Non-Gaussian Malliavin calculus on real Lie algebras

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Abstract

The non-commutative Malliavin calculus on the Heisenberg-Weyl algebra is extended to the affine algebra. A differential calculus and a non-commutative integration by parts are established. As an application we obtain sufficient conditions for the smoothness of Wigner type laws of non-commutative random variables with gamma or continuous binomial marginals.

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1 Introduction

Wigner densities [13] have various applications in time-frequency analysis, quantum optics and other fields, see e.g. [4] and the references given in [2]. In [7] a non-commutative Malliavin calculus has been introduced on the Heisenberg-Weyl algebra $\{\mathbf{p}, \mathbf{q}, I\}$, with $[\mathbf{p}, \mathbf{q}] = 2iI$, generalizing the Gaussian Malliavin calculus to Wigner densities, and allowing to prove the smoothness of Wigner laws with Gaussian marginals. In this paper we aim to treat other probability laws in a more general framework, in particular we will consider non-commutative couples of random variables with gamma and continuous binomial marginals. It is well known that gamma and continuous binomial non-commutative random variables can be constructed using representations of sl_2, or simply on the affine algebra viewed as a sub-algebra of sl_2. We will develop a functional calculus on the affine algebra, based on the general framework of [2], [3].

Before proceeding further, let us examine a situation where the gamma and continuous binomial laws appear naturally in a non-commutative framework related to integration by parts with respect to the gamma law. Let

$$\tilde{a}_\tau^- = \tau \partial_\tau ,$$

i.e. $\tilde{a}_\tau^- f(\tau) = \tau f'(\tau)$, $f \in \mathcal{C}_b^\infty(\mathbb{R})$. The adjoint $\tilde{a}_\tau^+$ of $\tilde{a}_\tau^-$ with respect to the gamma density $\gamma_\beta(\tau) = \mathbf{1}_{\{\tau \geq 0\}} \frac{\tau^{\beta-1}}{\Gamma(\beta)} e^{-\tau}$ on $\mathbb{R}$, $\beta > 0$, satisfies

$$\int_0^\infty g(\tau) \tilde{a}_\tau^- f(\tau) \gamma_\beta(\tau) d\tau = \int_0^\infty f(\tau) \tilde{a}_\tau^+ g(\tau) \gamma_\beta(\tau) d\tau, \quad f, g \in \mathcal{C}_b^\infty(\mathbb{R}), \tag{1.1}$$

and is given by

$$\tilde{a}_\tau^+ = (\tau - \beta) - \tilde{a}_\tau^- ,$$

i.e. $\tilde{a}_\tau^+ f(\tau) = (\tau - \beta) f(\tau) - \tau \partial f(\tau) = (\tau - \beta) f(\tau) - \tilde{a}_\tau^- f(\tau)$. The operator $\tilde{a}_\tau^\circ$ defined as

$$\tilde{a}_\tau^\circ = \tilde{a}_\tau^+ \partial_\tau = -(\beta - \tau)\partial - \tau \partial^2$$

has the Laguerre polynomials L_n^β with parameter β as eigenfunctions:

$$\tilde{a}_\tau^\circ L_n^\beta(\tau) = n L_n^\beta(\tau), \quad n \in \mathbb{N}.$$

The multiplication operator $\tilde{a}_\tau^- + \tilde{a}_\tau^+ = \tau - \beta$ has a compensated gamma law in the vacuum state $\mathbf{1}_{\mathbb{R}_+}$ in $L^2_{\mathbb{C}}(\mathbb{R}_+, \gamma_\beta(\tau) d\tau)$. In the Heisenberg-Weyl case, $\mathbf{q} = a^- + a^+$ and its conjugate $\mathbf{p} = i(a^- - a^+)$ both have Gaussian laws and can be constructed from the Boson annihilation and creation operators a^-, a^+. In [11], [12] it has been noticed that when $\beta = 1$, $i(\tilde{a}_\tau^- - \tilde{a}_\tau^+)$ has a continuous binomial law (or spectral measure) in the vacuum state, with hyperbolic cosine density $(2 \cosh \pi\xi/2)^{-1}$, in relation to a representation of the subgroup of sl_2 made of upper-triangular matrices. This construction extends to half-integer values of β, nevertheless this type of law can in fact be studied for every value of $\beta > 0$ in the more general framework of [1], starting from a representation $\{M, B^-, B^+\}$ of sl_2:

$$[B^-, B^+] = M, \quad [M, B^-] = -2B^-, \quad [M, B^+] = 2B^+,$$

which can be constructed as

$$M = \beta + 2\tilde{a}_\tau^\circ, \quad B^- = \tilde{a}_\tau^- - \tilde{a}_\tau^\circ, \quad B^+ = \tilde{a}_\tau^+ - \tilde{a}_\tau^\circ.$$

Letting

$$Q = B^- + B^+ = \tilde{a}_\tau^- + \tilde{a}_\tau^+ - 2\tilde{a}_\tau^\circ = (\tau - \beta) + 2(\beta - \tau)\partial + 2\tau\partial^2$$

and

$$P = i(B^- - B^+) = i(\tilde{a}_\tau^- - \tilde{a}_\tau^+) = 2i\tau\partial - i(\tau - \beta),$$

we have

$$[P, Q] = 2iM, \qquad [P, M] = 2iQ, \qquad [Q, M] = -2iP.$$

Now, $Q + M$ is a multiplication operator:

$$Q + M = \tau,$$

hence $Q + M$ has the gamma law with parameter β in the vacuum state $\Omega = \mathbf{1}_{\mathbb{R}_+}$ in $L^2_{\mathbb{C}}(\mathbb{R}_+, \gamma_\beta(\tau)d\tau)$. The law (or spectral measure) of $\alpha M + Q$ has been determined in [1], depending on the value of $\alpha \in \mathbb{R}$. When $\alpha = \pm 1$, $M + Q$ and $M - Q$ have gamma laws. For $|\alpha| < 1$, $Q + \alpha M$ has an absolutely continuous law and in particular for $\alpha = 0$, Q and P have continuous binomial laws. When $|\alpha| > 1$, $Q + \alpha M$ has a geometric distribution.

The Malliavin calculus on the Heisenberg-Weyl algebra $\{\mathbf{p}, \mathbf{q}, I\}$ of [6], [7] relies mainly on a functional calculus which allows to define the composition of a function with a couple of non-commutative random variables, and on a covariance identity which plays the role of integration by parts formula. In particular, a continuous map O from $L^p(\mathbb{R}^2)$, $p \geq 2$, into the space of bounded operators on $\mathcal{H}$ is defined via

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x, y) e^{ix\mathbf{p} + iy\mathbf{q}} dxdy,$$

where $\mathcal{F}$ denotes the Fourier transform, with the bound

$$\|O(f)\| \leq C_p \|f\|_{L^p(\mathbb{R}^2)},$$

and the relation

$$O(e^{iux+ivy}) = e^{iu\mathbf{p}+iv\mathbf{q}}, \qquad u, v \in \mathbb{R}.$$

In order to extend this construction to other probability laws we adopt the formalism of [2] which provides a functional calculus on more general Lie algebras. In particular, note that

$$X_1 = -\frac{i}{2}P \quad \text{and} \quad X_2 = i(Q + M),$$

form a representation of the affine algebra:

$$[X_1, X_2] = X_2.$$

Let $\mathcal{B}_2(\mathcal{H})$ denote the space of Hilbert-Schmidt operators on $\mathcal{H}$. Using results of [2] we show that a continuous map $O : L^2_{\mathbb{C}}(\mathbb{R}^2, d\xi_1 d\xi_2/|\xi_2|) \longrightarrow \mathcal{B}_2(\mathcal{H})$ can be defined as

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2) e^{-\frac{i}{2}x_1 P + ivx_2(Q+M)} dx_1 dx_2,$$

with the bound

$$\|O(f)\|_{\mathcal{B}_2(\mathcal{H})} \leq \|f\|_{L^2_{\mathbb{C}}(\mathbb{R}^2, \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})},$$

and the property

$$O(e^{-iu\xi_1 - iv\xi_2}) = e^{-\frac{i}{2}uP + iv(Q+M)}.$$

This allows to define a Wigner density $\tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2)$ which is the joint density of $(-\frac{1}{2}P, Q + M)$, with continuous binomial and gamma laws as marginals, such that

$$\langle\psi|e^{i\frac{u}{2}P - iv(Q+M)}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} e^{iu\xi_1 + iv\xi_2} \tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) d\xi_1 d\xi_2, \qquad \phi, \psi \in \mathcal{H}.$$

Using a non-commutative integration by parts formula, we are able to prove the smoothness of the joint density of $(P, Q + M)$.

We proceed as follows. In Sect. 2 we recall the main results of [2] on functional calculus on general Lie algebras, and give proofs not explicitly given in [2] of some particular results needed in our approach. In Sect. 3 we study in detail the particular case of the affine algebra and obtain a smoothness property for the joint density of $(P, Q + M)$. In Sect. 4 we state a non-commutative integration by parts formula on the affine algebra, which generalizes the classical integration by parts with respect to the gamma density. Finally in Sect. 5 we conclude with some remarks on the relation of our construction to the commutative case.

2 Functional calculus on Lie algebras

In this section we recall the main tools of functional calculus on general Lie algebras [2], and include some results and proofs not explicitly stated in [2]. Let G be a Lie group with Lie algebra $\mathcal{G}$ and let U be a unitary representation of G on some Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle_{\mathcal{H}}$. Let $\langle\cdot,\cdot\rangle_{\mathcal{G}^*,\mathcal{G}}$ denote the pairing between the Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$. We assume that U is irreducible, and square integrable i.e. there exists a non zero vector $\psi \in \mathcal{H}$ such that

$$\int_G |\langle U(g)\psi|\psi\rangle_{\mathcal{H}}|^2 d\mu(g) < \infty,$$

where μ denotes the left Haar measure on G. From [5] there exists a positive self-adjoint operator C on $\mathcal{H}$ such that

$$\int_G \overline{\langle U(g)\psi_1,\phi_1\rangle}_{\mathcal{H}}\langle U(g)\psi_2|\phi_2\rangle_{\mathcal{H}} d\mu(g) = \langle C\psi_2|C\psi_1\rangle_{\mathcal{H}}\langle\phi_1|\phi_2\rangle_{\mathcal{H}}. \qquad (2.1)$$

Moreover C is the identity if and only if G is unimodular, and Dom C^{-1} is dense in $\mathcal{H}$. We assume the existence of an open subset N_0 of $\mathcal{G}$, symmetric around the origin, whose image $\exp(N_0)$ by $\exp : \mathcal{G} \to G$ is dense in G with $\mu(G \setminus \exp(N_0)) = 0$. The image measure of μ on N_0 by $\exp^{-1} : \exp(N_0) \to N_0$ is called the Haar measure on $\mathcal{G}$, and we denote by $m(x)$ its density with respect to the Lebesgue measure dx on $\mathcal{G}$. Let $\sigma(\xi)$ denote the density in the decomposition of the Lebesgue measure $d\xi$ on $\mathcal{G}^*$:

$$d\xi = dk(\lambda)\sigma(\xi)d\Omega_\lambda(\xi),$$

where $dk(\lambda)$ is a measure on the parameter space of the co-adjoint orbits in $\mathcal{G}^*$ and $d\Omega_\lambda(\xi)$ is the invariant measure on the orbit $\mathcal{O}^*_\lambda$. Let $\mathcal{B}_2(\mathcal{H})$ denote the space of Hilbert-Schmidt operators equipped with the scalar product

$$\langle\rho_1|\rho_2\rangle_{\mathcal{B}} = \mathrm{Tr}\,[\rho_1^*\rho_2], \quad \rho_1,\rho_2 \in \mathcal{B}_2(H).$$

Let $(X_1,\ldots,X_n)$, resp. $(X_1^*,\ldots,X_n^*)$, denote a basis of $\mathcal{G}$, resp. $\mathcal{G}^*$.

Definition 2.1 *([2]) Given $(\phi,\psi) \in \mathcal{H} \times$ Dom C^{-1} the Wigner function $W_{|\phi\rangle\langle\psi|}$ is defined on $\mathcal{G}^*$ as:*

$$W_{|\phi\rangle\langle\psi|}(\xi) = \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}}\int_{N_0} e^{-i\langle\xi,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\langle U(e^{x_1X_1+\cdots+x_nX_n})C^{-1}\psi|\phi\rangle_{\mathcal{H}}\sqrt{m(x)}dx.$$

The following proposition extends the definition of W_ρ in $L^2_{\mathbb{C}}(\mathcal{G}^*;\frac{d\xi}{\sigma(\xi)})$ to $\rho \in \mathcal{B}_2(\mathcal{H})$.

Proposition 2.2 *([2]) The mapping*

$$\begin{array}{ll}\mathcal{H} \times \mathrm{Dom}\ C^{-1} & \longrightarrow L^2_{\mathbb{C}}(\mathcal{G}^*;\frac{d\xi}{\sigma(\xi)}) \\ \rho & \longmapsto W_\rho\end{array}$$

extends to an isometry on $\mathcal{B}_2(\mathcal{H})$:

$$\langle W_{\rho_1}|W_{\rho_2}\rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*;\frac{d\xi}{\sigma(\xi)})} = \langle \rho_1|\rho_2\rangle_{\mathcal{B}_2(\mathcal{H})}, \qquad \rho_1,\rho_2 \in \mathcal{B}_2(\mathcal{H}).$$

Proof. By a density argument it suffices to consider

$$\rho_1 = |\phi_1\rangle\langle\psi_1| \quad \text{and} \quad \rho_2 = |\phi_2\rangle\langle\psi_2|,$$

with $(\phi_1,\psi_1),(\phi_2,\psi_2) \in \mathcal{H} \times \operatorname{Dom} C^{-1}$. From the identity (2.1) and since

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle \xi, x-x'\rangle_{\mathcal{G}^*,\mathcal{G}}} d\xi dx' = \delta_x(dx'), \tag{2.2}$$

we have:

$$\begin{aligned}
&\langle W_{\rho_1}|W_{\rho_2}\rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*;\frac{d\xi}{\sigma(\xi)})} \\
&= \frac{1}{(2\pi)^n}\int_{\mathcal{G}^*}\Bigg(\overline{\int_{N_0} e^{-i\langle \xi,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\operatorname{Tr}\left[U(e^{-(x_1X_1+\cdots+x_nX_n)})\rho_1 C^{-1}\right]\sqrt{m(x)}dx} \\
&\quad \times \int_{N_0} e^{-i\langle \xi,x'\rangle_{\mathcal{G}^*,\mathcal{G}}}\operatorname{Tr}\left[U(e^{-(x'_1X_1+\cdots+x'_nX_n)})\rho_2 C^{-1}\right]\sqrt{m(x')}dx'\Bigg) d\xi \\
&= \int_{N_0} \overline{\operatorname{Tr}\left[U(e^{-(x_1X_1+\cdots+x_nX_n)})\rho_1 C^{-1}\right]}\operatorname{Tr}\left[U(e^{-(x_1X_1+\cdots+x_nX_n)})\rho_2 C^{-1}\right] m(x)dx \\
&= \int_{N_0} \overline{\langle U(e^{x_1X_1+\cdots+x_nX_n})C^{-1}\psi_1|\phi_1\rangle_{\mathcal{H}}}\langle U(e^{x_1X_1+\cdots+x_nX_n})C^{-1}\psi_2|\phi_2\rangle_{\mathcal{H}} m(x)dx \\
&= \int_G \overline{\langle U(g)C^{-1}\psi_1|\phi_1\rangle_{\mathcal{H}}}\langle U(g)C^{-1}\psi_2|\phi_2\rangle_{\mathcal{H}} d\mu(g) \\
&= \langle \psi_2|\psi_1\rangle_{\mathcal{H}}\langle\phi_1|\phi_2\rangle_{\mathcal{H}} \\
&= \langle \rho_2|\rho_1\rangle_{\mathcal{B}_2(\mathcal{H})},
\end{aligned}$$

where we used the relation

$$\begin{aligned}
\operatorname{Tr}(U(g)^*\rho C^{-1}) &= \operatorname{Tr}(U(g)^*|\phi\rangle\langle\psi|C^{-1}) = \operatorname{Tr}(C^{-1}U(g)^*|\phi\rangle\langle\psi|) \\
&= \langle \psi, C^{-1}U(g)^*\phi\rangle_{\mathcal{H}} = \langle U(g)C^{-1}\psi,\phi\rangle_{\mathcal{H}}.
\end{aligned}$$

□

As a result, the definition of $W_\rho(\xi)$ extends to $\rho \in \mathcal{B}_2(\mathcal{H})$ as:

$$W_\rho(\xi) = \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}}\int_{N_0} e^{-i\langle\xi,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\operatorname{Tr}\left[U(e^{-(x_1X_1+\cdots+x_nX_n)})\rho C^{-1}\right]\sqrt{m(x)}dx.$$

Definition 2.3 *Let $O : L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)}) \to \mathcal{B}_2(\mathcal{H})$ denote the dual of $\rho \mapsto W_\rho$, i.e.*

$$\langle \rho | O(f) \rangle_{\mathcal{B}_2(\mathcal{H})} = \int_{\mathcal{G}^*} \overline{W}_\rho(\xi) f(\xi) \frac{d\xi}{\sigma(\xi)}, \quad f \in L^2_{\mathbb{C}}\left(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)}\right), \quad \rho \in \mathcal{B}_2(\mathcal{H}).$$

Note that for $\rho = |\phi\rangle\langle\psi|$,

$$\begin{aligned} \langle \rho | O(f) \rangle_{\mathcal{B}_2(\mathcal{H})} &= \operatorname{Tr} |\phi\rangle\langle\psi|^* O(f) \\ &= \langle \phi | O(f) \psi \rangle_{\mathcal{H}} \\ &= \langle W_{|\phi\rangle\langle\psi|} | f \rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \\ &= \int_{\mathcal{G}^*} \overline{W}_{|\phi\rangle\langle\psi|}(\xi) f(\xi) \frac{d\xi}{\sigma(\xi)}. \end{aligned}$$

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined as

$$(\mathcal{F} f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle_{\mathcal{G}^*, \mathcal{G}}} f(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

and

$$(\mathcal{F}^{-1} f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle_{\mathcal{G}^*, \mathcal{G}}} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

The next proposition allows to extend O as a bounded operator from $L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})$ to $\mathcal{B}_2(\mathcal{H})$.

Proposition 2.4 *We have the bound*

$$\|O(f)\|_{\mathcal{B}_2(\mathcal{H})} \leq \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})}, \qquad f \in L^2_{\mathbb{C}}\left(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)}\right),$$

and the expression

$$O(f) = \int_{N_0} \sqrt{m(x)} \mathcal{F}\left(\frac{f}{\sqrt{\sigma}}\right)(x) U(e^{x_1 X_1 + \cdots + x_n X_n}) C^{-1} dx.$$

Proof. We have

$$\begin{aligned} |\langle O(f) | \rho \rangle_{\mathcal{B}_2(\mathcal{H})}| &= |\langle f | W_\rho \rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})}| \\ &\leq \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \|W_\rho\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \\ &\leq \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \|\rho\|_{\mathcal{B}_2(\mathcal{H})}, \end{aligned}$$

and

$$\langle \phi | O(f) \psi \rangle_{\mathcal{H}} = \operatorname{Tr} |\phi\rangle\langle\psi|^* O(f) = \int_{\mathcal{G}^*} \overline{W}_{|\phi\rangle\langle\psi|}(\xi) f(\xi) \frac{d\xi}{\sigma(\xi)}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{G}^*} \int_{N_0} e^{i\langle \xi, x\rangle_{\mathcal{G}^*,\mathcal{G}}} \overline{\mathrm{Tr}\, U(e^{-(x_1X_1+\cdots+x_nX_n)})|\phi\rangle\langle\psi|C^{-1}} \sqrt{\frac{m(x)}{\sigma(\xi)}} dx f(\xi) d\xi$$
$$= \int_{N_0} \mathcal{F}\left(\frac{f}{\sqrt{\sigma}}\right)(x)\langle\phi|U(e^{x_1X_1+\cdots+x_nX_n})C^{-1}\psi\rangle_{\mathcal{H}}\sqrt{m(x)}dx$$
$$= \left\langle \phi \middle| \int_{N_0} \mathcal{F}\left(\frac{f}{\sqrt{\sigma}}\right)(x) U(e^{x_1X_1+\cdots+x_nX_n})C^{-1}\sqrt{m(x)}dx\psi\right\rangle_{\mathcal{H}}.$$

□

In other terms we have

$$O(e^{-i\langle\cdot,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\sqrt{\sigma(\cdot)}) = (2\pi)^{n/2}\sqrt{m(x)}U(e^{x_1X_1+\cdots+x_nX_n})C^{-1} \tag{2.3}$$

and

$$O(f\sqrt{\sigma}) = \frac{1}{(2\pi)^{n/2}}\int_{N_0}(\mathcal{F}f)(x)O(e^{-i\langle\cdot,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\sqrt{\sigma})dx, \quad f \in L^2_{\mathbb{C}}(\mathcal{G}^*; d\xi).$$

Let $\mathrm{Ad}^\sharp_g\xi$, $\xi \in \mathcal{G}^*$, denote the co-adjoint action:

$$\langle \mathrm{Ad}^\sharp_g\xi, x\rangle_{\mathcal{G}^*,\mathcal{G}} = \langle \xi, \mathrm{Ad}_{g^{-1}}x\rangle_{\mathcal{G}^*,\mathcal{G}}, \qquad x \in \mathcal{G}.$$

Let $\widetilde{\mathrm{Ad}}_g$, $g \in G$, be defined for $f : \mathcal{G}^* \to \mathbb{C}$ as

$$\widetilde{\mathrm{Ad}}_g f = f \circ \mathrm{Ad}^\sharp_{g^{-1}},$$

and let $\widetilde{\mathrm{ad}}_x$ be the differential of $g \mapsto \widetilde{\mathrm{Ad}}_g$. The following proposition, called covariance property, will provide an analog of integration by parts formula.

Proposition 2.5 *We have for $x = (x_1, \ldots, x_n) \in \mathcal{G}$:*

$$[x_1U(X_1) + \cdots + x_nU(X_n), O(f)] = O(\widetilde{\mathrm{ad}}(x)f).$$

Proof. Using the relation

$$U(g)^*C^{-1}U(g) = \frac{C^{-1}}{\sqrt{\Delta(g^{-1})}}$$

and (34), (44), (56) in [2] we have

$$W_{U(g)\rho U(g)^*}(\xi)$$
$$= \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}}\int_{N_0}\sqrt{m(x)}e^{-i\langle\xi,x\rangle_{\mathcal{G}^*,\mathcal{G}}}\mathrm{Tr}\,[U(e^{-(x_1X_1+\cdots+x_nX_n)})U(g)\rho U(g)^*C^{-1}]dx$$

$$
\begin{aligned}
&= \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}} \int_{N_0} e^{-i\langle \xi, x\rangle_{\mathcal{G}^*,\mathcal{G}}} \mathrm{Tr}\, U(g^{-1}) U(e^{-(x_1X_1+\cdots+x_nX_n)}) U(g)\rho C^{-1} \sqrt{\frac{m(x)}{\Delta(g^{-1})}} dx \\
&= \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}} \int_{N_0} e^{-i\langle \xi, x\rangle_{\mathcal{G}^*,\mathcal{G}}} \mathrm{Tr}\, e^{-\mathrm{Ad}_{g^{-1}}x} \rho C^{-1} \sqrt{m(x)\Delta(g)} dx \\
&= \frac{\sqrt{\sigma(\xi)}}{(2\pi)^{n/2}} \int_{N_0} e^{-i\langle \xi, \mathrm{Ad}_g x\rangle_{\mathcal{G}^*,\mathcal{G}}} \mathrm{Tr}\, U(e^{-(x_1X_1+\cdots+x_nX_n)}) \rho C^{-1} \det(\mathrm{Ad}_g) \sqrt{m(\mathrm{Ad}_g x)\Delta(g)} dx \\
&= \frac{\sqrt{\sigma(\mathrm{Ad}^\sharp_{g^{-1}}\xi)}}{(2\pi)^{n/2}} \int_{N_0} e^{-i\langle \mathrm{Ad}^\sharp_{g^{-1}}\xi, x\rangle_{\mathcal{G}^*,\mathcal{G}}} \mathrm{Tr}\, U(e^{-(x_1X_1+\cdots+x_nX_n)}) \rho C^{-1} \sqrt{m(x)} dx \\
&= W_\rho(\mathrm{Ad}^\sharp_{g^{-1}}\xi).
\end{aligned}
$$

We proved the covariance property

$$
W_{U(g)\rho U(g)^*}(\xi) = W_\rho(\mathrm{Ad}^\sharp_{g^{-1}}\xi).
$$

By duality we have

$$
\begin{aligned}
\langle U(g)O(f)U(g)^* | \rho\rangle_{\mathcal{B}_2(\mathcal{H})} &= \mathrm{Tr}\,[(U(g)O(f)U(g)^*)^*\rho] \\
&= \mathrm{Tr}\,[U(g)O(f)^*U(g)^*\rho] \\
&= \mathrm{Tr}\,[O(f)^*U(g)^*\rho U(g)] \\
&= \langle O(f) | U(g)^*\rho U(g)\rangle_{\mathcal{B}_2(\mathcal{H})} \\
&= \langle f | W_{U(g)^*\rho U(g)}\rangle_{\mathcal{B}_2(\mathcal{H})} \\
&= \langle f | W_\rho \circ \mathrm{Ad}^\sharp_g\rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \\
&= \langle f \circ \mathrm{Ad}^\sharp_{g^{-1}} | W_\rho\rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi}{\sigma(\xi)})} \\
&= \langle O(f \circ \mathrm{Ad}^\sharp_{g^{-1}}) | \rho\rangle_{\mathcal{B}_2(\mathcal{H})},
\end{aligned}
$$

which implies

$$
U(g)O(f)U(g)^* = O(\widetilde{\mathrm{Ad}}_g f),
$$

The conclusion follows by differentiation. □

In [7] a quantum Malliavin calculus has been constructed on the Heisenberg-Weyl algebra $\{\mathbf{p}, \mathbf{q}, I\}$ with $[\mathbf{p}, \mathbf{q}] = 2iI$, generalizing to Wigner densities the Malliavin calculus with respect to a single Gaussian random variable. In this case the representation U is given on $\mathcal{H} = L^2(\mathbb{R}, dx)$ by

$$
U(x, y)\phi(t) = e^{2iyt+ixy}\phi(t + x), \quad \phi \in \mathcal{H}.
$$

Equivalently we can take $\mathbf{p}\phi(t) = \frac{2}{i}\phi'(t)$ and $\mathbf{q}\phi(t) = t\phi(t)$, $\phi \in \mathcal{S}(\mathbb{R})$. The group is unimodular, hence C is the identity, and $\sigma = m = 1$. We have

$$\begin{aligned} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^2} e^{-iy\xi_2 - ix\xi_1} \langle e^{-iy\mathbf{q}+ix\mathbf{p}}\psi|\phi\rangle_{\mathcal{H}} dx dy \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} e^{-it\xi_1} \overline{\psi}(\xi_2 - t)\phi(\xi_2 + t) dt. \end{aligned}$$

The marginals are given by

$$\int_{\mathbb{R}} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) d\xi_2 = \phi(\xi_1)\bar{\psi}(\xi_1), \qquad \xi_1 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) d\xi_1 = (\mathcal{F}\phi)(\xi_2)(\overline{\mathcal{F}\psi})(\xi_2) \qquad \xi_2 \in \mathbb{R}.$$

The operator $O(f)$ is defined by

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x, y) e^{ix\mathbf{p}+iy\mathbf{q}} dx dy,$$

with

$$O(e^{-iux-ivy}) = e^{iu\mathbf{p}+iv\mathbf{q}}, \quad u, v \in \mathbb{R},$$

and the bound

$$\|O(f)\|_{\mathcal{B}_2(\mathcal{H})} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}.$$

Hence

$$\langle \psi, e^{iu\mathbf{p}+iv\mathbf{q}}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} e^{iu\xi_1+iv\xi_2} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) d\xi_1 d\xi_2, \qquad u, v \in \mathbb{R},$$

i.e. $W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2)$ represents the Wigner density of $(\mathbf{p}, \mathbf{q})$ in the state $|\phi\rangle\langle\psi|$. In this case, the statement of Prop. 2.5 reads

$$\frac{i}{2}[u\mathbf{q} - v\mathbf{p}, O(f)] = O\left(u\partial_1 f + v\partial_2 f\right).$$

3 Malliavin calculus on the affine algebra

The affine algebra is generated by

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with $[X_1, X_2] = X_2$, and the affine group can be constructed as the group of 2×2 matrices of the form

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{x_1} & x_2 e^{\frac{x_1}{2}} \operatorname{sinch} \frac{x_1}{2} \\ 0 & 1 \end{pmatrix} = e^{x_1 X_1 + x_2 X_2}, \quad a > 0, \ b \in \mathbb{R}, \tag{3.1}$$

where

$$\operatorname{sinch} x = \frac{\sinh x}{x}, \quad x \in \mathbb{R}.$$

Consider the classical representation of the affine group on $L^2(\mathbb{R})$ given by

$$(U(g)\phi)(t) = a^{-1/2} \phi\left(\frac{t-b}{a}\right), \quad \phi \in L^2(\mathbb{R}), \quad g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \ a > 0, \ b \in \mathbb{R},$$

and the modified representation on $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}, \gamma_\beta(|\tau|)d\tau)$ defined by

$$(\hat{U}(g)\phi)(\tau) = \phi(a\tau) e^{ib\tau} e^{-(a-1)|\tau|/2} a^{\beta/2}, \quad \phi \in L^2_{\mathbb{C}}(\mathbb{R}, \gamma_\beta(|\tau|)d\tau), \quad g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

obtained by Fourier transformation and a change of measure. We have

$$\hat{U}(X_1)\phi(\tau) = \frac{d}{dt}\Big|_{t=0} \hat{U}(e^{itX_1})\phi(\tau) = \frac{1}{2}(\beta - |\tau|)\phi(\tau) + \tau\phi'(\tau) = -\frac{i}{2} P\phi(\tau),$$

$$\hat{U}(X_2)\phi(\tau) = \frac{d}{dt}\Big|_{t=0} \hat{U}(e^{itX_2})\phi(\tau) = i\tau\phi(\tau) = i(Q+M)\phi(\tau), \qquad \tau \in \mathbb{R},$$

i.e.

$$\hat{U}(X_1) = -\frac{i}{2} P \quad \text{and} \quad \hat{U}(X_2) = i(Q+M),$$

hence

$$\hat{U}(e^{x_1 X_1 + x_2 X_2}) = e^{-\frac{i}{2} x_1 P + i x_2 (Q+M)}.$$

Here $N_0 = \mathcal{G}$ is identified to $\mathbb{R}^2$ and

$$m(x_1, x_2) = e^{-\frac{x_1}{2}} \operatorname{sinch} \frac{x_1}{2}, \qquad x_1, x_2 \in \mathbb{R},$$

moreover from (92) in [2],

$$d\Omega_\pm(\xi_1, \xi_2) = \frac{1}{2\pi|\xi_2|} d\xi_1 d\xi_2,$$

hence

$$\sigma(\xi_1, \xi_2) = 2\pi|\xi_2|, \qquad \xi_1, \xi_2 \in \mathbb{R}, \tag{3.2}$$

and the operator C is given by

$$Cf(\tau) = \sqrt{\frac{2\pi}{|\tau|}} f(\tau), \qquad \tau \in \mathbb{R}.$$

Writing $\xi = \xi_1 X_1^* + \xi_2 X_2^* \in \mathcal{G}^*$, we have

$$W_\rho(\xi) = \frac{|\xi_2|^{1/2}}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-i\xi_1 x_1 - i\xi_2 x_2} \mathrm{Tr}\,[e^{-x_1X_1 - x_2X_2}\rho C^{-1}] \sqrt{e^{-\frac{x_1}{2}} \mathrm{sinch}\, \frac{x_1}{2}}\, dx_1 dx_2,$$

and for $\rho = |\phi\rangle\langle\psi|$,

$$\begin{aligned} W_{|\phi\rangle\langle\psi|}(\xi) &= \frac{|\xi_2|^{1/2}}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-i\xi_1 x_1 - i\xi_2 x_2} \langle \hat{U}(e^{x_1X_1 + x_2X_2}) C^{-1}\psi|\phi\rangle_{\mathcal{H}} \sqrt{e^{-\frac{x_1}{2}} \mathrm{sinch}\, \frac{x_1}{2}}\, dx_1 dx_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3} e^{-i\xi_1 x_1 - i\xi_2 x_2} \phi(e^{-x_1}\tau)\overline{\psi}(\tau) e^{-i\tau x_2 e^{-\frac{x_1}{2}} \mathrm{sinch}\, \frac{x_1}{2}} \\ &\quad \times e^{-(e^{-x_1}-1)|\tau|} e^{-\beta x_1/2} |\tau|^{\beta - 1/2} \sqrt{e^{-\frac{x_1}{2}} \mathrm{sinch}\, \frac{x_1}{2}}\, \frac{d\tau}{\Gamma(\beta)} dx_1 dx_2 \\ &= \int_{\mathbb{R}} \phi\left(\frac{\xi_2 e^{-\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) \frac{|\xi_2| e^{-ix\xi_1}}{\mathrm{sinch}\, \frac{x}{2}} \overline{\psi}\left(\frac{\xi_2 e^{\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) e^{-|\xi_2| \frac{\cosh \frac{x}{2}}{\mathrm{sinch}\, \frac{x}{2}}} \left(\frac{|\xi_2|}{\mathrm{sinch}\, \frac{x}{2}} \right)^{\beta - 1} \frac{dx}{\Gamma(\beta)}, \end{aligned}$$

as in (102) of [2]. Note that W_ρ takes real values when ρ is self-adjoint. As a consequence of Prop. 2.4 we have the bound

$$\|O(f)\|_{\mathcal{B}_2(\mathcal{H})} \le \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})}.$$

From (2.3) and (3.2) we have

$$e^{-\frac{i}{2}uP + iv(Q+M)} = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{u}{2}} \mathrm{sinch}\, \frac{u}{2} \right)^{-1/2} O(e^{-iu\xi_1 - iv\xi_2} \sqrt{|\xi_2|}) C,$$

i.e. from Relation (2.3):

$$O(e^{-i\langle \cdot, x\rangle_{\mathcal{G}^*,\mathcal{G}}} \sqrt{\sigma(\cdot)}) = (2\pi)^{n/2} \sqrt{m(x)} U(e^{x_1X_1 + \cdots + x_nX_n}) C^{-1}$$

The next proposition shows that these relations can be simplified, and that the Wigner function is directly related to the density of the couple $(P, Q+M)$.

Proposition 3.1 *We have*

$$O(e^{iu\xi_1 + iv\xi_2}) = e^{\frac{i}{2}uP - iv(Q+M)}, \qquad u, v \in \mathbb{R}. \tag{3.3}$$

Proof. We have for all $\phi, \psi \in \mathcal{H}$:

$$\begin{aligned}
\langle \phi | e^{-\frac{i}{2}uP+iv(Q+M)} \psi \rangle_{\mathcal{H}} &= \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{u}{2}} \mathrm{sinch}\, \frac{u}{2} \right)^{-1/2} \langle \phi, O(e^{-iu\xi_1 - iv\xi_2} \sqrt{|\xi_2|}) C\psi \rangle_{\mathcal{H}} \\
&= \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{u}{2}} \mathrm{sinch}\, \frac{u}{2} \right)^{-1/2} \langle W_{|\phi\rangle\langle C\psi|}(\xi_1, \xi_2) | e^{-iu\xi_1 - iv\xi_2} \sqrt{|\xi_2|} \rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} e^{-iu\xi_1 - iv\xi_2} \overline{\phi} \left(\frac{e^{-\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) \frac{e^{ix\xi_1}}{\mathrm{sinch}\, \frac{x}{2}} \sqrt{\frac{e^{-\frac{x}{2}} \mathrm{sinch}\, \frac{x}{2}}{e^{-\frac{u}{2}} \mathrm{sinch}\, \frac{u}{2}}} \\
&\quad \times \psi \left(\frac{\xi_2 e^{-\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) e^{-|\xi_2| \frac{\cosh \frac{x}{2}}{\mathrm{sinch}\, \frac{x}{2}}} \left(\frac{|\xi_2|}{\mathrm{sinch}\, \frac{x}{2}} \right)^{\beta-1} \frac{dx}{\Gamma(\beta)} d\xi_1 d\xi_2 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^3} e^{-iu\xi_1 - iv\xi_2} \overline{\phi} \left(\frac{\xi_2 e^{\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) \frac{e^{ix\xi_1}}{\mathrm{sinch}\, \frac{x}{2}} \psi \left(\frac{\xi_2 e^{\frac{x}{2}}}{\mathrm{sinch}\, \frac{x}{2}} \right) \\
&\quad \times e^{-|\xi_2| \frac{\cosh \frac{x}{2}}{\mathrm{sinch}\, \frac{x}{2}}} \left(\frac{|\xi_2|}{\mathrm{sinch}\, \frac{x}{2}} \right)^{\beta-1} \frac{dx}{\Gamma(\beta)} d\xi_1 d\xi_2 \\
&= \langle W_{|\phi\rangle\langle\psi|} | e^{-iu\xi_1 - iv\xi_2} \rangle_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})} \\
&= \langle \phi | O(e^{-iu\xi_1 - iv\xi_2}) \psi \rangle_{\mathcal{H}}.
\end{aligned}$$

□

As a consequence of (3.3), the operator $O(f)$ has the natural expression

$$\begin{aligned}
O(f) &= O\left(\int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2) e^{-ix_1\xi_1 - ix_2\xi_2} dx_1 dx_2 \right) \\
&= \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2) O(e^{-ix_1\xi_1 - ix_2\xi_2}) dx_1 dx_2 \\
&= \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2) e^{-\frac{i}{2}x_1 P + ix_2(Q+M)} dx_1 dx_2.
\end{aligned}$$

We also have the relations

$$\begin{aligned}
\langle \psi | O(f) \phi \rangle_{\mathcal{H}} &= \int_{\mathcal{G}^*} \overline{W}_{|\psi\rangle\langle\phi|}(\xi_1, \xi_2) f(\xi_1, \xi_2) \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|} \\
&= \int_{\mathcal{G}^*} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) f(\xi_1, \xi_2) \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|},
\end{aligned}$$

and

$$\langle \psi | e^{\frac{i}{2}uP - iv(Q+M)} \phi \rangle_{\mathcal{H}} = \int_{\mathcal{G}^*} e^{iu\xi_1 + iv\xi_2} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|},$$

which show that the density $\tilde{W}_{|\phi\rangle\langle\psi|}$ of $(\frac{1}{2}P, -(Q+M))$ in the state $|\phi\rangle\langle\psi|$ has the expression

$$\tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) = \frac{1}{2\pi|\xi_2|} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2)$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \phi\left(\frac{\xi_2 e^{-\frac{x}{2}}}{\mathrm{sinch}\,\frac{x}{2}}\right)\frac{e^{-ix\xi_1}}{\mathrm{sinch}\,\frac{x}{2}}\overline{\psi}\left(\frac{\xi_2 e^{\frac{x}{2}}}{\mathrm{sinch}\,\frac{x}{2}}\right)e^{-|\xi_2|\frac{\cosh\frac{x}{2}}{\mathrm{sinch}\,\frac{x}{2}}}\left(\frac{|\xi_2|}{\mathrm{sinch}\,\frac{x}{2}}\right)^{\beta-1}\frac{dx}{\Gamma(\beta)}. \quad (3.4)$$

Note that $\tilde{W}_{|\phi\rangle\langle\psi|}$ has the correct marginals since integrating in $d\xi_1$ in (3.4) we have using (2.2)

$$\frac{1}{2\pi|\xi_2|}\int_{\mathbb{R}} W_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)d\xi_1 = \gamma_\beta(|\xi_2|)\overline{\phi}(\xi_2)\psi(\xi_2),$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} W_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)\frac{d\xi_2}{|\xi_2|} = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\xi_1 x}\overline{\phi}(\omega e^{x/2})\psi(\omega e^{-x/2})e^{-|\omega|\cosh\frac{x}{2}}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}dxd\omega.$$

In the vacuum state, i.e. for $\phi = \psi = \Omega = \mathbf{1}_{\mathbb{R}_+}$, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}} W_{|\Omega\rangle\langle\Omega|}(\xi_1,\xi_2)\frac{d\xi_2}{\xi_2} &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty e^{-i\xi_1 x}\frac{\tau^{\beta-1}}{\Gamma(\beta)}e^{-\tau\cosh\frac{x}{2}}d\tau dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi_1 x}\frac{1}{(\cosh\frac{x}{2})^\beta}dx \\
&= c\left|\Gamma\left(\frac{\beta}{2}+\frac{i}{2}\xi_1\right)\right|^2,
\end{aligned}$$

where c is a normalization constant and Γ is the Gamma function. When $\beta = 1$ we have $c = 1/\pi$ and P has the hyperbolic cosine density in the vacuum state $\Omega = \mathbf{1}_{\mathbb{R}_+}$:

$$\xi_1 \mapsto \frac{1}{2\cosh\pi\xi_1/2}.$$

Proposition 3.2 *The characteristic function of $(P, Q+M)$ in the state $|\phi\rangle\langle\psi|$ is given by*

$$\langle\psi|e^{iuP+iv(Q+M)}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{iv\omega\mathrm{sinch}\,u}\overline{\psi}(\omega e^u)\phi(\omega e^{-u})e^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega.$$

In the vacuum state $\Omega = \mathbf{1}_{\mathbb{R}_+}$ we have

$$\langle\Omega|e^{iuP+iv(Q+M)}\Omega\rangle_{\mathcal{H}} = \frac{1}{(\cosh u - iv\mathrm{sinch}\,u)^\beta}, \qquad u, v \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned}
&\langle\psi|e^{-\frac{i}{2}uP+iv(Q+M)}\phi\rangle_{\mathcal{H}} = \left\langle\psi, \hat{U}\left(e^u, ve^{\frac{u}{2}}\mathrm{sinch}\,\frac{u}{2}\right)\phi\right\rangle_{\mathcal{H}} \\
&= \int_{\mathbb{R}}\overline{\psi}(\tau)\phi\left(\tau e^u\right)e^{iv\tau e^{\frac{u}{2}}\mathrm{sinch}\,\frac{u}{2}}e^{-(e^u-1)|\tau|/2}e^{\beta\frac{u}{2}}\frac{|\tau|^{\beta-1}}{\Gamma(\beta)}e^{-|\tau|}d\tau \\
&= \int_{\mathbb{R}} e^{iv\omega\mathrm{sinch}\,\frac{u}{2}}\overline{\psi}\left(\omega e^{-\frac{u}{2}}\right)\phi\left(\omega e^{\frac{u}{2}}\right)e^{-|\omega|\cosh\frac{u}{2}}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega.
\end{aligned}$$

In the vacuum state $|\Omega\rangle\langle\Omega|$ we have

$$\langle\Omega, e^{-\frac{i}{2}uP+iv(Q+M)}\Omega\rangle_{\mathcal{H}} = \int_0^\infty e^{i\omega \mathrm{sinch}\,\frac{u}{2}-|\omega|\cosh\frac{u}{2}} \frac{|\omega|^{\beta-1}}{\Gamma(\beta)} d\omega = \frac{1}{(\cosh\frac{u}{2} - iv\mathrm{sinch}\,\frac{u}{2})^\beta}.$$

□

In particular we have

$$\langle\psi|e^{iv(Q+M)}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{iv\omega}\overline{\psi}(\omega)\phi(\omega)e^{-|\omega|}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega$$

hence as expected, $Q + M$ has density $\overline{\psi}(\omega)\phi(\omega)\gamma_\beta(|\omega|)$, in particular a Gamma law in the vacuum state. On the other hand we have

$$\langle\psi|e^{iuP}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \overline{\psi}(\omega e^u)\phi(\omega e^{-u})e^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega,$$

which recovers the density of P:

$$\xi_1 \mapsto \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\xi_1 x}\overline{\psi}(\omega e^x)\phi(\omega e^{-x})e^{-|\omega|\cosh x}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}dx d\omega.$$

In the vacuum state we have

$$\langle\Omega|e^{iuP}\Omega\rangle_{\mathcal{H}} = \frac{1}{(\cosh u)^\beta}.$$

Next we define a gradient operator which will be useful in showing the smoothness of Wigner densities. Let $\mathcal{S}_{\mathcal{H}}$ denote the algebra of operators on $\mathcal{H}$ that leave the Schwartz space $\mathcal{S}(\mathbb{R})$ invariant.

Definition 3.3 *Fix $\kappa \in \mathbb{R}$. The gradient operator D is defined as*

$$D_x F = -\frac{i}{2}x_1[P, F] + \frac{i}{2}x_2[Q + \kappa M, F], \quad F \in \mathcal{S}_{\mathcal{H}},$$

with $x = (x_1, x_2) \in \mathbb{R}^2$.

Proposition 3.4 *Let $x = (x_1, x_2) \in \mathbb{R}^2$. The operator D_x is closable for the weak topology on the space $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$.*

Proof. Let $\phi, \psi \in \mathcal{S}(\mathbb{R})$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of operators in $\mathcal{S}_{\mathcal{H}} \cap \mathcal{B}(\mathcal{H})$ such that $D_x B_n \to B \in \mathcal{B}(\mathcal{H})$ in the weak topology. We have

$$\langle\psi|B\phi\rangle_{\mathcal{H}} = \lim_{n\to\infty}\langle\psi|D_x B_n\phi\rangle_{\mathcal{H}}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \langle\psi| - \frac{i}{2}x_1(PB_n\phi - B_nP\phi) + \frac{i}{2}x_2((Q+\kappa M)B_n\phi - B_n(Q+\kappa M)\phi)\rangle_{\mathcal{H}} \\
&= \lim_{n\to\infty} -\frac{i}{2}x_1(\langle P\psi|PB_n\phi\rangle_{\mathcal{H}} - \langle\psi|B_nP\phi\rangle_{\mathcal{H}}) \\
&\quad + \lim_{n\to\infty} -\frac{i}{2}x_2(\langle(Q+\kappa M)\psi|B_n\phi\rangle_{\mathcal{H}} - \langle\psi|B_n(Q+\kappa M)\phi\rangle_{\mathcal{H}}) = 0,
\end{aligned}
$$

hence $B = 0$. □

The following is the analog of the integration by parts (1.1).

Proposition 3.5 *Let $x = (x_1, x_2) \in \mathbb{R}^2$. We have*

$$
[x_1U(X_1) + x_2U(X_2), O(f)] = O(x_1\xi_2\partial_1 f(\xi_1,\xi_2) - x_2\xi_2\partial_2 f(\xi_1,\xi_2)).
$$

Proof. This is a consequence of the covariance property since from (3.1), the co-adjoint action is represented by the matrix

$$
\begin{pmatrix} 1 & ba^{-1} \\ 0 & a^{-1} \end{pmatrix},
$$

i.e.

$$
\widetilde{\mathrm{Ad}}_g f(\xi_1,\xi_2) = f \circ \mathrm{Ad}^\sharp_{g^{-1}}(\xi_1,\xi_2) = f(\xi_1 + ba^{-1}\xi_2, a^{-1}\xi_2).
$$

Hence

$$
\widetilde{\mathrm{ad}}_x f(\xi_1,\xi_2) = x_1\xi_2\partial_1 f(\xi_1,\xi_2) - x_2\xi_2\partial_2 f(\xi_1,\xi_2).
$$

□

For $\kappa = 1$, the integration by parts formula can also be written as

$$
D_{(x_1,2x_2)}O(f) = O(x_1\xi_2\partial_1 f - x_2\xi_2\partial_2 f).
$$

The Wigner density $\tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2) = \frac{1}{2\pi|\xi_2|}\overline{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)$ exists and we turn to proving its smoothness, more precisely we consider the smoothness of the Wigner function $W_{|\phi\rangle\langle\psi|}$. Let $H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))$ denote the Sobolev space with respect to the norm

$$
\begin{aligned}
&\|f\|^2_{H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))} \qquad (3.5)\\
&= \int_0^\infty \frac{1}{\xi_2}\int_{\mathbb{R}} |f(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2 + \int_0^\infty \xi_2 \int_{\mathbb{R}} (|\partial_1 f(\xi_1,\xi_2)|^2 + |\partial_2 f(\xi_1,\xi_2)|^2) d\xi_1 d\xi_2.
\end{aligned}
$$

Note that if ϕ, ψ have supports in $\mathbb{R}_+$, then $W_{|\phi\rangle\langle\psi|}$ has support in $\mathbb{R}\times(0,\infty)$, and the conclusion of Th. 3.6 below reads $W_{|\phi\rangle\langle\psi|} \in H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))$.

Theorem 3.6 *Let $\phi, \psi \in \mathrm{Dom}\ X_1 \cap \mathrm{Dom}\ X_2$. Then*

$$\mathbf{1}_{\mathbb{R}\times(0,\infty)} W_{|\phi\rangle\langle\psi|} \in H^\sigma_{1,2}(\mathbb{R}\times(0,\infty)).$$

Proof. We have, for $f \in \mathcal{C}^\infty_c(\mathbb{R}\times(0,\infty))$:

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} f(\xi_1,\xi_2)\overline{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)d\xi_1 d\xi_2 \right| &= 2\pi \left|\langle \phi | O(\xi_2 f(\xi_1,\xi_2))\psi\rangle_{\mathcal{H}}\right| \\
&\leq 2\pi \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \|O(\xi_2 f(\xi_1,\xi_2))\|_{\mathcal{B}_2(\mathcal{H})} \\
&\leq \sqrt{2\pi} \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \|\xi_2 f(\xi_1,\xi_2)\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi_1 d\xi_2}{|\xi_2|})} \\
&\leq \sqrt{2\pi} \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \xi_2 d\xi_1 d\xi_2)},
\end{aligned}$$

and for $x_1, x_2 \in \mathbb{R}$:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} (x_1\partial_1 f(\xi_1,\xi_2) + x_2\partial_2 f(\xi_1,\xi_2))\overline{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)d\xi_1 d\xi_2 \right| \\
&= 2\pi \left|\langle \phi | O(x_1\xi_2\partial_1 f(\xi_1,\xi_2) - x_2\xi_2\partial_2 f(\xi_1,\xi_2))\psi\rangle_{\mathcal{H}}\right| \\
&= 2\pi \left|\langle \phi | [x_1 U(X_1) + x_2 U(X_2), O(f)]\psi\rangle_{\mathcal{H}}\right| \\
&\leq \sqrt{2\pi} \|\phi\|_{\mathcal{H}} \|(x_1 U(X_1) + x_2 U(X_2))\psi\| \|f\|_{L^2_{\mathbb{C}}(\mathcal{G}^*; \frac{d\xi_1 d\xi_2}{|\xi_2|})}.
\end{aligned}$$

□

Under the same hypothesis we can show that $\mathbf{1}_{\mathbb{R}\times(-\infty,0)} W_{|\phi\rangle\langle\psi|}$ belongs to the Sobolev space $H^\sigma_{1,2}(\mathbb{R}\times(-\infty,0))$ which is defined in a way similar to (3.5). Note that the above result and the presence of $\sigma(\xi_1,\xi_2) = 2\pi|\xi_2|$ are consistent with the integrability properties of the gamma law, i.e. if $f(\xi_1,\xi_2) = |\xi_2| g(\xi_1)\gamma_\beta(\xi_2)$, $x_1 \in \mathbb{R}, \xi_2 > 0$, $g \neq 0$, then $f \in H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))$ if and only if $\beta > 0$.

4 Skorohod stochastic integration

The integration by parts formulas given in this section generalize the classical integration by parts formula (1.1) on $\mathbb{R}$. We define the expectation of X as

$$E[X] = \langle \Omega | X\Omega\rangle_{\mathcal{H}},$$

where $\Omega = \mathbf{1}_{\mathbb{R}_+}$ is the vacuum state in $\mathcal{H}$. The results of this section are in fact valid for any representation $\{M, B^-, B^+\}$ of sl_2 and any vector $\Omega \in \mathcal{H}$ such that $iP\Omega = Q\Omega$ and $M\Omega = \beta\Omega$.

Lemma 4.1 *Let $x = (x_1, x_2) \in \mathbb{R}^2$. We have*

$$E[D_x F] = \frac{1}{2} E\left[x_1 \{Q, F\} + x_2 \{P, F\}\right], \qquad F \in \mathcal{S}_{\mathcal{H}}.$$

Proof. We use the relation $iP\Omega = Q\Omega$:

$$\begin{aligned}
-E[[iP, F]] &= \langle \Omega, -iPF\Omega \rangle_{\mathcal{H}} - \langle \Omega, -iFP\Omega \rangle_{\mathcal{H}} \\
&= \langle iP\Omega, F\Omega \rangle_{\mathcal{H}} + \langle \Omega, FQ\Omega \rangle_{\mathcal{H}} \\
&= \langle Q\Omega, F\Omega \rangle_{\mathcal{H}} + \langle \Omega FQ\Omega \rangle_{\mathcal{H}} \\
&= \langle Q\Omega, F\Omega \rangle_{\mathcal{H}} + \langle \Omega, FQ\Omega \rangle_{\mathcal{H}} \\
&= E\left[\{Q, F\}\right],
\end{aligned}$$

$$\begin{aligned}
E[[iQ, F]] &= \langle \Omega, iQF\Omega \rangle_{\mathcal{H}} - \langle \Omega, iFQ\Omega \rangle_{\mathcal{H}} \\
&= -\langle iQ\Omega, F\Omega \rangle_{\mathcal{H}} + \langle \Omega, FP\Omega \rangle_{\mathcal{H}} \\
&= \langle P\Omega, F\Omega \rangle_{\mathcal{H}} + \langle \Omega, FP\Omega \rangle_{\mathcal{H}} \\
&= E\left[\{P, F\}\right],
\end{aligned}$$

and note that $E[[M, F]] = 0$. $\square$

Definition 4.2 *Fix $\alpha \in \mathbb{R}$ and let*

$$\delta(F \otimes x) = \frac{x_1}{2} \{Q + \alpha(M - \beta), F\} + \frac{x_2}{2} \{P, F\} - D_x F, \qquad F \in \mathcal{S}_{\mathcal{H}},$$

with $x = (x_1, x_2) \in \mathbb{R}^2$.

Note also that

$$\begin{aligned}
\delta(F \otimes x) = &\left(x_1 \frac{Q + iP + \alpha(M - \beta)}{2} + x_2 \frac{P - i(Q + \kappa M)}{2} \right) F \\
&+ F \left(x_1 \frac{Q - iP + \alpha(M - \beta)}{2} + x_2 \frac{P + i(Q + \kappa M)}{2} \right) \\
= \ & x_1 (B^+ F + F B^-) - i x_2 (B^+ F + F B^-) + \alpha \frac{x_1}{2} \{M - \beta, F\} - \frac{i}{2} x_2 \kappa [M, F] \\
= \ & (x_1 - i x_2)(B^+ F + F B^-) + \alpha \frac{x_1}{2} \{M - \beta, F\} - \frac{i}{2} x_2 \kappa [M, F].
\end{aligned}$$

The following Lemma shows that the divergence operator has expectation zero.

Lemma 4.3 *Let $x = (x_1, x_2) \in \mathbb{R}^2$. We have*

$$E[\delta(F \otimes x)] = 0, \qquad F \in \mathcal{S}_{\mathcal{H}}.$$

Proof. It suffices to apply Lemma 4.1 and to note that $\langle \Omega, M\Omega \rangle_{\mathcal{H}} = \beta$. □

Let for $F, U, V \in \mathcal{S}_{\mathcal{H}}$ and $x = (x_1, x_2) \in \mathbb{R}^2$:

$$U \overleftarrow{D}_x^F = (D_x U)F = -\frac{i}{2} x_1 [P, U]F + \frac{i}{2} x_2 [Q + \kappa M, U]F,$$

$$\overrightarrow{D}_x^F V = F D_x V = -\frac{i}{2} x_1 F[P, V] + \frac{i}{2} x_2 F[Q + \kappa M, V],$$

and define a two-sided gradient as

$$\begin{aligned} U \overleftrightarrow{D}_x^F V &= U \overleftarrow{D}_x^F V + U \overrightarrow{D}_x^F V \\ &= -\frac{i}{2} x_1 [P, U]FV - \frac{i}{2} x_1 UF[P, V] + \frac{i}{2} x_2 [Q + \kappa M, U]FV + \frac{i}{2} x_2 UF[Q + \kappa M, V]. \end{aligned}$$

Proposition 4.4 *Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $U, V \in \mathcal{S}_{\mathcal{H}}$. Assume that $x_1(Q + \alpha M) + x_2 P$ commutes with U and with V. We have*

$$E[U \overleftrightarrow{D}_x^F V] = E[U \delta(F \otimes x) V], \qquad F \in \mathcal{S}_{\mathcal{H}}.$$

Proof. From Lemma 4.3 we have

$$\begin{aligned} & E[U\delta(F \otimes x)V] \\ &= \frac{1}{2} E\left[U\left(\{x_1(Q + \alpha(M - \beta)) + x_2 P, F\} + i x_1 [P, F] - i x_2 [Q + \kappa M, F]\right) V\right] \\ &= \frac{1}{2} E[\{x_1(Q + \alpha(M - \beta)) + x_2 P, UFV\} + i x_1 U[P, F]V - i x_2 U[Q + \kappa M, F]V] \\ &= \frac{1}{2} E[\{x_1(Q + \alpha(M - \beta)) + x_2 P, UFV\} + i x_1 [P, UFV] \\ &\quad - i x_1 [P, U]FV] + E[-i x_1 UF[P, V] - i x_2 [Q + \kappa M, UFV] \\ &\quad + i x_2 [Q + \kappa M, U]FV + i x_2 UF[Q + \kappa M, V]] \\ &= E[\delta(UFV \otimes x)] + \frac{1}{2} E[-i x_1 [P, U]FV - i x_1 UF[P, V] \\ &\quad + i x_2 [Q + \kappa M, U]FV + i x_2 UF[Q + \kappa M, V]] \\ &= E[U \overleftrightarrow{D}_x^F V]. \end{aligned}$$

□

The closability of δ can be proved using the same argument as in Prop. 3.4. Next is a commutation relation between D and δ.

Proposition 4.5 *We have for* $\kappa = 0$ *and* $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$*:*

$$\begin{aligned}
&D_x\delta(F \otimes y) - \delta(D_x F \otimes y) \\
&= \frac{y_1 - iy_2}{2}(x_1\{M, F\} + ix_2[M, F]) + \alpha\frac{y_1}{2}(x_1\{Q, F\} + x_2\{P, F\}), \quad F \in \mathcal{S}_{\mathcal{H}}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
D_x\delta(F \otimes y) &= -\frac{i}{2}x_1[P, \delta(F \otimes y)] + \frac{i}{2}x_2[Q + \kappa M, \delta(F \otimes y)] \\
&= -\frac{i}{2}x_1[P, y_1(B^+F + FB^-) - iy_2(B^+F + FB^-) + \frac{y_1}{2}\alpha\{M - \beta, F\}] \\
&\quad +\frac{i}{2}x_2[Q + \kappa M, y_1(B^+F + FB^-) - iy_2(B^+F + FB^-) + \frac{y_1}{2}\alpha\{M - \beta, F\}] \\
&= \delta(D_x F \otimes y) - \frac{i}{2}x_1(y_1[P, B^+]F + y_1F[P, B^-] - iy_2[P, B^+]F - iy_2F[P, B^-] \\
&\quad +\frac{y_1}{2}\alpha[P, M]F + \frac{y_1}{2}\alpha F[P, M]) + \frac{i}{2}x_2(y_1[Q + \kappa M, B^+]F + y_1F[Q + \kappa M, B^-] \\
&\quad -iy_2[Q + \kappa M, B^+]F - iy_2F[Q + \kappa M, B^-] + \frac{y_1}{2}\alpha[Q, M]F + \frac{y_1}{2}\alpha F[Q, M]) \\
&= \delta(D_x F \otimes y) - \frac{i}{2}x_1(y_1\{iM, F\} - iy_2\{iM, F\} + \frac{y_1}{2}\alpha\{2iQ, F\}) \\
&\quad +\frac{i}{2}x_2(y_1[M, F] - iy_2[M, F] + iy_1\alpha\{P, F\}) \\
&= \delta(D_x F \otimes y) + \frac{1}{2}x_1y_1\{M + \alpha Q, F\} + x_2y_1\frac{i}{2}[M, F] + \frac{1}{2}x_2y_1\alpha\{P, F\} \\
&\quad -\frac{i}{2}x_1y_2\{M, F\} + \frac{1}{2}x_2y_2[M, F].
\end{aligned}$$

□

Proposition 4.6 *We have for* $F, G \in \mathcal{S}_{\mathcal{H}}$*:*

$$\delta(GF \otimes x) = G\delta(F) - G\overleftarrow{D}_F - \frac{x_1}{2}[Q + \alpha M, G]F - \frac{x_2}{2}[P, G]F,$$

and

$$\delta(FG \otimes x) = \delta(F)G - \overrightarrow{D}_F G - \frac{x_1}{2}F[Q + \alpha M, G] - \frac{x_2}{2}F[P, G].$$

Proof. We have

$$\delta(GF \otimes x) = \frac{x_1}{2}(Q + iP + \alpha(M - \beta))GF + \frac{x_1}{2}GF(Q - iP + \alpha(M - \beta))$$

$$
\begin{aligned}
& +\frac{x_2}{2}(P-iQ)GF+\frac{x_2}{2}GF(P+iQ) \\
= \;& \frac{x_1}{2}G(Q+iP+\alpha(M-\beta))F+\frac{x_1}{2}GF(Q-iP+\alpha M-\alpha/2) \\
& +\frac{x_2}{2}G(P-iQ)F+\frac{x_2}{2}GF(P+iQ) \\
& +\frac{i}{2}x_1[P,G]F-\frac{i}{2}x_2[Q,G]F-\frac{x_1}{2}[Q+\alpha M,G]F-\frac{x_2}{2}[P,G]F.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\delta(FG\otimes x) &= \frac{x_1}{2}(Q+iP+\alpha(M-\beta))FG+\frac{x_1}{2}FG(Q-iP+\alpha(M-\beta)) \\
& +\frac{x_2}{2}(P-iQ)FG+\frac{x_2}{2}FG(P+iQ) \\
&= \frac{x_1}{2}(Q+iP+\alpha(M-\beta))FG+\frac{x_1}{2}F(Q-iP+\alpha M-\alpha/2)G \\
& +\frac{x_2}{2}(P-iQ)FG+\frac{x_2}{2}F(P+iQ)G \\
& +\frac{i}{2}x_1F[P,G]-\frac{i}{2}x_2F[Q,G]-\frac{x_1}{2}F[Q+\alpha M,G]-\frac{x_2}{2}F[P,G].
\end{aligned}
$$

$\square$

5 Relation to the commutative case

Let $\mathbf{q}=a_x^-+a_x^+$, where $a_x^-=\frac{\partial}{\partial x}$ and $a_x^+=x-\frac{\partial}{\partial x}$, i.e. q is multiplication by x, and $\mathbf{p}=i(a_x^--a_x^+)$, with $[\mathbf{p},\mathbf{q}]=2iI$. When $\beta=1/2$, writing $\tau=\frac{1}{2}x^2$, we have the relations

$$
\tilde{a}_\tau^\circ=\frac{1}{2}a_x^+a_x^-,\quad \tilde{a}_\tau^-=\frac{1}{2}\mathbf{q}a_x^-,\quad \tilde{a}_\tau^+=\frac{1}{2}a_x^+\mathbf{q},
$$

i.e.

$$
\tilde{a}_\tau^\circ f(\tau)=\frac{1}{2}a_x^+a_x^-f\left(\frac{x^2}{2}\right),\quad \tilde{a}_\tau^- f(\tau)=\frac{1}{2}\mathbf{q}a_x^-f\left(\frac{x^2}{2}\right),\quad \tilde{a}_\tau^+ f(\tau)=\frac{1}{2}a_x^+\mathbf{q}f\left(\frac{x^2}{2}\right).
$$

These relations have been exploited in various contexts, see e.g. [8], [9], [10]. In [10], these relations have been used to construct a Malliavin calculus on Poisson space directly from the Gaussian case. In [9] they are used to prove logarithmic Sobolev inequalities for the exponential measure. From now on we take $\beta=1/2$. The representation $\{M,B^-,B^+\}$ of sl_2 can be constructed as

$$
M=\frac{1}{2}+2\tilde{a}_\tau^\circ=\frac{a_x^-a_x^++a_x^+a_x^-}{2}=\frac{\mathbf{p}^2+\mathbf{q}^2}{4},
$$

$$B^- = \tilde{a}_\tau^- - \tilde{a}_\tau^\circ = \frac{1}{2}(a_x^-)^2,$$
$$B^+ = \tilde{a}_\tau^+ - \tilde{a}_\tau^\circ = \frac{1}{2}(a_x^+)^2,$$

In fact, letting

$$Q = B^- + B^+ = \frac{1}{2}((a_x^-)^2 + (a_x^+)^2) = \frac{\mathbf{p}^2 - \mathbf{q}^2}{4},$$
$$P = i(B^- - B^+) = \frac{i}{2}((a_x^-)^2 - (a_x^+)^2) = \frac{\mathbf{pq} + \mathbf{qp}}{4},$$

we have

$$[P, Q] = 2iM, \qquad [P, M] = 2iQ, \qquad [Q, M] = -2iP.$$

We also have

$$Q + \alpha M = \frac{\alpha + 1}{2}\frac{\mathbf{p}^2}{2} + \frac{\alpha - 1}{2}\frac{\mathbf{q}^2}{2},$$

and

$$M + \alpha Q = \left(\frac{\alpha + 1}{2}\right)\frac{\mathbf{p}^2}{2} + \left(\frac{1 - \alpha}{2}\right)\frac{\mathbf{q}^2}{2}.$$

The commutative case is obtained with $\alpha = 1$ when considering functionals of $\frac{\mathbf{q}^2}{2}$ only, and with $\alpha = -1$ when considering functionals of $\frac{\mathbf{p}^2}{2}$ only. Other probability laws can be considered for different values of α. The law of $Q + \alpha M$ has been determined in [1], depending on the value of α. In particular when $|\alpha| = 1$,

$$Q + M = B^- + B^+ + M = \frac{\mathbf{p}^2}{2}, \quad Q - M = B^- + B^+ - M = -\frac{\mathbf{q}^2}{2},$$

i.e. $Q + M$ and $M - Q$ have gamma laws. For $|\alpha| < 1$, $Q + \alpha M$ has an absolutely continuous law and when $|\alpha| > 1$, $Q + \alpha M$ has a geometric law with parameter c^2 supported by

$$\{-1/2 - \mathrm{sgn}(\alpha)(c - 1/c)k \ : \ k \in \mathbb{N}\},$$

with $c = \alpha\mathrm{sgn}(\alpha) - \sqrt{\alpha^2 - 1}$. In particular the analogs of the classical integration by parts formula (1.1) are written as

$$E[D_{(1,0)}F] = \frac{1}{2}E\left[\left\{\frac{\mathbf{p}^2}{2}, F\right\} - F\right],$$

for $\alpha = 1$, and

$$E[D_{(1,0)}F] = \frac{1}{2}E\left[F - \left\{\frac{\mathbf{q}^2}{2}, F\right\}\right],$$

for $\alpha = -1$.

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