Wasserstein distance estimates for jump-diffusion processes

Jean-Christophe Breton*
Univ Rennes, CNRS, IRMAR - UMR 6625,
F-35000 Rennes, France

Nicolas Privault†
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link, Singapore 637371

February 29, 2024

Abstract

We derive Wasserstein distance bounds between the probability distributions of a stochastic integral (Itô) process with jumps $(X_t)_{t\in[0,T]}$ and a jump-diffusion process $(X_t^*)_{t\in[0,T]}$. Our bounds are expressed using the stochastic characteristics of $(X_t)_{t\in[0,T]}$ and the jump-diffusion coefficients of $(X_t^*)_{t\in[0,T]}$ evaluated in X_t, and apply in particular to the case of different jump characteristics. Our approach uses stochastic calculus arguments and L^p integrability results for the flow of stochastic differential equations with jumps, without relying on the Stein equation.

Keywords: Wasserstein distance, stochastic integrals, stochastic differential equations with jumps, Poisson random measures, stochastic flows.
Mathematics Subject Classification (2020): 60H05, 60H10, 60G57, 60G44, 60J60, 60J76.

1 Introduction

Comparison bounds on option prices with convex payoff functions have been obtained in [EJS98] in the continuous diffusion case, based on the classical Kolmogorov equation

*jean-christophe.breton@univ-rennes.fr
†nprivault@ntu.edu.sg

and the propagation of convexity property for Markov semigroups. For example, given $T > 0$ a fixed time horizon, Theorem 6.2 of [EJS98] states that

$$\mathbb{E}[\phi(X_T) \mid X_0 = x] \leq \mathbb{E}\left[\phi(X_T^*) \,\middle|\, X_0^* = x\right], \qquad x > 0, \tag{1.1}$$

for any convex function $\phi : \mathbb{R} \to \mathbb{R}$, provided that $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ are price processes of the form

$$\frac{dX_t}{X_t} = r_t dt + \sigma_t dB_t \quad \text{ and } \quad \frac{dX_t^*}{X_t^*} = r_t dt + \sigma^*(t, X_t^*) dB_t,$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, under the condition

$$|\sigma_t| \leq |\sigma^*(t, X_t)|, \qquad t \in [0, T],$$

allowing one to compare X_T and X_T^* in the convex order by comparing $|\sigma_t|$ to the evaluation of $\sigma^*(t, \cdot)$ at X_t, $t \in [0, T]$. The proof of (1.1) relies on stochastic calculus for the solution of a backward Kolmogorov equation, provided that the Markov semigroup of $(X_t^*)_{t\in[0,T]}$ propagates convexity.

Those results have been extended to jump-diffusion processes in several works, see [BJ00], [BR06], [ET07], under the propagation of convexity hypothesis. Note however that the propagation of convexity property is not always satisfied, for example in the (Markovian) jump-diffusion case, see e.g. Theorem 4.4 in [ET07]. In [BP08], lower and upper bounds on option prices have been obtained in one-dimensional jump-diffusion markets with point process components under different conditions. Related convex ordering results have been obtained for exponential jump-diffusion processes in [BP08] using forward-backward stochastic calculus. The case of random vectors admitting a predictable representation in terms of a Brownian motion and a non-necessarily independent jump component has been treated in [ABP08] using forward-backward stochastic calculus, extending the one-dimensional results of [KMP06], see also [BLP13] for stochastic integrals with jumps, [HY14] for Brownian stochastic integrals, and § 3 of [Pag16] for Lévy-Itô integrals. In [BP22], Wasserstein distance bounds have been derived for the distance between the probability distributions of stochastic integrals with jumps, based on the integrands appearing in their stochastic integral representations and using forward-backward stochastic calculus.

Let $(X_t)_{t\in[0,T]}$ be given as the stochastic integral (or Itô) process with jumps

$$X_t = X_0 + \int_0^t u_s\, ds + \int_0^t \sigma_s\, dB_s + \int_0^t \int_{-\infty}^{+\infty} y\big(\mu(ds,dy) - \nu_s(dy)ds\big), \tag{1.2}$$

where

- $(u_t)_{t\in[0,T]} \in L^1(\Omega \times [0,T])$, $(u_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T])$ are $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes,
- $\mu(dt,dy)$ is a jump measure with $(\mathcal{F}_t)_{t\in[0,T]}$-compensator $\nu_t(dy)dt$ such that

$$\mathbb{E}\left[\int_0^T \int_{-\infty}^{+\infty} y^2 \nu_t(dy)dt\right] < \infty, \tag{1.3}$$

and consider the jump-diffusion process $(X_t^*)_{t\in[0,T]}$ solving the Stochastic Differential Equation (SDE)

$$\begin{aligned} X_t^* = X_0 + \int_0^t u^*(s, X_s^*)\ ds + \int_0^t \sigma^*(s, X_s^*)\ dB_s \\ + \int_0^t \int_{-\infty}^{+\infty} g^*\big(s, X_{s^-}^*, y\big)\big(N^*(ds,dy) - \widehat{\nu}^*(s,dy)ds\big), \end{aligned} \tag{1.4}$$

where

- $u^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ are deterministic functions such that $x \mapsto u^*(t,x)$ and $x \mapsto \sigma^*(t,x)$ are Lipschitz, uniformly in $t \in [0,T]$,
- $g^* : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable deterministic function such that the function

$$x \mapsto \int_{-\infty}^{\infty} |g^*(t,x,y)|^2 \widehat{\nu}^*(t,dy)$$

  is Lipschitz in $x \in \mathbb{R}$, uniformly in $t \in [0,T]$,
- $N^*(dt,dy)$ is a Poisson random measure on $[0,T] \times \mathbb{R}$ with (deterministic) compensator $\widehat{\nu}^*(t,dy)dt$,

see Section 2 for details.

We will derive bounds on the difference $\mathbb{E}[\phi(X_T^*) \mid X_0^* = x] - \mathbb{E}[\phi(X_T) \mid X_0 = x]$ of expectations in (1.1), which allow us to estimate Wasserstein-type distances between the distribution $\mathscr{L}(X_T)$ of the terminal value of a stochastic integral process $(X_t)_{t\in[0,T]}$ as in (1.2) below and the distribution $\mathscr{L}(X_T^*)$ given by the terminal value of a jump-diffusion process $(X_t^*)_{t\in[0,T]}$ solution of the SDE (1.4). In the remaining of this paper we denote by $C > 0$ a finite positive constant whose value may change from statement to statement.

In Theorem 3.1, we obtain the following bound in smooth Wasserstein distance:

$$\begin{aligned} & d_{\mathrm{W}_3}(X_T, X_T^*) \\ & \leq \ C\, \mathbb{E}\left[\int_0^T \Big(\big|u^*(t,X_t) - u_t\big| + \big|\sigma^*(t,X_t)^2 - \sigma_t^2\big| + d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t,X_t,\cdot)\big)\Big)dt\right], \end{aligned} \tag{1.5}$$

for some $C > 0$, where

$$\widetilde{\nu}_t(dy) := y^2\nu_t(dy), \quad \widetilde{\nu}^*(t,x,dy) := y^2\nu^*(t,x,dy)$$

and

$$\nu^*(t,x,\cdot) := \widehat{\nu}^*(t,(g^*)^{-1}(t,x,\cdot)), \tag{1.6}$$

see the end of Section 2 for the definitions of the Fortet-Mourier distance d_{FM} and smooth Wasserstein distance d_{W_3}. In Theorem 3.3, by a smoothing argument on 1-Lipschitz functions we obtain the Wasserstein bound

$$\begin{aligned} & d_W(X_T, X_T^*) \\ & \leq C_K\left(\mathbb{E}\left[\int_0^T \big|u_t - u^*(t,X_t)\big|dt\right]\right)^{1/2} + C_K\left(\mathbb{E}\left[\int_0^T \big|\sigma_t^2 - \sigma^*(t,X_t)^2\big|dt\right]\right)^{1/2} \\ & \quad + C_K\left(\mathbb{E}\left[\int_0^T d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t,X_t,\cdot)\big)dt\right]\right)^{1/3}, \end{aligned} \tag{1.7}$$

provided that

$$\mathbb{E}\left[\int_0^T \Big(\big|u_t - u^*(t,X_t)\big| + \big|\sigma_t^2 - \sigma^*(t,X_t)^2\big| + d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t,X_t,\cdot)\big)\Big)dt\right] \leq K$$

for some $K > 0$. Bounds on the Wasserstein distance between random variables on the Wiener space and e.g. the normal or gamma distribution have been obtained in [NP09] by the Stein method, using the Malliavin calculus and covariance representations based on the Ornstein-Uhlenbeck operator. In contrast, our approach does not make use of the Stein equation and can be regarded as an alternative to the Stein method and to its semi-group version, see [Dec15].

The proof argument leading to (1.5)-(1.7) consists in expanding the difference $h(X^*_T) - h(X_T)$ for suitable functions $h : \mathbb{R} \to \mathbb{R}$ with the Itô formula and, taking the expectation, to bound the remaining terms with a suitable control of the characteristics of the related jump-diffusions. Consider the operator $\mathcal{L}$ and the generator $\mathcal{L}^*$ of $(X^*_t)_{t\in[0,T]}$, respectively given for $f \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ by

$$\begin{aligned}\mathcal{L}f(t,x) :=& u_t \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t,x) \\ &+ \int_{-\infty}^{+\infty} \left(f(t,x+y) - f(t,x) - y\frac{\partial f}{\partial x}(t,x)\right) \nu_t(dy),\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\mathcal{L}^*f(t,x) :=& u^*(t,x) \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^*(t,x)^2 \frac{\partial^2 f}{\partial x^2}(t,x) \\ &+ \int_{-\infty}^{+\infty} \left(f(t,x+y) - f(t,x) - y\frac{\partial f}{\partial x}(t,x)\right) \nu^*(t,x,dy),\end{aligned} \tag{1.9}$$

$(t,x) \in [0,T]\times\mathbb{R}$, where $\nu^*(t,x,\cdot)$ is the image measure

$$\nu^*(t,x,\cdot) := \widehat{\nu}^*(t,(g^*)^{-1}(t,x,\cdot)),$$

see Theorem 2 page 291 in [GS72]. In the sequel, we denote by $\mathcal{C}_b^k(\mathbb{R})$ the space of continuously differentiable functions whose derivatives of orders one to $k \geq 1$ are uniformly bounded on $\mathbb{R}$.

Following [EJS98], [BR07] and using $\mathcal{L}$ and $\mathcal{L}^*$, given $h \in \mathcal{C}_b^3(\mathbb{R})$ we represent the expected difference $\mathbb{E}[h(X_T)] - \mathbb{E}[h(X^*_T)]$ in terms of the solution

$$v^*(t,x) = \mathbb{E}[h(X^*_T) \mid X^*_t = x]$$

of the Kolmogorov equation of $(X^*_t)_{t\in[0,T]}$, as

$$\begin{aligned}&\mathbb{E}\left[h(X^*_T)\right] - \mathbb{E}\left[h(X_T)\right] = \mathbb{E}\left[\int_0^T \big(\mathcal{L}^* v^*(t,X_t) - \mathcal{L}v^*(t,X_t)\big)\,dt\right] \\ &= \mathbb{E}\left[\int_0^T \big(u^*(t,X_t) - u_t\big)\frac{\partial v^*}{\partial x}(t,X_t)dt\right] + \frac{1}{2}\,\mathbb{E}\left[\int_0^T \big(\sigma^*(t,X_t)^2 - (\sigma_t)^2\big)\,\frac{\partial^2 v^*}{\partial x^2}(t,X_t)dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T\int_0^1 (1-\tau)\int_{-\infty}^{+\infty} \frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)\big(\widetilde{\nu}^*(t,X_t,dy) - \widetilde{\nu}_t(dy)\big)d\tau dt\right].\end{aligned} \tag{1.10}$$

Here, the random measures $\widetilde{\nu}_t(dy)$ and $\widetilde{\nu}^*(t,x,dy)$ are defined in terms of the jump-characteristics of the jump-diffusions $\nu_t(dy)$, $\nu^*(t,x,dy)$ appearing in (1.8)-(1.9), see (1.6). Then, we proceed to show that the functions

$$y \mapsto \frac{\partial v^*}{\partial x}(s, X_s + \tau y) \quad \text{and} \quad y \mapsto \frac{\partial^2 v^*}{\partial x^2}(s, X_s + \tau y)$$

are Lipschitz using moment bounds from [BP20]. Due to the definitions of the relevant probability distances (see (2.8) and afterwards), this allows us to bound (1.10) by the Fortet-Mourier distance d_{FM} between $\nu_t(\cdot)$ and $\widetilde{\nu}^*(t, X_t, \cdot)$, which eventually leads to (1.5)-(1.7).

In contrast to [EJS98], [BR07], propagation of convexity is not required in our argument since no positivity is needed for the second derivative $\partial^2 v^*/\partial x^2$, which is only required to be a Lipschitz function in our argument.

We also note that in the case where both $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ share the same jump characteristics, the L^p norm $(\mathbb{E}[(X_T - X_T^*)^p])^{1/p}$ can be directly estimated using standard Gronwall-type arguments. This is the case in particular for the estimation of Euler discretization bounds, see e.g. [TT90] and [PT97]. In the absence of jumps, such comparison results between $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ can also be obtained by representing the Itô process $(X_t)_{t\in[0,T]}$ as a diffusion process under certain conditions, see [Gyö86] or Theorem 8.4.3 in [Øks03]. On the other hand, our method covers the case where $(X_t)_{t\in[0,T]}$ may not be written as a diffusion process and $(X_t)_{t\in[0,T]}$, $(X_t^*)_{t\in[0,T]}$ have different jump characteristics

We proceed as follows. In Section 2 we start by recalling the basics of characteristics for jump-diffusion processes and distances between probability measures. Wasserstein distance bounds between jump-diffusion processes and general stochastic integral processes are derived in Section 3, and specialized to jump-diffusion processes in Section 4. Technical results are gathered in the Appendix.

2 Preliminaries and notations

Jump-diffusion processes

Consider a standard Brownian motion $(B_t)_{t\in[0,T]}$ and a jump measure

$$\mu(dt,dy) := \sum_{s>0} 1_{\{\Delta M_s \neq 0\}} \delta_{(s,\Delta M_s)}(dt,dy),$$

generating a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, see e.g. [JM76], where $\delta_{(s,x)}$ is the Dirac measure at $(s,x)\in[0,T]\times\mathbb{R}$. We assume that $(B_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t)_{t\in[0,T]}$-standard Brownian motion and that the $(\mathcal{F}_t)_{t\in[0,T]}$-compensator $\nu(dt,dy)$ of $\mu(dt,dy)$ takes the form

$$\nu(dt,dy) = \nu_t(dy)dt.$$

We also assume that the (deterministic) compensator $\widehat{\nu}^*(t,dy)dt$ of the Poisson random measure N^* on $[0,T]\times\mathbb{R}$ is dominated by a (deterministic) measure η for any $t\in[0,T]$, in the sense that

$$\widehat{\nu}^*(t,A) \leq \eta(A), \qquad A\in\mathcal{B}(\mathbb{R}), \quad t\in[0,T], \tag{D}$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. In the sequel, our quantities of interest are the terminal value X_T of the stochastic integral process $(X_t)_{t\in[0,T]}$, given by (1.2), and the distribution $\mathscr{L}(X_T^*)$, to which $\mathscr{L}(X_T)$ will be compared, is given by the terminal value X_T^* of the solution $(X_t^*)_{t\in[0,T]}$ to the SDE (1.4). Setting

$$\nu^*(t,x,\cdot) := \widehat{\nu}^*(t,(g^*)^{-1}(t,x,\cdot)), \tag{2.1}$$

we note that (1.4) can be rewritten as

$$X_t^* = X_0^* + \int_0^t u^*(s,X_s^*)\,ds + \int_0^t \sigma^*(s,X_s^*)\,dB_s + \int_0^t \int_{-\infty}^{+\infty} y\big(\mu^*(ds,dy) - \nu^*(s,X_{s^-}^*,dy)ds\big)$$

as in (1.2), where $\mu^*(dt,dy)$ is the jump measure with $(\mathcal{F}_t)_{t\in[0,T]}$-compensator $\nu^*(t,X_{t^-}^*,dy)$. In the sequel, we use the operator $\mathcal{L}$ and the generator $\mathcal{L}^*$ of $(X_t^*)_{t\in[0,T]}$ given in (1.8) and (1.9), which can be rewritten in terms of $\widehat{\nu}_t^*$ as

$$\begin{aligned}\mathcal{L}^* f(t,x) &= u^*(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^*(t,x)^2\frac{\partial^2 f}{\partial x^2}(t,x)\\ &\quad + \int_{-\infty}^{+\infty}\left(f(t,x+g^*(t,x,z)) - f(t,x) - g^*(t,x,z)\frac{\partial f}{\partial x}(t,x)\right)\widehat{\nu}^*(t,dz).\end{aligned}$$

A crucial tool in our argument is the classical Kolmogorov equation, see Theorem 4 page 296 in [GS72], which can be extended to our setting as in the following lemma, by noting that the limit (5) page 291 of [GS72] remains valid when $\nu^*(t,dz)$ is time-dependent.

Lemma 2.1 *Let $h \in \mathcal{C}_b^2(\mathbb{R})$, and assume that for some $C \in (0,+\infty)$ we have*

$$\left(\frac{\partial^2 u^*}{\partial x^2}(t,x)\right)^2 + \left(\frac{\partial^2 \sigma^*}{\partial x^2}(t,x)\right)^2 + \int_{-\infty}^{\infty} \left(\frac{\partial^2 g^*}{\partial x^2}(t,x,y)\right)^2 \nu^*(t,dy) \leq C, \quad x \in \mathbb{R}, \ t \in [0,T].$$

Then, the function v^* defined by

$$v^*(t,x) = \mathbb{E}\left[h(X_T^*) | X_t^* = x\right] = \mathbb{E}\left[h(X_{t,T}^*(x))\right], \quad x \in \mathbb{R}, \quad t \in [0,T], \tag{2.2}$$

is in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, where $(X_{t,s}^*(x))_{s \geq t}$ is the solution of the SDE (1.4) started at $X_{t,t}^*(x) = x$ in time t. Moreover, v^* satisfies the Partial Differential Equation (PDE)

$$\begin{cases} \dfrac{\partial v^*}{\partial t}(t,x) + \mathcal{L}^* v^*(t,x) = 0, \\ \\ v^*(T,x) = h(x). \end{cases} \tag{2.3}$$

Regularity of the flow of jump SDEs

Our derivation of Wasserstein bounds relies on regularity and integrability results of [BP20], see Theorem 5.1 therein, and also Theorem 3.3 of [Kun04] in the case of first order differentiability. For that purpose, we assume further conditions on the jump-diffusion process $(X_t^*)_{t \in [0,T]}$ in (1.4), namely, we will make use of the following Assumption (A_n) on the coefficients σ^*, g^* of $(X_t^*)_{t \in [0,T]}$ in (1.4) for $n = 3$.

Assumption (A_n): *For every $t \in [0,T]$, the functions $\sigma^*(t,\cdot) : \mathbb{R} \to \mathbb{R}$ and $g^*(t,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^n$-differentiable and there is a constant $C > 0$ and a function $\theta \in \bigcap_{p \geq 2} L^p(\mathbb{R},\eta)$ such that*

$$\left|\frac{\partial^k \sigma^*}{\partial x^k}(t,x)\right| \leq C, \quad \left|\frac{\partial^{k+l} g^*}{\partial x^k \partial y^l}(t,x,y)\right| \leq C, \quad \left|\frac{\partial^k g^*}{\partial x^k}(t,x,y)\right| \leq C\theta(y),$$

for all $k,l = 1,\ldots,n$ with $1 \leq k+l \leq n$, $t \in [0,T]$, $x,y \in \mathbb{R}$.

Assumption (A_n) originates from Assumption (A'-r) in the time-homogeneous setting of [BGJ87], see page 60 therein. As noted in [BP20], the domination condition (D) allows us to apply the results of [BGJ87], in particular Lemma 5.1, Theorems 6-20, 6-24, 6-29 and 6-44 therein to the time-inhomogeneous case.

Let $n \geq 1$ and $p \geq 2$ be given. Under the domination condition (D) and Assumption (A_n), Theorem 5.1 in [BP20] ensures that for all $k = 1, \ldots, n$ the flow $X^*_{t,T}(x)$ of (1.2) is k-th differentiable in x with

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial^k}{\partial x^k}X^*_{t,T}(x)\right|^p\right] < +\infty, \tag{2.4}$$

i.e. the flow derivatives belong to $L^p(\Omega)$, uniformly in $(t,x) \in [0,T] \times \mathbb{R}$. In the sequel we shall use the following consequence of (2.4) for (joint) moments, which is a direct consequence of Proposition 4.1 and Theorem 5.1 in [BP20] applied with $n = 3$.

Lemma 2.2 *Assume that (A_3) holds together with the domination condition* (D). *Then, the flow $x \mapsto X^*_{t,T}(x)$ of the solution of SDE* (1.4) *is differentiable up to the order* 3 *and there exist constants $A_1, A_2, B_1, B_2, B_3 > 0$ depending on $T > 0$ such that uniformly in $x > 0$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial^2}{\partial x^2}X^*_{t,T}(x)\right|\right] \leq A_1,\ \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial}{\partial x}X^*_{t,T}(x)\right|^2\right] \leq A_2,\ \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial}{\partial x}X^*_{t,T}(x)\right|^3\right] \leq B_3, \tag{2.5}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial^3}{\partial x^3}X^*_{t,T}(x)\right|\right] \leq B_1, \quad \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial}{\partial x}X^*_{t,T}(x)\frac{\partial^2}{\partial x^2}X^*_{t,T}(x)\right|\right] \leq B_2. \tag{2.6}$$

Proof. Applying Proposition 4.1 in [BP20] with $n = 3$ ensures that $x \mapsto X^*_{t,T}(x)$ is differentiable up to the order 3. Next, by Theorem 5.1 in [BP20] we have

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{\partial^k}{\partial x^k}X^*_{t,T}(x)\right|^3\right] < +\infty, \qquad k = 1, 2, 3, \tag{2.7}$$

and we conclude by the Hölder inequality. □

Distances between measures

Given a set $\mathcal{H}$ of functions $h : \mathbb{R} \to \mathbb{R}$, we define the distance $d_{\mathcal{H}}$ between two measures μ, ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$d_{\mathcal{H}}(\mu,\nu) := \sup_{h\in\mathcal{H}}\left|\int_{-\infty}^{+\infty} h(x)\ \mu(dx) - \int_{-\infty}^{+\infty} h(x)\ \nu(dx)\right|, \tag{2.8}$$

provided that every $h \in \mathcal{H}$ is integrable with respect to μ and ν, and we write $d_{\mathcal{H}}(X,Y) = d_{\mathcal{H}}(\mu,\nu)$ when μ and ν are the probability distributions of the random variables X, Y.

- The Fortet-Mourier distance d_{FM} corresponds to the choice $\mathcal{H} = \mathcal{FM}$, where $\mathcal{FM}$ is the class of functions h such that $\|h\|_{BL} = \|h\|_L + \|h\|_\infty \leq 1$, where $\|\cdot\|_L$ denotes the Lipschitz semi-norm and $\|\cdot\|_\infty$ is the supremum norm.
- The Wasserstein distance d_{W} corresponds to $\mathcal{H} = \mathrm{Lip}(1)$, where $\mathrm{Lip}(1)$ is the class of functions h such that $\|h\|_L \leq 1$.
- The smooth Wasserstein distance d_{W_r}, $r \geq 0$, is obtained when $\mathcal{H} := \mathcal{H}_r$ is the set of continuous functions which are r-times continuously differentiable and such that $\|h^{(k)}\|_\infty \leq 1$, for all $0 \leq k \leq r$, where $h^{(0)} = h$, and where $h^{(k)}$, $k \geq 1$, is the k-th derivative of h.

The expression (2.8) can also be used to define the Kolmogorov distance when $\mathcal{H}$ is a set of indicator functions. It is easy to observe that $d_{\mathrm{FM}}(\cdot,\cdot) \leq d_{\mathrm{W}}(\cdot,\cdot)$ and the topology induced by d_{W} is stronger than the topology of convergence in distribution which is metrized by d_{FM}. Moreover, for the smooth Wasserstein distance $d_{\mathrm{W_r}}$ with $r > 1$, an approximation argument shows that

$$d_{\mathrm{W}_r}(X,Y) = \sup_{h \in C_c^\infty(\mathbb{R}) \cap \mathcal{H}_r} \big| \mathbb{E}[h(X)] - \mathbb{E}[h(Y)] \big|, \tag{2.9}$$

where $C_c^\infty(\mathbb{R})$ is the space of compactly supported, infinitely differentiable functions on $\mathbb{R}$, see Lemma A.3 in [AH19]. Note also that $d_{\mathrm{W}_{r-1}}(X,Y) \leq 3\sqrt{2 d_{\mathrm{W}_r}(X,Y)}$ and that the smooth Wasserstein distance d_{W_r} is weaker than the Wasserstein distance d_{W}, since

$$d_{\mathrm{W}_r}(X,Y) \leq d_{\mathrm{W}_1}(X,Y) \leq d_{\mathrm{W}}(X,Y),$$

see (2.16) in [AH19], to which we refer for further details in this direction, see also [Dud02].

3 Wasserstein bounds for stochastic integral processes

In this section, we bound the distance between the integral process $(X_t)_{t\in[0,T]}$ given in (1.2) and a process given by the jump-diffusion process $(X_t^*)_{t\in[0,T]}$ defined in (1.4). Our bounds use the difference $|\sigma_t - \sigma^*(t, X_t)|$ and the distance between the jump measure characteristics $\widetilde{\nu}_t(dy) = y^2\nu_t(dy)$ and

$$\widetilde{\nu}^*(t, x, dy) := y^2\nu^*(t, x, dy), \quad \text{with} \quad \nu^*(t, x, \cdot) := \widehat{\nu}^*\big(t, (g^*)^{-1}(t, x, \cdot)\big),$$

see (1.6). Recall that $\nu_t(dy)dt$ is the compensator of the random point measure $\mu(dt, dy)$ and $\widehat{\nu}^*(t, dy)dt$ is the compensator of $N^*(dt, dy)$, introduced in Section 2.

Theorem 3.1 (Smooth Wasserstein bound) *Let $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ be given by (1.2)–(1.4), with $X_0 = X_0^*$. Assume that (A_3) and the domination condition (D) hold true. Then, for some $C > 0$ we have*

$$\begin{aligned} & d_{\mathrm{W}_3}(X_T, X_T^*) \\ & \leq \ C\ \mathbb{E}\left[\int_0^T \Big(\big|u^*(t, X_t) - u_t\big| + \big|\sigma^*(t, X_t)^2 - \sigma_t^2\big| + d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big)\Big) dt\right], \end{aligned} \tag{3.1}$$

where $\widetilde{\nu}_t$ and $\widetilde{\nu}^*$ are given by (1.6).

Proof. Let $h \in \mathcal{C}_b^3(\mathbb{R})$ satisfy the conditions of Lemma 2.1. Applying first the Itô formula and then the Kolmogorov equation (2.3) in Lemma 2.1 to v^* in (2.2), we have

$$\begin{aligned} h(X_T) &= v^*(T, X_T) \\ &= v^*(0, X_0) + \int_0^T \sigma_t \frac{\partial v^*}{\partial x}(t, X_t) dB_t + \int_0^T \frac{\partial v^*}{\partial t}(t, X_t) dt + \int_0^T \mathcal{L}v^*(t, X_t) dt \\ &\quad + \int_0^T \int_{-\infty}^{+\infty} \Big(v^*(t, X_t + y) - v^*(t, X_t) - y\frac{\partial v^*}{\partial x}(t, X_t)\Big)(\mu(dt, dy) - \nu_t(dy)dt) \\ &= v^*(0, X_0) + \int_0^T \sigma_t \frac{\partial v^*}{\partial x}(t, X_t) dB_t + \int_0^T \big(\mathcal{L}v^*(t, X_t) - \mathcal{L}^*v^*(t, X_t)\big) dt \qquad (3.2) \\ &\quad + \int_0^T \int_{-\infty}^{+\infty} \Big(v^*(t, X_t + y) - v^*(t, X_t) - y\frac{\partial v^*}{\partial x}(t, X_t)\Big)(\mu(dt, dy) - \nu_t(dy)dt), \end{aligned}$$

where the above stochastic integrals are understood in the L^2 sense, as will be checked below. Since $h \in \mathcal{C}_b^3(\mathbb{R})$, using Lemma 2.2, we have

$$\frac{\partial v^*}{\partial x}(t, x) = \frac{\partial}{\partial x}\,\mathbb{E}[h(X_{t,T}^*(x)(x))] = \mathbb{E}\left[h'(X_{t,T}^*(x))\frac{\partial}{\partial x}X_{t,T}^*(x)\right], \tag{3.3}$$

and

$$\begin{aligned}\frac{\partial^2 v^*}{\partial x^2}(t,x) &= \frac{\partial^2}{\partial x^2}\,\mathbb{E}[h(X^*_{t,T}(x)(x))] \\ &= \frac{\partial}{\partial x}\,\mathbb{E}\left[h'(X^*_{t,T}(x))\frac{\partial}{\partial x}X^*_{t,T}(x)\right] \\ &= \mathbb{E}\left[h'(X^*_{t,T}(x))\frac{\partial^2}{\partial x^2}X^*_{t,T}(x) + h''(X^*_{t,T}(x))\left(\frac{\partial}{\partial x}X^*_{t,T}(x)\right)^2\right]. \quad (3.4)\end{aligned}$$

Hence by (2.7) we have

$$\sup_{x\in\mathbb{R},\ t\in[0,T]}\left|\frac{\partial v^*}{\partial x}(t,x)\right| < \infty, \qquad \sup_{x\in\mathbb{R},\ t\in[0,T]}\left|\frac{\partial^2 v^*}{\partial x^2}(t,x)\right| < \infty,$$

and Taylor's formula with integral remainder yields

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T\int_{-\infty}^{+\infty}\Big(v^*(t,X_t+y)-v^*(t,X_t)-y\frac{\partial v^*}{\partial x}(t,X_t)\Big)\big(\mu(dt,dy)-\nu_t(dy)dt\big)\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T\int_{-\infty}^{+\infty}\Big(v^*(t,X_t+y)-v^*(t,X_t)-y\frac{\partial v^*}{\partial x}(t,X_t)\Big)^2\nu_t(dy)dt\right] \\ &= \mathbb{E}\left[\int_0^T\int_{-\infty}^{+\infty}y^2\left(\int_0^1(1-\tau)\frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)d\tau\right)^2\nu_t(dy)dt\right] < \infty\end{aligned}$$

by (1.3). Therefore, the stochastic integrals

$$\int_0^T \sigma_t\frac{\partial v^*}{\partial x}(t,X_t)dB_t$$

and

$$\int_0^T\int_{-\infty}^{+\infty}\Big(v^*(t,X_t+y)-v^*(t,X_t)-y\frac{\partial v^*}{\partial x}(t,X_t)\Big)\big(\mu(dt,dy)-\nu_t(dy)dt\big)$$

are defined in $L^2(\Omega)$. As a consequence, we have

$$\mathbb{E}\left[\int_0^T \sigma_t\frac{\partial v^*}{\partial x}(t,X_t)dB_t\right] = 0$$

and

$$\mathbb{E}\left[\int_0^T\int_{-\infty}^{+\infty}\Big(v^*(t,X_t+y)-v^*(t,X_t)-y\frac{\partial v^*}{\partial x}(t,X_t)\Big)\big(\mu(dt,dy)-\nu_t(dy)dt\big)\right] = 0,$$

so that taking expectations in (3.2) yields

$$\mathbb{E}\big[h(X_T)\big] = \mathbb{E}\big[v^*(0,X_0)\big] + \mathbb{E}\left[\int_0^T\big(\mathcal{L}v^*(t,X_t)-\mathcal{L}^*v^*(t,X_t)\big)\,dt\right]. \qquad (3.5)$$

Given that the martingale property entails

$$\mathbb{E}\left[v^*(0,X_0^*)\right]=\mathbb{E}\left[v^*(T,X_T^*)\right]=\mathbb{E}\left[h(X_T^*)\right],$$

when $X_0=X_0^*$, we can rewrite (3.5) as

$$\mathbb{E}\left[h(X_T^*)\right]-\mathbb{E}\left[h(X_T)\right]=\mathbb{E}\left[\int_0^T\left(\mathcal{L}^*v^*(t,X_t)-\mathcal{L}v^*(t,X_t)\right)dt\right].\tag{3.6}$$

Next, using the following version of Taylor's formula

$$f(x+y)=f(x)+yf'(x)+y^2\int_0^1(1-\tau)f''(x+\tau y)d\tau$$

applied to $f\in\mathcal{C}^2(\mathbb{R})$, $x,y\in\mathbb{R}$, we have

$$\begin{aligned}
&\mathcal{L}^*v^*(t,X_t)-\mathcal{L}v^*(t,X_t)\\
&=u^*(t,X_t)\frac{\partial v^*}{\partial x}(t,X_t)+\frac{1}{2}\sigma^*(t,X_t)^2\frac{\partial^2 v^*}{\partial x^2}(t,X_t)\\
&\quad+\int_{-\infty}^{+\infty}y^2\int_0^1(1-\tau)\frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)d\tau\nu^*(t,X_t,dy)\\
&\quad-u_t\frac{\partial v^*}{\partial x}(t,X_t)-\frac{1}{2}\sigma_t^2\frac{\partial^2 v^*}{\partial x^2}(t,X_t)-\int_{-\infty}^{+\infty}y^2\int_0^1(1-\tau)\frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)d\tau\nu_t(dy)\\
&=\left(u^*(t,X_t)-u_t\right)\frac{\partial v^*}{\partial x}(t,X_t)+\frac{1}{2}\left(\sigma^*(t,X_t)^2-\sigma_t^2\right)\frac{\partial^2 v^*}{\partial x^2}(t,X_t)\\
&\quad+\int_0^1(1-\tau)\int_{-\infty}^{+\infty}\frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)\left(\widetilde{\nu}^*(t,X_t,dy)-\widetilde{\nu}_t(dy)\right)d\tau
\end{aligned}\tag{3.7}$$

where the measures $\widetilde{\nu}_t(dy)$ and $\widetilde{\nu}^*(t,x,dy)$ are defined in (1.6). Plugging the identity (3.7) in (3.6) yields

$$\begin{aligned}
\mathbb{E}\left[h(X_T^*)\right]-\mathbb{E}\left[h(X_T)\right]&=\left[\int_0^T\left(u^*(t,X_t)-u_t\right)\frac{\partial v^*}{\partial x}(t,X_t)dt\right]\\
&\quad+\frac{1}{2}\mathbb{E}\left[\int_0^T\left(\sigma^*(t,X_t)^2-\sigma_t^2\right)\frac{\partial^2 v^*}{\partial x^2}(t,X_t)dt\right]\\
&\quad+\mathbb{E}\left[\int_0^T\int_0^1(1-\tau)\int_{-\infty}^{+\infty}\frac{\partial^2 v^*}{\partial x^2}(t,X_t+\tau y)\left(\widetilde{\nu}^*(t,X_t,dy)-\widetilde{\nu}_t(dy)\right)d\tau dt\right].
\end{aligned}\tag{3.8}$$

We continue our argument by analyzing the integrand in (3.8). Recall that v^* is given in (2.2) in terms of the solution $(X_{t,s}^*(x))_{s\in[t,T]}$ of the SDE (1.4) started at $X_{t,t}^*(x)=x$. Lemma 2.2 ensures that $X_{t,T}^*(x)$ is differentiable in x up to the order 3, so that by (3.3) and (3.4) we find

$$\left|\frac{\partial v^*}{\partial x}(t,x)\right|\leq\mathbb{E}\left[\left|\frac{\partial}{\partial x}X_{t,T}^*(x)\right|\right]\|h'\|_\infty$$

and

$$\left|\frac{\partial^2 v^*}{\partial x^2}(t,x)\right| \le \mathbb{E}\left[\left|\frac{\partial^2}{\partial x^2}X^*_{t,T}(x)\right|\right]\|h'\|_\infty + \mathbb{E}\left[\left(\frac{\partial}{\partial x}X^*_{t,T}(x)\right)^2\right]\|h''\|_\infty,$$

and similarly

$$\begin{aligned}
&\left|\frac{\partial^3 v^*}{\partial x^3}(t,x)\right| \\
&= \left|\mathbb{E}\left[h'(X^*_{t,T}(x))\frac{\partial^3}{\partial x^3}X^*_{t,T}(x) + 3h''(X^*_{t,T}(x))\frac{\partial^2}{\partial x^2}X^*_{t,T}(x)\frac{\partial}{\partial x}X^*_{t,T}(x) + h^{(3)}(X^*_{t,T}(x))\left(\frac{\partial}{\partial x}X^*_{t,T}(x)\right)^3\right]\right| \\
&\le \|h'\|_\infty \mathbb{E}\left[\left|\frac{\partial^3}{\partial x^3}X^*_{t,T}(x)\right|\right] + 3\|h''\|_\infty \mathbb{E}\left[\left|\frac{\partial^2}{\partial x^2}X^*_{t,T}(x)\right|\left|\frac{\partial}{\partial x}X^*_{t,T}(x)\right|\right] + \|h^{(3)}\|_\infty \mathbb{E}\left[\left|\frac{\partial}{\partial x}X^*_{t,T}(x)\right|^3\right].
\end{aligned}$$

Next, using the bounds (2.5)–(2.6) in Lemma 2.2 (with its notations A_i, B_j), we have

$$\left|\frac{\partial v^*}{\partial x}(t,x)\right| \le \sqrt{A_2}\|h'\|_\infty \quad \text{and} \quad \left|\frac{\partial^2 v^*}{\partial x^2}(t,x)\right| \le A_1\|h'\|_\infty + A_2\|h''\|_\infty,$$

and

$$\left|\frac{\partial^3 v^*}{\partial x^3}(t,x)\right| \le B_1\|h'\|_\infty + 3B_2\|h''\|_\infty + B_3\|h^{(3)}\|_\infty, \quad (t,x)\in[0,T]\times\mathbb{R}.$$

Consequently, for every $\tau\in[0,1]$, the function

$$y \mapsto \frac{\partial v^*}{\partial x}(t, X_t + \tau y)$$

is bounded by $\sqrt{A_2}\|h'\|_\infty$ and is $\tau\big(A_1\|h'\|_\infty + A_2\|h''\|_\infty\big)$-Lipschitz. Similarly, for every $\tau\in[0,1]$, the function

$$y \mapsto \frac{\partial^2 v^*}{\partial x^2}(t, X_t + \tau y)$$

is bounded by $A_1\|h'\|_\infty + A_2\|h''\|_\infty$ and is $\tau(B_1\|h'\|_\infty + 3B_2\|h''\|_\infty + B_3\|h^{(3)}\|_\infty)$-Lipschitz.

Thus, by the definition (2.8) of the Fortet-Mourier distance d_{FM}, for all $\tau\in[0,1]$ we get

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty}\frac{\partial^2 v^*}{\partial x^2}(t, X_t+\tau y)\big(\widetilde{\nu}^*(t,X_t,dy) - \widetilde{\nu}_t(dy)\big)\right| \qquad (3.9)\\
&\le \big((A_1+\tau B_1)\|h'\|_\infty + (A_2+3\tau B_2)\|h''\|_\infty + \tau B_3\|h^{(3)}\|_\infty\big) d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t,X_t,\cdot)\big).
\end{aligned}$$

Plugging (3.9) into (3.8) yields the bound

$$\big|\mathbb{E}\left[h(X^*_T)\right] - \mathbb{E}\left[h(X_T)\right]\big| \le \sqrt{A_2}\|h'\|_\infty \mathbb{E}\left[\int_0^T \big|u_t - u^*(t,X_t)\big| dt\right]$$

$$+\frac{1}{2}\mathbb{E}\left[\int_0^T \big(A_1\|h'\|_\infty + A_2\|h''\|_\infty\big)\big|\sigma_t^2 - \sigma^*(t,X_t)^2\big|dt\right]$$
$$+\int_0^1 (1-\tau)\big((A_1+\tau B_1)\|h'\|_\infty + (A_2+3\tau B_2)\|h''\|_\infty + \tau B_3\|h^{(3)}\|_\infty\big)\,d\tau$$
$$\times\,\mathbb{E}\left[\int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)dt\right]$$
$$= \sqrt{A_2}\|h'\|_\infty\,\mathbb{E}\left[\int_0^T \big|u_t - u^*(t,X_t)\big|dt\right] \tag{3.10}$$
$$+\frac{1}{2}\mathbb{E}\left[\int_0^T \big(A_1\|h'\|_\infty + A_2\|h''\|_\infty\big)\big|\sigma_t^2 - \sigma^*(t,X_t)^2\big|dt\right]$$
$$+\frac{1}{2}\left(\left(A_1+\frac{B_1}{3}\right)\|h'\|_\infty + (A_2+B_2)\|h''\|_\infty + \frac{B_3}{3}\|h^{(3)}\|_\infty\right)\mathbb{E}\left[\int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)dt\right],$$

for $h\in\mathcal{C}_b^3(\mathbb{R})$. Finally, using the expression (2.9) of the smooth Wasserstein distance d_{W_3}, the bound (3.1) follows from (3.10) with

$$C := \max\Big(\sqrt{A_2}, (A_1+A_2)/2, (A_1+B_1/3+A_2+B_2+B_3/3)/2\Big).$$

□

Continuing the proof of Theorem 3.1 with a regularization argument, we obtain the following bound in Wasserstein distance.

Proposition 3.2 (Wasserstein bound) *Let $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ be the integral and jump-diffusion processes given in (1.2)–(1.4), with $X_0 = X_0^*$. Assume that (A_3) and the domination condition (D) hold true. Then, for a finite constant $C>0$, we have*

$$d_W(X_T,X_T^*) \le \tag{3.11}$$
$$C\max\left(\left(\mathbb{E}\left[\int_0^T \Big(\big|u_t-u^*(t,X_t)\big| + \big|\sigma_t^2-\sigma^*(t,X_t)^2\big| + d_{\rm FM}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)\Big)dt\right]\right)^{1/3},\right.$$
$$\left.\mathbb{E}\left[\int_0^T \Big(\big|u_t-u^*(t,X_t)\big| + \big|\sigma_t^2-\sigma^*(t,X_t)^2\big| + d_{\rm FM}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)\Big)dt\right]\right)$$

where the measures $\widetilde{\nu}_t(dy)$ and $\widetilde{\nu}^*(t,x,dy)$ are defined in (1.6).

Proof. We extend the bound (3.10) from $h\in\mathcal{C}_b^3(\mathbb{R})$ to $h\in{\rm Lip}(1)$ using the approximation

$$h_\alpha(x) = \int_{-\infty}^{+\infty} h\big(x+y\sqrt{\alpha}\big)\phi(y)\,dy, \qquad \alpha>0, \tag{3.12}$$

of $h \in \mathrm{Lip}(1)$, where ϕ is the standard $\mathcal{N}(0,1)$ probability density function. By the bound (A.2) in Lemma A.1 in Appendix we know that $h_\alpha \in \mathcal{C}_b^\infty(\mathbb{R})$ satisfies the conditions of Lemma 2.1, hence by (3.10) and the bound (A.1), we have

$$\begin{aligned}
&\big|\,\mathbb{E}[h(X_T^*)] - \mathbb{E}[h(X_T)]\big| \\
&\leq 2\|h-h_\alpha\|_\infty + \big|\,\mathbb{E}[h_\alpha(X_T^*)] - \mathbb{E}[h_\alpha(X_T)]\big| \\
&\leq 2\sqrt{\frac{2\alpha}{\pi}} + \sqrt{A_2}\|h'_\alpha\|_\infty\,\mathbb{E}\left[\int_0^T \big|u_t - u^*(t,X_t)\big|dt\right] \\
&+ \frac{A_1\|h'_\alpha\|_\infty + A_2\|h''_\alpha\|_\infty}{2}\,\mathbb{E}\left[\int_0^T \big|\sigma_t^2 - \sigma^*(t,X_t)^2\big|dt\right] \\
&+ \frac{1}{2}\left(\left(A_1+\frac{B_1}{3}\right)\|h'_\alpha\|_\infty + (A_2+B_2)\|h''_\alpha\|_\infty + \frac{B_3}{3}\|h_\alpha^{(3)}\|_\infty\right)\mathbb{E}\left[\int_0^T d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)dt\right].
\end{aligned}$$

Next, using (A.2) in Lemma A.1 in Appendix and optimizing in $h \in \mathrm{Lip}(1)$, we find

$$d_{\mathrm{W}}(X_T, X_T^*) \leq D_0\sqrt{\alpha} + D_1 + \frac{D_2}{\sqrt{\alpha}} + \frac{D_3}{\alpha}, \tag{3.13}$$

where

$$D_0 = 2\sqrt{\frac{2}{\pi}}, \quad D_1 = \frac{C_1}{2}\left(A_1 + 2\sqrt{A_2} + \frac{B_1}{3}\right)\Theta, \quad D_2 = \frac{C_2}{2}(A_2+B_2)\Theta, \quad D_3 = C_3\frac{B_3}{6}\Theta,$$

and

$$C_n := \int_{-\infty}^{+\infty} |\phi^{(n-1)}(y)|\,dy, \qquad n \geq 1, \tag{3.14}$$

with

$$\Theta := \mathbb{E}\left[\int_0^T \big|u_t - u^*(t,X_t)\big|dt + \int_0^T \big|\sigma_t^2 - \sigma^*(t,X_t)^2\big|dt + \int_0^T d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)dt\right].$$

Next, we optimize (3.13) in $\alpha > 0$ using Lemma A.2 in Appendix. Using the notation $a \wedge b := \min(a,b)$, the inequality (3.13) and the bound (A.10) give

$$d_W(X_T, X_T^*) \leq D_1 + \left(1+\frac{2}{\sqrt{3}}\right)\sqrt{D_0D_2}\left(\frac{(D_2)^3}{D_0(D_3)^2}\wedge 27\right)^{1/6} + \frac{D_0D_3}{D_2}\left(\frac{(D_2)^3}{D_0(D_3)^2}\wedge 27\right)^{1/3}$$

$$\leq \frac{C_1}{2}\left(A_1 + 2\sqrt{A_2} + \frac{B_1}{3}\right)\Theta \tag{3.15}$$

$$+ \left(1+\frac{2}{\sqrt{3}}\right)\frac{\sqrt[3]{3}}{(\pi/2)^{1/4}}\sqrt{C_2(A_2+B_2)\Theta}\left(\frac{(A_2+B_2)^3(C_2)^3\Theta}{2D_0(C_3)^2(B_3)^2}\wedge 3\right)^{1/6} \tag{3.16}$$

$$+ \frac{2\sqrt[3]{9}C_3B_3}{3C_2(A_2+B_2)\sqrt{\pi/2}}\left(\frac{(A_2+B_2)^3(C_2)^3\Theta}{2D_0(C_3)^2(B_3)^2}\wedge 3\right)^{1/3}. \tag{3.17}$$

When Θ is small, the order of the bound (3.15)–(3.17) is given by the third term (3.17), which yields for some constant $C \in (0, +\infty)$:

$$d_{\rm W}(X_T, X_T^*) \leq \tag{3.18}$$

$$C \left(\mathbb{E}\left[\int_0^T \big|u_t - u^*(t, X_t)\big| dt + \int_0^T \big|\sigma_t^2 - \sigma^*(t, X_t)^2\big| dt + \int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) dt \right] \right)^{1/3}.$$

On the other hand, when Θ is large, the order of the bound (3.15)–(3.17) is given by the first term (3.15), which yields for some constant $C \in (0, +\infty)$:

$$d_{\rm W}(X_T, X_T^*) \leq$$

$$C\, \mathbb{E}\left[\int_0^T \big|u_t - u^*(t, X_t)\big| dt + \int_0^T \big|\sigma_t^2 - \sigma^*(t, X_t)^2\big| dt + \int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) dt \right].$$

□

The bound (3.11) is simpler than e.g. the inequality (4.2) in [Pri15] with $\sigma^* = 1$, however it involves a power 1/2. In the next result we improve the bound (3.11) via a better rate 1/2 on the continuous component, under the additional condition (3.19).

Theorem 3.3 (Wasserstein bound) *Let $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ be the integral and jump-diffusion processes given in (1.2)–(1.4), with $X_0 = X_0^*$. Assume that (A_3) and the domination condition (D) hold true. If for some $K > 0$ we have*

$$\mathbb{E}\left[\int_0^T \Big(\big|u_t - u^*(t, X_t)\big| + \big|\sigma_t^2 - \sigma^*(t, X_t)^2\big| + d_{\rm FM}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) \Big) dt \right] \leq K, \tag{3.19}$$

then there exists some finite constant $C_K > 0$ such that

$$d_W(X_T, X_T^*) \leq C_K \max\left(\mathbb{E}\left[\int_0^T \big|u_t - u^*(t, X_t)\big| dt \right], \left(\mathbb{E}\left[\int_0^T \big|\sigma_t^2 - \sigma^*(t, X_t)^2\big| dt \right] \right)^{1/2}, \left(\mathbb{E}\left[\int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) dt \right] \right)^{1/3} \right), \tag{3.20}$$

where the measures $\widetilde{\nu}_t(dy)$ and $\widetilde{\nu}^*(t, x, dy)$ are defined in (1.6).

Proof. In this proof, we set

$$\theta_u := \mathbb{E}\left[\int_0^T \big|u_t - u^*(t, X_t)\big| dt \right]$$

$$\theta_\sigma := \mathbb{E}\left[\int_0^T \left|\sigma_t^2 - \sigma^*(t,X_t)^2\right| dt\right]$$

$$\theta_\nu := \mathbb{E}\left[\int_0^T d_{\rm FM}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t,X_t,\cdot)\big) dt\right]$$

so that, with the notations of the proof of Proposition 3.2, we have $\Theta = \theta_u + \theta_\sigma + \theta_\nu$, which by (3.19) is assumed to be bounded by some $K > 0$.

First, we refine (3.13) into

$$d_{\rm W}(X_T, X_T^*) \le D_0\sqrt{\alpha} + D_1' + \frac{D_2'}{\sqrt{\alpha}} + \frac{D_3'}{\alpha}, \quad \alpha > 0, \tag{3.21}$$

with

$$D_0 = 2\sqrt{\frac{2}{\pi}}, \qquad D_2' = \frac{C_2}{2}(A_2\theta_\sigma + (A_2+B_2)\theta_\nu),$$

$$D_1' = \frac{C_1}{2}\Big(2\sqrt{A_2}\theta_u + \Big(A_1 + \frac{B_3}{3}\Big)\theta_\sigma + A_1\theta_\nu\Big), \qquad D_3' = C_3\frac{B_3}{6}\theta_\nu,$$

where C_n is defined in (3.14) for any $n \ge 1$. Optimizing (3.21) in $\alpha > 0$ as done previously using Lemma A.2 in Appendix, the inequality (3.21) and the bound (A.10) yield

$$\begin{aligned} & d_W(X_T, X_T^*) \qquad (3.22)\\ \le\ & D_1' + \Big(1+\frac{2}{\sqrt{3}}\Big)\sqrt{D_0 D_2'}\left(\frac{(D_2')^3}{D_0(D_3')^2}\wedge 27\right)^{1/6} + \frac{D_0 D_3'}{D_2'}\left(\frac{(D_2')^3}{D_0(D_3')^2}\wedge 27\right)^{1/3}\\ :=\ & F(\theta_u, \theta_\sigma, \theta_\nu), \end{aligned}$$

where

$$F(\theta_u, \theta_\sigma, \theta_\nu) \tag{3.23}$$

$$= \frac{C_1}{2}\Big(2\sqrt{A_2}\theta_u + \Big(A_1+\frac{B_3}{3}\Big)\theta_\sigma + A_1\theta_\nu\Big) \tag{3.24}$$

$$+ \Big(1+\frac{2}{\sqrt{3}}\Big)\frac{\sqrt[3]{3}}{(\pi/2)^{1/4}}\sqrt{C_2(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)}\left(\frac{(C_2)^3(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)^3}{2(C_3)^2\theta_\nu^2}\wedge 3\right)^{1/6} \tag{3.25}$$

$$+ \frac{2\sqrt[3]{9}}{\sqrt{\pi/2}}\frac{C_3 B_3\theta_\nu}{3C_2(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)}\left(\frac{(C_2)^3(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)^3}{2(C_3)^2\theta_\nu^2}\wedge 3\right)^{1/3}. \tag{3.26}$$

A careful analysis of the order of the terms in (3.24)–(3.26) as θ_u, θ_σ, θ_ν tend to zero shows that for some constant $C \in (0, +\infty)$ and $\gamma_u, \gamma_\sigma, \gamma_\nu > 0$ such that for all

$$(\theta_u, \theta_\sigma, \theta_\nu) \in [0, \gamma_u] \times [0, \gamma_\sigma] \times [0, \gamma_\nu] \tag{3.27}$$

we have:

$$F(\theta_u, \theta_\sigma, \theta_\nu) \leq C \max\left(\theta_u, \theta_\sigma^{1/2}, \theta_\nu^{1/3}\right), \tag{3.28}$$

see Lemma A.3 for details. Hence (3.22) and (3.28) ensures (3.20) under (3.27). When (3.27) does not hold, then $\max(\theta_u, \theta_\sigma^{1/2}, \theta_\nu^{1/3}) \geq \min(\gamma_u, \gamma_\sigma^{1/2}, \gamma_\nu^{1/3})$. But Condition (3.19) and Proposition 3.2 imply $d_W(X_T, X_T^*) \leq C \max(K, K^{1/3})$, so that (3.20) stills holds in this case with $C_K = C \max(K, K^{1/3}) / \min(\gamma_u, \gamma_\sigma^{1/2}, \gamma_\nu^{1/3})$. □

4 Application to jump-diffusion processes

Theorems 3.1 and 3.3 allow us to control the d_{W_3} and d_W-distances between X_T and X_T^* based on the closeness of the diffusion and jump characteristics of $(X_t)_{t\in[0,T]}$ and of $(X_t^*)_{t\in[0,T]}$. In this section, we focus on jump components and illustrate the bounds (3.1) and (3.18) by examining the impact of the jump measures $\widetilde{\nu}_t(dy)$ and $\widetilde{\nu}^*(t, x, dy)$ on the term

$$\mathbb{E}\left[\int_0^T d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big)\, dt\right], \tag{4.1}$$

which involves a combination of the jump intensity $\widehat{\nu}^*(t, dy)$ and jump sizes $g^*(t, x, y)$ of $(X_t^*)_{t\in[0,T]}$ via the measure $\nu^*(t, x, \cdot)$ given in (2.1), see (1.6) for the definitions of $\widetilde{\nu}_t(\cdot)$ and $\widetilde{\nu}^*(t, x, \cdot)$. Namely, we show how $d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big)$ in (3.1) and (3.18) can be bounded in terms of the driving parameters of $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$, which allows us to make the bounds of Theorems 3.1 and 3.3 more explicit.

We consider the case where the stochastic integral process $(X_t)_{t\in[0,T]}$ in (1.2) is solution of a SDE similar to (1.4), i.e. $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ solve SDEs of the form

$$dX_t = \sigma_t\, dB_t + \int_{-\infty}^{+\infty} g_t(X_{t^-}, y)\big(N(dt, dy) - \widehat{\nu}(t, dy)dt\big), \tag{4.2}$$

where $g_t(x, y)$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process, and

$$dX_t^* = \sigma^*(t, X_t^*)\, dB_t + \int_{-\infty}^{+\infty} g^*(t, X_{t^-}^*, y)\big(N^*(dt, dy) - \widehat{\nu}^*(t, dy)dt\big), \tag{4.3}$$

where $N(dt, dy)$ and $N^*(dt, dy)$ are Poisson random measures on $[0, T] \times \mathbb{R}$ with (deterministic) compensators $\widehat{\nu}(t, dy)dt$ and $\widehat{\nu}^*(t, dy)dt$. In this setting, we provide an explicit bound on the distance $d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big)$ in the term (4.1) appearing in

Theorems 3.1 and 3.3. Given μ a measure on $\mathbb{R}$ we denote by $\|\mu\|$ the total variation measure of μ, defined as $\mu(A) = \mu^+(A) - \mu^-(A)$, $A \in \mathcal{B}(\mathbb{R})$, where μ^+ and μ^- are the upper and lower variations in the Hahn-Jordan decomposition of μ.

Proposition 4.1 *Let $(X_t)_{t\in[0,T]}$ and $(X_t^*)_{t\in[0,T]}$ be the integral and jump-diffusion processes given by (4.2) and (4.3). Then, we have*

$$
\begin{aligned}
d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) \leq & \int_{-\infty}^{+\infty} \big|g_t(X_t, y)^2 - g^*(t, X_t, y)^2\big| \widehat{\nu}(t, dy) \\
& + \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 \big|g_t(X_t, y) - g^*(t, X_t, y)\big| \widehat{\nu}^*(t, dy) \\
& + \int_{-\infty}^{\infty} g^*(t, X_t, y)^2 \big\|\widehat{\nu}(t, dy) - \widehat{\nu}^*(t, dy)\big\|,
\end{aligned}
$$

where $\big\|\widehat{\nu}(t, dy) - \widehat{\nu}^*(t, dy)\big\|$ denotes the total variation measure of $\widehat{\nu}(t, dy) - \widehat{\nu}^*(t, dy)$.

Proof. First, we note that $(X_t)_{t\in[0,T]}$ in (4.2) can be written as in (1.2) by taking $\sigma_t := \sigma(t, X_t)$ and the jump measure $\mu(dt, dy)$ with $(\mathcal{F}_t)_{t\in[0,T]}$-compensator

$$
\nu_t := \widehat{\nu}(t, \cdot) \circ g_t^{-1}(X_{t^-}, \cdot). \tag{4.4}
$$

Using $\widetilde{\nu}_t(\cdot)$ and $\widetilde{\nu}^*(t, x, \cdot)$ defined in (1.6) from (4.4) and $\nu^*(t, x, \cdot)$ defined in (2.1), we have

$$
\begin{aligned}
& d_{\mathrm{FM}}\big(\widetilde{\nu}_t(\cdot), \widetilde{\nu}^*(t, X_t, \cdot)\big) = \sup_{h\in\mathcal{FM}} \left| \int_{-\infty}^{+\infty} h(x)\widetilde{\nu}_t(dx) - \int_{-\infty}^{+\infty} h(x)\widetilde{\nu}_t^*(dx) \right| \\
& = \sup_{h\in\mathcal{FM}} \left| \int_{-\infty}^{+\infty} g_t(X_t, y)^2 h(g_t(X_t, y))\widehat{\nu}(t, dy) - \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g^*(t, X_t, y))\widehat{\nu}^*(t, dy) \right|,
\end{aligned}
$$

and, for all $h \in \mathcal{FM}$,

$$
\begin{aligned}
& \Big| \int_{-\infty}^{+\infty} g_t(X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}(t, dy) - \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g^*(t, X_t, y))\, \widehat{\nu}^*(t, dy) \Big| \\
& \leq \Big| \int_{-\infty}^{+\infty} g_t(X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}(t, dy) - \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}(t, dy) \Big| \\
& \quad + \Big| \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}(t, dy) - \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}^*(t, dy) \Big| \\
& \quad + \Big| \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g_t(X_t, y))\, \widehat{\nu}^*(t, dy) - \int_{-\infty}^{+\infty} g^*(t, X_t, y)^2 h(g^*(t, X_t, y))\, \widehat{\nu}^*(t, dy) \Big| \\
& \leq \int_{-\infty}^{+\infty} |g_t(X_t, y)^2 - g^*(t, X_t, y)^2||h(g_t(X_t, y))|\, \widehat{\nu}(t, dy)
\end{aligned}
$$

$$+\Big|\int_{-\infty}^{+\infty} g^*(t,X_t,y)^2 h(g_t(X_t,y))\ \big(\widehat{\nu}(t,dy)-\widehat{\nu}^*(t,dy)\big)\Big|$$
$$+\int_{-\infty}^{+\infty} g^*(t,X_t,y)^2 |h(g_t(X_t,y))-h(g^*(t,X_t,y))|\ \widehat{\nu}^*(t,dy)$$

from which the conclusion derives. □

Examples

Assume that the processes u_t, σ_t, $g_t(X_t,y)$ take the forms

$$u_t = u(t,X_t), \quad \sigma_t = \sigma(t,X_t), \quad g_t(X_t,y) = g(t,X_t,y),$$

where $u(t,x)$, $\sigma(t,x)$, $g(t,x,y)$ are measurable deterministic functions on $[0,T]\times\mathbb{R}$ and $[0,T]\times\mathbb{R}^2$ respectively, such that for some deterministic $c_u(t), c_\sigma(t), c_\nu(t)>0$ we have

$$|u(t,x)-u^*(t,x)| \le c_u(t)|x|, \quad \big|\sigma(t,x)^2-\sigma^*(t,x)^2\big| \le c_\sigma(t)|x|^2,$$

and

$$|g^*(t,x,y)|^2 \le c_\nu^*(t)|x|^2, \quad \big|g(t,x,y)^p - g^*(t,x,y)^p\big| \le c_\nu(t)|x|^p,$$

$(t,x,y)\in[0,T]\times\mathbb{R}^2$, $p=1,2$. Then, by Proposition 4.1 we have

$$\begin{aligned} &d_{\rm FM}\big(\widetilde{\nu}_t(\cdot),\widetilde{\nu}^*(t,X_t,\cdot)\big)\\ &\quad\le\ c_\nu(t)\widehat{\nu}(t,\mathbb{R})|X_t|^2 + c_\nu^*(t)c_\nu(t)|X_t|^3\widehat{\nu}^*(t,\mathbb{R}) + c_\nu^*(t)|X_t|^2\big\|\widehat{\nu}(t,\mathbb{R})-\widehat{\nu}^*(t,\mathbb{R})\big\|, \end{aligned}$$

and Theorem 3.3 yields the bound

$$\begin{aligned} d_W(X_T,X_T^*) &\le C\int_0^T \mathbb{E}[|X_t|]c_u(t)dt + C\left(\int_0^T \mathbb{E}[|X_t|^2]c_\sigma(t)dt\right)^{1/2}\\ &+C\left(\int_0^T \big(c_\nu(t)\widehat{\nu}(t,\mathbb{R})|X_t|^2 + c_\nu^*(t)c_\nu(t)|X_t|^3\widehat{\nu}^*(t,\mathbb{R}) + c_\nu^*(t)|X_t|^2\big\|\widehat{\nu}(t,\mathbb{R})-\widehat{\nu}^*(t,\mathbb{R})\big\|\big)dt\right)^{1/3} \end{aligned} \tag{4.5}$$

for some constant $C\in(0,+\infty)$. We note that explicit bounds on the moments of the solution X_t are available in the literature, see for example Theorem 3.1 in [BP20] and its proof.

For example, if $\widehat{\nu}(t,dy) := \mathbf{1}_{(0,\infty)}(y)e^{-\alpha(t)y}dy/y$ and $\widehat{\nu}^*(t,dy) := \mathbf{1}_{(0,\infty)}(y)e^{-\beta(t)y}dy/y$ are gamma Lévy measures with time-dependent parameters $\alpha(t), \beta(t) > 0$, then by Frullani's identity the total variation term in (4.5) reads

$$\left\|\widehat{\nu}(t,\mathbb{R}) - \widehat{\nu}^*(t,\mathbb{R})\right\| = \int_0^\infty \left|e^{-\alpha(t)y} - e^{-\beta(t)y}\right| \frac{dy}{y} = \left|\log \frac{\beta(t)}{\alpha(t)}\right|, \quad t \in [0,T].$$

In the particular case of Poisson processes with deterministic compensators

$$\widehat{\nu}(t,dy) = a(t)\delta_1(dy) \quad \text{and} \quad \widehat{\nu}^*(t,dy) = a^*(t)\delta_1(dy),$$

we find

$$d_W(X_T, X_T^*) \le C\int_0^T \mathbb{E}[|X_t|]c_u(t)dt + C\left(\int_0^T \mathbb{E}[|X_t|^2]c_\sigma(t)dt\right)^{1/2}$$
$$+ C\left(\int_0^T \left(c_\nu(t)a(t)\,\mathbb{E}[|X_t|^2] + c_\nu^*(t)c_\nu(t)a^*(t)\,\mathbb{E}[|X_t|^3] + c_\nu^*(t)\,\mathbb{E}[|X_t|^2]|a(t) - a^*(t)|\right)dt\right)^{1/3}.$$

More specifically, in the case of geometric jump-diffusion processes solving SDEs of the form

$$dX_t = u(t)X_tdt + \sigma(t)X_tdB_t + \eta(t)X_{t^-}(N(dt) - a(t)dt)$$

and

$$dX_t^* = u^*(t)X_t^*dt + \sigma^*(t)X_t^*\ dB_t + \eta^*(t)X_{t^-}^*(N^*(dt) - a^*(t)dt),$$

taking $g_t(x) := \eta(t)x$ and $g^*(t,x) := \eta^*(t)x$, we obtain

$$d_W(X_T, X_T^*) \le C\int_0^T |u(t) - u^*(t)|\,\mathbb{E}[|X_t|]dt + C\left(\int_0^T |\sigma(t) - \sigma^*(t)|^2\,\mathbb{E}[|X_t|^2]dt\right)^{1/2}$$
$$+ C\left(\int_0^T \Big(\big(a(t)\big|\eta(t)^2 - \eta^*(t)^2\big|\,\mathbb{E}[|X_t|^2] + a^*(t)\eta^*(t)^2\big|\eta(t) - \eta^*(t)\big|\,\mathbb{E}[|X_t^3|]$$
$$+ \eta^*(t)^2|a(t) - a^*(t)|\big)\,\mathbb{E}[X_t^2]\Big)dt\right)^{1/3}. \tag{4.6}$$

A Appendix

Lemma A.1 (Approximation) *Let* $\alpha > 0$, $h \in \mathrm{Lip}(1)$, *and consider the function* h_α *defined in* (3.12)*. Then we have*

$$\|h_\alpha - h\|_\infty \le \sqrt{\frac{2\alpha}{\pi}}. \tag{A.1}$$

Moreover, we have $h_\alpha \in \mathcal{C}_b^\infty(\mathbb{R})$, and

$$\|h_\alpha^{(n)}\|_\infty \leq \alpha^{-(n-1)/2} \int_{-\infty}^{+\infty} |\phi^{(n-1)}(y)|\, dy, \qquad n \geq 1. \tag{A.2}$$

Proof. The bound (A.1) follows from the Lipschitz property of h:

$$\begin{aligned}
\big\|h_\alpha - h\big\|_\infty &= \sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{+\infty} \big(h\big(\sqrt{\alpha}y + x\big) - h(x)\big)\phi(y)\, dy \right| \\
&\leq \int_{-\infty}^{+\infty} \sqrt{\alpha}|y|\phi(y)\, dy = \sqrt{\frac{2\alpha}{\pi}}.
\end{aligned}$$

Next, since the function h is differentiable almost everywhere with $\|h'\|_\infty = \|h\|_L \leq 1$, the function $\phi(y)h'\big(x + y\sqrt{\alpha}\big)$ is dominated by the integrable function $\phi(y)$. Thus, we have

$$\begin{aligned}
h'_\alpha(x) &= \int_{-\infty}^{+\infty} h'\big(x + y\sqrt{\alpha}\big)\phi(y)\, dy \qquad (A.3)\\
&= -\frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} h\big(x + y\sqrt{\alpha}\big)\phi'(y)\, dy, \qquad (A.4)
\end{aligned}$$

with $\|h'_\alpha\|_\infty \leq 1$ from (A.3) (note that since the Lipschitz function h is sub-linear, the bracket in the integration by part (A.4) is indeed zero). By induction from (A.4) and a similar domination argument, we get

$$\begin{aligned}
h_\alpha^{(n)}(x) &= (-1)^{n-1}\alpha^{-(n-1)/2} \int_{-\infty}^{+\infty} h'\big(x + y\sqrt{\alpha}\big)\phi^{(n-1)}(y)\, dy \\
&= (-1)^n \alpha^{-n/2} \int_{-\infty}^{+\infty} h\big(x + y\sqrt{\alpha}\big)\phi^{(n)}(y)\, dy,
\end{aligned}$$

from which we derive (A.2). □

The following lemma is based on Cardan's formula.

Lemma A.2 (Cardan type estimate) *Let*

$$G(\alpha) := D_0\sqrt{\alpha} + D_1 + \frac{D_2}{\sqrt{\alpha}} + \frac{D_3}{\alpha}, \qquad \alpha > 0, \tag{A.5}$$

where $D_0, D_1, D_2, D_3 > 0$ are positive constants.

a) Assume that $(D_2)^3 \leq 27D_0(D_3)^2$. The function $G(\alpha)$ reaches its minimum at

$$\alpha_* := \left(\frac{D_3}{D_0}\right)^{2/3} \left(\left(1 - \sqrt{1 - \frac{(D_2)^3}{27D_0(D_3)^2}}\right)^{1/3} + \left(1 + \sqrt{1 - \frac{(D_2)^3}{27D_0(D_3)^2}}\right)^{1/3} \right)^2 \tag{A.6}$$

and this minimum is upper bounded by

$$D_1 + 2D_2\sqrt[3]{\frac{D_0}{D_3}} + 3\sqrt[3]{D_0^2 D_3}. \tag{A.7}$$

b) Assume that $(D_2)^3 > 27D_0(D_3)^2$*. The function* $G(\alpha)$ *reaches its minimum in*

$$\alpha_* = \frac{4D_2}{3D_0}\cos^2\left(\frac{1}{3}\arccos\left(\sqrt{\frac{27(D_3)^2D_0}{(D_2)^3}}\right)\right), \tag{A.8}$$

and this minimum is upper bounded by

$$D_1 + \left(1 + \frac{2}{\sqrt{3}}\right)\sqrt{D_0D_2} + \frac{D_0D_3}{D_2}. \tag{A.9}$$

c) In general, the minimum of $G(\alpha)$ *is upper bounded by*

$$D_1 + \left(1 + \frac{2}{\sqrt{3}}\right)\sqrt{D_0D_2}\left(\frac{(D_2)^3}{D_0(D_3)^2} \wedge 27\right)^{1/6} + \frac{D_0D_3}{D_2}\left(\frac{(D_2)^3}{D_0(D_3)^2} \wedge 27\right)^{1/3}. \tag{A.10}$$

Proof. We set $\beta := \sqrt{\alpha}$ and study the variations of the function $\beta \mapsto D_0\beta + D_2/\beta + D_3/\beta^2$ by considering the sign of $D_0\beta^3 - D_2\beta - 2D_3$ in its derivative $D_0 - D_2/\beta^2 - 2D_3/\beta^3 = (D_0\beta^3 - D_2\beta - 2D_3)/\beta^3$. For this, as seen below, it suffices to discuss the position of $(D_2)^3/(27D_0(D_3)^2)$ with respect to 1.

a) When $(D_2)^3 \leq 27D_0(D_3)^2$, the derivative admits a unique zero β^* given from Cardan's formula for cubic equations, see, e.g., [EMS12], by

$$\beta_* = \sqrt[3]{\frac{D_3}{D_0}}\left(\left(1 - \sqrt{1 - \frac{(D_2)^3}{27D_0(D_3)^2}}\right)^{1/3} + \left(1 + \sqrt{1 - \frac{(D_2)^3}{27D_0(D_3)^2}}\right)^{1/3}\right),$$

which yields (A.6). Since the quantity inside the above bracket lies within the interval $[1, (1 + \sqrt[3]{2})]$, we have

$$\sqrt[3]{\frac{D_3}{D_0}} \leq \beta_* \leq (1 + \sqrt[3]{2})\sqrt[3]{\frac{D_3}{D_0}},$$

and the bound for the minimum in (A.7) follows easily.

b) When $(D_2)^3 > 27D_0(D_3)^2$, the derivative admits three distinct zeros given from Cardan's formula by

$$\beta_k = 2\sqrt{\frac{D_2}{3D_0}}\cos\left(\frac{1}{3}\arccos\left(\sqrt{\frac{27(D_3)^2D_0}{(D_2)^3}}\right)+\frac{2k\pi}{3}\right), \qquad k=0,1,2.$$

Since $D_0\beta^3 - D_2\beta - 2D_3$ is negative when $\beta = 0$, either all three zeros are positive, or only one of them is positive. Setting $\varphi := 3^{-1}\arccos\left(\sqrt{27(D_3)^2D_0/(D_2)^3}\right) \in [0,\pi/6]$, we note that

$$\begin{aligned}
\frac{\beta_0}{2}\sqrt{\frac{3D_0}{D_2}} &= \cos(\varphi),\\
\frac{\beta_1}{2}\sqrt{\frac{3D_0}{D_2}} &= \cos\left(\varphi + 2\frac{\pi}{3}\right) = -\frac{1}{2}\cos(\varphi) - \frac{\sqrt{3}}{2}\sin(\varphi) < 0,\\
\frac{\beta_2}{2}\sqrt{\frac{3D_0}{D_2}} &= \cos\left(\varphi - 2\frac{\pi}{3}\right) = -\frac{1}{2}\cos(\varphi) + \frac{\sqrt{3}}{2}\sin(\varphi),
\end{aligned}$$

and $-\cos(\varphi)+\sqrt{3}\sin(\varphi) \le 2\cos(\varphi)$, since this is equivalent to $\tan(\varphi) \le 1/\sqrt{3}$ and $\varphi \in [0,\pi/6]$. As a consequence, the minimum of $D_0\beta + D_2/\beta + D_3/\beta^2$ is reached at $\beta = \beta_0$, which yields (A.8). Next, we note that (A.8) implies

$$\frac{D_2}{D_0} \le \alpha_* \le \frac{4D_2}{3D_0},$$

hence by (A.5) we have

$$G(\alpha_*) \le D_1 + \left(1+\frac{2}{\sqrt{3}}\right)\sqrt{D_0D_2} + \frac{D_0D_3}{D_2}, \qquad \alpha > 0,$$

from which (A.9) follows.

c) The last point stems from the comparisons of both the second terms in (A.7) and in (A.9) and of their third terms (observe that when $(D_2)^3 > 27D_0(D_3)^2$, we are losing a factor $\sqrt{3}$ for the second term.

□

In the sequel we use the notation $\theta \ll \theta'$, resp. $\Delta \sim \Delta'$, to denote $\theta/\theta' \to 0$, resp. $\Delta/\Delta' \to 1$, as θ' tends to zero and we use the notations of the proof of Theorem 3.3.

Lemma A.3 *The function $F(\theta_u, \theta_\sigma, \theta_\nu)$ in (3.23) is of order* $\max(\theta_u, \theta_\sigma^{1/2}, \theta_\nu^{1/3})$ *as θ_u, θ_σ and θ_ν tend to zero.*

Proof. Rewriting (3.24)–(3.26) as

$$\begin{aligned} d_W(X_T, X_T^*) \leq\ & \frac{C_1}{2}\Big(2\sqrt{A_2}\theta_u + \Big(A_1 + \frac{B_3}{3}\Big)\theta_\sigma + A_1\theta_\nu\Big) \\ & + \left(1 + \frac{2}{\sqrt{3}}\right)\frac{\sqrt[3]{3}\sqrt{C_2}}{(\pi/2)^{1/4}}\sqrt{A_2\theta_\sigma + (A_2+B_2)\theta_\nu}\left(\frac{C_2^3\Delta}{2C_3^2} \wedge 3\right)^{1/6} \\ & + \frac{2^{3/2}}{3^{1/3}\sqrt{\pi}}\frac{C_3 B_3 \theta_\nu}{C_2(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)}\left(\frac{C_2^3\Delta}{2C_3^2} \wedge 3\right)^{1/3} \end{aligned}$$

with

$$\Delta := \frac{(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)^3}{\theta_\nu^2}.$$

We note that as $\theta_\sigma, \theta_\nu \to 0$, the order of the bound depends on the order of the quantity

$$\begin{aligned} \Delta &:= \frac{(A_2\theta_\sigma + (A_2+B_2)\theta_\nu)^3}{(\theta_\nu)^2} \\ &= (A_2)^3\frac{\theta_\sigma^3}{(\theta_\nu)^2} + 3(A_2)^2(A_2+B_2)\frac{\theta_\sigma^2}{\theta_\nu} + 3A_2(A_2+B_2)^2\theta_\sigma + (A_2+B_2)^3\theta_\nu. \end{aligned}$$

When $\theta_\sigma \ll \theta_\nu$ we have

$$\frac{\theta_\sigma^3}{\theta_\nu^2} \ll \frac{\theta_\sigma^2}{\theta_\nu} \ll \theta_\nu,$$

hence $\Delta \sim (A_2+B_2)^3\theta_\nu$ as θ_ν tends to zero, whereas when $\theta_\nu \ll \theta_\sigma$, we find

$$\theta_\nu \ll \frac{\theta_\sigma^2}{\theta_\nu} \ll \frac{\theta_\sigma^3}{\theta_\nu^2} \quad \text{and} \quad \theta_\sigma \ll \frac{\theta_\sigma^3}{\theta_\nu^2},$$

hence $\Delta \sim A_2^3\theta_\sigma^3/\theta_\nu^2$ as θ_ν tends to zero. Thus, we can consider the following cases:

- if $\theta_\sigma \ll \theta_\nu$ or $\theta_\nu \ll \theta_\sigma \ll \theta_\nu^{2/3}$ then $\Delta \to 0$ and the terms between parentheses in (3.25)–(3.26) are of order $\Delta \sim \max\big((A_2+B_2)^3\theta_\nu, A_2^3\theta_\sigma^3/\theta_\nu^2\big)$;
- if $\theta_\nu^{2/3} \ll \theta_\sigma$ then $\Delta \to +\infty$ and the terms between parentheses in (3.25)–(3.26) are equal to 3.

Namely, we have the following:

a) If $\theta_\nu^{2/3} \ll \theta_\sigma$, then

   - (3.24) is equivalent to $C_1\big(2\sqrt{A_2}\theta_u + (A_1 + B_3/3)\theta_\sigma\big)/2$ with $\theta_\sigma \ll \theta_\sigma^{1/2}$,

- (3.25) is equivalent to $\dfrac{(2+\sqrt{3})\sqrt{C_2A_2\theta_\sigma}}{(\pi/2)^{1/4}}$,
- (3.26) is of order $\theta_\nu/\theta_\sigma \ll \theta_\sigma^{1/2}$,

so that $F(\theta_u,\theta_\sigma,\theta_\nu) \sim \max\left(C_1\sqrt{A_2}\theta_u, \dfrac{(2+\sqrt{3})\sqrt{C_2A_2\theta_\sigma}}{(\pi/2)^{1/4}}\right)$ when $\theta_u,\theta_\sigma,\theta_\nu \to 0$.

b) If $\theta_\sigma \ll \theta_\nu$, then $\Delta \sim (A_2+B_2)^3\theta_\nu \to 0$ and

- (3.24) is equivalent to $C_1\big(2\sqrt{A_2}\theta_u + A_1\theta_\nu\big)/2$ with $\theta_\nu \ll \theta_\nu^{1/3}$,
- (3.25) is of order $\theta_\nu^{2/3}$,
- (3.26) is equivalent to $\dfrac{6^{2/3}B_3}{\sqrt{\pi/2}}(C_3\theta_\nu)^{1/3}$,

so that $F(\theta_u,\theta_\sigma,\theta_\nu) \sim \max\left(C_1\sqrt{A_2}\theta_u, \dfrac{6^{2/3}(C_3\theta_\nu)^{1/3}B_3}{\sqrt{\pi/2}}\right)$ when $\theta_u,\theta_\sigma,\theta_\nu \to 0$.

c) If $\theta_\nu \ll \theta_\sigma \ll \theta_\nu^{2/3}$, then $\Delta \sim A_2^3\theta_\sigma^3/\theta_\nu^2 \to 0$,

- (3.24) is equivalent to $C_1(2\sqrt{A_2}\theta_u + (A_1+B_3/3)\theta_\sigma)/2$ with $\theta_\sigma \ll \theta_\nu^{1/3}$,
- (3.25) is of order $\theta_\sigma^{1/2} \times \left(\dfrac{\theta_\sigma^3}{\theta_\nu^2}\right)^{1/6} = \dfrac{\theta_\sigma}{\theta_\nu^{1/3}} \ll \theta_\nu^{1/3}$,
- (3.26) is equivalent to $\dfrac{2^{3/2}C_3B_3}{3^{1/3}\sqrt{\pi}C_2A_2}\dfrac{\theta_\nu}{\theta_\sigma} \times \left(\dfrac{C_2^3A_2^3}{2C_3^2}\dfrac{\theta_\sigma^3}{\theta_\nu^2}\right)^{1/3} = \dfrac{(4C_3\theta_\nu)^{1/3}B_3}{3^{1/3}\sqrt{\pi/2}}$,

so that $F(\theta_u,\theta_\sigma,\theta_\nu) \sim \max\left(C_1\sqrt{A_2}\theta_u, \dfrac{(4C_3\theta_\nu)^{1/3}B_3}{3^{1/3}\sqrt{\pi/2}}\right)$ when $\theta_u,\theta_\sigma,\theta_\nu \to 0$.

In conclusion, $F(\theta_u,\theta_\sigma,\theta_\nu)$ is of order $\max(\theta_u, \theta_\sigma^{1/2}, \theta_\nu^{1/3})$ when $\theta_u,\theta_\sigma,\theta_\nu \to 0$. □

Acknowledgement

We thank two anonymous referees for useful suggestions. The first author conducted this work within the France 2030 framework program, the Centre Henri Lebesgue ANR-11-LABX-0020-01. The second author is supported by the Ministry of Education, Singapore, under its Tier 2 Grant MOE-T2EP20120-0005.

References

[ABP08] M. Arnaudon, J.-C. Breton, and N. Privault. Convex ordering for random vectors using predictable representation. *Potential Anal.*, 29(4):327–349, 2008.

[AH19] B. Arras and C. Houdré. *On Stein's Method for Infinitely Divisible Laws with Finite First Moment*. SpringerBriefs in Probability and Mathematical Statistics. Springer, New York, 2019.

[BGJ87] K. Bichteler, J.-B. Gravereaux, and J. Jacod. *Malliavin Calculus for Processes with Jumps*, volume 2 of *Stochastics Monographs*. Gordon and Breach, 1987.

[BJ00] N. Bellamy and M. Jeanblanc. Incompleteness of markets driven by a mixed diffusion. *Finance and Stochastics*, 4(2):209–222, 2000.

[BLP13] J.-C. Breton, B. Laquerrière, and N. Privault. Convex comparison inequalities for non-Markovian stochastic integrals. *Stochastics*, 85(5):789–806, 2013.

[BP08] J.-C. Breton and N. Privault. Bounds on option prices in point process diffusion models. *Int. J. Theor. Appl. Finance*, 11(6):597–610, 2008.

[BP20] J.-C. Breton and N. Privault. Integrability and regularity of the flow of stochastic differential equations with jumps. *Theory Probab. Appl.*, 65(1):82–101, 2020.

[BP22] J.-C. Breton and N. Privault. Wasserstein distance estimates for stochastic integrals by forward-backward stochastic calculus. *Potential Analysis*, 56(1):1–20, 2022.

[BR06] J. Bergenthum and L. Rüschendorf. Comparison of option prices in semimartingale models. *Finance and Stochastics*, 10(2):229–249, 2006.

[BR07] J. Bergenthum and L. Rüschendorf. Comparison of semimartingales and Lévy processes. *Ann. Probab.*, 35(1), 2007.

[Dec15] L. Decreusefond. The Stein-Dirichlet-Malliavin method. *ESAIM Probab. Statist.*, 51:49–59, 2015.

[Dud02] R.M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[EJS98] N. El Karoui, M. Jeanblanc, and S. Shreve. Robustness of the Black and Scholes formula. *Math. Finance*, 8(2):93–126, 1998.

[EMS12] EMS. Cardano formula, Encyclopedia of Mathematics, European Mathematical Society. https://encyclopediaofmath.org/index.php?title=Cardano_formula, 2012. Accessed: 2022-10-25.

[ET07] E. Ekström and J. Tysk. Properties of option prices in models with jumps. *Math. Finance*, 17(3):381–397, 2007.

[GS72] I.I. Gihman and A.V. Skorokhod. *Stochastic Differential Equations*. Springer, 1972.

[Gyö86] I. Gyöngy. Mimicking the one-dimensional marginal distributions of processes having an Itô differential. *Probab. Theory Relat. Fields*, 71(4):501–516, 1986.

[HY14] F. Hirsch and M. Yor. Comparing Brownian stochastic integrals for the convex order. In *Modern stochastics and applications*, volume 90 of *Springer Optim. Appl.*, pages 3–19. Springer, Cham, 2014.

[JM76] J. Jacod and J. Mémin. Caractéristiques locales et conditions de continuité absolue pour les semi-martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 35(1):1–37, 1976.

[KMP06] Th. Klein, Y. Ma, and N. Privault. Convex concentration inequalities via forward-backward stochastic calculus. *Electron. J. Probab.*, 11:27 pp., 2006.

[Kun04] H. Kunita. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and stochastic analysis*, Trends Math., pages 305–373. Birkhäuser Boston, Boston, MA, 2004.

[NP09] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.

[Øks03] B. Øksendal. *Stochastic differential equations.* Universitext. Springer-Verlag, Berlin, sixth edition, 2003.

[Pag16] G. Pagès. Convex order for path-dependent derivatives: a dynamic programming approach. In *Séminaire de Probabilités XLVIII*, volume 2168 of *Lecture Notes in Math.*, pages 33–96. Springer, Cham, 2016.

[Pri15] N. Privault. Stein approximation for Itô and Skorohod integrals by Edgeworth type expansions. *Electron. Comm. Probab.*, 20:Article 35, 2015.

[PT97] P. Protter and D. Talay. The Euler scheme for Lévy driven stochastic differential equations. *Ann. Probab.*, 25(1):393–423, 1997.

[TT90] D. Talay and L. Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Anal. Appl.*, 8(4):483–509, 1990.