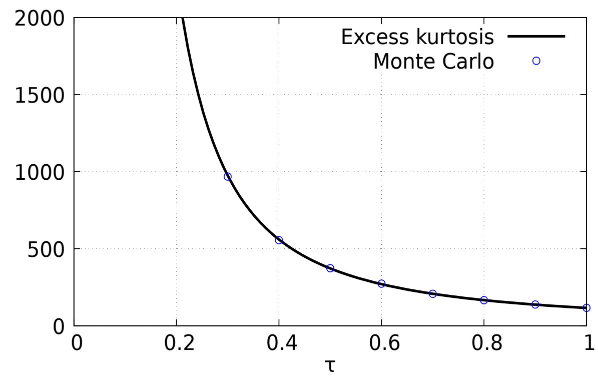


(b) Fourth cumulant $c_{4,4}^{(\lambda)}(\tau;\tau;\tau;\tau)$.

Figure 7: Third and fourth cumulants of the 4-hop count $\sigma_4(4-\tau)$.



(a) Skewness of $\sigma_4(4-\tau)$.



(b) Kurtosis of $\sigma_4(4-\tau)$.

Figure 8: Skewness and kurtosis of the 4-hop count $\sigma_4(4-\tau)$.

7 Moment and cumulant bounds

In this section we take $\lambda_l(ds) := \lambda ds$, $l = 1, \ldots, k$, $\lambda > 0$, and we write $f(\lambda) = O(\lambda^n)$ if there exist $C_n > 0$ and $\lambda_n > 0$ such that $|f(\lambda)| \leqslant C_n \lambda^n$ for any $\lambda > \lambda_n$. The moment bound in Proposition 7.1 is obtained by induction from Proposition 5.1.

Proposition 7.1 *Moment bound. For any $n \geqslant 0$, we have*

$$\mathbb{E}_\lambda[(\sigma_k(kr-\tau))^n] \leqslant (\mathbb{E}[(N_{\lambda\tau})^n])^{k-1} = O\big((\lambda\tau)^{(k-1)n}\big)$$

as λ tends to infinity, for $k \geqslant 2$ and $\tau \in [0, r]$.

Proof. We show by induction on $k \geqslant 1$ that

$$m_{k,n}^{(\lambda)}(\tau_1, \ldots, \tau_n) \leqslant (\mathbb{E}[(N_{\lambda\tau})^n])^{k-1}, \qquad 0 \leqslant \tau_1, \ldots, \tau_n \leqslant \tau.$$

where, denoting by $S(n, l)$ the Stirling number of the second kind,

$$\mathbb{E}[(N_{\lambda\tau})^n] = \sum_{l=1}^{n} S(n, l)(\lambda\tau)^l = O(\lambda^n).$$

The case $k = 1$ is covered by the fact that $\sigma_1(r - \tau) = 1$ and $m_{1,n}^{(\lambda)}(\tau) = 1$, $n \geqslant 0$. Next, by the recurrence relation (5.2) we have

$$\begin{aligned} m_{k+1,n}^{(\lambda)}(\tau, \ldots, \tau) &\leqslant \sum_{l=1}^{n} (\lambda\tau)^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}} \sup_{0 \leqslant \tau_1, \ldots, \tau_l \leqslant \tau} m_{k,n}^{(\lambda)}(\tau_1, \ldots, \tau_l) \\ &= \sup_{0 \leqslant \tau_1, \ldots, \tau_l \leqslant \tau} m_{k,n}^{(\lambda)}(\tau_1, \ldots, \tau_l) \sum_{l=1}^{n} S(n, l)(\lambda\tau)^l \\ &= \mathbb{E}[(N_{\lambda\tau})^n] m_{k,n}^{(\lambda)}(\tau, \ldots, \tau), \qquad k \geqslant 1. \end{aligned}$$

□

The bound on joint cumulants in Proposition 7.2 is obtained by induction from Proposition 6.2. Here, B_n denotes the Bell number of order $n \geqslant 1$, see (1.8).

Proposition 7.2 *For any $\tau \in [0, r]$, $k \geqslant 2$ and $l_1, \ldots, l_p \geqslant 0$, $p \geqslant 1$, we have the joint cumulant bound*

$$\kappa_\lambda\big((\sigma_k(kr - \tau))^{l_1}, \ldots, (\sigma_k(kr - \tau))^{l_p}\big) \leqslant (2(\lambda + 1)\tau)^{(k-1)(l_1 + \cdots + l_p) + 1 - p} (B_n)^{k-2}. \tag{7.1}$$

In particular, we have

$$c_{k,n}^{(\lambda)}(\tau; \ldots; \tau) \leqslant (2(\lambda + 1)\tau)^{1 + (k-2)n} (B_n)^{k-2}, \quad \tau \in [0, r]. \tag{7.2}$$

Proof. We note that from (6.2) we have $c_{2,p}^{(\lambda)}(\tau_1; \ldots; \tau_p) = \min(\tau_1, \ldots, \tau_p)$, hence by induction on $l_1, \ldots, l_p \geqslant 1$, Proposition 6.2 yields

$$\kappa_\lambda\big((\sigma_2(2r - \tau_1))^{l_1}, \ldots, (\sigma_2(2r - \tau_p))^{l_p}\big) \leqslant (2(\lambda + 1)\tau)^{l_1 + \cdots + l_p + 1 - p},$$

$0 \leqslant \tau_1, \ldots, \tau_p \leqslant \tau$, which is (7.1) for $k = 2$. Next, using Proposition 6.1, using induction on (7.1) for $k \geqslant 2$, we have

$$\begin{aligned} & \left|\kappa_\lambda\big(\sigma_{k+1}((k+1)r - \tau), \ldots, \sigma_{k+1}((k+1)r - \tau)\big)\right| \\ & \quad \leqslant \sum_{l=1}^n \lambda^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} \int_{[0,\tau]^l} |c_{k,l}^{(\lambda)}(\overline{\tau}_{\pi_1}; \ldots; \overline{\tau}_{\pi_l})| d\tau_1 \cdots d\tau_l \\ & \quad = (B_n)^{k-2} \sum_{l=1}^n (\lambda\tau)^l \sum_{\pi_1 \cup \cdots \cup \pi_l = \{1,\ldots,n\}} (2(\lambda+1)\tau)^{(k-1)n+1-l} \\ & \quad \leqslant (B_n)^{k-2} (2(\lambda+1)\tau)^{(k-1)n+1} \sum_{l=1}^n S(n,l) \\ & \quad = (B_n)^{k-1} (2(\lambda+1)\tau)^{(k-1)n+1} \end{aligned} \tag{7.3}$$

for $\lambda \geqslant 1$, which yields

$$\kappa_\lambda\big((\sigma_{k+1}((k+1)r - \tau_1))^{l_1}, \ldots, (\sigma_{k+1}((k+1)r - \tau_p))^{l_p}\big) \leqslant (B_n)^{k-1} (2(\lambda+1)\tau)^{k(l_1+\cdots+l_p)+1-p},$$

$0 \leqslant \tau_1, \ldots, \tau_p \leqslant \tau$, by induction from Proposition 6.2. □

8 Berry-Esseen bounds

In this section we will use the Wasserstein and Kolmogorov distances $d_W(X,Y)$ and $d_K(X,Y)$ between the distributions of random variables X, Y, defined as

$$d_W(X,Y) := \sup_{h \in \mathrm{Lip}(1)} |\mathrm{E}[h(X)] - \mathrm{E}[h(Y)]|,$$

where Lip(1) denotes the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1, and

$$d_K(X,Y) := \sup_{x \in \mathrm{R}} |\mathbb{P}(X \leqslant x) - \mathbb{P}(Y \leqslant x)|.$$

For $\lambda > 0$ and $t \in \mathbb{R}_+$ we let

$$\begin{aligned} \sigma_k^{(\lambda)}(t) &:= \int_0^{\lambda t} \cdots \int_0^{\lambda t} f_k(s_1/\lambda, \ldots, s_{k-1}/\lambda) dN_{s_1} \cdots dN_{s_{k-1}} \\ &= \frac{1}{(k-1)!} \sum_{l=0}^{k-1} \lambda^{k-1-l} \binom{k-1}{l} I_l \left(\int_0^t \cdots \int_0^t f_{k-1}(*, s_{l+1}, \ldots, s_{k-1}) ds_{l+1} \cdots ds_{k-1} \right) \end{aligned} \tag{8.1}$$

according to (2.1) and (2.6), so that the distribution of $\sigma_k^{(\lambda)}(t)$ under $\mathbb{P}$ is the distribution of $\sigma_k(t)$ under the distribution $\mathbb{P}_\lambda$ of the 1D unit disk model with constant Poisson intensity $\lambda > 0$.

In addition, given $k \geqslant 2$ and $t \in [(k-1)r, kr)$, we consider the renormalized k-hop count

$$\tilde{\sigma}_k^{(\lambda)}(t) := \frac{\sigma_k^{(\lambda)}(t) - \mathbb{E}_\lambda[\sigma_k(t)]}{\sqrt{\mathrm{Var}_\lambda[\sigma_k(t)]}}.$$

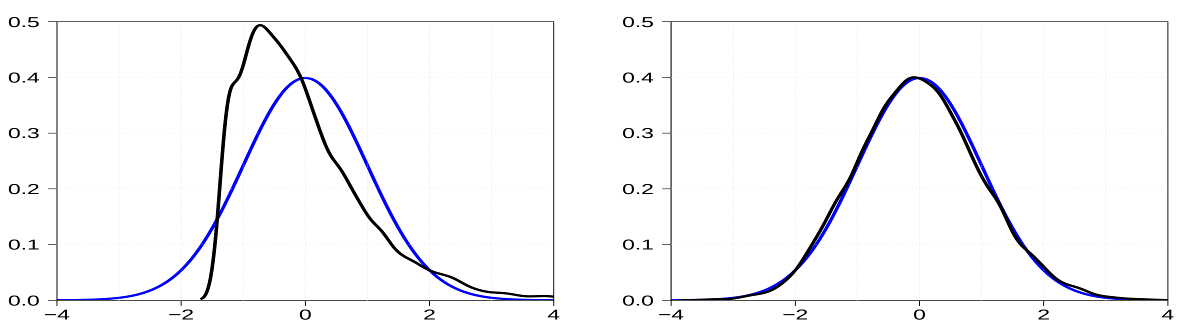
From Proposition 7.2, for any $t \in [(k-1)r, kr)$, the skewness of $\sigma_k(t)$ satisfies

$$\frac{\mathbb{E}_\lambda[(\sigma_k(t) - \mathbb{E}[\sigma_k(t)])^3]}{(\mathrm{Var}_\lambda[\sigma_k(t)])^{3/2}} = \mathbb{E}\left[\left(\tilde{\sigma}_k^{(\lambda)}(t)\right)^3\right] = \frac{c_{k,3}^{(\lambda)}(kr-t; kr-t; kr-t)}{(\mathrm{Var}_\lambda[\sigma_k(t)])^{3/2}} \approx \frac{1}{\sqrt{\lambda}}.$$

More generally, by (4.4) and (7.2) we have

$$\frac{c_{k,n}^{(\lambda)}(kr-t; \ldots; kr-t)}{(\mathrm{Var}_\lambda[\sigma_k(t)])^{n/2}} = (B_n)^{k-2} O(\lambda^{1-n/2}) \leqslant (n!)^{k-2} O(\lambda^{1-n/2}), \quad n \geqslant 2, \tag{8.2}$$

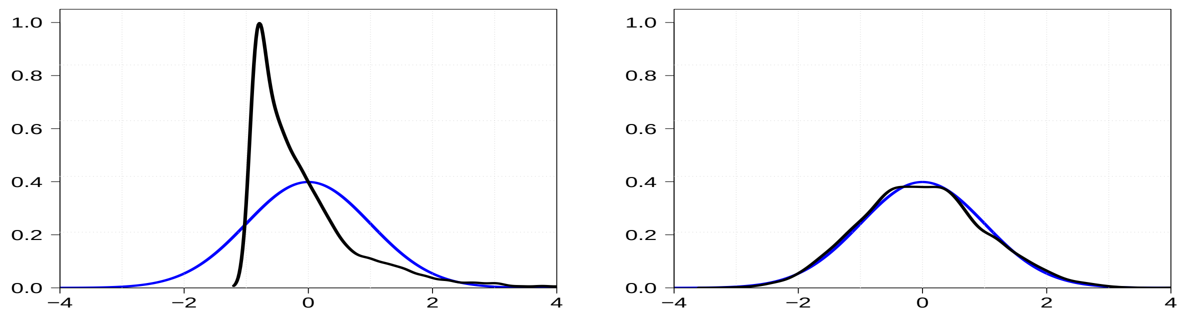
hence by Theorem 1 in [Jan88], $\tilde{\sigma}_k^{(\lambda)}(t)$ converges in distribution to the standard normal distribution $\mathcal{N}(0,1)$ as λ tends to infinity, as illustrated in Figures 9-10 using empirical probability density plots.



(a) $\lambda = 5$.

(b) $\lambda = 400$.

Figure 9: Convergence of 3-hop counts using probability density functions.



(a) $\lambda = 5$.

(b) $\lambda = 400$.

Figure 10: Convergence of 4-hop counts using probability density functions.

In addition, (8.2) shows that the Statulevičius condition, see § 1.3 of [DJS22], is satisfied with $\gamma := k - 3$ and $\Delta := \sqrt{\lambda}$. By [RSS78], Corollary 2.1 in § 1.3 in [SS91], see also Theorem 2.4 in [DJS22], this yields the Kolmogorov distance bound

$$d_K\big(\widetilde{\sigma}_k^{(\lambda)}(t), \mathcal{N}\big) \leqslant \frac{C(k,r)}{\lambda^{1/(2+4(k-3))}} \tag{8.3}$$

for $t \in [(k-1)r, kr)$, as λ tends to infinity. Moreover, (8.3) can be improved as as Berry-Esseen bound in the following proposition.

Proposition 8.1 *Let $k \geqslant 2$ and $t \in [(k-1)r, kr)$. The renormalized k-hop count $\widetilde{\sigma}_k^{(\lambda)}(t)$ satisfies the Wasserstein and Kolmogorov bounds*

$$d_{K/W}\big(\widetilde{\sigma}_k^{(\lambda)}(t), \mathcal{N}\big) \leqslant \frac{C(k,r)}{\sqrt{\lambda}} \tag{8.4}$$

for some constant $C(k,r) > 0$, as λ tends to infinity.

Proof. The kurtosis of $\sigma_k^{(\lambda)}(t)$ satisfies

$$\begin{aligned}
\frac{\mathbb{E}_\lambda[(\sigma_k(t) - \mathbb{E}_\lambda[\sigma_k(t)])^4]}{(\mathrm{Var}_\lambda[\sigma_k(t)])^2} - 3 &= \mathbb{E}_\lambda\left[\left(\widetilde{\sigma}_k^{(\lambda)}(t)\right)^4\right] - 3 \\
&= \frac{c_{k,4}^{(\lambda)}(kr-t; kr-t; kr-t; kr-t)}{(\mathrm{Var}_\lambda[\sigma_k(t)])^2} \\
&= (B_4)^{k-2} O(\lambda^{-1}),
\end{aligned}$$

as λ tends to infinity. The Kolmogorov distance bound in (8.4) then follows from the fourth moment theorem for U-statistics and sums of multiple stochastic integrals Corollary 4.10 in [ET14] applied to (8.1), see also Theorem 3 in [LRR16].

Regarding the Wasserstein distance bound, according to (3.2) we can the represent $\sigma_k^{(\lambda)}(t)$ as the U-statistics

$$\sigma_k^{(\lambda)}(t) = \sum_{\substack{((x_1,l_1),\ldots,((x_{k-1},l_{k-1}))\in\omega^{k-1} \\ (x_i,l_i)\neq(x_j,l_j),1\leqslant i\neq j\leqslant d}} \tilde{f}_{\lambda(kr-t)}(x_1/\lambda, l_1; \ldots; x_{k-1}/\lambda, l_{k-1})$$

of order $k-1$, where $\tilde{f}_t : ([0,r] \times \{1,\ldots,k-1\})^{k-1} \to \{0,1\}$, given by

$$\tilde{f}_t(x_1, l_1; \ldots, x_{k-1}; l_{k-1}) := \frac{1}{(k-1)!} \prod_{i=0}^{k-1} \mathbf{1}_{\{(l_{i+1}-l_i)x_i<(l_{i+1}-l_i)x_{i+1}\}},$$

$((x_1,l_1),\ldots,(x_{k-1},l_{k-1})) \in ([0,r]\times\{1,\ldots,k-1\})^{k-1}$ is the symmetrization in $k-1$ variables in $[0,r] \times \{1,\ldots,k-1\}$ of f_τ. Theorem 4.7 in [RS13] yields the bound

$$d_W\big(\tilde{\sigma}_k^{(\lambda)}, \mathcal{N}\big) \leqslant \sum_{1\leqslant i\leqslant j<k} \frac{\sqrt{M_{i,j}}}{\mathrm{Var}_\lambda[\sigma_k(t)]}$$

where $M_{i,j}$ is defined in (14) therein satisfies

$$M_{1,1} \leqslant (k-1)^4(\lambda r)^{4(k-1)-3}, \qquad i,j=1.$$

and

$$M_{i,j} \leqslant \binom{k-1}{i}^2 \binom{k-1}{j}^2 (\lambda r)^{4(k-1)-i-j}, \qquad 2 \leqslant i \leqslant j < k.$$

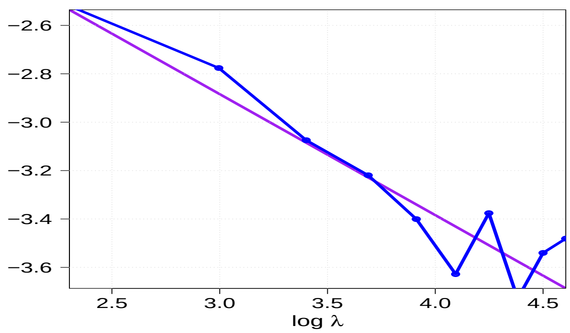
Hence by (4.4) and (4.3) we have

$$\begin{aligned} & d_W\big(\tilde{\sigma}_k^{(\lambda)}, \mathcal{N}\big) \\ \leqslant \; & \frac{1}{\mathrm{Var}_\lambda[\sigma_k(t)]} \left((k-1)^2(\lambda r)^{2(k-1)-3/2} + \sum_{2\leqslant i\leqslant j<k} \binom{k-1}{i}\binom{k-1}{j} (\lambda r)^{2(k-1)-i/2-j/2} \right) \\ \leqslant \; & \frac{C(k)}{\sqrt{\lambda r}} + C(k,r) \sum_{2\leqslant i\leqslant j<k} (\lambda r)^{1-i/2-j/2}. \end{aligned}$$

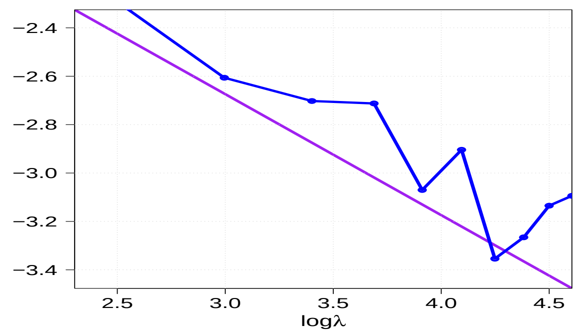
The above conclusions can also be reached by noting that $\tilde{\sigma}_k^{(\lambda)}$ admits a Hoeffding decomposition and by applying Theorem 1.3 in [DP17] for the Wasserstein distance, or Theorem 6.3 in [PS22] for the Kolmogorov distance, which refine the central limit theorem of [dJ90].

□

Figure 11 presents numerical estimates that are consistent with the rate in (8.4), by plotting $\log d_K\big(\widetilde{\sigma}_k^{(\lambda)}(t), \mathcal{N}\big)$ against $\log \lambda$ and their comparison with the line of slope $-1/2$. Kolmogorov distances d_K have been estimated in R using the distrEx package.



(a) Three-hop counts.



(b) Four-hop counts.

Figure 11: Log-log plots of Kolmogorov distances.

A Computer codes

Computation of joint moments

Explicit moment expressions are obtained using the following Mathematica code which implements the recursion (5.2) of Proposition 5.1 in two steps. The computation of the multiple integral with respect to $\lambda_k(du_1)\cdots\lambda_k(du_l)$ and the summation over $\pi_1 \cup \cdots \cup \pi_l = \{1, \ldots, n\}$ and $l = 1, \ldots, n$ are implemented in the following code.

```
Needs["Combinatorica`"]
fseq[n_] := fseq[n] = (Module[{k, tmp}, tmp = {};
   If[n == 1, Return[{{1}}], For[k = 1, k <= n, k++, Do[tmp = Prepend[tmp, Sort[Append[a, k]]], {a,
       fseq[n - 1]}]]; tmp]])
r2[x___, t1___, t2___, t3___, p_, q___, f_] := r2[x, t1, t2, t3, p, q, f] = (Module[{s}, If [p == 1,
    Integrate[f[Join @@ MapThread[Table, {Prepend[x, s], q}]], {s, 0,
     t1[[2]]}],
    If[t2[[p]] === t2[[p + 1]],
     Integrate[r2[Prepend[x, s], ReplacePart[t1, p -> s], t2, t3, p - 1, q, f], {s, t3[[p]], t1[[p +
         1]]}],
     Integrate[r2[Prepend[x, s], t2, t2, t3, p - 1, q, f], {s, t3[[p]], t1[[p + 1]]}]]]])
mk[t___, lambda___] := mk[t, lambda] = (Module[{b, z1, q, n, m, z, zz, tmp},
   If[lambda == {}, Return[1]]; m = Length[t];
   If[Length[lambda] == 1, b[l___] = 1, b[l___] := (c[d___] := mk[d, Drop[lambda, -1]];
     With[{e = c[Array[s, Length[t]]]},
      h[z_] := Block[{s}, s[i_] := z[[i]]; e]; Return[h[l]]])];
   tmp = 0; Do[n = Length[pp]; q = Map[Length, pp]; z = t[[Map[Min, pp]]];
    zz = Prepend[Drop[z, -1], 0];
    Do[Do[If[r == r[[Ordering@p0]], z1 = Prepend[z[[r]], 0];
      tmp += Last[lambda]^n*r2[{}, z1, z1, zz[[r]], n, q[[Ordering@p0]], b]], {r, fseq[n]}], {p0,
          Permutations[Range[n]]}], {pp, SetPartitions[m]}]; Return[Expand[Flatten[{tmp}][[1]]]]])
mk[{t1, t2}, {l1, l2}]
```

Mathematica Code 1.

The joint moment $\mathbb{E}[\sigma_k(kr-\tau_1)\cdots\sigma_k(kr-\tau_n)]$ of order $n \geqslant 1$ is then computed from the command $\mathrm{mk}[\{\tau_1,\ldots,\tau_n\},\{\lambda_1,\ldots,\lambda_{k-1}]$, with $0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant r$.

Computation of joint cumulants

Explicing cumulant expressions are obtained using the following Mathematica code which implements the recursion (6.1) of Proposition 6.1 in two steps. First, the computation of the multiple integral with respect to $\lambda_k(du_1)\cdots\lambda_k(du_l)$ in the following code.

```
Needs["Combinatorica`"]
fseq2[n_] := fseq2[n] = (Module[{k, tmp, tmp2}, tmp = {};
   If[n == 1, Return[{{1}}],
    For[k = 1, k <= n, k++, Do[tmp2 = Join[{k}, a]; tmp = Append[tmp, ReverseSort[tmp2]], {a, fseq2[n -
         1]}]]; tmp]])
r1c[x___, t___, t2___, t3___, p_, f_] := (Module[{s},
  If [p == 1, Integrate[f[Prepend[x, s]], {s, 0, t[[2]]}],
   If[t2[[p]] === t2[[p + 1]], Integrate[r1c[Prepend[x, s], ReplacePart[t, p -> s], t2, t3, p - 1, f],
       {s, t3[[p]], t[[p + 1]]}], Integrate[r1c[Prepend[x, s], t2, t2, t3, p - 1, f], {s, t3[[p]],
       t[[p + 1]]}]]]])
```

Mathematica Code 2.

This is followed by the recursive computation of $c_{k,l}^{(\lambda)}(\overline{s}_{\pi_1};\ldots;\overline{s}_{\pi_l})$ by the induction relation of Proposition 6.2, and the summation (6.1) over $\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}$ and $l=1,\ldots,n$ in the following code

```
ck[t__, l3__, lambda__] := ck[t, l3, lambda] = (Catch[
   Module[{b, l4, k, n, l5, l6, t5, tmp3, tmp4, oo, z, zz, z1, z2, u, q}, oo = 0; n = Length[t];
    Do[oo += 1; If[l0 > 1, l4 = l3; l4[[oo]] += -1;
      l5 = Join[Range[oo], {oo}, Range[oo + 1, n + 1]];
      l6 = Join[l4[[Range[oo]]], {1}, l4[[Range[oo + 1, n]]]];
      t5 = Join[t[[Range[oo]]], {t[[oo]]}, t[[Range[oo + 1, n]]]];
      tmp4 = 0; Do[If[MemberQ[l5[[pi[[1]]]], oo] && MemberQ[l5[[pi[[2]]]], oo],
         tmp4 += ck[t5[[pi[[1]]]], l6[[pi[[1]]]], lambda]*ck[t5[[pi[[2]]]], l6[[pi[[2]]]], lambda]],
              {pi, KSetPartitions[Range[n + 1], 2]}]; tmp4 += ck[t5, l6, lambda]; Throw[tmp4]], {l0,
              l3}]; k = Length[lambda]; If[k == 1, Throw[lambda[[1]]*t[[1]]]];
    If [Length[lambda] == 1, b[l__] = 1, b[l__] := (c[d__] := ck[d, q0, Drop[lambda, -1]];
       With[{e = c[Array[s4s, Length[l]]]}, fl[z7_] := Block[{s4s}, s4s[i_] := z7[[i]]; e];
        Return[fl[l]]])]; m = Length[t]; q0 = {m};
    tmp3 = lambda[[1]]*r1c[{}, {0, t[[1]]}, {0, t[[1]]}, {0}, 1, b];
    For[n = 2, n <= m, n++, Do[zz = {0}; q = Map[Length, pp];
      z = t[[Map[Min, pp]]]; zz = Prepend[Drop[z, -1], 0];
      Do[q0 = q[[Ordering@p0]]; Do[If[bb == bb[[Ordering@p0]], z1 = Prepend[z[[Reverse[bb]]], 0];
         tmp3 += lambda[[1]]^n*r1c[{}, z1, z1, zz[[Reverse[bb]]], n, b]], {bb,
         fseq2[n]}], {p0, Permutations[Range[n]]}], {pp, KSetPartitions[m, n]}]];
    Return[Expand[Flatten[{tmp3}][[1]]]]]])
ck[{t, t, t}, {1, 1, 1}, {1, 1, 1}]
```

Mathematica Code 3.

The joint cumulant $c^{(\lambda)}_{k,n}(\overline{\tau}_{\pi_1};\ldots;\overline{\tau}_{\pi_l}) = \kappa_\lambda\big((\sigma_k(kr-\tau_1))^{|\pi_1|},\ldots,(\sigma_k(kr-\tau_l))^{|\pi_l|}\big)$ is computed from the command $\mathrm{ck}[\{\tau_1,\ldots,\tau_l\},\{|\pi_1|,\ldots,|\pi_l|\},\{\lambda_1,\ldots,\lambda_{k-1}\}]$, and the joint cumulant $c^{(\lambda)}_{k,n}(\tau_1;\ldots;\tau_n) = \kappa_\lambda\big(\sigma_k(kr-\tau_1),\ldots,\sigma_k(kr-\tau_n)\big)$ is computed from the command $\mathrm{ck}[\{\tau_1,\ldots,\tau_n\},\{1,\ldots,1\},\{\lambda_1,\ldots,\lambda_{k-1}\}]$, with $0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant r$.

B Moments of Poisson stochastic integrals

In this section we review the background results on the moments of Poisson stochastic integrals that are used in this paper. Let $\omega(dx)$ denote a Poisson point process with intensity measure $\mu(dx)$ on a measure space X. The next proposition, see Proposition 3.1 in [Pri12] or Theorem 1 and Proposition 7 in [Pri16], provides a moment identity for Poisson stochastic integrals with random integrands using sums over partitions.

Proposition B.1 *Let $f_1,\ldots,f_n : X \longrightarrow \mathbb{R}$ be deterministic functions, $n \geqslant 1$. We have*

$$\mathbb{E}\left[\int_X f_1(x)\omega(dx)\cdots\int_X f_n(x)\omega(dx)\right]$$
$$= \sum_{l=1}^{n}\sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}}\int_{X^l}\left(\prod_{j=1}^{l}\prod_{i\in\pi_j} f_i(x_j)\right)\mu(dx_1)\cdots\mu(dx_l),$$

where the sum runs over all partitions $\pi_1,\ldots,\pi_l$ of $\{1,\ldots,n\}$ of sizes and $|\pi_l|$ denotes the cardinality of each block π_l, $l = 1,\ldots,n$.

Proposition B.1 extends in Proposition B.2 as a moment identity for the stochastic integral of functions $f(z_1, \dots z_p)$ of p variables $z_1, \dots z_p \in X^p$, see in Theorem 3.1 of [BRSW17]. For this, let $\Pi[n \times p]$ denote the set of partitions of the set

$$[n \times p] := \{1, \dots, n\} \times \{1, \dots, p\} = \big\{(i,j) \ : \ i = 1, \dots, n, \ j = 1, \dots, p\big\},$$

identified to $\{1, \dots, np\}$, and let $\rho := (\rho_1, \dots, \rho_n) \in \Pi[n \times p]$ denote the partition made of the n blocks $\rho_i := \{(i,1), \dots, (i,p)\}$ of size p, for $i = 1, \dots, n$.

Proposition B.2 *For $f : X^p \longrightarrow \mathbb{R}$ a sufficiently integrable function of p variables, we have*

$$\mathbb{E}\left[\left(\int_X \cdots \int_X f(x_1, \dots, x_p; \omega)\omega(dx_1)\cdots\omega(dx_p)\right)^n\right]$$
$$= \sum_{\pi \in \Pi[n\times r]} \int_X \cdots \int_X \prod_{l=1}^n f\big(x_{\zeta^\pi_{l,1}}, \dots, x_{\zeta^\pi_{l,p}}\big)\mu(dx_1)\cdots\mu(dx_{|\pi|}),$$

where $\zeta^\pi_{i,j}$ is the index of the block of π that contains (i,j).

Proposition B.2 can also be extended as a joint moment identity for multiparameter processes in the next proposition.

Proposition B.3 *Let $f_i : X^p \longrightarrow \mathbb{R}$, $i = 1, \dots, n$ be sufficiently integrable functions of p variables. We have*

$$\mathbb{E}\left[\prod_{i=1}^n \int_X \cdots \int_X f_i(x_1, \dots, x_p)\omega(dx_1)\cdots\omega(dx_p)\right]$$
$$= \sum_{\pi \in \Pi[n\times r]} \int_X \cdots \int_X \prod_{j=1}^{|\pi|} \prod_{i\in\pi_j} f_i\big(x_{\zeta^\pi_{l,1}}, \dots, x_{\zeta^\pi_{l,p}}\big)\mu(dx_1)\cdots\mu(dx_{|\pi|}),$$

where $\zeta^\pi_{i,j}$ is the index of the block of π that contains (i,j).

In case the function $f(x_1, \dots, x_p; \omega)$ vanishes on the diagonals in X^p, the integral of f rewrites as the U-statistics

$$\int_{X^r} f(x_1, \dots, x_p; \omega)\omega(dx_1)\cdots\omega(dx_p) = \sum_{\substack{(x_1,\dots,x_p)\in\omega^p \\ x_i \neq x_j, 1 \leqslant i \neq j \leqslant p}} f(x_1, \dots, x_p; \omega), \qquad \text{(B.1)}$$

we have the following corollary of Proposition B.3. For this, we let $\rho := (\rho_1, \ldots, \rho_n)$ be the partition of $[n \times p]$ made of the blocks $\rho_i = ((i,j))_{j=1,\ldots,p}$, $i = 1, \ldots, n$, and we say that a partition π of $[n \times p]$ is *non-flat*, i.e. $\pi \wedge \rho = \widehat{0}$, if every block of π contains at most one element of ρ_i, $i = 1, \ldots, n$, where $\widehat{\mathbf{0}} := \{\{1\}, \ldots, \{n\}\}$ is the n-block partition of $\{1, \ldots, n\}$.

Corollary B.4 *Let $f_i : X^p \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$ be sufficiently integrable functions of p variables such that the function $f_i(x_1, \ldots, x_p; \omega)$ vanishes on the diagonals in X^p, $i = 1, \ldots, n$. Then, the joint moments of the U-statistics (B.1) can be computed as*

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{n} \int_X \cdots \int_X f_i(x_1, \ldots, x_p) \omega(dx_1) \cdots \omega(dx_p)\right] \\
&= \sum_{\substack{\pi \in \Pi[n \times p] \\ \pi \wedge \rho = \hat{0}}} \int_X \cdots \int_X \prod_{j=1}^{|\pi|} \prod_{i \in \rho_j} f_i\big(x_{\zeta^\pi_{i,1}}, \ldots, x_{\zeta^\pi_{i,p}}\big) \mu(dx_1) \cdots \mu(dx_{|\pi|}).
\end{aligned}$$

C Explicit moment recursions

In this section we confirm the joint moment induction of Proposition 5.1 via explicit calculations for $n = 1, 2, 3$.

First moment recursion

By (5.3) we have

$$\begin{aligned}
m^{(\lambda)}_{k+1,1}(\tau) &= \mathbb{E}_\lambda[\sigma_{k+1}((k+1)r - \tau)] \\
&= \mathbb{E}_\lambda\left[Z^{(k+1)}_\tau\right] \\
&= \int_0^\tau \mathbb{E}_\lambda\left[Z^{(k)}_s\right] \lambda_k(ds) \\
&= \int_0^\tau m^{(\lambda)}_{k,1}(s) \lambda_k(ds),
\end{aligned}$$

which recovers (4.2) as

$$m^{(\lambda)}_{k,1}(\tau) = \int_0^\tau \int_0^{s_{k-1}} \cdots \int_0^{s_2} \lambda_1(ds_1) \cdots \lambda_{k-1}(ds_{k-1}).$$

Second joint moment recursion

For $n = 2$, by (5.3) the induction relation (5.4) reads

$$\begin{aligned}
m^{(\lambda)}_{k+1,2}(\tau_1,\tau_2) &= \mathbb{E}_\lambda[\sigma_{k+1}((k+1)r-\tau_1)\sigma_k((k+1)r-\tau_2)] \\
&= \mathbb{E}_\lambda\left[Z^{(k+1)}_{\tau_1}Z^{(k+1)}_{\tau_2}\right] \\
&= \mathbb{E}_\lambda\left[\int_0^{\tau_1} Z^{(k)}_{s_1}dN^{(k)}_{s_1}\int_0^{\tau_2} Z^{(k)}_{s_2}dN^{(k)}_{s_2}\right] \\
&= \int_0^{\tau_1}\mathbb{E}_\lambda\left[\left(Z^{(k)}_{s_1}\right)^2\right]\lambda_k(ds_1)+\int_0^{\tau_2}\int_0^{\tau_1}\mathbb{E}_\lambda\left[Z^{(k)}_{s_1}Z^{(k)}_{s_2}\right]\lambda_k(ds_1)\lambda_k(ds_2) \\
&= \int_0^{\tau_1} m^{(\lambda)}_{k,2}(u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_2}\int_0^{\tau_1} m^{(\lambda)}_{k,2}(u_1,u_2)\lambda_k(du_1)\lambda_k(du_2).
\end{aligned}$$

Third joint moment recursion

For $\tau_1,\tau_2,\tau_3 \in [0,r]$ the recursion (5.4) reads

$$\begin{aligned}
&m^{(\lambda)}_{k+1,3}(\tau_1,\tau_2,\tau_3) = \mathbb{E}_\lambda[\sigma_{k+1}((k+1)r-\tau_1)\sigma_k((k+1)r-\tau_2)\sigma_k((k+1)r-\tau_3)] \\
&= \mathbb{E}_\lambda\left[Z^{(k+1)}_{\tau_1}Z^{(k+1)}_{\tau_2}Z^{(k+1)}_{\tau_3}\right] \\
&= \mathbb{E}_\lambda\left[\int_0^{\tau_1} Z^{(k)}_{u}dN^{(k)}_{u}\int_0^{\tau_2} Z^{(k)}_{u}dN^{(k)}_{u}\int_0^{\tau_3} Z^{(k)}_{u}dN^{(k)}_{u}\right] \\
&= \int_0^{\tau_1\wedge\tau_2\wedge\tau_3}\mathbb{E}\left[\left(Z^{(k)}_{s_1}\right)^3\right]\lambda_k(ds_1) \\
&\quad+\int_0^{\tau_3}\int_0^{\tau_1\wedge\tau_2}\mathbb{E}_\lambda\left[\left(Z^{(k)}_{s_1}\right)^2Z^{(k)}_{s_3}\right]\lambda_k(ds_1)\lambda_k(ds_3)+\int_0^{\tau_2\wedge\tau_3}\int_0^{\tau_1}\mathbb{E}_\lambda\left[\left(Z^{(k)}_{s_1}\right)^2Z^{(k)}_{s_2}\right]\lambda_k(ds_1)\lambda_k(ds_2) \\
&\quad+\int_0^{\tau_2\wedge\tau_3}\int_0^{\tau_1}\mathbb{E}_\lambda\left[Z^{(k)}_{s_1}\left(Z^{(k)}_{s_2}\right)^2\right]\lambda_k(ds_1)\lambda_k(ds_2)+\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1}\mathbb{E}_\lambda\left[Z^{(k)}_{s_1}Z^{(k)}_{s_2}Z^{(k)}_{s_3}\right]\lambda_k(ds_1)\lambda_k(ds_2)\lambda_k(ds_3) \\
&= \int_0^{\tau_1\wedge\tau_2\wedge\tau_3} m^{(\lambda)}_{k,3}(u_1,u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_3}\int_0^{\tau_1\wedge\tau_2} m^{(\lambda)}_{k,n}(u_1,u_1,u_3)\lambda_k(du_1)\lambda_k(du_3) \\
&\quad+\int_0^{\tau_2}\int_0^{\tau_1\wedge\tau_3} m^{(\lambda)}_{k,n}(u_1,u_1,u_2)\lambda_k(du_1)\lambda_k(du_2)+\int_0^{\tau_2\wedge\tau_3}\int_0^{\tau_1} m^{(\lambda)}_{k,n}(u_1,u_2,u_2)\lambda_k(du_1)(du_2) \\
&\quad+\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1} m^{(\lambda)}_{k,n}(u_1,u_2,u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3) \\
&= \int_0^{\tau_1\wedge\tau_2\wedge\tau_3} m^{(\lambda)}_{k,3}(u_1,u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_3}\int_0^{\tau_1\wedge\tau_2} m^{(\lambda)}_{k,n}(u_1,u_1,u_3)\lambda_k(du_1)\lambda_k(du_3) \\
&\quad+\int_0^{\tau_2}\int_0^{\tau_1\wedge\tau_3} m^{(\lambda)}_{k,n}(u_1,u_2,u_1)\lambda_k(du_1)\lambda_k(du_2)+\int_0^{\tau_2\wedge\tau_3}\int_0^{\tau_1} m^{(\lambda)}_{k,n}(u_1,u_2,u_2)\lambda_k(du_1)(du_2) \\
&\quad+\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1} m^{(\lambda)}_{k,n}(u_1,u_2,u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3).
\end{aligned}$$

D Explicit cumulant recursions

In this section we confirm the joint moment induction of Proposition 6.1 via explicit calculations for $n = 2, 3, 4$.

Second cumulant recursion

By Proposition 5.1 and the joint cumulant inversion relation, we have the second cumulant recursion

$$\begin{aligned}
&c^{(\lambda)}_{k+1,2}(\tau_1;\tau_2) = m^{(\lambda)}_{k+1,2}(\tau_1,\tau_2) - m^{(\lambda)}_{k+1,1}(\tau_1)m^{(\lambda)}_{k+1,1}(\tau_2)\\
&= \int_0^{\tau_1} m^{(\lambda)}_{k,2}(u_1,u_1)\lambda_k(du_1) + \int_0^{\tau_1}\int_0^{\tau_2} m^{(\lambda)}_{k,2}(u_1,u_2)\lambda_k(du_1)\lambda_k(du_2)\\
&\quad - \int_0^{\tau_1} m^{(\lambda)}_{k+1,1}(s_1)ds_1 \int_0^{\tau_2} m^{(\lambda)}_{k+1,1}(s_2)ds_2\\
&= \int_0^{\tau_1} m^{(\lambda)}_{k,2}(u_1,u_1)\lambda_k(du_1) + \int_0^{\tau_1}\int_0^{\tau_2} (m^{(\lambda)}_{k,2}(u_1,u_2) - m^{(\lambda)}_{k,1}(u_1)m^{(\lambda)}_{k,1}(u_2))\lambda_k(du_1)\lambda_k(du_2)\\
&= \int_0^{\tau_1} c^{(\lambda)}_{k,2}(u_1,u_1)\lambda_k(du_1) + \int_0^{\tau_1}\int_0^{\tau_2} c^{(\lambda)}_{k,2}(u_1;u_2)\lambda_k(du_1)\lambda_k(du_2).
\end{aligned}$$

Third cumulant recursion

At the third order, we have the next cumulant expression.

Proposition D.1 *For $k \geqslant 1$ we have the third cumulant recursion*

$$\begin{aligned}
&c^{(\lambda)}_{k+1,3}(\tau_1;\tau_2;\tau_3) = \int_0^{\tau_1} c^{(\lambda)}_{k,3}(u_1,u_1,u_1)\lambda_k(du_1) + \int_0^{\tau_1}\int_0^{\tau_3} c^{(\lambda)}_{k,3}(u_1,u_1;u_3)\lambda_k(du_1)\lambda_k(du_3)\\
&+ \int_0^{\tau_1}\int_0^{\tau_2} c^{(\lambda)}_{k,3}(u_1,u_1;u_2)\lambda_k(du_1)\lambda_k(du_2) + \int_0^{\tau_1}\int_0^{\tau_2} c^{(\lambda)}_{k,3}(u_1;u_2,u_2)\lambda_k(du_1)(du_2)\\
&+ \int_0^{\tau_1}\int_0^{\tau_2}\int_0^{\tau_3} c^{(\lambda)}_{k,3}(u_1;u_2;u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3).
\end{aligned}$$

Proof. By the joint cumulant-moment relationship

$$c^{(\lambda)}_{k+1,3}(\tau_1;\tau_2;\tau_3) = \sum_{l=1}^{n}(l-1)!(-1)^{l-1} \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \prod_{j=1}^{l} m^{(\lambda)}_{k+1,|\pi_j|}(\overline{\tau}_{\pi_j}) \tag{D.1}$$

$$= m^{(\lambda)}_{k+1,3}(\tau_1,\tau_2,\tau_3) - \sum_{\pi_1\cup\pi_2=\{1,2,3\}} m^{(\lambda)}_{k+1,|\pi_1|}(\overline{\tau}_{\pi_1})m^{(\lambda)}_{k+1,|\pi_2|}(\overline{\tau}_{\pi_2}) + 2m^{(\lambda)}_{k+1,1}(\tau_1)m^{(\lambda)}_{k+1,1}(\tau_2)m^{(\lambda)}_{k+1,1}(\tau_3)$$

and Proposition 5.1, we have

$$
\begin{aligned}
&c_{k+1,3}^{(\lambda)}(\tau_1;\tau_2;\tau_3)\\
&=\int_0^{\tau_1\wedge\tau_2\wedge\tau_3} m_{k,3}^{(\lambda)}(u_1,u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_1\wedge\tau_2}\int_0^{\tau_3} m_{k,3}^{(\lambda)}(u_1,u_1,u_3)\lambda_k(du_3)\lambda_k(du_1)\\
&+\int_0^{\tau_1\wedge\tau_3}\int_0^{\tau_2} m_{k,3}^{(\lambda)}(u_1,u_1,u_2)\lambda_k(du_2)\lambda_k(du_1)+\int_0^{\tau_1}\int_0^{\tau_2\wedge\tau_3} m_{k,3}^{(\lambda)}(u_1,u_2,u2)\lambda_k(du_2)\lambda_k(du_1)\\
&+\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1} m_{k,3}^{(\lambda)}(u_1,u_2,u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3)\\
&-\int_0^{\tau_1} m_{k,1}^{(\lambda)}(u_1)\lambda_k(du_1)\left(\int_0^{\tau_2\wedge\tau_3} m_{k,2}^{(\lambda)}(u_2,u_2)\lambda_k(du_2)+\int_0^{\tau_3}\int_0^{\tau_2} m_{k,2}^{(\lambda)}(u_2,u_3)\lambda_k(du_2)\lambda_k(du_3)\right)\\
&-\int_0^{\tau_2} m_{k,1}^{(\lambda)}(u_2)\lambda_k(du_2)\left(\int_0^{\tau_1\wedge\tau_3} m_{k,2}^{(\lambda)}(u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_3}\int_0^{\tau_1} m_{k,2}^{(\lambda)}(u_1,u_3)\lambda_k(du_1)\lambda_k(du_3)\right)\\
&-\int_0^{\tau_3} m_{k,1}^{(\lambda)}(u_3)\lambda_k(du_3)\left(\int_0^{\tau_1\wedge\tau_2} m_{k,2}^{(\lambda)}(u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_2}\int_0^{\tau_1} m_{k,2}^{(\lambda)}(u_1,u_2)\lambda_k(du_1)\lambda_k(du_2)\right)\\
&+2\int_0^{\tau_1} m_{k,1}^{(\lambda)}(u_1)\lambda_k(du_1)\int_0^{\tau_2} m_{k,1}^{(\lambda)}(u_2)\lambda_k(du_2)\int_0^{\tau_3} m_{k,1}^{(\lambda)}(u_3)\lambda_k(du_3). \qquad \text{(D.2)}
\end{aligned}
$$

Applying (D.1) to k-hops, i.e.

$$
\begin{aligned}
c_{k,3}^{(\lambda)}(u_1;u_2;u_3) \;=\;& m_{k,3}^{(\lambda)}(u_1,u_2;u_3)\\
&-m_{k,1}^{(\lambda)}(u_1)m_{k,2}^{(\lambda)}(u_1;u_2)-m_{k,1}^{(\lambda)}(u_1)m_{k,2}^{(\lambda)}(u_1;u_2)-m_{k,1}^{(\lambda)}(u_1)m_{k,2}^{(\lambda)}(u_1;u_2)\\
&+2m_{k,1}^{(\lambda)}(u_1)m_{k,1}^{(\lambda)}(u_2)m_{k,1}^{(\lambda)}(u_3),
\end{aligned}
$$

allows us to simplify (D.2) to

$$
\begin{aligned}
&c_{k+1,3}^{(\lambda)}(\tau_1;\tau_2;\tau_3)\\
&=\int_0^{\tau_1\wedge\tau_2\wedge\tau_3} m_{k,3}^{(\lambda)}(u_1,u_1,u_1)\lambda_k(du_1)+\int_0^{\tau_1\wedge\tau_2}\int_0^{\tau_3} m_{k,3}^{(\lambda)}(u_1,u_1,u_3)\lambda_k(du_3)\lambda_k(du_1)\\
&+\int_0^{\tau_1\wedge\tau_3}\int_0^{\tau_2} m_{k,3}^{(\lambda)}(u_1,u_1,u_2)\lambda_k(du_2)\lambda_k(du_1)+\int_0^{\tau_1}\int_0^{\tau_2\wedge\tau_3} m_{k,3}^{(\lambda)}(u_1,u_2,u_2)\lambda_k(du_2)\lambda_k(du_1)\\
&+\int_0^{\tau_1}\int_0^{\tau_2}\int_0^{\tau_3} c_{k,3}^{(\lambda)}(u_1;u_2;u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3)\\
&-\int_0^{\tau_1} m_{k,1}^{(\lambda)}(u_1)\lambda_k(du_1)\int_0^{\tau_2\wedge\tau_3} m_{k,2}^{(\lambda)}(u_2,u_2)\lambda_k(du_2)\\
&-\int_0^{\tau_2} m_{k,1}^{(\lambda)}(u_2)\lambda_k(du_2)\int_0^{\tau_1\wedge\tau_3} m_{k,2}^{(\lambda)}(u_1,u_1)\lambda_k(du_1)\\
&-\int_0^{\tau_3} m_{k,1}^{(\lambda)}(u_3)\lambda_k(du_3)\int_0^{\tau_1\wedge\tau_2} m_{k,2}^{(\lambda)}(u_1,u_1)\lambda_k(du_1)
\end{aligned}
$$

$$= \int_0^{\tau_1\wedge\tau_2\wedge\tau_3} m_{k,3}^{(\lambda)}(u_1,u_1,u_1)\lambda_k(du_1) + \int_0^{\tau_1\wedge\tau_2}\int_0^{\tau_3} c_{k,3}^{(\lambda)}(u_1,u_1;u_3)\lambda_k(du_3)\lambda_k(du_1)$$
$$+ \int_0^{\tau_1\wedge\tau_3}\int_0^{\tau_2} c_{k,3}^{(\lambda)}(u_1,u_1;u_2)\lambda_k(du_2)\lambda_k(du_1) + \int_0^{\tau_1}\int_0^{\tau_2\wedge\tau_3} c_{k,3}^{(\lambda)}(u_1;u_2,u_2)\lambda_k(du_2)\lambda_k(du_1)$$
$$+ \int_0^{\tau_1}\int_0^{\tau_2}\int_0^{\tau_3} c_{k,3}^{(\lambda)}(u_1;u_2;u_3)\lambda_k(du_1)\lambda_k(du_2)\lambda_k(du_3).$$

□

Fourth cumulant recursion

Taking $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$ for simplicity, we have

$$c_{k+1,4}^{(\lambda)}(\tau;\tau;\tau;\tau) = \sum_{l=1}^{n}(l-1)!(-1)^{l-1}\sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}}\prod_{j=1}^{l} m_{k+1,|\pi_j|}^{(\lambda)}(\widehat{s}_{\pi_j})$$
$$= m_{k+1,4}^{(\lambda)}(\tau,\tau,\tau,\tau) - 4m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,3}^{(\lambda)}(\tau,\tau,\tau) - 3m_{k+1,2}^{(\lambda)}(\tau,\tau)m_{k+1,2}^{(\lambda)}(\tau,\tau)$$
$$+ 6m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,2}^{(\lambda)}(\tau,\tau) - m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)$$
$$= \int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_1,u_1,u_1)\lambda_k(du_1) + 4\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_2,u_2)du_1du_2$$
$$+ 3\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_1,u_2,u_2)du_1du_2$$
$$+ 6\int_0^{\tau}\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_3)du_1du_2du_3 + \int_0^{\tau}\int_0^{\tau}\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_4)du_1du_2du_3du_4$$
$$- 4m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,3}^{(\lambda)}(\tau,\tau,\tau) - 3m_{k+1,2}^{(\lambda)}(\tau,\tau)m_{k+1,2}^{(\lambda)}(\tau,\tau)$$
$$+ 6m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,2}^{(\lambda)}(\tau,\tau) - m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)$$
$$= \int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_1,u_1,u_1)\lambda_k(du_1) + 4\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_2,u_2)du_1du_2$$
$$+ 3\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_1,u_2,u_2)du_1du_2$$
$$+ 6\int_0^{\tau}\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_3)du_1du_2du_3 + \int_0^{\tau}\int_0^{\tau}\int_0^{\tau}\int_0^{\tau} m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_4)du_1du_2du_3du_4$$
$$- 4\int_0^{\tau} m_{k,1}^{(\lambda)}(u)du\int_0^{\tau} m_{k,3}^{(\lambda)}(u_1,u_1,u_1)du_1 - 12\int_0^{\tau} m_{k,1}^{(\lambda)}(u)du\int_0^{\tau}\int_0^{\tau} m_{k,3}^{(\lambda)}(u_1,u_2,u_2)du_1du_2$$
$$- 4\int_0^{\tau} m_{k,1}^{(\lambda)}(u)du\int_0^{\tau}\int_0^{\tau}\int_0^{\tau} m_{k,3}^{(\lambda)}(u_1,u_2,u_3)du_1du_2du_3 - 3\left(\int_0^{\tau} m_{k,2}^{(\lambda)}(u_1,u_1)du_1\right)^2$$
$$- 6\int_0^{\tau} m_{k,2}^{(\lambda)}(u_1,u_1)du_1\int_0^{\tau}\int_0^{\tau} m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2 - 3\left(\int_0^{\tau}\int_0^{\tau} m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2\right)^2$$

$$
\begin{aligned}
&+ 6\left(\int_0^\tau m_{k,1}^{(\lambda)}(u)du\right)^2 \left(\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1 + \int_0^\tau\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2\right)\\
&- m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)m_{k+1,1}^{(\lambda)}(\tau)\\
&= \int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_1,u_1,u_1)\lambda_k(du_1) + 4\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_2,u_2,u_2)du_1du_2\\
&+ 3\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_1,u_2,u_2)du_1du_2\\
&+ 6\int_0^\tau\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_3)du_1du_2du_3 + \int_0^\tau\int_0^\tau\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2;u_3;u_4)du_1du_2du_3du_4\\
&- 4\int_0^\tau m_{k,1}^{(\lambda)}(u)du\int_0^\tau m_{k,3}^{(\lambda)}(u_1,u_1,u_1)du_1 - 12\int_0^\tau m_{k,1}^{(\lambda)}(u)du\int_0^\tau\int_0^\tau m_{k,3}^{(\lambda)}(u_1,u_2,u_2)du_1du_2\\
&- 3\left(\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1\right)^2 - 6\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1\int_0^\tau\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2\\
&+ 6\left(\int_0^\tau m_{k,1}^{(\lambda)}(u)du\right)^2\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1\\
&= \int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_1,u_1,u_1)\lambda_k(du_1) + 4\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2,u_2,u_2)du_1du_2\\
&+ 3\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1,u_1;u_2,u_2)du_1du_2\\
&+ 6\int_0^\tau\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_3)du_1du_2du_3 + \int_0^\tau\int_0^\tau\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2;u_3;u_4)du_1du_2du_3du_4\\
&- 12\int_0^\tau m_{k,1}^{(\lambda)}(u)du\int_0^\tau\int_0^\tau m_{k,3}^{(\lambda)}(u_1,u_2,u_2)du_1du_2\\
&- 6\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1\int_0^\tau\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2\\
&+ 6\left(\int_0^\tau m_{k,1}^{(\lambda)}(u)du\right)^2\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_1)du_1\\
&= \int_0^\tau c_{k,4}^{(\lambda)}(u_1,u_1,u_1,u_1)\lambda_k(du_1) + 4\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2,u_2,u_2)du_1du_2\\
&+ 3\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1,u_1;u_2,u_2)du_1du_2\\
&+ 6\int_0^\tau\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2;u_3,u_3)du_1du_2du_3 + \int_0^\tau\int_0^\tau\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2;u_3;u_4)du_1du_2du_3du_4.
\end{aligned}
$$

In order to reach we above conclusion we used the relations

$$
\begin{aligned}
&4\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2,u_2,u_2)du_1du_2\\
&= 4\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_2,u_2,u_2)du_1du_2 - 4\int_0^\tau\int_0^\tau m_{k,1}^{(\lambda)}(u_1)m_{k,3}^{(\lambda)}(u_2,u_2,u_2)du_1du_2,
\end{aligned}
$$

$$
\begin{aligned}
&3\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1,u_1;u_3,u_3)du_1du_2\\
&\quad = \quad 3\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_1,u_3,u_3)du_1du_2 - 3\int_0^\tau\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_2)m_{k,2}^{(\lambda)}(u_3,u_3)du_1du_3,
\end{aligned}
$$

and

$$
\begin{aligned}
&6\int_0^\tau\int_0^\tau\int_0^\tau c_{k,4}^{(\lambda)}(u_1;u_2;u_3,u_3)du_1du_2du_3\\
&= \quad 6\int_0^\tau\int_0^\tau\int_0^\tau m_{k,4}^{(\lambda)}(u_1,u_2,u_3,u_3)du_1du_2du_3 - 12\int_0^\tau m_{k,1}^{(\lambda)}(u_1)du_1\int_0^\tau\int_0^\tau m_{k,3}^{(\lambda)}(u_2,u_3,u_3)du_2du_3\\
&\quad -6\int_0^\tau m_{k,2}^{(\lambda)}(u_3,u_3)du_3\int_0^\tau\int_0^\tau m_{k,2}^{(\lambda)}(u_1,u_2)du_1du_2\\
&\quad +6\int_0^\tau\int_0^\tau\int_0^\tau m_{k,1}^{(\lambda)}(u_1)m_{k,1}^{(\lambda)}(u_2)m_{k,2}^{(\lambda)}(u_3,u_3)du_1du_2du_3.
\end{aligned}
$$

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